

ON THE CONVERGENCE OF THE ORTHOGONAL SPECTRAL SEQUENCE

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Abstract

We show that the orthogonal spectral sequence introduced by the second author is strongly convergent in Voevodsky’s triangulated category of motives DM over a field k . In the context of the Morel–Voevodsky \mathbb{A}^1 -stable homotopy category we provide concrete examples where the spectral sequence is not strongly convergent, and give a criterion under which the strong convergence still holds. This criterion holds for Voevodsky’s slices, and as a consequence we obtain a spectral sequence which converges strongly to the E_1 -term of Voevodsky’s slice spectral sequence.

1. Introduction

1.1.

In order to study finite filtrations on the Chow groups of a smooth projective variety Y over a field k which satisfy some of the properties of the still conjectural Bloch–Beilinson–Murre filtration [Beĭ87], [Blo80], [Mur93], the second author introduced a tower of triangulated functors $bc_{\leq \bullet}$ [Pel17]:

$$\cdots \rightarrow bc_{\leq n-1} \rightarrow bc_{\leq n} \rightarrow bc_{\leq n+1} \cdots \rightarrow \quad (1.1.1)$$

in Voevodsky’s triangulated category of motives DM .

The filtration on the Chow groups with coefficients in a commutative ring R is defined by evaluating the tower (1.1.1) in the motive of a point $\mathbf{1}_R$ and then mapping $M(Y)(-q)[-2q]$ into $bc_{\leq \bullet}(\mathbf{1}_R)$, where $M(Y)$ is the motive of Y , and $(-q)$ (resp. $[-2q]$) is defined in terms of the Tate twist (resp. suspension) in DM , see §2.5. This process gives a filtration in the Chow groups since [Voe02a]:

$$CH^q(Y)_R \cong \mathrm{Hom}_{DM}(M(Y)(-q)[-2q], \mathbf{1}_R).$$

Given $A, B \in DM$, one may as well evaluate the tower (1.1.1) in A and then map B into $bc_{\leq \bullet}(A)$. Then one obtains a spectral sequence:

$$E_{p,q}^1 = \mathrm{Hom}_{DM}(B, (bc_{p/p-1}A)[q-p]) \Rightarrow \mathrm{Hom}_{DM}(B, A), \quad (1.1.2)$$

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where $bc_{p/p-1}A$ is defined in terms of a canonical distinguished triangle in DM [Pel17, 3.2.8]:

$$bc_{\leq p-1}A \rightarrow bc_{\leq p}A \rightarrow bc_{p/p-1}A.$$

The goal of this paper is to study the convergence properties of the spectral sequence (1.1.2).

Our main result (4.3) shows that the spectral sequence (1.1.2) is strongly convergent for $B = M(X)(r)[s]$, with $r, s \in \mathbb{Z}$ and X an arbitrary smooth scheme of finite type over k . We show as well that the analogous result does not hold in the Morel–Voevodsky \mathbb{A}^1 -stable homotopy category \mathcal{SH} by providing explicit counterexamples (5.1)–(5.3). On the other hand, we verify that under suitable conditions for $A \in \mathcal{SH}$ the spectral sequence is strongly convergent (5.4), and that this conditions are satisfied for Voevodsky’s slices $s_m G \in \mathcal{SH}$. As a direct consequence we obtain a spectral sequence converging strongly to the E_1 -term of Voevodsky’s slice spectral sequence (5.5).

In a future work, we will apply the spectral sequence (1.1.2) and its convergence properties to describe the higher terms for the filtration [Pel17, 6.14] mentioned above on the Chow groups of smooth projective varieties with rational coefficients.

We refer the reader to [Koh19] and [BK21] for filtrations which are constructed by a related process.

2. Preliminaries

In this section we fix the notation that will be used throughout the rest of the paper and collect together facts from the literature that will be necessary to establish our results.

2.1. Definitions and notation

We fix a base field k . We will write Sch_k for the category of k -schemes of finite type and Sm_k for the full subcategory of Sch_k consisting of smooth k -schemes regarded as a site with the Nisnevich topology.

We will use the following notation in all the categories under consideration: 0 will denote the zero object, and \cong will denote that a map (resp. a functor) is an isomorphism (resp. an equivalence of categories).

We shall use freely the language of triangulated categories. Our main reference will be [Nee01]. Given a triangulated category, we will write $[1]$ (resp. $[-1]$) to denote its suspension (resp. desuspension) functor; and for $n > 0$, the composition of $[1]$ (resp. $[-1]$) iterated n -times will be $[n]$ (resp. $[-n]$). If $n = 0$, then $[0]$ will be the identity functor. Given an inductive system $\cdots T_n \rightarrow T_{n+1} \rightarrow \cdots$, its homotopy colimit, $\text{hocolim}_{n \rightarrow \infty} T_n$ will be defined as in [Nee01].

2.2. Triangulated categories

Let \mathcal{T} be a compactly generated triangulated category in the sense of Neeman [Nee96, Def. 1.7] with set of compact generators \mathcal{G} . For $\mathcal{G}' \subseteq \mathcal{G}$, let $Loc(\mathcal{G}')$ denote the smallest full triangulated subcategory of \mathcal{T} which contains \mathcal{G}' and is closed under arbitrary (infinite) coproducts.

