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THE HOMOTOPY TYPES OF $Sp(n)$-GAUGE GROUPS OVER $\mathbb{C}P^2$

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Abstract

Let $n > 2$ and $\mathcal{G}_k(\mathbb{C}P^2)$ be the gauge groups of the principal $Sp(n)$-bundles over $\mathbb{C}P^2$. In this article we partially classify the homotopy types of $\mathcal{G}_k(\mathbb{C}P^2)$ by showing that if there is a homotopy equivalence $\mathcal{G}_k(\mathbb{C}P^2) \simeq \mathcal{G}_{k'}(\mathbb{C}P^2)$ then $(k, 4n(2n+1)) = (k', 4n(2n+1))$.

In memory of Professor Mohammad Ali Asadi-Golmankhaneh.

1. Introduction

Let M be a simply-connected closed four-manifold and G be a topological group. Let $P \to M$ be a principal G-bundle over M. The gauge group of this principal G-bundle, denote by $\mathcal{G}(P)$, is the topological group of automorphisms of P, where an automorphism of P is a G-equivariant self map of P covering the identity map of M. The main problem is to classify the homotopy types of $\mathcal{G}(P)$ as P ranges over all principal G-bundles over M for fixed G and M.

Let G be a simply-connected, simple compact Lie group. As $[M, BG] = \mathbb{Z}$, there are countably many equivalence classes of principal G-bundles over M. Each has a gauge group, so there are potentially countably many distinct gauge groups. While there are countably many inequivalent principal G-bundles, Crabb and Sutherland in [3] showed that their gauge groups have only finitely many distinct homotopy types. Let $P_k \to M$ represent the equivalence class of principal G-bundle whose second Chern class is k and $\mathcal{G}_k(M)$ be the gauge group of this principal G-bundle. In recent years there has been considerable interest in determining the precise number of homotopy types of these gauge groups and explicit classification results have been obtained. When M is a spin 4-manifold, Theriault in [20] showed that there is a homotopy equivalence

$$\mathcal{G}_k(M) \simeq \mathcal{G}_k(S^4) \times \prod_{i=1}^{t} \Omega^2 G,$$

where t is the second Betti number of M. Thus the homotopy type of $\mathcal{G}_k(M)$ depends on the special case $\mathcal{G}_k(S^4)$. Let (a, b) be the their greatest common divisor of two integers a and b. The first classification was done by Kono [9] for $G = SU(2)$. He

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showed that there is a homotopy equivalence $\mathcal{G}_k \simeq \mathcal{G}_{k'}$ if and only if $(k, 12) = (k', 12)$. Results formally similar to that of Kono have been obtained for principal bundles over S^4 with different structure groups. In the following, we mention some results

- $SU(3)$-gauge group [**6**];
- $SU(5)$-gauge group [**23**];
- $Sp(2)$-gauge group [**21**];
- $Sp(3)$-gauge group [**2**].

There are also several classification results for gauge groups of principal bundles with base spaces other than S^4 as follow

- $SU(3)$-gauge groups over S^6 [**7**];
- $SU(n)$-gauge groups over S^6 [**14**];
- $Sp(2)$-gauge groups over S^8 [**8**];
- $SU(4)$-gauge groups over S^8 [**13**];
- $SU(n)$-gauge groups over S^{2m} [**12**].

Furthermore, when M is a non-spin 4-manifold, So [**17**] showed that there is a homotopy equivalence

$$\mathcal{G}_k(M) \simeq \mathcal{G}_k(\mathbb{C}P^2) \times \prod_{i=1}^{t-1} \Omega^2 G,$$

therefore to study the homotopy type of $\mathcal{G}_k(M)$ it suffices to study $\mathcal{G}_k(\mathbb{C}P^2)$. Only four cases of the homotopy types of gauge groups over simply-connected non-spin four-manifolds have been studied, which are

- $SU(2)$-gauge groups [**11**];
- $SU(3)$-gauge groups [**22**];
- $SU(n)$-gauge groups [**18**];
- $Sp(2)$-gauge groups [**19**].

