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PATH HOMOLOGY OF DIRECTED HYPERGRAPHS

YURI MURANOV, ANNA SZCZEPKOWSKA AND VLADIMIR VERSHININ

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Abstract

We describe various path homology theories constructed for a directed hypergraph. We introduce the category of directed hypergraphs and the notion of a homotopy in this category. Also, we investigate the functoriality and the homotopy invariance of the introduced path homology groups. We provide examples of computation of these homology groups.

1. Introduction

Directed hypergraphs are the generalization of digraphs and have been widely used in discrete mathematics and computer science, see e.g. [1], [2], [5], and [7]. In particular, the directed hypergraphs give effective tools for the investigation of databases and structures on complicated discrete objects.

Recently, the topological properties of digraphs, hypergraphs, multigraphs, and quivers have been studied using various (co)homology theories, consult e.g. [3], [4], [6], [16], [15], [14], [10], [13].

In this paper, we construct several functorial and homotopy invariant homology theories on the category of directed hypergraphs using the path homology theory introduced in [8], [10], [12], [13], and [14].

A rich structure of a directed hypergraph gives a number of opportunities to define functorially a path complex for the category of hypergraphs which we construct in the paper. We describe these constructions in Section 3. We introduce also a notion of a homotopy in the category of directed hypergraphs and describe functorial relations between homotopy categories of directed hypergraphs, digraphs, and path complexes.

The essential difference from the situation of the category of digraphs is the existence of the notion of the density of the path complex that we introduce for the two of the introduced path complexes of directed hypergraphs. This notion gives an opportunity to define a filtration on the corresponding path complex and hence a filtration on its path homology groups. We consider all homology groups with coefficients in a unitary commutative ring R.

In Section 2, we define a category of directed hypergraphs and introduce the notion of homotopy in this category.

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In Section 3, we construct several path homology theories on the category of directed hypergraphs.

2. Path complexes and homotopy of directed hypergraphs

Let $\Pi = (V, P)$ be a path complex with $V = \{0, \dots, n\}$ and $J = \{0, 1\}$ be a set, see [10], [13]. Define a path complex $\Pi' = (V', P')$ where $V' = \{0', \dots, n'\}$ and $p' = (i'_0 \dots i'_n) \in P'$ iff $p = (i_0 \dots i_n) \in P$. We identify $V \times J = V \times \{0\} \amalg V \times \{1\}$ with $V \amalg V'$. Define a path complex $\Pi^{\uparrow} = (V \times J, P^{\uparrow})$ where

$$P^{\uparrow} = P \cup P' \cup P^{\#},$$
$$P^{\#} = \{q_k^{\#} = (i_0 \dots i_k i'_k i'_{k+1} \dots i'_n) | q = (i_0 \dots i_k i_{k+1} \dots i_n) \in P\}.$$

We have morphisms $i_{\bullet} \colon \Pi \to \Pi^{\uparrow}$ and $j_{\bullet} \colon \Pi \to \Pi^{\uparrow}$ that are induced by the natural inclusion V onto $V \times \{0\}$ and onto $V \times \{1\}$, respectively.

Definition 2.1. (i) A *hypergraph* is a pair $G = (V, E)$ consisting of a non-empty set V and a set $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of distinct and non-ordered subsets of V such that $\bigcup_{i=1}^{n} \mathbf{e}_i = V$ and every $\mathbf{e}_i$ contains strictly more than one element. The elements of V are called *vertices* and the elements of E are called *edges*.

(ii) A *directed hypergraph* G is a pair (V, E) consisting of a set V and a set $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ where $\mathbf{e}_i \in E$ is an ordered pair (A_i, B_i) of disjoint non-empty subsets of the set V such that $V = \bigcup_{\mathbf{e}_i \in E} (A_i \cup B_i)$. The elements of V are called *vertices* and the elements of E are called *arrows*. The set $A = \operatorname{orig}(A \to B)$ is called the *origin of the arrow* and the set $B = \operatorname{end}(A \to B)$ is called the *end of the arrow*. The elements of A are called the *initial vertices* of $A \to B$ and the elements of B are called its *terminal vertices*.

For a finite set X let $\mathbf{P}(X)$ denote as usual the power set. We define a set $\mathbb{P}(X) := \{\mathbf{P}(X) \setminus \emptyset\} \times \{\mathbf{P}(X) \setminus \emptyset\}$ consisting of ordered pairs of non-empty subsets of X. Every map of finite sets $f \colon V \to W$ induces a map $\mathbb{P}(f) \colon \mathbb{P}(V) \to \mathbb{P}(W)$. For a directed hypergraph $G = (V, E)$, by Definition 2.1, we have the natural map $\varphi_G \colon E \to \mathbb{P}(V)$ defined by $\varphi_G(A \to B) := (A, B)$.

Definition 2.2. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two directed hypergraphs. A *morphism* $f \colon G \to H$ is given by a pair of maps $f_V \colon V_G \to V_H$ and $f_E \colon E_G \to E_H$ such that the following diagram

$$\begin{array}{ccc} E_G & \xrightarrow{\varphi_G} & \mathbb{P}(V_G) \\ \downarrow{f_E} & & \downarrow{\mathbb{P}(f_V)} \\ E_H & \xrightarrow{\varphi_H} & \mathbb{P}(V_H) \end{array}$$

is commutative.

Let us denote by $\mathcal{DH}$ the category whose objects are directed hypergraphs and whose morphisms are morphisms of directed hypergraphs.

For a directed hypergraph $G = (V_G, E_G)$, we can consider subsets

$$\mathbf{P}_0(G) \subset \mathbf{P}(V_G) \setminus \emptyset, \mathbf{P}_1(G) \subset \mathbf{P}(V_G) \setminus \emptyset \text{ and } \mathbf{P}_{01}(G) = \mathbf{P}_0(G) \cup \mathbf{P}_1(G)$$

by setting

$$\mathbf{P}_0(G) = \{A \in \mathbf{P}(V_G) \setminus \emptyset | \exists B \in \mathbf{P}(V_G) \setminus \emptyset \colon A \to B \in E_G\},$$
$$\mathbf{P}_1(G) = \{B \in \mathbf{P}(V_G) \setminus \emptyset | \exists A \in \mathbf{P}(V_G) \setminus \emptyset \colon A \to B \in E_G\}.$$

Definition 2.3. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be directed hypergraphs. We define the *box product* $G \Box H$ as a directed hypergraph with the set of vertices $V_{G\Box H} = V_G \times V_H$ and the set of arrows $E_{G\Box H}$ consisting of the union of arrows $\{A \times C \to B \times C\}$ with $(A \to B) \in E_G$, $C \in \mathbf{P}_{01}(H)$ and $\{A \times C \to A \times D\}$ with $(C \to D) \in E_H$, $A \in \mathbf{P}_{01}(G)$.

Every connected digraph $H = (V_H, E_H)$ can be considered as a directed hypergraph with the same set of vertices and of a set of arrows of the form $\{v\} \to \{w\}$ whith $(v \to w) \in E_H$. Hence, Definition 2.3 gives naturally a box product $G \Box H$ of a directed hypergraph G and a connected digraph H. Note that a line digraph I_n defined for example in [**11**, Sec. 3.1] is connected and that we have two digraphs I_1, namely $0 \to 1$ and $1 \to 0$.