Definition 2.1. Let $\mathcal{T}' \subseteq \mathcal{T}$ be a triangulated subcategory. We will write \mathcal{T}'^\perp for the full subcategory of \mathcal{T} consisting of the objects $E \in \mathcal{T}$ such that for every $K \in \mathcal{T}'$: $\text{Hom}_{\mathcal{T}}(K, E) = 0$.

If $\mathcal{T}' = \text{Loc}(\mathcal{G}')$ and $E \in \mathcal{T}'^\perp$, we will say that E is \mathcal{G}' -orthogonal.

2.3. Slice and orthogonal towers

As in (2.2), consider a decreasing family of subsets of \mathcal{G} : $\mathcal{S} = \{\mathcal{G}_n\}_{n \in \mathbb{Z}}$ such that $\mathcal{G}_{n+1} \subseteq \mathcal{G}_n \subseteq \mathcal{G}$ for every $n \in \mathbb{Z}$.

Thus, we obtain a tower of full triangulated subcategories of \mathcal{T} :

$$\cdots \subseteq \text{Loc}(\mathcal{G}_{n+1}) \subseteq \text{Loc}(\mathcal{G}_n) \subseteq \text{Loc}(\mathcal{G}_{n-1}) \subseteq \cdots \tag{2.3.1}$$

We will call (2.3.1) the *slice tower* determined by \mathcal{S} . The reason for this terminology is [Voe02b], [HK06], [Voe10b, p. 18]. If we consider the orthogonal categories $\text{Loc}(\mathcal{G}_n)^\perp$ (see Def. 2.1), we obtain a tower of full triangulated subcategories of \mathcal{T} :

$$\cdots \subseteq \text{Loc}(\mathcal{G}_{n-1})^\perp \subseteq \text{Loc}(\mathcal{G}_n)^\perp \subseteq \text{Loc}(\mathcal{G}_{n+1})^\perp \subseteq \cdots \tag{2.3.2}$$

2.3.1. Orthogonal covers

Recall [Nee96, Thm. 4.1], [Pel17, 2.1.7(3)] that the inclusion, $j_n: \text{Loc}(\mathcal{G}_n)^\perp \rightarrow \mathcal{T}$ admits a right adjoint:

$$p_n: \mathcal{T} \rightarrow \text{Loc}(\mathcal{G}_n)^\perp,$$

which is also a triangulated functor. We define $bc_{\leq n} = j_{n+1} \circ p_{n+1}$, where bc stands for birational cover [Pel17, 3.2].

2.3.2.

The counit $bc_{\leq n} = j_{n+1} p_{n+1} \xrightarrow{\theta_n} id$ of the adjunction in (2.3.1) satisfies the following universal property (by an argument parallel to [Pel17, 3.2.4]):

For any A in \mathcal{T} and for any $B \in \text{Loc}(\mathcal{G}_{n+1})^\perp$, the map $\theta_n^A: bc_{\leq n} A \rightarrow A$ in \mathcal{T} induces an isomorphism of abelian groups:

$$\text{Hom}_{\mathcal{T}}(B, bc_{\leq n} A) \xrightarrow[\cong]{\theta_n^A} \text{Hom}_{\mathcal{T}}(B, A) .$$

2.3.3.

Observe that by construction $bc_{\leq n} A$ is in $\text{Loc}(\mathcal{G}_{n+1})^\perp$ (2.3.1) and in addition we have $\text{Loc}(\mathcal{G}_{n+1})^\perp \subseteq \text{Loc}(\mathcal{G}_{n+2})^\perp$ (2.3.2). Thus, it follows from (2.3.2) that there exists a canonical natural transformation $bc_{\leq n} \rightarrow bc_{\leq n+1}$ and that $bc_{\leq n} \circ bc_{\leq n+1} \cong bc_{\leq n}$ [Pel17, 3.2.6] (the argument works for any compactly generated triangulated category). Hence, for every A in \mathcal{T} there is a functorial tower in \mathcal{T} [Pel17, 3.2.14, 3.2.15]:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & bc_{\leq n} A & \longrightarrow & bc_{\leq n+1} A & \longrightarrow & \cdots \longrightarrow \text{hocolim}_{n \rightarrow \infty} bc_{\leq n} A \\ & & \searrow \theta_n^A & & \downarrow \theta_{n+1}^A & & \swarrow c \\ & & & & A & & \end{array} \tag{2.3.3}$$

where all the triangles commute. We will call (2.3.3) the orthogonal tower of A .

