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HYPERPLANE RESTRICTIONS OF INDECOMPOSABLE n-DIMENSIONAL PERSISTENCE MODULES

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(*communicated by Peter Bubenik*)

Abstract

Understanding the structure of indecomposable n-dimensional persistence modules is a difficult problem, yet is foundational for studying multipersistence. To this end, Buchet and Escolar showed that any finitely presented rectangular $(n-1)$-dimensional persistence module with finite support is a hyperplane restriction of an indecomposable n-dimensional persistence module. We extend this result to the following: If M is any finitely presented $(n-1)$-dimensional persistence module with finite support, then there exists an indecomposable n-dimensional persistence module M' such that M is the restriction of M' to a hyperplane. We also show that any finite zigzag persistence module is the restriction of some indecomposable 3-dimensional persistence module to a path.

1. Introduction

Understanding the structure of multiparameter persistence modules is an important foundational problem in the area of persistent homology. Every finite multiparameter persistence module has a decomposition into indecomposable multiparameter persistence modules. Thus, to understand the structure of all multiparameter persistence modules, it suffices to understand the structure of the indecomposables. But the structure of the indecomposable n-dimensional persistence modules is rich when $n > 1$; such modules have no complete discrete invariant [**CZ**]. One might still hope that finite indecomposable nD modules would have simple 'substructures' in some sense. For example, it would be convenient if the hyperplane restrictions of such modules are in some restricted set of $(n-1)$-dimensional persistence modules. Buchet and Escolar proved that this is not true when $n = 2$; any finite 1-dimensional persistence module is a hyperplane restriction of an indecomposable 2D persistence module (Theorem 1.1 of [**BE2**]). Their work was motivated largely by [**LW**], which introduced the software RIVET — an important tool for studying 2D persistence

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modules. In particular, one of RIVET's functions is the ability to find the barcode of any path in a finite 2D persistence module.

Buchet and Escolar further found that any finite rectangular nD persistence module M is a hyperplane restriction of some indecomposable $(n+1)$ dimensional persistence module M' (Theorem 7.7 of [**BE2**]). To prove this result, Buchet and Escolar provided three constructions for M'. In this paper, we extend their result to further shed light on how rich the structure of $(n+1)$-dimensional persistence modules can be. In particular, our primary results are as follows:

- **Theorem 3.1** Let M be any finite n-dimensional persistence module. Then there exists an indecomposable $(n+1)$-dimensional persistence module M' such that M is the restriction of M' to a hyperplane.[1]
- **Corollary 3.2** Let M be a finite zigzag persistence module. Then there exists an indecomposable 3-dimensional persistence module M' such that M is the restriction of M' to a path.

The structure of the paper is as follows: Sections 2.1 and 2.2 cover the basic definitions needed from multipersistence, as well as an important lemma about the homomorphisms between interval multiparameter persistence modules. In Section 2.3, we discuss the classification of nD persistence modules and state the constructions/proof for Theorem 7.7 of [**BE2**], as our results rely heavily on these. In Section 3, we provide constructions and proofs for our results stated above and discuss potential future directions for research.

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2. Background and previous results

2.1. Initial definitions

Let $\{e_j\}_{j=1}^n$ denote the standard basis of $\mathbb{N}^n$. An **n-dimensional persistence module** M over a field K is an assignment of a K-vector space M_α to each $\alpha \in \mathbb{N}^n$ and a homomorphism

$$\phi^M_{\alpha,\alpha+e_j} \colon M_\alpha \to M_{\alpha+e_j} \text{ for each } \alpha \in \mathbb{N}^n \text{ and each } j \in [1,n]$$

such that the resulting grid-like diagram commutes. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ are points in $\mathbb{N}^n$ such that $\alpha_j \leqslant \beta_j$ for all j, then we write $\alpha \leqslant \beta$. This yields a partial ordering on $\mathbb{N}^n$. Notice that the definition of an n-dimensional persistence module M implies that there is a well-defined map $\phi^M_{\alpha,\beta} \colon M_\alpha \to M_\beta$ whenever $\alpha \leqslant \beta$.

A **homomorphism** $f \colon M \to N$ between n-dimensional persistence modules M and N is a collection of homomorphisms $f_\alpha \colon M_\alpha \to N_\alpha$ such that

$$\phi^N_{\alpha,\alpha+e_j} \circ f_\alpha = f_{\alpha+e_j} \circ \phi^M_{\alpha,\alpha+e_j} \text{ for all } \alpha \in \mathbb{N}^n \text{ and all } j \in [1,n].$$

Let $V = \{v_i\}_i$ be a set of homogeneous elements in M (meaning that for each i, $v_i \in M_{\alpha_i}$ for some $\alpha_i \in \mathbb{N}^n$). We say that V is a **generating set** of M if for all $\beta \in \mathbb{N}^n$

[1] Buchet and Escolar separately proved this result recently, using a similar method [**BE3**].

and $v \in M_\beta$, there exist coefficients $c_i \in K$ such that $v = \bigoplus_i c_i \phi^M_{\alpha_i,\beta}(v_i)$. If M has a generating set V with finite cardinality, then we say that M is **finitely generated**. All of the nD persistence modules in this paper are assumed to be **finite**, meaning they are finitely presented and have finite support (where the **support** of M, denoted by $\mathrm{Supp}(M)$, is defined to be $\{\alpha \in \mathbb{N}^n | M_\alpha \neq 0\}$).

There is a bijective correspondence between finitely generated n-dimensional persistence modules M and finitely generated multigraded modules over the polynomial ring $K[x_1, x_2, \cdots, x_n]$; given a finitely generated n-dimensional persistence module M, define the finitely generated multigraded module

$$M' := \bigoplus_{\alpha \in \mathbb{N}^n} M_\alpha \text{ over } K[x_1, x_2, \cdots, x_n],$$

whose $K[x_1, x_2, \cdots, x_n]$-action is given by $x_j \cdot v \colon = \phi^M_{\alpha,\alpha+e_j}(v)$ for all $v \in M_\alpha$ and $j \in [1, n]$. This bijection is in fact an equivalence of categories [**CZ**], which allows us to utilize definitions and results from commutative algebra.