Definition 2.4. i) Two morphisms $f_0, f_1 \colon G \to H$ of directed hypergraphs are called *one-step homotopic* if there exists a line digraph I_1 and a morphism $F \colon G \Box I_1 \to H$, such that

$$F|_{G\Box\{0\}} = f_0 \colon G\Box\{0\} \to H, \quad F|_{G\Box\{1\}} = f_1 \colon G\Box\{1\} \to H.$$

If the appropriate morphism F called a one-step homotopy exists, we write $f_0 \simeq_1 f_1$.

ii) Two morphisms $f, g \colon G \to H$ of directed hypergraphs are called *homotopic*, which we denote $f \simeq g$ if there exists a sequence of morphisms $f_i \colon G \to H$ for $i = 0, \dots, n$ such that $f = f_0 \simeq_1 f_1 \simeq_1 \cdots \simeq_1 f_n = g$.

iii) Two directed hypergraphs G and H are *homotopy equivalent* if there exist morphisms $f \colon G \to H$ and $g \colon H \to G$ such that $fg \simeq \mathrm{Id}_H$ and $gf \simeq \mathrm{Id}_G$. In such a case, we write $G \simeq H$ and call the morphisms f, g *homotopy inverses* of each other.

Proposition 2.5. *Two morphisms $f, g \colon G \to H$ of directed hypergraphs are homotopic if and only if there is a line digraph I_n with $n \geqslant 0$ and a morphism*

$F \colon G\Box I_n \to H$ such that

$$F|_{G\Box\{0\}} = f_0 \colon G\Box\{0\} \to H, \quad F|_{G\Box\{n\}} = g \colon G\Box\{n\} \to H. \quad \blacksquare$$

The relation “to be homotopic” is an equivalence relation on the set of morphisms between two directed hypergraphs, and homotopy equivalence is an equivalence relation on the set of directed hypergraphs. Thus, we can consider a category $h\mathcal{DH}$ whose objects are directed hypergraphs and morphisms are the classes of homotopic morphisms. We shall call the category $h\mathcal{DH}$ by *homotopy category of directed hypergraphs*.

3. Path homology of directed hypergraphs

3.1. k-connective path homology

For a directed hypergraph $G = (V, E)$ and $c = 1, 2, 3, \dots$ define a path complex [**13**, S3.1] $\mathfrak{C}^c(G) = (V^c, P^c_G)$ where $V^c = V$ and a path $(i_0 \dots i_n) \in P_V$ lies in P^c_G iff for any pair of consequent vertices (i_k, i_{k+1}) of the path, we have $i_k = i_{k+1}$ or there are at least c different edges $\mathbf{e}_1 = (A_1 \to B_1), \dots, \mathbf{e}_c = (A_c \to B_c)$ such that the vertex i_k is the initial vertex and the vertex i_{k+1} is the terminal vertex of every edge $\mathbf{e}_i$. The number c is called the *density* of the path complex $\mathfrak{C}^c(G)$. It is clear that we have a filtration

$$\mathfrak{C}(G) = \mathfrak{C}^1(G) \supset \mathfrak{C}^2(G) \supset \mathfrak{C}^3(G) \supset \dots \quad . \tag{1}$$

Proposition 3.1. *For every morphism of directed hypergraphs $f\colon G \to H$ define a morphism*

$$\mathfrak{C}(f) = (f_V^1, f_p^1)\colon \mathfrak{C}(G) \to \mathfrak{C}(H)$$

of path complexes putting $f_V^1 := f_V$ and $f_p^1 := f_p|_{P_G^1}\colon P_G^1 \to P_H^1$ where f_p is defined by $f_p(i_0 \dots i_n) = (f(i_0) \dots f(i_n))$. Then we have the functor $\mathfrak{C}$ from the category $\mathcal{DH}$ of directed hypergraphs to the category $\mathcal{P}$ of path complexes. ■

The functor $\mathfrak{C}$ provides the functorial path homology theory on the category $\mathcal{DH}$ of directed hypergraphs. For any directed hypergraph G and $k \in \mathbb{N}$, we set $H_*^{\mathbf{c}(k)}(G) := H_*(\mathfrak{C}^k(G))$ as *regular path homology groups* of path complex $\mathfrak{C}^k(G)$, see [**10**, S2]. We denote $H_*^{\mathbf{c}}(G) := H_n^{\mathbf{c}(1)}(G)$.

We call these homology groups *the connective path homology groups* and for $k \geqslant 2$ *the k-connective path homology groups* of the directed hypergraph G, respectively. The connective path homology theory is functorial by Proposition 3.1. However the k-connective homology theory $H_n^{\mathbf{c}(k)}(G)$ is not functorial for $k \geqslant 2$ as it follows from Example 3.2 below. For any directed hypergraph G the filtration in (1) induces homomorphisms

$$H_n^{\mathbf{c}}(G) = H_n^{\mathbf{c}(1)}(G) \longleftarrow H_n^{\mathbf{c}(2)}(G) \longleftarrow H_n^{\mathbf{c}(3)}(G) \longleftarrow \dots \quad .$$

Let $\mathcal{D}$ be a category of digraphs without loops [**11**, S2]. A category $\mathcal{G}$ of graphs is defined similarly [**11**, S6].

Let $G = (V, E)$ be a directed hypergraph. Define a digraph $\mathfrak{G}(G) = \left(V_G^d, E_G^d\right)$ where $V_G^d = V$ and an arrow $v \to w$ lies in E_G^d iff there is a hyperedge $(A \to B) \in E$ such that $v \in A, w \in B$.

Example 3.2. i) Let $G = (V, E)$ be a directed hypergraph such that V is the union $A \cup B$ of two non-empty sets with empty intersection and the set E consists of one element $\mathbf{e} = (A \to B)$. Then $\mathfrak{G}(G)$ is a complete bipartite digraph with arrows from vertices lying in A to vertices lying in B.

ii) Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two directed hypergraphs with

$$V_G = \{1, 2, 3, 4\}, E_G = \{\mathbf{e}_1 = (\{1\} \to \{2, 3\}), \mathbf{e}_2 = (\{1\} \to \{2.4\})\},$$

$$V_H = \{a, b, c\}, E_H = \{\mathbf{e}_1' = (\{a\} \to \{b, c\})\}.$$

The map $f_V\colon V_G \to V_H$, given by $f_V(1) = a, f_V(2) = b, f_V(3) = f_V(4) = c$, induces a morphism f of directed hypergraphs. However the map f does not induce a morphism from $\mathfrak{C}^2(G)$ to $\mathfrak{C}^2(H)$.

For every morphism $f = (f_V, f_E)\colon G \to H$ of directed hypergraphs, define a map $\mathfrak{G}(f)\colon V_G^d \to V_H^d$ by $\mathfrak{G}(f) = f_V$. For any arrow $(v \to w) \in E_G^d$, we have an arrow $(f_V(v) \to f_V(w)) \in E_H^d$ and the morphism $\mathfrak{G}(f)$ of digraphs is well defined. Thus we have a functor $\mathfrak{G}$ from the category $\mathcal{DH}$ of directed hypergraphs to the category $\mathcal{D}$ of digraphs. *Regular path homology of digraphs* was constructed in [9], [15]. It is based on the natural functor $\mathfrak{D}$ from the category $\mathcal{D}$ of digraphs to the category $\mathcal{P}$ of path complexes.