Definition 2.2. For A, B in \mathcal{T} we consider the increasing filtration F_\bullet of the Abelian group $\text{Hom}_{\mathcal{T}}(B, A)$ (resp. $\text{Hom}_{\mathcal{T}}(B, \text{hocolim}_{n \rightarrow \infty} bc_{\leq n} A)$), where F_p is given by the image of

$$\begin{aligned} \theta_{p*}^A &: \text{Hom}_{\mathcal{T}}(B, bc_{\leq p} A) \rightarrow \text{Hom}_{\mathcal{T}}(B, A), \\ (\text{resp. } \lambda_{p,*}^A &: \text{Hom}_{\mathcal{T}}(B, bc_{\leq p} A) \rightarrow \text{Hom}_{\mathcal{T}}(B, \text{hocolim}_{n \rightarrow \infty} bc_{\leq n} A)), \end{aligned}$$

where $\lambda_p^A: bc_{\leq p} A \rightarrow \text{hocolim}_{n \rightarrow \infty} bc_{\leq n} A$ is the canonical map into the homotopy colimit (2.3.3).

2.4. The orthogonal spectral sequence

Let A, B be in \mathcal{T} . By an argument parallel to [Pel17, Thm. 3.2.8] for $p \in \mathbb{Z}$ there exist canonical triangulated functors $bc_{p/p-1}: \mathcal{T} \rightarrow \mathcal{T}$, equipped with natural transformations $bc_{\leq p} \rightarrow bc_{p/p-1}$ and $bc_{p/p-1} \rightarrow [1] \circ bc_{\leq p-1}$, which fit in a natural distinguished triangle in \mathcal{T} :

$$bc_{\leq p-1} A \rightarrow bc_{\leq p} A \rightarrow bc_{p/p-1} A \rightarrow bc_{\leq p-1} A[1]. \tag{2.4.1}$$

Then (2.3.3) induces a spectral sequence of homological type [Pel17, Thm. 3.2.16]:

$$E_{p,q}^1 = \text{Hom}_{\mathcal{T}}(B, (bc_{p/p-1} A)[q-p]) \Rightarrow \text{Hom}_{\mathcal{T}}(B, A) \tag{2.4.2}$$

with differentials $d_r: E_{p,q}^r \rightarrow E_{p-r,q-r+1}^r$ and where the abutment is given by the associated graded group for the increasing filtration (2.2) F_\bullet of $\text{Hom}_{\mathcal{T}}(B, A)$.

Similarly, the horizontal row in (2.3.3) induces a spectral sequence of homological type:

$$E_{p,q}^1 = \text{Hom}_{\mathcal{T}}(B, (bc_{p/p-1} A)[q-p]) \Rightarrow \text{Hom}_{\mathcal{T}}(B, \text{hocolim}_{n \rightarrow \infty} bc_{\leq n} A) \tag{2.4.3}$$

with exactly the same differentials as (2.4.2) and where the abutment is given by the associated graded group for the increasing filtration (2.2) F_\bullet of the Abelian group $\text{Hom}_{\mathcal{T}}(B, \text{hocolim}_{n \rightarrow \infty} bc_{\leq n} A)$.

Now, we observe that the map c in (2.3.3) induces a map of spectral sequences (2.4.3) \rightarrow (2.4.2) which is the identity on the E_1 -terms:

$$\begin{array}{ccc} E_{p,q}^1 = \text{Hom}_{\mathcal{T}}(B, (bc_{p/p-1} A)[q-p]) & \Longrightarrow & \text{Hom}_{\mathcal{T}}(B, \text{hocolim}_{n \rightarrow \infty} bc_{\leq n} A) \\ \parallel & & \downarrow c_* \\ E_{p,q}^1 = \text{Hom}_{\mathcal{T}}(B, (bc_{p/p-1} A)[q-p]) & \Longrightarrow & \text{Hom}_{\mathcal{T}}(B, A). \end{array} \tag{2.4.4}$$

2.5. Voevodsky’s triangulated category of motives

We will only consider motives with R -coefficients, where $R = \mathbb{Z}[\frac{1}{p}]$ and p is the exponential characteristic of the base field k .

Let Cor_k be the Suslin–Voevodsky category of finite correspondences over k , i.e. the category with the same objects as Sm_k and morphisms $c(U, V)$ given by the R -module of finite relative cycles on $U \times_k V$ over U [SV00] with composition as in [Voe10a, p. 673 diagram (2.1)]. The graph of a morphism in Sm_k induces a functor $\Gamma: Sm_k \rightarrow Cor_k$. A Nisnevich sheaf with transfers is an additive contravariant

functor \mathcal{F} from Cor_k to the category of R -modules such that the restriction $\mathcal{F} \circ \Gamma$ is a Nisnevich sheaf. Let Shv^{tr} be the category of Nisnevich sheaves with transfers which is an abelian category [MVW06, 13.1]. Given $X \in Sm_k$, we will write $\mathbb{Z}_{tr}(X)$ for the Nisnevich sheaf with transfers represented by X [MVW06, 2.8 and 6.2].

Consider the category $K(Shv^{tr})$ of chain complexes (unbounded) on this Shv^{tr} , equipped with the injective model structure [Bek00, Prop. 3.13], and let $D(Shv^{tr})$ be its homotopy category. Let $K^{\mathbb{A}^1}(Shv^{tr})$ be the left Bousfield localization [Hir03, 3.3] of $K(Shv^{tr})$ with respect to the set of maps

$$\{\mathbb{Z}_{tr}(X \times_k \mathbb{A}^1)[n] \rightarrow \mathbb{Z}_{tr}(X)[n] : X \in Sm_k; n \in \mathbb{Z}\}$$

induced by the projections $p: X \times_k \mathbb{A}^1 \rightarrow X$. Voevodsky’s triangulated category of effective motives DM^{eff} is the homotopy category of $K^{\mathbb{A}^1}(Shv^{tr})$ [Voe00].