The **direct sum** of two n-dimensional persistence modules M and N is the n-dimensional persistence module $M \bigoplus N$, whose vector spaces are given entrywise by $(M \bigoplus N)_\alpha := M_\alpha \bigoplus N_\alpha$ and whose maps are given by $\phi^{M \bigoplus N}_{\alpha,\beta} = \phi^M_{\alpha,\beta} \bigoplus \phi^N_{\alpha,\beta}$ for all $\alpha \leqslant \beta \in \mathbb{N}^n$. An n-dimensional persistence module M is **indecomposable** if M cannot be written as $M = M_1 \bigoplus M_2$ where M_1, M_2 are both nonzero n-dimensional persistence modules. By the Krull–Schmidt Theorem, any finite nD persistence module M has an **indecomposable decomposition**, meaning there exist indecomposable n-dimensional persistence modules I_i such that $M = \bigoplus_{i=1}^m I_i$. The following lemma will be important for proving our main results:

Lemma 2.1 (Corollary 4.8 of [**ASS**])**.** *Let M be a finite n-dimensional persistence module. Then* $\mathrm{End}(M)$ *is local if and only if M is indecomposable.*

2.2. Interval n-dimensional persistence modules

Let $M = \bigoplus_{i=1}^m I_i$ be the decomposition of an n-dimensional persistence module M into its indecomposable summands. Then M is called an **interval** n-dimensional persistence module if $\dim((I_i)_\alpha) \in \{0, 1\}$ for all $\alpha \in \mathbb{N}^n$ and

$$\phi^{I_i}_{\alpha,\beta} = \begin{cases} \mathrm{Id}_K & \text{if } \alpha \leqslant \beta \text{ and } (I_i)_\alpha = (I_i)_\beta = K, \\ 0 & \text{otherwise.} \end{cases}$$

Interval nD persistence modules are some of the simplest multiparameter persistence modules, and they will be fundamental to proving the results in Sections 2.3 and 3. Thus, we use this section to explore the properties of such modules. For additional properties of interval multiparameter persistence modules, see [**ABENY, AENY, DX**].

Remark 2.2. Let I be an indecomposable n-dimensional interval persistence module. We claim that $\alpha, \beta \in \mathrm{Supp}(I)$ implies $\gamma \in \mathrm{Supp}(I)$ whenever $\alpha \leqslant \gamma \leqslant \beta$. Indeed, $\alpha, \beta \in \mathrm{Supp}(I)$ implies that $\phi^I_{\alpha,\beta} = \mathrm{Id}_K$ by above. Let $\gamma \in \mathbb{N}^n$ such that $\alpha \leqslant \gamma \leqslant \beta$. Then $\mathrm{Id}_K = \phi^I_{\alpha,\beta} = \phi^I_{\gamma,\beta} \circ \phi^I_{\alpha,\gamma}$ since the ϕ^I maps must commute with each other.

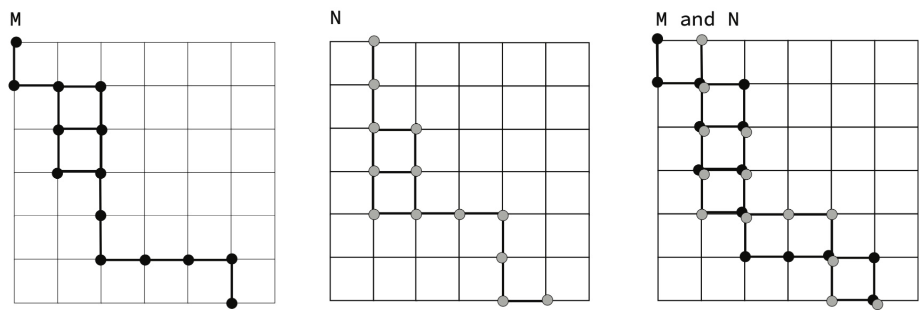

Figure 1: The first two images in this figure are 2-dimensional interval persistence modules M and N. In each module, each dot at $\alpha \in \mathbb{N}^2$ represents a basis element of the α-vector space. Thick edges represent the identity map on colored components (i.e. gray basis elements map to gray basis elements while black basis elements map to black basis elements), while the thinner edges represent the zero map. The third image shows M and N overlaid, which makes it clear that there are three components in $\mathrm{Supp}(M) \cap \mathrm{Supp}(N)$. Of these components, only the rightmost is (M, N) viable. The leftmost component C is (M, N) non-viable because $(1, 6) \in \mathrm{Supp}(N) \setminus \mathrm{Supp}(M)$ and $(1, 5) \in C$ have the property that $(1, 5) < (1, 6)$. Meanwhile, the middle component C is (M, N) non-viable because $(3, 1) \in \mathrm{Supp}(M) \setminus \mathrm{Supp}(N)$ and $(4, 1) \in C$ have the property that $(3, 1) < (4, 1)$.

Thus $\phi^I_{\alpha,\gamma} \neq 0$, implying (by the definition of interval nD persistence modules) that $\gamma \in \mathrm{Supp}(I)$.

An indecomposable nD interval persistence module I is **rectangular** if its support is an m-dimensional box for some $m \leqslant n$. Such modules are denoted by $K[\alpha, \beta]$ where $\alpha, \beta \in \mathrm{Supp}(I)$ such that $\alpha \leqslant \gamma \leqslant \beta$ for all $\gamma \in \mathrm{Supp}(I)$. An n-dimensional interval persistence module M is **rectangular** if each of its indecomposable summands is.

Suppose M and N are finite n-dimensional interval persistence modules. Let C_i be an edge-connected component of $\mathrm{Supp}(M) \cap \mathrm{Supp}(N)$. Call C_i $(\mathbf{M}, \mathbf{N})$ **non-viable** if there exists $\beta \in \mathrm{Supp}(M) \setminus \mathrm{Supp}(N)$ and $\alpha \in C_i$ such that $\beta < \alpha$, or if there exists $\beta \in \mathrm{Supp}(N) \setminus \mathrm{Supp}(M)$ and $\alpha \in C_i$ such that $\alpha < \beta$. Otherwise, refer to C_i as $(\mathbf{M}, \mathbf{N})$ **viable**. For an example of these concepts, see Fig. 1.