Theorem 3.3. *For every directed hypergraph G there is an isomorphism $H_*^{\mathbf{c}}(G) \cong H_*(\mathfrak{D} \circ \mathfrak{G}(G))$ of path homology groups.*

Proof. The path complexes $\mathfrak{C}(G)$ and $\mathfrak{D} \circ \mathfrak{G}(G)$ coincide. □

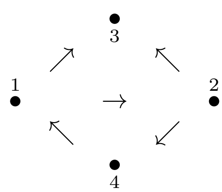
Example 3.4. The following example illustrates the technique of computations of the connective path homology groups $H_*^{\mathbf{c}(k)}(G)$. For $k \geqslant 3$ in the presented case, there is nothing to compute. Let $R = \mathbb{R}$ be the ring of coefficients. Consider a hypergraph $G = (V_G, E_G)$ for which

$$V_G = \{1,2,3,4\}, \quad E_G = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\},$$

$$\mathbf{e}_1 = (\{1\} \to \{2\}), \mathbf{e}_2 = (\{2\} \to \{3,4\}), \mathbf{e}_3 = (\{4\} \to \{1\}),$$

$$\mathbf{e}_4 = (\{1\} \to \{2,3\}), \mathbf{e}_5 = (\{2\} \to \{3\}), \mathbf{e}_6 = (\{2\} \to \{4\}).$$

We compute homology of the path complex $\mathfrak{C}^c(G) = (V^c, P_G^c)$ as in [10]. We have $\mathcal{R}_0^{reg} = \langle 1,2,3,4 \rangle = \Omega_0$, $\mathcal{R}_1^{reg} = \langle e_{12}, e_{13}, e_{23}, e_{24}, e_{41} \rangle$. We get $\partial(e_{ij}) \in \mathcal{R}_0^{reg}$ for all basic elements $e_{ij} \in \mathcal{R}_1^{reg}$, so $\Omega_1 = \mathcal{R}_1^{reg}$. Thus, Ω_1 is generated by all directed edges of the digraph $\mathfrak{G}(G)$ presented below



Similarly, we obtain $\Omega_2 = \langle e_{123} \rangle$. From the definition of $\mathfrak{G}(G)$, it follows that $\Omega_i = 0$ for $i \geqslant 3$ and the homology of the chain complex Ω_* coincides with the regular path homology $\mathfrak{G}(G)$. Hence $H_0^{\mathbf{c}(1)}(G) = H_1^{\mathbf{c}(1)}(G) = \mathbb{R}$ and $H_i^{\mathbf{c}(1)}(G) = 0$ for $i \geqslant 2$.

For $H_i^{\mathbf{c}(2)}(G)$, we have $\Omega_0 = \langle 1,2,3,4 \rangle$ and by definition $\Omega_1 = \langle e_{12}, e_{23}, e_{24} \rangle$. Moreover, $\Omega_i = 0$ for $i \geqslant 2$. Thus, homology groups $H_i^{\mathbf{c}(2)}(G)$ coincide with the homology groups of the digraph which has the set of vertices $V_{\mathfrak{G}(G)}$ and the set of arrows obtained from $E_{\mathfrak{G}(G)}$ by deleting arrows $(1 \to 3)$ and $(4 \to 1)$. Hence, $H_0^{\mathbf{c}(2)}(G) = \mathbb{R}$ and $H_i^{\mathbf{c}(2)}(G) = 0$ for $i \geqslant 1$. For $k \geqslant 3$ we have $H_0^{\mathbf{c}(k)} = \mathbb{R}^4$ and $H_i^{\mathbf{c}(k)}(G) = 0$ for $i \geqslant 1$.

Lemma 3.5. *Let $G = (V, E)$ be a directed hypergraph and $I_1 = (0 \to 1)$. We have a natural isomorphism $\mathfrak{C}(G \Box I_1) \cong [\mathfrak{C}(G)]^{\uparrow}$ of path complexes.*

Proof. By Definition 2.3 a directed hypergraph $G \Box I_1 = (V_{G\Box I_1}, E_{G\Box I_1})$ has the set of vertices $V_{G\Box I_1} = V \times J = V \times \{0,1\}$ which we identify with $V \cup V'$, where $V =$

$\{0,\dots,n\}$, $V'=\{0',\dots,n'\}$ and the set of edges $E_{G\Box I_1}$ is the union $E^0\cup E^1\cup E^{01}$ of sets

$$E^i=\{A\times\{i\}\to B\times\{i\}\} \text{ with } (A\to B)\in E_G \text{ for } i=0,1,$$

and

$$E^{01}=\{C\times\{0\}\to C\times\{1\}\} \text{ with } C\in\mathbb{S}_{01}(G).$$

Let $q=(i_0\dots i_n)$ be a path lying in $\mathfrak{C}(G\Box I_1)$. It follows from definition, that there are only three possibilities, namely

(1) all the vertices $i_j\in V\times\{0\}$ and, hence, q determines the unique path in $\mathfrak{C}(G)$,

(2) all the vertices $i_j\in V\times\{1\}$ and, hence, q determines the unique path in $[\mathfrak{C}(G)]'$,

(3) there exists exactly one pair (i_k,i_{k+1}) of consequent vertices in q such that $i_k\in C\times\{0\}, i_{k+1}\in C\times\{1\}$ for $C\in\mathbb{S}_{01}(G)$.

Thus, the union of paths from (1)–(3) on the set of vertices $V\times J$ defines the path complex $[\mathfrak{C}(G)]^{\uparrow}$ and vice versa. □

Theorem 3.6. *For a directed hypergraph G, the connective path homology groups $H^{\mathbf{c}}(G)$ are homotopy invariant.*

Proof. By Definition 2.4, it is sufficient to prove homotopy invariance for a one-step homotopy. Then the result follows from Lemma 3.5 and [**10**, Th. 3.4]. □

3.2. Bold path homology

Let $p=(i_0\dots i_n)$ and $q=(j_0\dots j_m)$ be two paths of a path complex Π with $i_n=j_0$. The *concatenation* $p\vee q$ of these paths is a path given by $p\vee q=(i_0\dots i_nj_1\dots j_m)$. The concatenation is well defined only if $i_n=j_0$.

For a directed hypergraph $G=(V,E)$, define a path complex $\mathfrak{B}(G)=(V_G^b,P_G^b)$ where $V_G^b=V$ and a path $q=(i_0\dots i_n)\in P_V$ lies in P_G^b iff there is a sequence of hyperedges $(A_0\to B_0),\dots,(A_r\to B_r)$ in E such that $B_i\cap A_{i+1}\neq\emptyset$ for $0\leqslant i\leqslant r-1$ and the path q has the presentation

$$(p_0\vee v_0w_0\vee p_1\vee v_1w_1\vee p_2\vee\dots\vee p_r\vee v_rw_r\vee p_{r+1})\,, \tag{2}$$

where $p_0\in P_{A_0}$, $p_{r+1}\in P_{B_r}$, $v_i\in A_i$, $w_i\in B_i$, $p_i\in P_{B_{i-1}}\cap P_{A_i}$ for $1\leqslant i\leqslant r$ and all concatenations in (2) are well defined. Note, that in the case of empty sequence of edges $A_i\to B_i$ every path $q\in P_A$ and every path $q\in P_B$ for an edge $A\to B$ lies in P_G^b.

Proposition 3.7. *Let $f\colon G=(V_G,E_G)\to H=(V_H,E_H)$ be a morphism of directed hypergraphs. Define a morphism of path complexes*

$$\mathfrak{B}(f)=(f_V^b,f_p^b)\colon(V_G^b,P_G^b)\to(V_H^b,P_H^b)$$

by $f_V^b=f\colon V_G^b=V_G\to V_H=V_H^b$ and $f_p^b=f_p|_{P_G^b}$, where f_p is defined as in Proposition 3.1. Thus, we obtain a functor $\mathfrak{B}$ from the category $\mathcal{DH}$ of directed hypergraphs to the category $\mathcal{P}$ of path complexes. ■

Let us define *the bold path homology groups* of directed hypergraph G by

$$H^{\mathbf{b}}_*(G) := H_*(\mathfrak{B}(G)).$$

By Proposition 3.7, we obtain a functorial path homology theory on the category $\mathcal{DH}$ of directed hypergraphs.