Let $T \in K^{\mathbb{A}^1}(Shv^{tr})$ denote the chain complex of the form $\mathbb{Z}_{tr}(\mathbb{G}_m)[1]$ [MVW06, 2.12], where \mathbb{G}_m is the k -scheme $\mathbb{A}^1 \setminus \{0\}$ pointed by 1. We consider the category $Spt_T(Shv^{tr})$ of symmetric T -spectra on $K^{\mathbb{A}^1}(Shv^{tr})$, equipped with the model structure defined in [Hov01, 8.7 and 8.11], [Ayo07, Def. 4.3.29]. Voevodsky’s triangulated category of motives DM is the homotopy category of $Spt_T(Shv^{tr})$ [Voe00].

Given $X \in Sm_k$, we will write $M(X)$ for the image of $\mathbb{Z}_{tr}(X) \in D(Shv^{tr})$, under the \mathbb{A}^1 -localization map $D(Shv^{tr}) \rightarrow DM^{eff}$. Let $\Sigma^\infty: DM^{eff} \rightarrow DM$ be the suspension functor [Hov01, 7.3] (denoted by F_0 in *loc.cit.*), we will abuse notation and simply write E for $\Sigma^\infty E$, where $E \in DM^{eff}$. Given a map $f: X \rightarrow Y$ in Sm_k , we will write $f: M(X) \rightarrow M(Y)$ for the map induced by f in DM .

Notice that, DM^{eff} and DM are tensor triangulated categories [Ayo07, Thm. 4.3.76 and Prop. 4.3.77] with unit $\mathbf{1} = M(\text{Spec}(k))$. Given $A \in DM$, we will write $A(1)$ for $A \otimes \mathbb{Z}_{tr}(\mathbb{G}_m)[-1]$, and inductively $A(n) = (A(n-1))(1)$, when $n > 0$. We observe that $A \mapsto A(1)$ defines a functor $DM \rightarrow DM$, which is an equivalence of categories [Hov01, 8.10], [Ayo07, Thm. 4.3.38]; we will write $A \mapsto A(-1)$ for its inverse, and inductively $A(-n) = (A(-n+1))(-1)$, when $n > 0$. By convention $A(0) = A$ for $A \in DM$.

2.5.1. Generators

It is well known that DM is a compactly generated triangulated category (2.2) with compact generators [Ayo07, Thm. 4.5.67]:

$$\mathcal{G}_{DM} = \{M(X)(p) : X \in Sm_k; p \in \mathbb{Z}\}. \tag{2.5.1}$$

Let $\mathcal{G}^{eff} \subseteq \mathcal{G}_{DM}$ be the set consisting of compact objects of the form:

$$\mathcal{G}^{eff} = \{M(X)(p) : X \in Sm_k; p \geq 0\}. \tag{2.5.2}$$

If $n \in \mathbb{Z}$, we will write $\mathcal{G}^{eff}(n) \subseteq \mathcal{G}_{DM}$ for the set consisting of compact objects of the form:

$$\mathcal{G}^{eff}(n) = \{M(X)(p) : X \in Sm_k; p \geq n\}. \tag{2.5.3}$$

2.5.2.

By Voevodsky’s cancellation theorem [Voe10a], the suspension functor induces an equivalence of categories $\Sigma^\infty: DM^{eff} \rightarrow DM$ between DM^{eff} and the full triangulated subcategory $Loc(\mathcal{G}^{eff})$ of DM (2.2). We will abuse notation and write DM^{eff} for

$Loc(\mathcal{G}^{\text{eff}})$. Strictly speaking [Voe10a, Cor. 4.10] is only stated for perfect base fields, but by the work of Suslin [Sus17, Cor. 4.13, Thm. 4.12 and Thm. 5.1] it follows that the result holds as well for non-perfect base fields.

2.5.3.

We will write $DM^{\text{eff}}(n)$ for the full triangulated subcategory $Loc(\mathcal{G}^{\text{eff}}(n))$ of DM (2.2), and $DM^\perp(n)$ for the orthogonal category $Loc(\mathcal{G}^{\text{eff}}(n))^\perp$ (see Def. 2.1). Notice that $DM^{\text{eff}}(n)$ is compactly generated with the set of generators $\mathcal{G}^{\text{eff}}(n)$ [Nee96, Thm. 2.1(2.1.1)].

2.5.4.

We will consider (2.3.1), (2.3.2), (2.3.3), (2.2), (2.4.2), (2.4.3) and (2.4.4) in DM for the family $\mathcal{S} = \{\mathcal{G}^{\text{eff}}(n)\}_{n \in \mathbb{Z}}$ of subsets of \mathcal{G}_{DM} (2.5.3).