Lemma 2.3. *Suppose M and N are finite n-dimensional indecomposable interval persistence modules. Let m denote the number of (M, N) viable components of the intersection Supp$(M) \cap \mathrm{Supp}(N)$. Then*

$$\mathrm{Hom}(M, N) \cong \begin{cases} K^m & \text{if } m > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\{C_i\}_i$ denote the set of (possibly (M, N) non-viable) components of the intersection $\mathrm{Supp}(M) \cap \mathrm{Supp}(N)$ and suppose $f \in \mathrm{Hom}(M, N)$. Then $f|_\alpha$ is trivially zero whenever $\alpha \notin \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$. On the other hand, $f|_\alpha = c_\alpha \,\mathrm{Id}_K$ for some $c_\alpha \in K$ whenever $\alpha \in \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$, as $M_\alpha = N_\alpha = K$. Furthermore, because f must commute with the maps ϕ^M and ϕ^N, it must be the case that $c_\alpha = c_\beta$ whenever

α, β are in the same component of $\text{Supp}(M) \cap \text{Supp}(N)$. That is, f is fully determined by $\{c_i \in K\}_i$, where $c_i := c_\alpha$ if $\alpha \in C_i$.

We claim that if C_i is (M, N) non-viable, then c_i must be zero. That is, for all $\alpha \in C_i$, $f|_\alpha$ is the zero map. Suppose there exists $\beta \in \text{Supp}(M) \setminus \text{Supp}(N)$ and $\alpha \in C_i$ such that $\beta < \alpha$. Then $\beta \notin \text{Supp}(N)$ implies $f|_\beta$ is the zero map. Because f must commute with the maps ϕ^M, ϕ^N, it must be the case that $f|_\alpha \circ \phi^M_{\beta,\alpha} = \phi^N_{\beta,\alpha} \circ f|_\beta = 0$. We know that $\phi^M_{\beta,\alpha} = \text{Id}_K$ since M is an indecomposable interval n-dimensional persistence module. Thus it must be the case that $f|_\alpha = 0$. On the other hand, suppose there exists $\beta \in \text{Supp}(N) \setminus \text{Supp}(M)$ and $\alpha \in C_i$ such that $\alpha < \beta$. Then $M_\beta = 0$ implies $f|_\beta = 0$ and commutativity of the f, ϕ^M, and ϕ^N maps yields that $f|_\alpha = f|_\beta = 0$.

Alternatively, if C_i is (M, N) viable, then we claim that the f, ϕ^M, and ϕ^N maps are able to commute even if c_i is nonzero. Let $\alpha \in C_i$. Notice that $\alpha + e_j$ must be an element of C_i or $\mathbb{N}^n \setminus \text{Supp}(N)$ since C_i is (M, N) viable. If $\alpha + e_j \in C_i$ then $f|_{\alpha+ej} \circ \phi^M_{\alpha,\alpha+e_j} = c_i \text{Id}_K = \phi^N_{\alpha,\alpha+e_j} \circ f|_\alpha$, as desired. On the other hand, if $\alpha + e_j \in \mathbb{N}^n \setminus \text{Supp}(N)$, then $f|_{\alpha+ej} \circ \phi^M_{\alpha,\alpha+e_j}$ and $\phi^N_{\alpha,\alpha+e_j} \circ f|_\alpha$ are maps with codomain equal to zero, so both compositions are trivially the zero map.

Thus c_i may be nonzero whenever C_i is an (M, N) viable component, implying that any $f \in \text{Hom}(M, N)$ is determined by $\{c_i \in K | C_i \text{ is } (M, N) \text{ viable and } f|_\alpha = c_i \text{Id}_K \text{ for all } \alpha \in C_i\}$. $\square$

We will primarily work with pairs of indecomposable interval n-dimensional persistence modules I and J such that $\text{Hom}(I, J) \cong K$ or 0. In the setting that $\text{Hom}(I, J) \cong K$, we use the notation f_I^J to indicate a fixed natural nonzero homomorphism from I to J. Note that it is possible that $f_J^L \circ f_I^J = 0$ (consider the case where $I \cap L = \emptyset$).

2.3. The structure of indecomposables

Recall that every finite nD persistence module has an indecomposable decomposition. Thus, to understand the structure all n-dimensional persistence modules, it suffices to understand the set of indecomposable n-dimensional persistence modules. When $n = 1$, the indecomposable persistence modules are of a simple form (see [**G**]); M is indecomposable if and only if there is an interval $[\alpha, \beta]$ (with β possibly infinite) such that

$$\dim(M_\gamma) = \begin{cases} 1 & \text{if } \alpha \leqslant \gamma \leqslant \beta, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\phi^M_{\gamma,\delta} = \begin{cases} \text{Id}_K & \text{if } \alpha \leqslant \gamma \leqslant \delta \leqslant \beta, \\ 0 & \text{otherwise.} \end{cases}$$

That is, every indecomposable 1-dimensional persistence module is rectangular. Unfortunately, for $n > 1$, the full set of indecomposable n-dimensional persistence modules do not have such a simple structure.

Theorem 2.4 ([**CZ**])**.** *There does not exist a complete discrete invariant for the set of indecomposable n-dimensional persistence modules whenever $n > 1$.*

In other words, there does not exist a finitely parameterized invariant that can distinguish between all indecomposable n-dimensional persistence modules. This was proved in [**CZ**] by showing the existence of a continuously parameterized family of non-isomorphic indecomposable n-dimensional persistence modules. Buchet and Escolar have found other continuously parameterized families of non-isomorphic indecomposable nD persistence modules, such as the family shown in Fig. 2 [**BE1**]. The following theorem gives insight to how complicated the structure of indecomposable $(n+1)$-dimensional persistence modules can be.

$$
\begin{tikzcd}
K^d \arrow[r, "\begin{bmatrix} I \\ 0 \end{bmatrix}"] & K^{2d} \arrow[r, "\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}"] & K^{2d} \arrow[r, "\begin{bmatrix} I & 0 \end{bmatrix}"] & K^d \arrow[r] & 0 \\
0 \arrow[u] \arrow[r] & K^d \arrow[u, "\begin{bmatrix} I \\ J_d(\lambda) \end{bmatrix}"] \arrow[r, "\begin{bmatrix} 0 \\ I \end{bmatrix}"'] & K^{2d} \arrow[u, "\begin{bmatrix} I & I \\ I & J_d(\lambda) \end{bmatrix}"] \arrow[r, "\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}"'] & K^{2d} \arrow[u, "\begin{bmatrix} I & I \end{bmatrix}"] \arrow[r, "\begin{bmatrix} 0 & I \end{bmatrix}"'] & K^d \arrow[u]
\end{tikzcd}
$$

Figure 2: For each choice of $d \in \mathbb{N}$, shown is a continuous family of non-isomorphic indecomposable 2-dimensional persistence modules, where continuity comes from allowing $\lambda \in K = \mathbb{R}$ to vary. The maps ϕ^M are written in matrix form with respect to the bases $K[1,4]^d \bigoplus K[2,3]^d$ of layer 0 and $K[0,3]^d \bigoplus K[1,2]^d$ of layer 1. The matrix denoted by $J_d(\lambda)$ is the $d \times d$ Jordan block with λ along its diagonal. This family was introduced in [**BE1**].