Example 3.8. Let $G = (V, E)$ be a directed hypergraph such that for every edge $\mathbf{e} = (A \to B) \in E$ the sets A and B are one-vertex sets, $A = \{v\}, B = \{w\}, v, w \in V$. We can consider the hypergraph G as a digraph and $H^{\mathbf{c}}_*(G) \cong H^{\mathbf{b}}_*(G)$. On the category of connected digraphs that can be considered as the subcategory of directed hypergraphs, the bold path homology groups are naturally isomorphic to the connective path homology groups and to the regular path homology groups $H_*(G)$ defined in [**9**].

Example 3.9. Now we compute the bold path homology groups $H^{\mathbf{b}}_*(G)$ of the directed hypergraph G from Example 3.4 in dimensions 0,1,2 for $R = \mathbb{R}$. First, we describe the modules $\mathcal{R}^{reg}_n(\mathfrak{B}(G))$ for $0 \leqslant n \leqslant 4$. We have

$$\mathcal{R}^{reg}_0 = \langle e_1, e_2, e_3, e_4\rangle, \quad \mathcal{R}^{reg}_1 = \langle e_{12}, e_{13}, e_{23}, e_{24}, e_{32}, e_{34}, e_{43}, e_{41}\rangle,$$

$$\mathcal{R}^{reg}_2 = \langle e_{123}, e_{124}, e_{132}, e_{232}, e_{234}, e_{241}, e_{243}, e_{323}, e_{343}, e_{434}, e_{412}, e_{413}\rangle,$$

$$\mathcal{R}^{reg}_3 = \langle e_{1232}, e_{1234}, e_{1241}, e_{1243}, e_{1323}, e_{2323}, e_{2343},\\ e_{2412}, e_{2413}, e_{2434}, e_{3232}, e_{3434}, e_{4343}, e_{4123}, e_{4124}, e_{4132}\rangle,$$

$$\mathcal{R}^{reg}_4 = \langle e_{12323}, e_{12343}, e_{12412}, e_{12413}, e_{12434}, e_{13232}, e_{23232}, e_{23434},\\ e_{24123}, e_{24132}, e_{24343}, e_{32323}, e_{34343}, e_{43434}, e_{41232}, e_{41234}, e_{41243}, e_{41323}\rangle,$$

$$\Omega_n = \mathcal{R}^{reg}_n \quad \text{for} \quad n = 0, 1.$$

Thus Ω_0 is generated by all the vertices and Ω_1 is generated by all directed edges of the digraph H on Fig. 1.

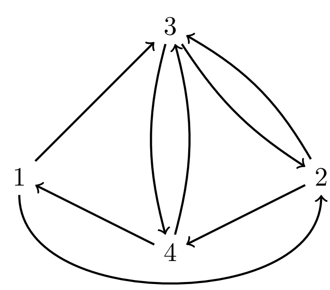


Figure 1: The digraph H.

As it follows from the path homology theory of digraphs, the rank of the image of $\partial\colon \Omega_1 \to \Omega_0$ is equal to 3, the rank of the kernel of ∂ is equal to 5, and hence $H^{\mathbf{b}}_0(G) = \mathbb{R}$.

By the direct computation Ω_2 is the vector space with the following basis:

$$\{e_{123}, e_{132}, e_{232}, e_{234}, e_{243}, e_{323}, e_{343}, e_{434}, e_{413}\}.$$

In this basis the matrix of the homomorphism $\partial\colon \Omega_2 \to \Omega_1$ has the form:

$$\begin{pmatrix} & e_{12} & e_{13} & e_{23} & e_{24} & e_{32} & e_{34} & e_{43} & e_{41} \\ e_{123} & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ e_{132} & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ e_{232} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ e_{234} & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ e_{243} & 0 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ e_{323} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ e_{343} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ e_{434} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ e_{413} & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}.$$

Its rank is equal to 5. Hence the rank of the image of ∂ is equal to 5, the rank of the kernel of ∂ is equal to 4, and hence $H_1^{\mathbf{b}}(G) = 0$.

We have $\Omega_3 = \langle e_{1232}, e_{1323}, e_{2323}, e_{2343}, e_{2434}, e_{3232}, e_{3434}, e_{4343}\rangle$. Similar to the previous calculation, the rank of the image of $\partial\colon \Omega_3 \to \Omega_2$ is equal to 4, the rank of the kernel of ∂ is equal to 4, and hence $H_2^{\mathbf{b}}(G) = 0$.

We have $\Omega_4 = \langle e_{12323}, e_{13232}, e_{23232}, e_{23434}, e_{24343}, e_{32323}, e_{34343}, e_{43434}\rangle$ and, similar to the previous calculation, the rank of the image of $\partial\colon \Omega_4 \to \Omega_3$ is equal to 4, the rank of the kernel of ∂ is equal to 4. Hence $H_3^{\mathbf{b}}(G) = 0$.

Lemma 3.10. *Let $G = (V, E)$ be a directed hypergraph and $I_1 = (0 \to 1)$ the digraph. There is an inclusion $\lambda\colon [\mathfrak{B}(G)]^{\uparrow} \to \mathfrak{B}(G\Box I_1)$ of path complexes. The restrictions of λ to the images of the morphisms $i_{\bullet}$ and $j_{\bullet}$, defined in Section 2, are the natural identifications.*

Proof. By definition in Section 2, we have

$$[\mathfrak{B}(G)]^{\uparrow} = (V \times J, [P_G^b]^{\uparrow}),$$

where $[P_G^b]^{\uparrow} = P_G^b \cup [P_G^b]' \cup [P_G^b]^{\#}$. We have

$$V_{G\Box I_1} = V \times J = V \times \{0, 1\} = V \cup V' \text{ with}$$

$$V = \{0, \dots, n\},\ V' = \{0', \dots, n'\}$$

and $E_{G\Box I_1}$ is the union of sets $E^0 \cup E^1 \cup E^{01}$, where

$$E^i = \{A \times \{i\} \to B \times \{i\} \,|\, (A \to B) \in E_G\}$$

for $i = 0, 1$ and

$$E^{01} = \{C \times \{0\} \to C \times \{1\} \,|\, C \in \mathbf{P}_{01}(G)\}.$$

Now it follows that

$$\mathfrak{B}(G) = \mathfrak{B}(G\Box\{0\}), \mathfrak{B}(G)' = \mathfrak{B}(G\Box\{1\}), \text{ where } \mathfrak{B}(G\Box\{0\}), \mathfrak{B}(G\Box\{1\}) \subset \mathfrak{B}(G\Box I_1).$$

Let $q = (i_0 \dots i_n)$ be n-path in $\mathfrak{B}(G) = \mathfrak{B}(G\Box\{0\})$. Consider its presentation in the form (2) and let A_i, B_i be the corresponding sets of vertices. For $0 \leqslant k \leqslant n$, consider a path $q_k^\# = \big(i_0 \dots i_k i'_k i'_{k+1} \dots i'_n\big) \in [P^b_G]^\#$. We will prove now that this path in $P^b_{G\Box I_1}$. There are following possibilities for the path q.