2.6. The Morel–Voevodsky \mathbb{A}^1 -stable homotopy category

We refer the reader to [Jar00, §4.2, Thm. 4.15] for the construction of the stable model structure on the category of symmetric T -spectra. We will write \mathcal{SH} for its homotopy category, which is the Morel–Voevodsky \mathbb{A}^1 -stable homotopy category.

Given $X \in Sm_k$, let $\Sigma_T^\infty X_+ \in \mathcal{SH}$ denote the infinite suspension of the simplicial presheaf represented by X with a disjoint base point (written $F_0(X_+)$ in [Jar00, p. 506]). By [Jar00, Prop. 4.19], \mathcal{SH} is a tensor triangulated category with unit $\mathbf{1} = \Sigma_T^\infty \text{Spec } k_+$. When $E \in \mathcal{SH}$, we will write $E(1)$ for $E \otimes \Sigma_T^\infty(\mathbb{G}_m)[-1]$, and inductively $E(n) = (E(n-1))(1)$, if $n > 0$. We observe that $E \mapsto E(1)$ then defines a functor $\mathcal{SH} \rightarrow \mathcal{SH}$, which is an equivalence of categories [Hov01, 8.10], [Ayo07, Thm. 4.3.38]; we will write $E \mapsto E(-1)$ for its inverse, and inductively we write $E(-n) = (E(-n+1))(-1)$, when $n > 0$. By convention $E(0) = E$ for $E \in \mathcal{SH}$.

As in the case of DM , it follows from [Ayo07, Thm. 4.5.67] that \mathcal{SH} is a compactly generated triangulated category (2.2) with compact generators:

$$\mathcal{G}_{\mathcal{SH}} = \{\Sigma_T^\infty X_+(p) : X \in Sm_k; p \in \mathbb{Z}\}. \tag{2.6.1}$$

For $n \in \mathbb{Z}$, we will write $\mathcal{G}_{\mathcal{SH}}^{\text{eff}}(n) \subseteq \mathcal{G}_{\mathcal{SH}}$ for the set consisting of compact objects of the form:

$$\mathcal{G}_{\mathcal{SH}}^{\text{eff}}(n) = \{\Sigma_T^\infty X_+(p) : X \in Sm_k; p \geq n\}. \tag{2.6.2}$$

2.6.1.

Let $\mathcal{SH}^{\text{eff}}(n)$ be the full triangulated subcategory $Loc(\mathcal{G}_{\mathcal{SH}}^{\text{eff}}(n))$ of \mathcal{SH} (2.2), and $\mathcal{SH}^\perp(n)$ be the orthogonal category $Loc(\mathcal{G}_{\mathcal{SH}}^{\text{eff}}(n))^\perp$ (see Def. (2.1)). Notice that $\mathcal{SH}^{\text{eff}}(n)$ is compactly generated with the set of generators $\mathcal{G}_{\mathcal{SH}}^{\text{eff}}(n)$ [Nee96, Thm. 2.1(2.1.1)].

2.6.2.

We will consider (2.3.1), (2.3.2), (2.3.3), (2.2), (2.4.2), (2.4.3) and (2.4.4) in \mathcal{SH} for the family $\mathcal{S} = \{\mathcal{G}_{\mathcal{SH}}^{\text{eff}}(n)\}_{n \in \mathbb{Z}}$ of subsets of $\mathcal{G}_{\mathcal{SH}}$ (2.6.2). These were constructed in [Pel14].

3. Orthogonality and duality

3.1.

Recall that we are working with $\mathbb{Z}[\frac{1}{p}]$ -coefficients (2.5). In this section we will consider $Y \in Sm_k$ connected of dimension d , and $s, t \in \mathbb{Z}$.

Proposition 3.1. *With the notation and conditions of §3.1. Then:*

$$M(Y)(s)[t] \in DM^\perp(d + s + 1).$$

(see (2.5.3) and Def. 2.1).

Proof. By [Pel17, 2.1.2] it suffices to show that $\text{Hom}_{DM}(M(X)(a)[b], M(Y)(s)[t]) = 0$, for every $X \in Sm_k$, $a, b \in \mathbb{Z}$ such that $a \geq d + s + 1$. So, by [Sus17, Cor. 4.13, Thm. 4.12 and Thm. 5.1] we may assume that the base field k is perfect. Now, if the base field k admits resolution of singularities, it follows from [Voe00, Thm. 4.3.7] that:

$$\text{Hom}_{DM}(M(X)(a)[b], M(Y)(s)[t]) \cong \text{Hom}_{DM}(M(X) \otimes M^c(Y)(e)[f], \mathbf{1}),$$

where $M^c(Y) \in DM^{\text{eff}}$ is the motive of Y with compact supports [Voe00, §4.1, Cor. 4.1.6], $e = a - s - d$ and $f = b - t - 2d$. For an arbitrary perfect base field, we obtain the same conclusion by [Kel17, Thm. 5.3.14 and Lem. 5.3.6].