Theorem 2.5 ([**BE2**]). *Let $M = \bigoplus_{i=1}^{m} K[\alpha_i, \beta_i]$ be a finite n-dimensional rectangular persistence module. Then there exists an indecomposable finite $(n+1)$-dimensional persistence module M' such that M is a hyperplane restriction of M'.*

In fact, Buchet and Escolar provided 3 methods for constructing such M' from M. We repeat these constructions and the proof of indecomposability, as our results rely heavily on such. Before we begin, we state two key facts which will allow us to easily define the maps $\phi^{M'}$. First, let M, N be finitely generated n-dimensional persistence modules and suppose $M = \bigoplus_{i=1}^{m} I_i^M$ and $N = \bigoplus_{j=1}^{p} I_j^N$ are the indecomposable summands of M and N. Then any homomorphism $F\colon M \to N$ can be summarized by a collection of maps $F_i^j\colon I_i^M \to I_j^N$. Second, Lemma 2.3 implies the following for n-dimensional rectangular persistence modules:

$$\operatorname{Hom}(K[\alpha_1, \beta_1], K[\alpha_2, \beta_2]) \cong \begin{cases} K & \text{if } \alpha_2 \leqslant \alpha_1 \leqslant \beta_2 \leqslant \beta_1, \\ 0 & \text{otherwise.} \end{cases}$$

Let the j^{th} **layer** of an n-dimensional persistence module M be the restriction of M to the plane with final coordinate j. We call the first construction from [**BE2**] the **main rectangular construction**, which is as follows: Layer 3 of M' is $M = \bigoplus_{i=1}^{m} K[\alpha_i, \beta_i] =: \bigoplus_{i=1}^{m} I_{3i}$. In layer 2, place $\bigoplus_{i=1}^{m} K[\alpha_i, \beta_i'] =: \bigoplus_{i=1}^{m} I_{2i}$ where $\{\beta_i'\}_i$ are chosen such that $\beta_i \leqslant \beta'_i < \beta'_{i+1}$ for all i and $\frac{\alpha_i+\beta'_i}{2} \leqslant \beta'_j$ for all i,j (where our partial ordering on $\mathbb{N}^n$ is extended to all of $(\frac{1}{2}\mathbb{N})^n$ in the obvious way). The maps $\phi^{M'}$ between layer 2 and layer 3 may be summarized by the direct sum

$$\phi_2^{M'} = \bigoplus_{i=1}^{m} f_{2i}^{3i},$$

where f_{ri}^{sj} is shorthand for $f_{I_{ri}}^{I_{sj}}$ (c.f. notation at the end of Section 2.2).

Let α^j denote the j^{th} coordinate of $\alpha \in \mathbb{N}^n$. Define $\mu \in \mathbb{N}^n$ by $\mu^j = \max_i(\alpha_i + \beta'_i)^j$. In layer 1 of M' place $\bigoplus_{i=1}^{m} K[\alpha'_i, \beta'_i] =: \bigoplus_{i=1}^{m} I_{1i}$ where $\alpha'_i := \mu - \beta'_i$. Notice that this causes $\alpha_i \leqslant \alpha'_i < \beta'_j$ for all i,j. Combining this with the fact that $\beta'_i < \beta'_{i+1}$ implies that $\alpha'_{i+1} < \alpha'_i < \beta'_i < \beta'_{i+1}$ for all i. We thus have strict inclusions $\mathrm{Supp}(I_{1i}) \subsetneq \mathrm{Supp}(I_{1(i+1)})$ for all i, a detail that will be critical later. The maps $\phi^{M'}$ between layer 1 and layer 2 are defined by

$$\phi_1^{M'} = \bigoplus_{i=1}^{m} f_{1i}^{2i}.$$

Layer 0 of M' is $K[\max_i(\alpha'_i), \max_i(\beta'_i)] = I_{01}$, and the maps between layer 0 and layer 1 are given by

$$\phi_0^{M'} = \bigoplus_{i=1}^{m} f_{01}^{1i}.$$

The second construction from [**BE2**], which we call the **dual rectangular construction**, is as follows: Layer 0 of M' is $M = \bigoplus_{i=1}^{m} K[\alpha_i, \beta_i] =: \bigoplus_{i=1}^{m} I_{0i}$. In layer 1, place $\bigoplus_{i=1}^{m} K[\alpha'_i, \beta_i] = \bigoplus_{i=1}^{m} I_{1i}$ where $\{\alpha'_i\}_i$ are chosen to be distinct elements of $\mathbb{N}^n$ such that $\alpha'_i \leqslant \alpha_i$ for all i and $\alpha'_i \leqslant \frac{\alpha'_j+\beta_j}{2}$ for all i,j (where our partial ordering on $\mathbb{N}^n$ has again been extended to a partial ordering on $(\frac{1}{2}\mathbb{N})^n$). Distinctness implies that, without loss of generality, $\alpha'_i < \alpha'_{i+1}$ for all i.

Remark 2.6. Notice that such restrictions on $\{\alpha'_i\}_i$ may force some α'_i to be negative, which is not allowed. In this case, simply shift M' to the right until all $\{\alpha'_i\}_i$ are non-negative. Due to the triviality of such shifting issues, we will not mention them in the constructions given henceforth.