(1) Vertices $i_k, i_{k+1} \in p_s$ for $1 \leqslant s \leqslant r+1$ in presentation (2). Then we write path $q_k^\#$ in the form

$$q_k^\# = \Big(p_0^\# \vee v_0^\# w_0^\# \vee p_1^\# \vee \dots \vee p_{r+1}^\# \vee v_{r+1}^\# w_{r+1}^\# \vee p_{r+2}^\#\Big) \tag{3}$$

putting

$$A_i^\# = \begin{cases} A_i \times \{0\} & \text{for } i \leqslant s-1, \\ B_{s-1} \times \{0\} & \text{for } i = s, \\ A_{s-1} \times \{1\} & \text{for } i \geqslant s+1, \end{cases} \qquad B_i^\# = \begin{cases} B_i \times \{0\} & \text{for } i \leqslant s-1, \\ B_{s-1} \times \{1\} & \text{for } i \geqslant s. \end{cases}$$

We have the following arrows in $E_{G\Box I_1}$:

$$(A_i^\# \to B_i^\#) = (A_i \times \{0\} \to B_i \times \{0\}) \ \text{ for } 0 \leqslant i \leqslant s-1,$$

$$(A_s^\# \to B_s^\#) = (B_{s-1} \times \{0\} \to B_{s-1} \times \{1\}),$$

$$(A_i^\# \to B_i^\#) = (A_{i-1} \times \{1\} \to B_{i-1} \times \{1\}) \ \text{ for } s+1 \leqslant i \leqslant r+2.$$

Using identifications $\mathfrak{B}(G) = \mathfrak{B}(G\Box\{0\}), \mathfrak{B}(G)' = \mathfrak{B}(G\Box\{1\})$, we obtain

$$p_i^\# = \begin{cases} p_i & \text{for } i \leqslant s-1, \\ p'_i & \text{for } s+2 \leqslant i \leqslant r+2, \\ (w_{s-1} \dots i_s) & \text{for } i = s, \\ (i'_s i'_{s+1} \dots v'_s) & \text{for } i = s+1, \end{cases} \tag{4}$$

where

$$(w_{s-1} \dots i_s) \in P_{B_{s-1}^\# \cap A_s^\#} = P_{B_{s-1} \times \{0\}},$$

$$\big(i'_s i'_{s+1} \dots v'_s\big) \in P_{B_s^\# \cap A_{s+1}^\#} = P_{(B_{s-1} \times \{1\}) \cap (A_s \times \{1\})}.$$

Paths $p_i^\#$ in (4) define vertices $v_i^\#, w_i^\#$ in (3). Hence, (3) gives a presentation of $q_k^\#$ in the form (2) for the hypergraph $G\Box I_1$ and $q_k^\# \in P^b_{G\Box I_1}$ in the considered case.

(2) Vertices $i_k, i_{k+1} \in p_0$ in presentation (2). Then we write path $p_k^\#$ in the form (3) putting

$$A_i^\# = \begin{cases} A_0 \times \{0\} & \text{for } i = 0, \\ A_{i-1} \times \{1\} & \text{for } 1 \leqslant i \leqslant r+2, \end{cases}$$

$$B_i^\# = \begin{cases} A_0 \times \{1\} & \text{for } i = 1, \\ B_{i-1} \times \{1\} & \text{for } 2 \leqslant i \leqslant r+2 \end{cases}$$

and

$$p_i^\# = \begin{cases} (i_0 \dots i_k) & \text{for } i = 0, \\ (i'_k \dots v'_0) & \text{for } i = 1, \\ p'_{i-1} & \text{for } 2 \leqslant i \leqslant r+2, \end{cases}$$

where $(i_0 \dots i_k) \in P_{A_0^{\#}}$, $(i'_k \dots v'_0) \in P_{B_0^{\#} \cap A_1^{\#}} = P_{A_0 \times \{1\}}$ Hence, (3) gives a presentation of $q_k^{\#}$ in the form (2) in the hypergraph $G\Box I_1$ and $q_k^{\#} \in P^b_{G\Box I_1}$ in the considered case.

(3) Let $i_k = v_s, i_{k+1} = w_s$ for $0 \leqslant s \leqslant r$ in the presentation (2). Then we write path $q_k^{\#}$ in the form (3) putting

$$A_i^{\#} = \begin{cases} A_i \times \{0\} & \text{for } i \leqslant s, \\ A_{s+1} \times \{1\} & \text{for } i = s+1, \\ A_{s-1} \times \{1\} & \text{for } i \geqslant s+2, \end{cases} \quad B_i^{\#} = \begin{cases} B_i \times \{0\} & \text{for } i \leqslant s-1, \\ A_s \times \{1\} & \text{for } i = s, \\ B_{s-1} \times \{1\} & \text{for } i \geqslant s+1. \end{cases}$$

We have the following arrows in $E_{G\Box I_1}$:

$$(A_i^{\#} \to B_i^{\#}) = (A_i \times \{0\} \to B_i \times \{0\}) \ \text{ for } 0 \leqslant i \leqslant s-1,$$

$$(A_s^{\#} \to B_s^{\#}) = (A_s \times \{0\} \to A_s \times \{1\}),$$

$$(A_i^{\#} \to B_i^{\#}) = (A_{i-1} \times \{1\} \to B_{i-1} \times \{1\}) \ \text{ for } s+1 \leqslant i \leqslant r+2.$$

Similarly to case (1), we have

$$p_i^{\#} = \begin{cases} p_i & \text{for } i \leqslant s, \\ (v'_s) & \text{for } i = s+1, \\ p'_{i-1} & \text{for } s+2 \leqslant i \leqslant r+2, \end{cases}$$

where $w_s^{\#} = v_s, v_s^{\#} = v'_s, w_{s+1}^{\#} = v'_s$. Hence, (3) gives a presentation of $q_k^{\#}$ in the form (2) in the hypergraph $G\Box I_1$ and $q_k^{\#} \in P^b_{G\Box I_1}$ in the considered case. □

Theorem 3.11. *Let G be a directed hypergraph. The bold path homology groups $H_*^{\mathbf{b}}(G)$ are homotopy invariant.*

Proof. By Definition 2.4, it is sufficient to prove homotopy invariance for the one-step homotopy. Let $f_0, f_1 \colon G \to H$ be one-step homotopic morphisms of directed hypergraphs with homotopy $F \colon G\Box I_1 \to H$, where $I_1 = (0 \to 1)$. Since $\mathfrak{B}$ is a functor, we obtain morphisms of path complexes

$$\mathfrak{B}(f_0),\, \mathfrak{B}(f_1) \colon \mathfrak{B}(G) \to \mathfrak{B}(H)$$

and

$$\mathfrak{B}(F) \colon \mathfrak{B}(G\Box I_1) \to \mathfrak{B}(H).$$

Consider the composition

$$[\mathfrak{B}(G)]^{\uparrow} \xrightarrow{\lambda} \mathfrak{B}(G\Box I_1) \xrightarrow{\mathfrak{B}(F)} \mathfrak{B}(H),$$

which gives a homotopy between morphisms $\mathfrak{B}(f_0)$ and $\mathfrak{B}(f_1)$ of path complexes by using identifications of the top and the bottom of $[\mathfrak{B}(G)]^{\uparrow}$ described in Lemma 3.10. Now the result follows from [**10**, Th. 3.4]. □

3.3. Non-directed path homology

In this subsection, we describe several path homology theories on the category of directed hypergraphs $\mathcal{DH}$ that are based on functorial relations between hypergraphs and directed hypergraphs.

For a hypergraph $G = (V_G, E_G)$, we have a natural map $\phi_G\colon E_G \to \mathbf{P}(V_G) \setminus \emptyset$. A *morphism* of hypergraphs $f\colon G \to H = (V_H, E_H)$ is given by the pair of maps $f_V : V_G \to V_H$ and $f_E\colon E_G \to E_H$ such that $\mathbf{P}(f_V) \circ \phi_G = \phi_H \circ f_E$, where

$$\mathbf{P}(f_V)\colon \mathbf{P}(V_G) \setminus \emptyset \to \mathbf{P}(V_H) \setminus \emptyset$$

is the map induced by f_V. So we may turn to the category of hypergraphs $\mathcal{H}$ in [10].