In this article, we will study the homotopy types of $Sp(n)$-gauge groups over $\mathbb{C}P^2$, for $n > 2$. Let $\mathcal{G}_k(\mathbb{C}P^2)$ be the gauge group of the principal $Sp(n)$-bundles over $\mathbb{C}P^2$ with second Chern class k. We partially classify the homotopy types of $\mathcal{G}_k(\mathbb{C}P^2)$ by using unstable K-theory to give a better lower bound for the number of homotopy types. We will prove the following theorem.

Theorem 1.1. *Let $n > 2$. If there is a homotopy equivalence $\mathcal{G}_k(\mathbb{C}P^2) \simeq \mathcal{G}_{k'}(\mathbb{C}P^2)$ then we have $(k, 4n(2n+1)) = (k', 4n(2n+1))$.*

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2. Preliminaries

Let BG and $B\mathcal{G}_k(M)$ be the classifying spaces of G and $\mathcal{G}_k(M)$ respectively. Let $Map_k(M, BG)$ be the component of the space of continuous unbased maps from M to BG which contains the map inducing P, similarly let $Map^*_k(M, BG)$ be the component of the space of pointed continuous maps from M to BG which contains the map inducing P. We observe that there is a fibration

$$Map^*_k(M, BG) \to Map_k(M, BG) \xrightarrow{ev} BG,$$

where the map ev is evaluation map at the basepoint of M. Atiyah, Bott and Gottlieb [**1, 4**] showed that there is a homotopy equivalence

$$B\mathcal{G}_k(M) \simeq Map_k(M, BG).$$

The evaluation fibration therefore determines a homotopy fibration sequence

$$G \longrightarrow Map^*_k(M, BG) \to B\mathcal{G}_k(M) \xrightarrow{ev} BG. \tag{1}$$

According to [**16**], for any $k \in \mathbb{Z}$, there exists a homotopy equivalence connecting $Map^*_k(M, BG)$ and $Map^*_0(M, BG)$. So for $G = Sp(n)$ and $M = \mathbb{C}P^2$, we rewrite (1) as a homotopy fibration sequence

$$Sp(n) \xrightarrow{\beta_k} Map^*_0(\mathbb{C}P^2, BSp(n)) \to B\mathcal{G}_k(\mathbb{C}P^2) \xrightarrow{ev} BSp(n), \tag{2}$$

where β_k is the fibration connecting map. Note that when $M = S^n$, $Map^*_0(M, BG)$ is an H-group so $[G, Map^*_0(M, BG)]$ is a group and we can discuss the order of the map $G \longrightarrow Map^*_0(M, BG)$ that is important for finding the homotopy types of $\mathcal{G}_k(S^n)$. But when $M = \mathbb{C}P^n$ then $Map^*_0(M, BG)$ is not an H-space so $[G, Map^*_0(M, BG)]$ is not a group and the order of the map $G \longrightarrow Map^*_0(M, BG)$ makes no sense. However, Theriault in [**22**] defined the "order" of the map $G \longrightarrow Map^*_0(M, BG)$, for $M = \mathbb{C}P^2$. In this paper, we study the classification of the homotopy types of the gauge groups of the principal $Sp(n)$-bundles over $\mathbb{C}P^2$, for $n > 2$ and will give a lower bound for the number of homotopy types of this gauge groups. Since to find the "order" of the map β_k is very hard, we do not prove the converse.

Let $Q_2 = S^3 \cup e^7$ be the symplectic quasi-projective space for $Sp(2)$. This article is organized as follows. In Section 3, in separate cases where n is even and n is odd we calculate $[\mathbb{C}P^2 \wedge A, Sp(n)]$, where $A = \Sigma^{4n-9}Q_2$ and $n > 2$. In Section 4 we compute $[\Sigma A, B\mathcal{G}_k(\mathbb{C}P^2)]$ and prove Theorem 1.1.