The maps $\phi^{M'}$ between layer 0 and layer 1 are given by

$$\phi_0^{M'} = \bigoplus_{i=1}^{m} f_{0i}^{1i}.$$

Define $\mu \in \mathbb{N}^n$ by $\mu^j := \min_i(\alpha'_i + \beta_i)^j$. Layer 2 is given by $\bigoplus_{i=1}^{m} K[\alpha'_i, \beta'_i] =: \bigoplus_{i=1}^{m} I_{2i}$, where $\beta'_i := \mu - \alpha'_i$. This implies that $\alpha'_i < \alpha'_{i+1} < \beta'_{i+1} < \beta'_i$ for all i since $\alpha'_i < \alpha'_{i+1}$. In particular, we have that $\mathrm{Supp}(I_{2(i+1)}) \subsetneq \mathrm{Supp}(I_{2i})$ for all i. The maps $\phi^{M'}$ between

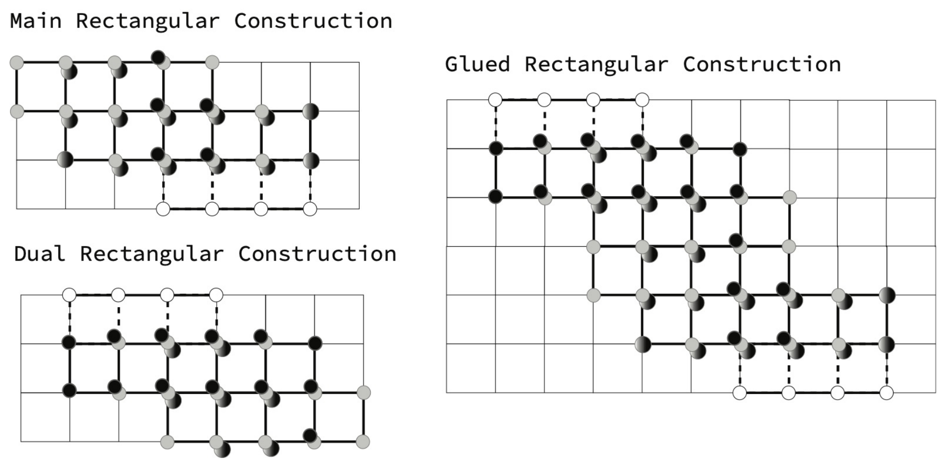

Figure 3: This is an example of the three constructions given in [**BE2**] for creating an indecomposable 2-dimensional persistence module M' from a finite 1-dimensional persistence module M such that M is a hyperplane restriction of M'. In particular, these 2-dimensional persistence modules come from the 1-dimensional persistence module $M = K[3,3] \bigoplus K[0,4] \bigoplus K[1,2]$. In the main rectangular construction, dashed edges represent the map which sends the white basis vector to $(1,1,1), (1,1)$, or 1, as applicable. In the dual rectangular construction, dashed edges represent the map which sends each of the gray, black, and ombre basis vectors to the white basis vector.

layer 1 and layer 2 are summarized by

$$\phi_1^{M'} = \bigoplus_{i=1}^{m} f_{1i}^{2i}.$$

Layer 3 of M' is $K[\min_i(\alpha_i'), \min_i(\beta_i')] =: I_{31}$. The maps from layer 2 to layer 3 are given by

$$\phi_2^{M'} = \bigoplus_{i=1}^{m} f_{2i}^{31}.$$

The third construction from [**BE2**], which we call the **glued rectangular construction**, is to place M in layer 3, then place the main rectangular construction for M in layers $0-2$ and the dual rectangular construction for M in layers $4-6$. For an example of these three constructions, see Figure 3.

Proof. We prove that the main rectangular construction results in a finite indecomposable $(n+1)$-dimensional persistence module M' when $M = \bigoplus_{i=1}^{m} K[\alpha_i, \beta_i]$ is a finite n-dimensional (rectangular) persistence module. The proofs involving the other constructions are similar.

Finiteness of M' is clear. By Lemma 2.1, to show that M' is indecomposable, it suffices to prove that $\mathrm{End}(M') \cong K$. Let $f \in \mathrm{End}(M')$. Then f is equivalent to a set of endomorphisms f_j on each layer j of M' such that the maps $\phi_i^{M'}$ and f_j commute.

In particular, we need $\phi^{M'}_{j-1} \circ f_{j-1} = f_j \circ \phi^{M'}_{j-1}$ for all j. By Lemma 2.3, $f_0 = cf^{01}_{01}$ for some $c \in K$. We wish to show that

$$f_j = \bigoplus_{i=1}^{m} f^{ji}_{ji}$$

for all $j \geqslant 1$, which implies that f is determined fully by the choice of $c \in K$.

Note that

$$\phi^{M'}_0 \circ f_0 = \bigoplus_{i=1}^{m} cf^{1i}_{01}.$$

By Lemma 2.3,

$$\mathrm{Hom}(I_{1i}, I_{1j}) \cong \begin{cases} K & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

as $I_{1i} = K[\alpha'_i, \beta'_i]$ and $\alpha'_{i+1} < \alpha'_i < \beta'_i < \beta'_{i+1}$ for all i. As such,

$$f_1 = \bigoplus_{i=1}^{m} c^i_i f^{1i}_{1i}$$

for some constants $c^i_i \in K$, which implies that

$$f_1 \circ \phi^{M'}_0 = \bigoplus_{i=1}^{m} c^i_i f^{1i}_{01}.$$

Since $f_1 \circ \phi^{M'}_0 = \phi^{M'}_0 \circ f_0$, we may conclude that

$$f_1 = \bigoplus_{i=1}^{m} cf^{1i}_{1i},$$

as claimed.

Now we wish to show that

$$f_2 = \bigoplus_{i=1}^{m} cf^{2i}_{2i}.$$

By Lemma 2.3, $f_2 = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m} c^j_i g^{2j}_{2i}$, where

$$g^{2j}_{2i} = \begin{cases} f^{2j}_{2i} & \text{if } \mathrm{Hom}(I_{2i}, I_{2j}) \cong K, \\ 0 & \text{if } \mathrm{Hom}(I_{2i}, I_{2j}) = 0 \end{cases}$$

and each $c^j_i \in K$. The same lemma also guarantees that $g^{2j}_{2j} = f^{2j}_{2j}$ for all j. Then

$$\phi^{M'}_1 \circ f_1 = \bigoplus_{i=1}^{m} cf^{2i}_{1i} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{m} c^j_i g^{2j}_{2i} \circ f^{2i}_{1i} = f_2 \circ \phi^{M'}_1$$

implies that $c^j_i g^{2j}_{2i} \circ f^{2i}_{1i} = c\delta_{ij} g^{2j}_{1i}$ for all i, j. Thus $c^i_i = c$ for all i.