Therefore, by [Pel17, 5.1.1] it suffices to check that we have $M(X) \otimes M^c(Y)(e)[f] \in DM^{\text{eff}}(1)$ (2.5.3), which holds by hypothesis: $e = a - s - d \geq 1$. \square

Corollary 3.2. *With the notation and conditions of §3.1. Let $E = M(Y)(s)[t] \in DM$. Then:*

1. The natural map (2.3.2):

$$\theta_{d+s}^E: bc_{\leq d+s}E \rightarrow E$$

is an isomorphism in DM .

2. For any $A \in DM$, and any map $f: E \rightarrow A$ in DM , there exists a unique lifting $g: E \rightarrow bc_{\leq d+s}A$ such that the following diagram commutes in DM :

$$\begin{array}{ccc}
 & bc_{\leq d+s}A & \\
 g \nearrow & & \downarrow \theta_{d+s}^A \\
 E & \xrightarrow{f} & A.
 \end{array} \tag{3.1.1}$$

3. The map f in (3.1.1) is zero if and only if the map g in (3.1.1) is zero.

Proof. (1): This follows directly by combining Prop. 3.1 with [Pel17, 3.2.7].

(2) and (3) follow from Prop. 3.1 and the universal property of θ_{d+s}^A (2.3.2). \square

4. Convergence

4.1.

In this section we will consider objects $A, B \in DM$, where B is of the form $B = M(X)(s)[t]$ for $X \in Sm_k$ and $s, t \in \mathbb{Z}$.

Theorem 4.1. *With the notation and conditions of §4.1. Then the spectral sequence (2.4.3) is strongly convergent [Boa99, Def. 5.2(iii)].*

Proof. Since $B = M(X)(s)[t]$ is compact in DM (2.5.1), it follows from [Nee96, Lem. 2.8] that

$$\mathrm{Hom}_{DM}(B, \mathrm{hocolim}_{n \rightarrow \infty} bc_{\leq n} A) \cong \mathrm{colim}_{n \rightarrow \infty} \mathrm{Hom}_{DM}(B, bc_{\leq n} A),$$

which implies that the filtration (2.2) F_\bullet on $\mathrm{Hom}_{DM}(B, \mathrm{hocolim}_{n \rightarrow \infty} bc_{\leq n} A)$ is exhaustive.

Now, we observe that $B \in DM^{\mathrm{eff}}(s)$ for every $t \in \mathbb{Z}$ (2.5.3) and by construction $bc_{\leq n} A \in DM^\perp(n+1)$ (2.3.1), so we deduce that $\mathrm{Hom}_{DM}(B, bc_{\leq n} A) = 0$ for all $n \leq s-1$ and every $t \in \mathbb{Z}$ (see Def. 2.1). Hence, applying the distinguished triangle (2.4.1) we conclude that $E_{p,q}^1 = 0$ for $p \leq s-1$. Then, [Boa99, Thm. 6.1(a)] implies that the spectral sequence is strongly convergent since the differentials are of the form $d_r: E_{p,q}^r \rightarrow E_{p-r,q-r+1}^r$ (notice that our notation is homological while Boardman’s is cohomological, see [Boa99, (12.1) and Thm. 12.2] for an explicit comparison). \square

Corollary 4.2. *With the notation and conditions of §4.1. Assume that the canonical map $c: \mathrm{hocolim}_{n \rightarrow \infty} bc_{\leq n} A \rightarrow A$ (2.3.3) induces an isomorphism of abelian groups:*

$$c_*: \mathrm{Hom}_{DM}(B, \mathrm{hocolim}_{n \rightarrow \infty} bc_{\leq n} A) \xrightarrow{\cong} \mathrm{Hom}_{DM}(B, A).$$

Then the spectral sequence (2.4.2) is strongly convergent [Boa99, Def. 5.2(iii)].

Proof. Follows directly by combining (2.4.4) with Thm. 4.1. \square

The following is the main theorem:

Theorem 4.3. *With the notation and conditions of §4.1. Then the canonical map*

$$c: \mathrm{hocolim}_{n \rightarrow \infty} bc_{\leq n} A \xrightarrow{\cong} A$$

is an isomorphism in DM . Hence, the spectral sequence (2.4.2) is strongly convergent.

Proof. By (4.2) it is enough to show that c is an isomorphism in DM . In order to prove this, it suffices to see (2.5.1) that for every $a, b \in \mathbb{Z}$ and every connected $Y \in Sm_k$ the induced map:

$$c_*: \mathrm{Hom}_{DM}(M(Y)(a)[b], \mathrm{hocolim}_{n \rightarrow \infty} bc_{\leq n} A) \rightarrow \mathrm{Hom}_{DM}(M(Y)(a)[b], A)$$

is an isomorphism of abelian groups.

First we show that c_* is surjective. In effect given $f: M(Y)(a)[b] \rightarrow A$ we obtain a lifting by 3.2(2):

$$\begin{array}{ccc} & & bc_{\leq d+a} A \\ & \nearrow & \downarrow \theta_{d+a} \\ M(Y)(a)[b] & \xrightarrow{f} & A, \end{array}$$

where d is the dimension of Y . Then the surjectivity follows by (2.3.3).