First, define a functor from category $\mathcal{DH}$ to category $\mathcal{H}$. For a finite set X, define a map $\sigma_X : \mathbb{P}(X) \to \mathbf{P}(X)$ by setting $\sigma_X(A, B) = A \cup B$. Let $G = (V, E)$ be a directed hypergraph. Define a hypergraph $\mathfrak{E}(G) = (V^e, E^e)$ where $V^e = V$ and

$$E^e = \{C \in \mathbf{P}(V) \setminus \emptyset \,|\, C = A \cup B, (A \to B) \in E\}. \tag{5}$$

Recall that in Section 2, for a directed hypergraph $G = (V, E)$ we defined a map $\varphi_G\colon E \to \mathbb{P}(V)$ by $\varphi_G(A \to B) = (A, B)$.

Proposition 3.12. *Let $f = (f_V, f_E)\colon G = (V_G, E_G) \to H = (V_H, E_H)$ be a morphism of directed hypergraphs. Define a map*

$$f^e_E\colon E^e_G \to \mathbf{P}(V_H)$$

putting

$$f^e_E(C) = [\mathbf{P}(f_V)](C)$$

for every $C = A \cup B \in E^e_G$. Then the map f^e_E is a well defined map $E^e_G \to E^e_H$ and the pair

$$(f^e_V, f^e_E) \text{ with } f^e_V = f_V$$

defines a morphism

$$\mathfrak{E}(f)\colon (V^e_G, E^e_G) \to (V^e_H, E^e_H)$$

of hypergraphs. Thus, we obtain a functor $\mathfrak{E}$ from the category $\mathcal{DH}$ of directed hypergraphs to the category $\mathcal{H}$ of hypergraphs.

Proof. The map f^e_E is well defined. Now we prove that its image lies in E^e_H. Let $C \in E^e_G$, $C = \sigma_{V_G} \circ \varphi_G(A \to B) = A \cup B$ and $f_{E_G}(A \to B) = (A' \to B') \in E_H$. Then, by Definition 2.2, $[\mathbb{P}(f_{V_G})](A, B) = (A', B') \in \mathbb{P}(V_G)$ and, hence, $A' = [\mathbf{P}(f_V)](A), B' = [\mathbf{P}(f_V)](B)$. We have

$$[\mathbf{P}(f_V)](C) = [\mathbf{P}(f_V)](A \cup B) = \{[\mathbf{P}(f_V)](A)\} \cup \{[\mathbf{P}(f_V)](B)\} = A' \cup B'.$$

However, $A' \cup B' = \sigma_{V_H} \circ \varphi_H(A' \to B') \in E^e_H$ and the claim that morphism f^e_E is well defined is proved. The functoriality is evident. □

For a hypergraph $G = (V, E)$, define a path complex $\mathfrak{H}^q(G) = (V^q, P^q_G)$ of *density* $q \geqslant 1$ where $V^q = V$ and a path $(i_0 \dots i_n) \in P_V$ lies $\in P^q_G$ iff every q consequent vertices of this path lie in a hyperedge $\mathbf{e}$, see [10]. Thus, we obtain a collection of functors $\mathfrak{H}^q$ from the category $\mathcal{H}$ to the category $\mathcal{P}$. Composition $\mathfrak{H}^q \circ \mathfrak{E}$ gives collection of functors from category $\mathcal{DH}$ to category $\mathcal{P}$. For a directed hypergraph G define

$$H^{\mathbf{e}(q)}_*(G) \,:=\, H_*(\mathfrak{H}^q \circ \mathfrak{E}(G)) \text{ for } \ q = 1, 2, \dots \ .$$

We call these groups by the *non-directed path homology groups of density* q of a directed hypergraph G. We denote $H^{\mathbf{e}}_*(G) \,:=\, H^{\mathbf{e}(1)}_*(G)$.

Proposition 3.13. *Let $G = (V, E)$ be a directed hypergraph and Π_V be a path complex of all paths on the set V. Then $H^{\mathbf{e}}_*(G) = H_*(\Pi_V)$.*

Proof. By Definition 2.1, $V = \cup_{\mathbf{e}_i \in E}(A_i \cup B_i)$ and every vertex $v \in V^e = V$ lies in an edge $e \in E^e$. So path complexes Π_V and $\mathfrak{H}^1 \circ \mathfrak{E}(G)$ coincide. □

Example 3.14. Now we compute path homology groups $H^{\mathbf{e}(q)}_*(G)$ of density $q = 1, 2, 3$ with coefficients in $\mathbb{R}$ of the directed hypergraph G with

$$V_G = \{1, 2, 3, 4, 5, 6\}, \quad E_G = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5\},$$

where

$$\mathbf{e}_1 = (\{1\} \to \{2\}), \mathbf{e}_2 = (\{1\} \to \{3\}), \mathbf{e}_3 = (\{2\} \to \{4, 6\}),$$

$$\mathbf{e}_4 = (\{3\} \to \{5\}), \mathbf{e}_5 = (\{4\} \to \{5, 6\}).$$

Then the hypergraph $\mathfrak{E}(G)$ has the set of vertices $V^e_G = \{1, 2, 3, 4, 5, 6\}$ and the set of hyperedges

$$E^e_G = \{\mathbf{e}'_1 = \{1, 2\}, \mathbf{e}'_2 = \{1, 3\}, \mathbf{e}'_3 = \{2, 4, 6\}, \mathbf{e}'_4 = \{3, 5\}, \mathbf{e}'_5 = \{4, 5, 6\}\}.$$

In the case of $q = 1$, the homology groups $H^{\mathbf{e}}_*(G)$ coincide with the path homology group of the complete digraph $D = (V_D, E_D)$ which has six vertices and for every two vertices $v, w \in V_D$ there are two arrows $(v \to w), (w \to v) \in E_D$. This digraph is contractible, and hence, see [**11**, S3.3], $H^{\mathbf{e}}_0(G) = \mathbb{R}$ and groups $H^{\mathbf{e}}_i(G)$ are trivial for $i \geqslant 1$.

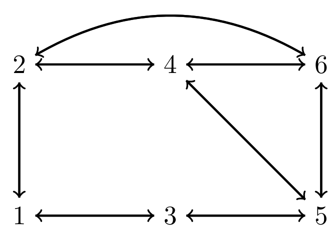


Figure 2: The digraph D_2 for $q = 2$.

If $q = 2$, homology groups $H^{\mathbf{e}(2)}_*(G)$ coincide with path homology group of the digraph D_2 on Fig. 2, where two-sided arrow $a \longleftrightarrow b$ means that there are arrows $a \to b$ and $b \to a$. The digraph D is homotopy equivalent to the induced sub-digraph $D'_2 \subset D_2$ with the set of vertices $\{1, 2, 3, 4, 5\}$. We compute directly the path homology of D'_2 and we obtain $H^{\mathbf{e}(2)}_0(G) = H^{\mathbf{e}(2)}_1(G) = \mathbb{R}$ and trivial groups $H^{\mathbf{e}}_i(G)$ for $i \geqslant 2$.