We claim that $c^j_i g^{2j}_{2i} = 0$ if $i \neq j$. If $g^{2j}_{2i} = 0$, then we are done. Otherwise, we have that $g^{2j}_{2i} = f^{2j}_{2i} \neq 0$ but $c^j_i f^{2j}_{2i} \circ f^{2i}_{1i} = 0$. We aim to show that $f^{2j}_{2i} \circ f^{2i}_{1i} \neq 0$, which would imply $c_i^j = 0$. Recall that $I_{2i} = K[\alpha_i, \beta_i']$ for each i. By Lemma 2.3, $f_{2i}^{2j} \neq 0$ implies that $\alpha_j \leqslant \alpha_i \leqslant \beta_j' \leqslant \beta_i'$. Recall that $I_{1i} = K[\alpha_i', \beta_i']$ where $\alpha_i \leqslant \alpha_i'$. The definition of α_i' further implies that $\alpha_j < \alpha_i' < \beta_j' \leqslant \beta_i'$. By Lemma 2.3, there thus exists a nonzero homomorphism from I_{1i} into I_{2j}. As such, $f_{2i}^{2j} \circ f_{1i}^{2i} = f_{1i}^{2j} \neq 0$. Thus $c_i^j f_{2i}^{2j} \circ f_{1i}^{2i} = 0$ may only occur if $c_i^j = 0$, as desired.

It can similarly be shown that

$$f_3 = \bigoplus_{i=1}^{m} c f_{3i}^{3i}$$

by paying careful attention to the relationships between the intervals I_{2i}, I_{3i} and I_{3j} for $i \neq j$. Thus $f \in \mathrm{End}(M')$ is uniquely determined by the choice of $c \in K$. $\square$

3. Generalizations of Buchet and Escolar's result

We may now prove our main result:

Theorem 3.1. *Let M be any finite n-dimensional persistence module. Then there exists an indecomposable $(n+1)$-dimensional persistence module M' such that M is the restriction of M' to a hyperplane.*

We have one construction for M', called the **main construction**. Buchet and Escolar also introduced two other constructions for this setting in [**BE3**], which generalize their dual and glued rectangular constructions.

3.1. Main construction

Our main construction is defined as follows: Place $M = \bigoplus_{i=1}^{m} I_{5i}$ in layer 5 of M', where each I_{5i} is an indecomposable nD persistence module. Choose a minimal generating set $\{g_{5ij}\}_{j=1}^{k_i}$ for I_{5i}, and let $g_{5ij} \in (I_{5i})_{\alpha_{ij}}$ for all i, j. Let I_{4ij} denote the span of g_{5ij} in I_{5i}, meaning

$$(I_{4ij})_\alpha \colon = \begin{cases} \mathrm{Span}(\phi^{I_{5i}}_{\alpha_{ij},\alpha}(g_{5ij})) \subset (I_{5i})_\alpha & \text{if } \alpha_{ij} \leqslant \alpha, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\phi^{I_{4ij}}_{\alpha,\beta} = \begin{cases} \mathrm{Id}_K & \text{if } (I_{4ij})_\alpha \cong (I_{4ij})_\beta \cong K, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that I_{4ij} is a finite indecomposable interval n-dimensional persistence module with a single generator g_{4ij}. For each i, j define a homomorphism $A^{5i}_{4ij} \colon I_{4ij} \to I_{5i}$ by $A^{5i}_{4ij}(g_{4ij}) = g_{5ij}$. In layer 4 of M', place $\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_i} I_{4ij}$ and define the maps $\phi^{M'}$ between layers 4 and 5 of M' by

$$\phi_4^{M'} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_i} A^{5i}_{4ij}.$$

For $\alpha \in \mathbb{N}^n$, let α^k denote the k^{th} coordinate of α. For each i, j, define $\beta_{i,j} \in \mathbb{N}^n$ by $\beta^k_{i,j} = \max\limits_{\alpha \in \mathrm{Supp}(I_{4ij})} (\alpha^k)$. It follows that $\alpha \leqslant \beta_{i,j}$ for all $\alpha \in \mathrm{Supp}(I_{4ij})$. Define I_{3ij}

to be the indecomposable n-dimensional interval persistence module whose support is given by

$$\mathrm{Supp}(I_{3ij}) = \{\gamma \in \mathbb{N}^n | \text{ there exists } \alpha \in \mathrm{Supp}(I_{4ij}) \text{ satisfying } \alpha \leqslant \gamma \leqslant \beta_{i,j}\}.$$

We claim that $\mathrm{Hom}(I_{3ij}, I_{4ij}) \cong K$ for every i, j, which would allow us to define the maps $\phi^{M'}$ from layer 3 to layer 4 via

$$\phi_3^{M'} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_i} f_{3ij}^{4ij}.$$

By Lemma 2.3, it suffices to show that $\mathrm{Supp}(I_{4ij}) = \mathrm{Supp}(I_{3ij}) \cap \mathrm{Supp}(I_{4ij})$ is an (I_{3ij}, I_{4ij}) viable component. Suppose there exists $\gamma \in \mathrm{Supp}(I_{3ij}) \setminus \mathrm{Supp}(I_{4ij})$ and $\delta \in \mathrm{Supp}(I_{4ij})$ such that $\gamma < \delta$. By the definition of $\mathrm{Supp}(I_{3ij})$, $\gamma \in \mathrm{Supp}(I_{3ij})$ implies that there exists $\alpha \in \mathrm{Supp}(I_{4ij})$ such that $\alpha \leqslant \gamma < \delta$. By Remark 2.2, it follows that $\gamma \in \mathrm{Supp}(I_{4ij})$, which is a contradiction. On the other hand, there are no points $\gamma \in \mathrm{Supp}(I_{4ij}) \setminus \mathrm{Supp}(I_{3ij})$, so the second condition for (I_{3ij}, I_{4ij}) non-viability also cannot exist in $\mathrm{Supp}(I_{3ij}) \cap \mathrm{Supp}(I_{4ij})$. For an example of constructing layers 3 and 4 of M' from M, see Figure 4.