3. The group $[\mathbb{C}P^2 \wedge A, Sp(n)]$

Our main goal in this section to compute the group $[\mathbb{C}P^2 \wedge \Sigma^{4n-9}Q_2, Sp(n)]$, where $n > 2$. We denote $Sp(\infty)/Sp(n)$ by X_n and $[X, Sp(n)]$ by $Sp_n(X)$. We recall that the symplectic quasi projective space Q_2 has the cellular structure

$$Q_2 = S^3 \cup_{v_1} e^7,$$

where $v_1 \in \pi_6(S^3) \cong \mathbb{Z}_{12}$. Put $X = \mathbb{C}P^2 \wedge A$, where $A = \Sigma^{4n-9}Q_2$. Note that X has a cellular structure

$$X \simeq S^{4n-4} \cup e^{4n-2} \cup e^{4n} \cup e^{4n+2}.$$

Recall that as an algebra

$$H^*(Sp(n);\mathbb{Z}) = \bigwedge(y_3, y_7, \dots, y_{4n-1}),$$
$$H^*(Sp(\infty);\mathbb{Z}) = \bigwedge(y_3, y_7, \dots),$$
$$H^*(BSp(\infty);\mathbb{Z}) = \mathbb{Z}[q_1, q_2, \dots],$$

where $y_{4i-1} = \sigma q_i$, σ is the cohomology suspension and q_i is the $i-$th universal symplectic Pontrjagin class. Consider the projection map $\pi : Sp(\infty) \to X_n$, as an algebra we have

$$H^*(X_n;\mathbb{Z}) = \bigwedge(\bar{y}_{4n+3}, \bar{y}_{4n+7}, \dots),$$
$$H^*(\Omega X_n;\mathbb{Z}) = \mathbb{Z}\{b_{4n+2}, b_{4n+6}, \dots, b_{8n+2}\} \quad (* \leqslant 8n+2),$$

where $\pi^*(\bar{y}_{4i+3}) = y_{4i+3}$ and $b_{4n+4j-2} = \sigma(\bar{y}_{4n+4j-1})$. Consider the following fibre sequence

$$\Omega Sp(\infty) \xrightarrow{\Omega\pi} \Omega X_n \xrightarrow{\delta} Sp(n) \xrightarrow{j} Sp(\infty) \xrightarrow{\pi} X_n. \tag{3}$$

Note that for $n \geqslant 3$, A is a suspension, implying that X is a suspension as well. Therefore, applying the functor $[X, -]$ to fibration (3), there is an exact sequence of groups

$$[X, \Omega Sp(\infty)] \xrightarrow{(\Omega\pi)_*} [X, \Omega X_n] \xrightarrow{\delta_*} Sp_n(X) \xrightarrow{j_*} [X, Sp(\infty)] \xrightarrow{\pi_*} [X, X_n]. \tag{$*$}$$

Note that X_n has a cellular structure as following

$$X_n \simeq S^{4n+3} \cup_{\eta'} e^{4n+7} \cup e^{4n+11} \cup \cdots,$$

where η' is the generator of $\pi_{4n+6}(S^{4n+3})$ and

$$\Omega X_n \simeq S^{4n+2} \cup e^{4n+6} \cup e^{4n+10} \cup \cdots.$$

According to the CW-structure of X_n we have the following isomorphisms

$$\pi_i(X_n) = 0 \quad (\text{for} \quad i \leqslant 4n+2), \qquad \pi_{4n+3}(X_n) \cong \mathbb{Z}.$$

Observe that

$$[X, Sp(\infty)] \cong [\Sigma X, BSp(\infty)] \cong \widetilde{KSp}^{-1}(X).$$

Since $\widetilde{KSp}^{-1}(S^{4m-2}) = 0$, for every $m \geqslant 1$, applying $\widetilde{KSp}^{-1}$ to the homotopy cofibration $\Sigma^{4n-6}\mathbb{C}P^2 \to X \to \Sigma^{4n-2}\mathbb{C}P^2$ shows that $\widetilde{KSp}^{-1}(X) = 0$. On the other hand we know that ΩX_n is $(4n+1)$-connected and $H^{4n+2}(\Omega X_n) \cong \mathbb{Z}$ which is generated by $b_{4n+2} = \sigma(\bar{y}_{4n+3})$. The map $b_{4n+2} : \Omega X_n \to K(\mathbb{Z}, 4n+2)$ is a loop map and is a $(4n+3)$-equivalence. Since $\dim X \leqslant 4n+2$, it follows that the postcomposition map $(b_{4n+2})_* : [X, \Omega X_n] \to H^{4n+2}(X)$ is an isomorphism of groups. Thus we rewrite $(*)$ as the following exact sequence