Finally we consider the injectivity. Let $f: M(Y)(a)[b] \rightarrow \operatorname{hocolim}_{n \rightarrow \infty} bc_{\leq n}A$ such that the composition $c_*(f): M(Y)(a)[b] \rightarrow A$ is zero. Since $M(Y)(a)[b]$ is compact in DM (2.5.1), we conclude that [Nee96, Lem. 2.8]:

$$\operatorname{Hom}_{DM}(M(Y)(a)[b], \operatorname{hocolim}_{n \rightarrow \infty} bc_{\leq n}A) \cong \operatorname{colim}_{n \rightarrow \infty} \operatorname{Hom}_{DM}(M(Y)(a)[b], bc_{\leq n}A).$$

Thus we may assume that f factors as:

$$\begin{array}{ccc} & & bc_{\leq n}A \\ & \nearrow f' & \downarrow \\ M(Y)(a)[b] & \xrightarrow{f} & \operatorname{hocolim}_{n \rightarrow \infty} bc_{\leq n}A \end{array}$$

for some $n \geq d + a$. Now, applying again 3.2(2) we may factor f' as follows:

$$\begin{array}{ccc} & & bc_{\leq d+a}A \cong bc_{\leq d+a}(bc_{\leq n}A) \\ & \nearrow f'' & \downarrow \\ M(Y)(a)[b] & \xrightarrow{f'} & bc_{\leq n}A, \end{array}$$

where the isomorphism follows from [Pel17, 3.2.6] since $d + a \leq n$. Thus it suffices to show that f'' is zero. But this follows from 3.2(3) since

$$\theta_{d+a}^A \circ f'' = c_*(f) = 0,$$

where the first equality follows from (2.3.3) and the two commutative triangles above while the second equality follows by hypothesis. \square

Remark 4.4. We observe that Thm. 4.1 and (4.2) hold for a compactly generated triangulated category \mathcal{T} with compact generators \mathcal{G} and any choice of a family of subsets of \mathcal{G} : $\mathcal{S} = \{\mathcal{G}_n\}_{n \in \mathbb{Z}}$ satisfying the conditions in (2.3) and in addition $\cup_{n \in \mathbb{Z}} \mathcal{G}_n = \mathcal{G}$.

However, (4.3) does not hold for a general compactly generated triangulated category as we will see in the next section.

5. The \mathbb{A}^1 -stable homotopy category

5.1.

In this section we show that (4.3) does not hold for the sphere spectrum in \mathcal{SH} . On the other hand, we show that (4.3) holds for objects in $\mathcal{SH}^\perp(n)$, $n \in \mathbb{Z}$. As a direct consequence we obtain a spectral sequence converging to the E_1 -term of Voevodsky’s slice spectral sequence.

Proposition 5.1. *The canonical map $c: \operatorname{hocolim}_{n \rightarrow \infty} bc_{\leq n}\mathbf{1} \rightarrow \mathbf{1}$ (2.3.3) is not an isomorphism in \mathcal{SH} .*

Proof. We proceed by contradiction, and assume that c is an isomorphism in \mathcal{SH} .

Then, since $\mathbf{1} \in \mathcal{SH}$ is compact we conclude [Nee96, Lem. 2.8]:

$$\operatorname{colim}_{n \rightarrow \infty} \operatorname{Hom}_{\mathcal{SH}}(\mathbf{1}, bc_{\leq n} \mathbf{1}) \cong \operatorname{Hom}_{\mathcal{SH}}(\mathbf{1}, \operatorname{hocolim}_{n \rightarrow \infty} bc_{\leq n} \mathbf{1}) \xrightarrow[\cong]{c_*} \operatorname{Hom}_{\mathcal{SH}}(\mathbf{1}, \mathbf{1}).$$

Thus, for some $n \in \mathbb{Z}$ the identity map for $\mathbf{1}$ factors as in the following commutative diagram in \mathcal{SH} :

$$\begin{array}{ccc} & & bc_{\leq n} \mathbf{1} \\ & \nearrow & \downarrow \theta_n^{\mathbf{1}} \\ \mathbf{1} & \xrightarrow{id} & \mathbf{1}. \end{array}$$

Hence, $\mathbf{1} \oplus E \cong bc_{\leq n} \mathbf{1} \in \mathcal{SH}^{\perp}(n+1)$ (2.3.1) for some $E \in \mathcal{SH}$. Since $\mathcal{SH}^{\perp}(n+1)$ is closed under direct summands (see Def. 2.1), we deduce that $\mathbf{1} \in \mathcal{SH}^{\perp}(n+1)$. But this is a contradiction since it implies that for every $m \geq n$, the slice functors of Voevodsky [Voe02b, Thm. 2.2] vanish $s_m \mathbf{1} = 0$ for the sphere spectrum, which is not the case [Voe02b, Conj. 9], [Lev14, p. 350]. \square

Remark 5.2. The argument above shows that Prop. 5.1 holds for any compact object $A \in \mathcal{SH}$ such that for every $n \in \mathbb{Z}$ there exists $m \geq n$ with $s_m A \neq 0$.