Now we consider the case of $\mathbf{e}(3)$. We have $\Omega^{\mathbf{e}(3)}_n = \Omega^{\mathbf{e}(2)}_n$ for $n = 0, 1$ and this equality is also true for all $n \geqslant 0$. We have

$$\mathcal{R}^{\mathbf{e}(3)^{reg}}_2 = A \oplus A_{246} \oplus A_{456},$$

where $A = \langle e_{121}, e_{212}, e_{131}, e_{313}, e_{353}, e_{535} \rangle$ and A_{abc} is the module generated by all regular paths with three vertices in the full digraph with vertices a, b, c. Hence $\Omega^{\mathbf{e}(3)}_2 = \mathcal{R}^{\mathbf{e}(3)^{reg}}_2$. Considering the digraph D_2, we obtain that $\Omega^{\mathbf{e}(3)}_2 = \Omega^{\mathbf{e}(2)}_2$. The cases with $n \geqslant 4$ are similar and $\Omega^{\mathbf{e}(3)}_n = \Omega^{\mathbf{e}(2)}_n$ for $n \geqslant 4$. Hence, $H^{\mathbf{e}(2)}_n(G) = H^{\mathbf{e}(3)}_n(G)$ for $n \geqslant 0$.

Proposition 3.15. *Let $G = (V, E)$ be a directed hypergraph, $I_1 = (0 \to 1)$, and $I = (V_I, E_I)$ be the hypergraph with the set of vertices $V_I = \{0, 1\}$ and the set of edges $E_I = \{\mathbf{e}'_0 = \{0\}, \mathbf{e}'_1 = \{1\}, \mathbf{e}'_2 = \{0, 1\}\}$. There is a natural inclusion of path complexes*

$$\mathfrak{H}^q[\mathfrak{E}(G\Box I_1)] \subset \mathfrak{H}^q[\mathfrak{E}(G) \times I] \tag{6}$$

for $q \geqslant 2$. Moreover, in general case complexes in (6) are not equal.

Proof. Recall that the product " $\times$ " of hypergraphs is defined in [**5**], [**10**]. Then the directed hypergraph $G\Box I_1 = (V_{G\Box I_1}, E_{G\Box I_1})$ has as its set of vertices $V_{G\Box I_1} = V \times \{0, 1\}$ and the set of edges that can be presented as the union $E_0 \cup E_1 \cup E_{01}$ of three pairwise disjoint sets

$$\begin{aligned} E_0 &= \{(C \times \{0\} \to D \times \{0\}) | (C \to D) \in E\}, \\ E_1 &= \{(C \times \{1\} \to D \times \{1\}) | (C \to D) \in E\}, \\ E_{01} &= \{A \times \{0\} \to A \times \{1\} | A \subset \mathbf{P}_{01}(G)\}. \end{aligned}$$

Hence, the hypergraph

$$\mathfrak{E}(G\Box I_1) = \big(V_{\mathfrak{E}(G\Box I_1)}, E_{\mathfrak{E}(G\Box I_1)}\big)$$

has the set of vertices $V_{\mathfrak{E}(G\Box I_1)} = V_{G\Box I_1} = V \times \{0, 1\}$ and the set of edges that can be presented as a union of three pairwise disjoint sets $E'_0 \cup E'_1 \cup E'_{01}$ where

$$\begin{aligned} E'_0 &= \{(C \times \{0\}) \cup (D \times \{0\}) | (C \to D) \in E\}, \\ E'_1 &= \{(C \times \{1\}) \cup (D \times \{1\}) | (C \to D) \in E\}, \\ E'_{01} &= \{(A \times \{0\}) \cup (A \times \{1\}) | A \subset \mathbf{P}_{01}(G)\}. \end{aligned} \tag{7}$$

By definition of a hypergraph $\mathfrak{E}(G) = (V^e, E^e)$, we obtain that the hypergraph

$$\mathfrak{E}(G) \times I = (V_{\mathfrak{E}(G)\times I}, E_{\mathfrak{E}(G)\times I})$$

has the set of vertices $V_{\mathfrak{E}(G)\times I} = V \times \{0, 1\}$ and the set of edges that can be presented as a union of three pairwise disjoint sets $E''_0 \cup E''_1 \cup E''_{01}$ where

$$\begin{aligned} E''_0 &= \{((C \cup D) \times \{0\}, C \cup D, \{0\}) \,|\, (C \to D) \in E\}, \\ E''_1 &= \{((C \cup D) \times \{1\}, C \cup D, \{0\}) \,|\, (C \to D) \in E\}, \\ E''_{01} &= \{(A, C \cup D, \{0, 1\}) \,|\, (C \to D) \in E, A \subset (C \cup D) \times \{0, 1\}\}. \end{aligned} \tag{8}$$

Let $p_1 \colon V \times \{0, 1\} \to V, p_2 \colon V \times \{0, 1\} \to \{0, 1\}$ be natural projections. Then $p_1(A) = C \cup D$ and $p_2(A) = \{0, 1\}$ by definition of the product of hypergraphs. Thus, path complexes

$$\mathfrak{H}^q\left[\mathfrak{E}(\Box I_1)\right]$$

and

$$\mathfrak{H}^q\left[\mathfrak{E}(G) \times I\right]$$

have the same vertex set and, by (7) and (8), $E'_0 = E''_0,\ E'_1 = E''_1,\ E'_{01} \subset E''_{01}$ and (6) follows.

Now we prove that in general case of (6) there is no equality. Let $(C \to D) \in E$ be a directed edge and $v \in C, w \in D$ be such vertices that the pair (v, w) does not lie in a set $A \in \mathbf{P}_{01}(G)$. Then the two vertex path $(v \times \{0\}), (w \times \{1\}) \in \mathfrak{H}^q\left[\mathfrak{E}(G) \times I\right]$ for $q = 2$ lies in E''_{01} and does not lie in $E'_0 \cup E'_1 \cup E'_{01}$. $\Box$

Lemma 3.16. *Let $G = (V, E)$ be a directed hypergraph, $I_1 = (0 \to 1)$. There is the inclusion $\mu\colon [\mathfrak{H}^2 \circ \mathfrak{E}(G)]^{\uparrow} \to \mathfrak{H}^2 \circ \mathfrak{E}(G \Box I_1)$ of path complexes.*

Proof. By definition of the hypergraph $\mathfrak{E}(G) = (V^e, E^e)$ and the functor $\mathfrak{H}^2$, we obtain that the set $P^2_{\mathfrak{E}(G)}$ of $\mathfrak{H}^2 \circ \mathfrak{E}(G)$ consists of paths $p = (i_0 \dots i_n)$ on the set V such that every two consequent vertices $i_s, i_{s+1} \in p$ lie in a common set of the form $\{C \cup D \,|\, (C \to D) \in E\}$. By definition, the set of paths of the path complex $[\mathfrak{H}^2 \circ \mathfrak{E}(G)]^{\uparrow}$ is a union of paths

$$P^2_{\mathfrak{E}(G)} \cup [P^2_{\mathfrak{E}(G)}]' \cup [P^2_{\mathfrak{E}(G)}]^{\#} \tag{9}$$

on the set $V \times \{0, 1\} = V \cup V'$. A path $p = (i_0 \dots i_n)$ on the set $V \times \{0, 1\}$ lies in $P^2_{\mathfrak{E}(G \Box I_1)}$ if any two consequent vertices lie in the exactly one of the sets E'_0, E'_1, E'_{01} defined in (7). From definition of the functor $\mathfrak{E}$, we conclude that in (9) any pair of consequent vertices of a path from $P^2_{\mathfrak{E}(G)}$ lies in an edge from E'_0, any pair of consequent vertices of a path from $[P^2_{\mathfrak{E}(G)}]'$ lies in an edge from E'_1, and any pair of consequent vertices of a path from $[P^2_{\mathfrak{E}(G)}]^{\#}$ lies in an edge from $E'_0 \cup E'_1 \cup E'_{01}$. □

Theorem 3.17. *For a directed hypergraph G, the non-directed path homology groups $H^{\mathfrak{e}(2)}_*(G)$ of density two are homotopy invariant.*