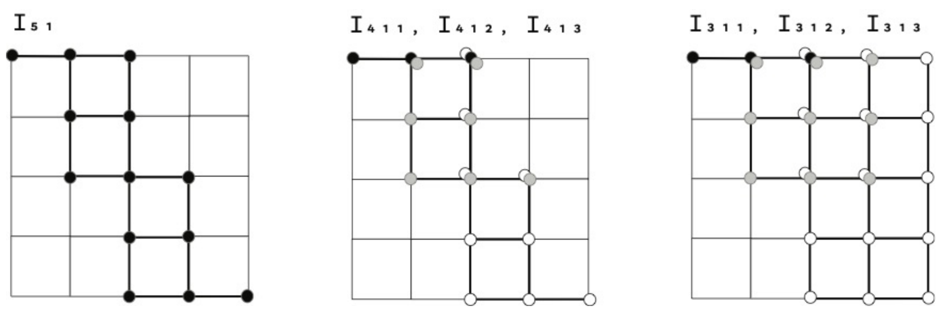

Figure 4: These are the top three layers of M' in the main interval for $M = I_{51}$. In layer 5, we have M. In layer 4, we have three interval persistence modules corresponding to the span of each of the three generators of I_{5i}. In the third layer, we extend each of the interval persistence modules from layer 4 to include the supremum of their respective supports.

Notice that layer 3 is a finite rectangular n-dimensional persistence module (since M was finite), so we may perform the main rectangular construction for layer 3 in layers $0-2$ of M'. This finishes the main construction of M'.

3.2. Proof of our main result

Proof of Theorem 3.1. We now show that the main construction yields a finite indecomposable $(n+1)$-dimensional persistence module M'. Finiteness is trivial. Let $f \in \mathrm{End}(M')$. Then f is equivalent to a set of endomorphisms f_ℓ on each layer ℓ of M' such that $\phi_{\ell-1}^{M'} \circ f_{\ell-1} = f_\ell \circ \phi_{\ell-1}^{M'}$. By Lemma 2.3, $f_0 = cf_{01}^{01}$ for some $c \in K$. The proof of Theorem 2.5 then implies that $f_\ell = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_i} cf_{\ell ij}^{\ell ij}$ for each $\ell \in [1, 3]$.

We now show that $f_4 = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_i} c f_{4ij}^{4ij}$. Because each I_{4ij} in layer 4 is generated by a single element, $\mathrm{Hom}(I_{4ij}, I_{4k\ell})$ is isomorphic to either 0 or K by Lemma 2.3, as $I_{4ij} \cap I_{4k\ell}$ will have at most one connected component. Let

$$g_{4ij}^{4k\ell} : = \begin{cases} 0 & \text{if } \mathrm{Hom}(I_{4ij}, I_{4k\ell}) = 0, \\ f_{4ij}^{4k\ell} & \text{if } \mathrm{Hom}(I_{4ij}, I_{4k\ell}) \cong K. \end{cases}$$

Thus $f_4 = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_i} \bigoplus_{k=1}^{m} \bigoplus_{\ell=1}^{k_k} c_{ij}^{k\ell} g_{4ij}^{4k\ell}$ for some choice of coefficients $c_{ij}^{k\ell} \in K$. This implies

$$f_4 \circ \phi_3^{M'} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_i} \bigoplus_{k=1}^{m} \bigoplus_{\ell=1}^{k_k} c_{ij}^{k\ell} g_{4ij}^{4k\ell} \circ f_{3ij}^{4ij}.$$

On the other hand,

$$f_4 \circ \phi_3^{M'} = \phi_3^{M'} \circ f_3 = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_i} c f_{3ij}^{4ij}.$$

Comparing the last two equations implies that $c_{ij}^{ij} = c$ for all $i \in [1, m]$ and $j \in [1, k_i]$ and that $c_{ij}^{k\ell} g_{4ij}^{4k\ell} \circ f_{3ij}^{4ij} = 0$ whenever $(i, j) \neq (k, \ell)$. We wish to show that when $(i, j) \neq (k, \ell)$ we have $c_{ij}^{k\ell} g_{4ij}^{4k\ell} = 0$. If $g_{4ij}^{4k\ell} = 0$, we are done. If $g_{4ij}^{4k\ell} = f_{4ij}^{4k\ell}$, consider the following: Let g_{3ij} be a generator of I_{3ij}. Then $f_{3ij}^{4ij}(g_{3ij}) = g_{4ij}$ for some generator g_{4ij} of I_{4ij}. Because $g_{4ij}^{4k\ell} = f_{4ij}^{4k\ell}$ and g_{4ij} is a generator, it follows that $g_{4ij}^{4k\ell}(g_{4ij}) \neq 0$. Thus $g_{4ij}^{4k\ell} \circ f_{3ij}^{4ij} \neq 0$, but $c_{ij}^{k\ell} g_{4ij}^{4k\ell} \circ f_{3ij}^{4ij} = 0$. We conclude that $c_{ij}^{k\ell} = 0$, and determine that $f_4 = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_i} c f_{4ij}^{4ij}$, as desired.

Now we wish to show that we have $f_5 = \bigoplus_{i=1}^{m} c \,\mathrm{Id}_{I_{5i}} = c \,\mathrm{Id}_M$. We may write $f_5 = \bigoplus_{i=1}^{m} \bigoplus_{k=1}^{m} c_i^k G_{5i}^{5k}$, where $G_{5i}^{5k} \in \mathrm{Hom}(I_{5i}^{5k})$ and $c_i^k \in K$ for all i, k.

We thus have $\phi_4^{M'} \circ f_4 = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{k_i} c A_{4ij}^{5i} = f_5 \circ \phi_4^{M'} = \bigoplus_{i=1}^{m} \bigoplus_{k=1}^{m} \bigoplus_{j=1}^{k_i} c_i^k G_{5i}^{5k} \circ A_{4ij}^{5i}$. We conclude that

$$c_i^i G_{5i}^{5i} \circ A_{4ij}^{5i} = c A_{4ij}^{5i} \tag{1}$$

for all i and

$$c_i^k G_{5i}^{5k} \circ A_{4ij}^{5i} = 0 \tag{2}$$

whenever $i \neq k$ and $j \in [1, k_i]$.