Corollary 5.3. *Consider the spectral sequence (2.4.2) in \mathcal{SH} for $A = \mathbf{1}$. Then the spectral sequence is not strongly convergent for every $B = \Sigma_T^{\infty} X_+(s)[t]$, $X \in Sm_k$, $s, t \in \mathbb{Z}$.*

Proof. We proceed by contradiction and assume that the spectral sequence is strongly convergent for every B as above. Then combining (2.4.4) and Thm. 4.1 (which also holds in \mathcal{SH} (4.4)) we conclude that

$$c_* : \operatorname{Hom}_{\mathcal{SH}}(B, \operatorname{hocolim}_{n \rightarrow \infty} bc_{\leq n} \mathbf{1}) \rightarrow \operatorname{Hom}_{\mathcal{SH}}(B, \mathbf{1})$$

is an isomorphism. But this implies that c is an isomorphism since \mathcal{SH} is a compactly generated category with generators $\mathcal{G}_{\mathcal{SH}}$ (2.6.1). \square

However, the spectral sequence (2.4.2) is strongly convergent for a large class of objects in \mathcal{SH} :

Proposition 5.4. *Let $A \in \mathcal{SH}$ such that for some $r \in \mathbb{Z}$, $A \in \mathcal{SH}^{\perp}(r)$ (2.6.1). Then the canonical map*

$$c : \operatorname{hocolim}_{n \rightarrow \infty} bc_{\leq n} A \xrightarrow{\cong} A$$

is an isomorphism in \mathcal{SH} . Hence, the spectral sequence (2.4.2) is strongly convergent for every $B = \Sigma_T^{\infty} X_+(s)[t]$, $X \in Sm_k$, $s, t \in \mathbb{Z}$.

Proof. By (4.2) (which holds as well in \mathcal{SH} (4.4)) it is enough to show that c is an isomorphism in \mathcal{SH} .

Let $m \geq r$ be an arbitrary integer. It follows from (2.3.2) that $A \in \mathcal{SH}^{\perp}(m)$, so by the universal property (2.3.2) we conclude that $\theta_m^A : bc_{\leq m} A \rightarrow A$ is an isomorphism in \mathcal{SH} (see [Pel17, 2.3.7]). But this implies that the canonical map $c : \operatorname{hocolim}_{n \rightarrow \infty} bc_{\leq n} A \rightarrow A$ (2.3.3) is an isomorphism in \mathcal{SH} . \square

5.1.1.

Let $B \in \mathcal{SH}$ with $B = \Sigma_T^\infty X_+(s)[t]$, $X \in Sm_k$, $s, t \in \mathbb{Z}$. Consider Voevodsky’s slice spectral sequence [Voe02b, §7] for $G \in \mathcal{SH}$:

$$E_1^{m,n} = \text{Hom}_{\mathcal{SH}}(B, s_m G[m+n]) \Rightarrow \text{Hom}_{\mathcal{SH}}(B, G).$$

A direct consequence of (5.4) is the fact that we obtain a spectral sequence which converges strongly to the E_1 -term of Voevodsky’s slice spectral sequence and which is compatible with the differentials $d_1: E_1^{m,n} \rightarrow E_1^{m+1,n}$:

Corollary 5.5. *With the notation and conditions of (5.1.1). Then the spectral sequence (2.4.2) for $A = s_m G[m+n]$ converges strongly to the E_1 -term of Voevodsky’s slice spectral sequence $E_1^{m,n} = \text{Hom}_{\mathcal{SH}}(B, s_m G[m+n])$. In addition, the differential $d_1: E_1^{m,n} \rightarrow E_1^{m+1,n}$ in Voevodsky’s slice spectral sequence induces a map between the spectral sequences:*

$$\begin{array}{ccc} E_{p,q}^1 = \text{Hom}_{\mathcal{SH}}(B, (bc_{p/p-1} s_m G[m+n])[q-p]) & \Longrightarrow & E_1^{m,n} \\ \downarrow d_{1*} & & \downarrow d_1 \\ E_{p,q}^1 = \text{Hom}_{\mathcal{SH}}(B, (bc_{p/p-1} s_{m+1} G[m+1+n])[q-p]) & \Longrightarrow & E_1^{m+1,n}. \end{array} \tag{5.1.1}$$

Proof. By construction $s_m G \in \mathcal{SH}^-(m+1)$ [Voe02b, Thm. 2.2(3)]. Thus the strong convergence of (2.4.2) for $A = s_m G[m+n]$ follows directly from (5.4).

We observe that the differential $d_1: E_1^{m,n} \rightarrow E_1^{m+1,n}$ in the slice spectral sequence is induced by the map $\partial[m+n]$ in \mathcal{SH} where ∂ is the following composition [Voe02b, Thm. 2.2(1)]:

$$s_m G \xrightarrow{\sigma_m} f_{m+1} G[1] \xrightarrow{\pi_{m+1}[1]} s_{m+1} G[1].$$

Since the tower (2.3.3) is functorial in \mathcal{SH} we conclude that $\partial[m+n]$ induces the desired map of spectral sequences (5.1.1). □

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