Proof. It is sufficient to prove homotopy invariance for the one-step homotopy. Let f_0, $f_1\colon G \to H$ be one-step homotopic morphisms of directed hypergraphs with a homotopy $F\colon G \Box I_1 \to H$. Since $\mathfrak{H}^2 \circ \mathfrak{E}$ is a functor, we obtain morphisms of path complexes

$$\mathfrak{H}^2 \circ \mathfrak{E}(f_0),\, \mathfrak{H}^2 \circ \mathfrak{E}(f_1)\colon \mathfrak{H}^2 \circ \mathfrak{E}(G) \to \mathfrak{H}^2 \circ \mathfrak{E}(H),$$

and

$$\mathfrak{H}^2 \circ \mathfrak{E}(F)\colon \mathfrak{H}^2 \circ \mathfrak{E}(G \Box I_1) \to \mathfrak{H}^2 \circ \mathfrak{E}(H).$$

Using Lemma 3.16, we can consider the composition

$$[\mathfrak{H}^2 \circ \mathfrak{E}(G)]^{\uparrow} \xrightarrow{\mu} \mathfrak{H}^2 \circ \mathfrak{E}(G \Box I_1) \xrightarrow{\mathfrak{H}^2 \circ \mathfrak{E}(F)} \mathfrak{H}^2 \circ \mathfrak{E}(H),$$

which gives a homotopy between morphisms $\mathfrak{H}^2 \circ \mathfrak{E}(f_0)$ and $\mathfrak{H}^2 \circ \mathfrak{E}(f_1)$. Now the result follows from [**10**, Th. 3.4]. □

3.4. Natural path homology

Let $G = (V, E)$ be a directed hypergraph. Define a digraph $\mathfrak{N}(G) = (V^n_G, E^n_G)$ where

$$V^n_G = \{C \in \mathbf{P}(V) \setminus \emptyset | C \in \mathbf{P}_{01}(G)\}$$

and

$$E^n_G = \{A \to B | (A \to B) \in E\}.$$

Thus a set $X \in \mathbf{P}(V) \setminus \emptyset$ is a vertex of the digraph $\mathfrak{N}(G)$ iff X is an origin or an end of an arrow $\mathbf{e} \in E$. Any arrow $\mathbf{e} = (A \to B) \in E$ gives an arrow $(A \to B) \in E^n$.

Proposition 3.18. *Every morphism of directed hypergraphs $f\colon G \to H$ defines a morphism of digraphs*

$$[\mathfrak{N}(f)] = (f_V^n, f_E^n)\colon (V_G^n, E_G^n) \to (V_H^n, E_H^n)$$

by

$$f_V^n(C) := [\mathbf{P}(f)] \circ \phi_G(C)$$

and

$$f_E^n(A \to B) = (f_V(A) \to f_V(B)) \in E_H^n.$$

Moreover, $\mathfrak{N}$ is a functor from the category $\mathcal{DH}$ to the category $\mathcal{D}$ of digraphs. ■

The composition $\mathfrak{D} \circ \mathfrak{N}$ gives a functor from $\mathcal{DH}$ to the category of path complexes. For a directed hypergraph G, we set $H_*^{\mathbf{n}}(G) := H_*(\mathfrak{D} \circ \mathfrak{N}(G))$. These homology groups will be called the *natural path homology groups* of G.

Example 3.19. Now we compute $H_*^{\mathbf{n}}(G)$ with coefficients in $\mathbb{R}$ of directed hypergraph $G = (V_G, E_G)$:

$$V_G = \{1,2,3,4,5,6,7,8\}, \quad E_G = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7, \mathbf{e}_8, \mathbf{e}_9\},$$

$$\mathbf{e}_1 = (\{1\} \to \{3,4\}), \mathbf{e}_2 = (\{1\} \to \{5,6\}), \mathbf{e}_3 = (\{1\} \to \{7,8\}),$$

$$\mathbf{e}_4 = (\{2\} \to \{3,4\}), \mathbf{e}_5 = (\{2\} \to \{5,6\}), \mathbf{e}_6 = (\{2\} \to \{7,8\}),$$

$$\mathbf{e}_7 = (\{3,4\} \to \{5,6\}), \mathbf{e}_8 = (\{5,6\} \to \{7,8\}), \mathbf{e}_9 = (\{7,8\} \to \{3,4\}).$$

The groups $H_*^{\mathbf{n}}(G)$ coincide with the regular path homology groups of the digraph given on Fig. 3.

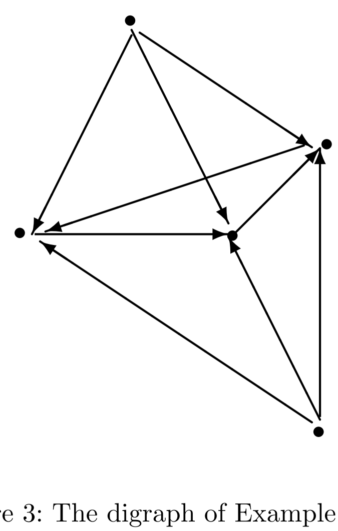


Figure 3: The digraph of Example 3.19.

Computation gives: $H_0^{\mathbf{n}}(G) = H_2^{\mathbf{n}}(G) = \mathbb{R}$ and $H_i^{\mathbf{n}}(G) = 0$ for other i.

Lemma 3.20. *Let $G = (V, E)$ be a directed hypergraph and $I_1 = (0 \to 1)$. There is an equality $\mathfrak{N}(G)\Box I_1 = \mathfrak{N}(G\Box I_1)$ of digraphs.*

Proof. The digraph $\mathfrak{N}(G)\Box I_1$ has

$$V_{\mathfrak{N}(G)\Box I_1} = \{C \times \{0,1\} |\, C \in \mathbf{P}_{01}(G)\}$$

and

$$E_{\mathfrak{N}(G)\Box I_1} = E_{0,1} \cup E_{0\to 1},$$

where

$$E_{0,1} = \{A \times \{i\} \to B \times \{i\} |\, A \to B \in E; i = 0, 1\},$$

$$E_{0\to 1} = \{C \times \{0\} \to C \times \{1\} |\, C \in \mathbf{P}_{01}(G)\}\,.$$

The digraph $\mathfrak{N}(G\Box I_1)$ has the set of vertices

$$V_{\mathfrak{N}(G\Box I_1)} = \{C \times \{0,1\} |\, C \in \mathbf{P}_{01}(G)\}\,,$$

which coincides with $V_{\mathfrak{N}(G)\Box I_1}$ and the set of edges $E_{\mathfrak{N}(G\Box I_1)} = E_{0,1} \cup E_{0\to 1}$ which coincides with $E_{\mathfrak{N}(G)\Box I_1}$. □

Theorem 3.21. *For a directed hypergraph G, the natural path homology groups $H_*^{\mathbf{n}}(G)$ are homotopy invariant.*

Proof. The path homology groups defined on the category of digraphs are homotopy invariant [**10**], [**11**] and thus, the result follows from Lemma 3.20. □

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Yuri Muranov muranov@matman.uwm.edu.pl

Faculty of Mathematics and Computer Science, University of Warmia and Mazury in Olsztyn, Słoneczna 54, 10-710 Olsztyn, Poland

Anna Szczepkowska anna.szczepkowska@matman.uwm.edu.pl

Faculty of Mathematics and Computer Science, University of Warmia and Mazury in Olsztyn, Słoneczna 54, 10-710 Olsztyn, Poland

Vladimir Vershinin vladimir.verchinine@univ-montp2.fr

Département des Sciences Mathématiques, Université de Montpellier, Place Eugéne Bataillon, 34095 Montpellier, cedex 5, France