We first wish to show that $c_i^i G_{5i}^{5i} = c \,\mathrm{Id}_{I_{5i}}$ for all i. By Equation (1), we have that $c_i^i G_{5i}^{5i} \circ A_{4ij}^{5i}(g_{4ij}) = c A_{4ij}^{5i}(g_{4ij})$ for all i, j. By the definition of A_{4ij}^{5i}, this implies $c_i^i G_{5i}^{5i}(g_{5ij}) = c g_{5ij}$ for all i, j. Because $\{g_{5ij}\}_j$ generates I_{5i}, we may conclude that $c_i^i G_{5i}^{5i} = c \,\mathrm{Id}_{I_{5i}}$, as desired.

We also claim that $c_i^k G_{5i}^{5k} = 0$ for all $i \neq k$. If $G_{5i}^{5k} = 0$, we are done. Otherwise, there is a generator g_{5ij} of I_{5i} such that $G_{5i}^{5k}(g_{5ij}) \neq 0$. Thus $G_{5i}^{5k} \circ A_{4ij}^{5i}(g_{4ij}) \neq 0$. Comparing to Equation (2) yields that $c_i^k = 0$. Thus $c_i^k G_{5i}^{5k} = 0$ for all $i \neq k$.

Thus $f_5 = c\,\mathrm{Id}_M$ and f is fully determined by the choice of $c \in K$, implying $\mathrm{End}(M') \cong K$. By Lemma 2.1, M' is indecomposable. □

3.3. Implications for zigzag persistence modules

Theorem 3.1 has an interesting consequence on zigzag persistence modules, which are a generalization of 1-dimensional persistence modules that were first introduced in [**CdS**].

A **zigzag persistence module** M over K is an assignment of a vector space M_α to each $\alpha \in \mathbb{N}$ and for each $\alpha \in \mathbb{N}$, a homomorphism of the form

$$\phi^M_{\alpha,\alpha+1}\colon M_\alpha \to M_{\alpha+1} \text{ or of the form } \phi^M_{\alpha+1,\alpha}\colon M_{\alpha+1} \to M_\alpha.$$

A map of the form $\phi^M_{\alpha,\alpha+1}\colon M_\alpha \to M_{\alpha+1}$ is said to be **forwards-oriented**, whereas a map of the form $\phi^M_{\alpha+1,\alpha}\colon M_{\alpha+1} \to M_\alpha$ is **backwards-oriented**.

Thus a 1-dimensional persistence module is a zigzag module in which all maps are forward-oriented. The definitions of **finitely generated** and **finite** zigzag persistence modules are analogous to the definitions in the nD persistence module case. Finite zigzag modules also have indecomposable decompositions (which are again defined analogously to the nD persistence module case). By [**G**], a zigzag module M is indecomposable if and only if there exist some α, β such that

$$\dim(M_\gamma) = \begin{cases} 1 & \text{if } \gamma \in [\alpha,\beta], \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \phi^M_{\gamma,\delta} = \begin{cases} \mathrm{Id}_K & \text{if } \gamma,\delta \in [\alpha,\beta], \\ 0 & \text{otherwise.} \end{cases}$$

Such a zigzag persistence module is denoted by $K[\alpha,\beta]$.

Corollary 3.2. *Let $M = \bigoplus_{i=1}^{m} K[\alpha_i,\beta_i]$ be a finite zigzag persistence module. Then there exists an indecomposable 3-dimensional persistence module M' such that M is the restriction of M' to a path.*

Proof. Let Q_2 denote the directed graph whose vertices are the elements of $\mathbb{N}^2$ and whose edges are of the form $\alpha \to \alpha + e_i$ for all $\alpha \in \mathbb{N}^2$ and $i \in [1,2]$. Place M in Q_2 such that the backwards oriented arrows of M go along arrows of the form $\alpha \to \alpha + e_2$ in Q_2 and forwards oriented arrows in M go along arrows of the form $\alpha \to \alpha + e_1$ in Q_2. See Fig. 5 for examples of these placements. The result is a finite 2-dimensional interval persistence module. Apply Theorem 3.1. □

3.4. Future research directions

Of course, it may be possible to strengthen the results in this paper. Some open questions are:

1. Can Corollary 3.2 be strengthened to a statement that any finite zigzag persistence module can be embedded into an indecomposable 2-dimensional persistence module, rather than a 3D persistence module?
2. Given a finite n-dimensional persistence module M, what is the smallest support possible for an indecomposable $(n+1)$-dimensional persistence module M' which has M as a hyperplane restriction?

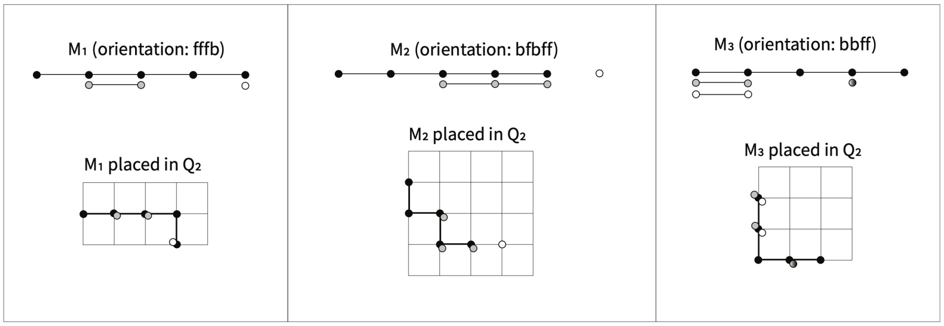

Figure 5: Shown are three examples of how to place a zigzag module M_i onto Q_2. The orientation of each M_i is given, with 'f' indicating a forwards-oriented arrow, and 'b' indicating a backwards-oriented arrow. For example, the first four maps in M_1 are forwards-oriented, but the map between the final two coordinates is backwards-oriented. Similar to our depictions of n-dimensional persistence modules, each dot at α represents a basis vector in M_α. The homomorphisms in M_i are denoted by edges; an edge between dots indicates which basic vectors map to each other.

3. Is Theorem 3.1 generalizable to nD persistence modules over $\mathbb{R}^n$ rather than $\mathbb{N}^n$? Note that nD persistence modules over $\mathbb{R}^n$ have very different properties than the n-dimensional persistence modules over $\mathbb{N}^n$, so such a generalization may be highly nontrivial [**M**].

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