$$\widetilde{KSp}^{-2}(X) \xrightarrow{\psi} H^{4n+2}(X) \to Sp_n(X) \to 0, \tag{4}$$

where we use the isomorphism

$$\widetilde{KSp}^{-i}(X) \cong [\Sigma^i X, BSp(\infty)].$$

So we have the exact sequence

$$0 \to \operatorname{Coker}\psi \xrightarrow{\iota} Sp_n(X) \to 0.$$

Therefore we get the following lemma.

Lemma 3.1. $Sp_n(X) \cong \operatorname{Coker}\psi$. □

In the following we will calculate the image of ψ.

Let Y be a CW-complex with $\dim Y \leqslant 4n+2$, we will denote $[Y, U(2n+1)]$ by $U_{2n+1}(Y)$. By [**5**, Theorem 1.1] there is an exact sequence

$$\tilde{K}^{-2}(Y) \xrightarrow{\varphi} H^{4n+2}(Y) \to U_{2n+1}(Y) \to \tilde{K}^{-1}(Y) \to 0,$$

where, for any $f \in \tilde{K}^{-2}(Y)$, the map φ is defined by

$$\varphi(f) = (2n+1)!ch_{2n+1}(f),$$

where $ch_{4n+2}(f)$ is the $4n+2$-th part of $ch(f)$. Also, we use the isomorphism

$$\tilde{K}^{-i}(Y) \cong [\Sigma^i Y, BU(\infty)].$$

The map of $\widetilde{KSp}^*(X) \to \tilde{K}^*(X)$ is induced by the map $Sp(\infty) \to U(\infty)$ obtained by taking the direct limit of the maps $Sp(n) \to U(2n)$ as n increases. In this paper, we use the same symbol c' for the canonical inclusion $Sp(n) \hookrightarrow U(2n)$ and the induced map $\widetilde{KSp}^*(X) \to \tilde{K}^*(X)$. By [**15**, Theorem 1.3] there is a commutative diagram

$$\begin{array}{ccc} \widetilde{KSp}^{-2}(X) & \xrightarrow{\psi} & H^{4n+2}(X) \\ {\scriptstyle c'}\downarrow & & \downarrow{\scriptstyle (-1)^{n+1}} \\ \tilde{K}^{-2}(X) & \xrightarrow{\varphi} & H^{4n+2}(X) \end{array} \tag{5}$$

Therefore to calculate the image of ψ we first calculate the image of φ. The calculation of the image of φ will appear as part of the proof of Proposition 3.3. We denote the free abelian group with a basis $e_1, e_2, \ldots,$ by $\mathbb{Z}\{e_1, e_2, \ldots\}$. We have the following lemma.

Lemma 3.2. *The following hold*:

(a): $\widetilde{KSp}^{-2}(X)$ *is a free abelian group that includes the subgroup generated by* ξ_2, ξ_4, *where*

$$\xi_2 \in \widetilde{KSp}^{-2}(S^{4n-2}) \quad \text{and} \quad \xi_4 \in \widetilde{KSp}^{-2}(S^{4n+2}),$$

(b): $\tilde{K}^{-2}(X) = \mathbb{Z}\{\xi_1', \xi_2', \xi_3', \xi_4'\}$, *where*

$$\xi_1' \in \tilde{K}^{-2}(S^{4n-4}), \quad \xi_2' \in \tilde{K}^{-2}(S^{4n-2}),$$
$$\xi_3' \in \tilde{K}^{-2}(S^{4n}) \quad \text{and} \quad \xi_4' \in \tilde{K}^{-2}(S^{4n+2}),$$

(c):

$$\begin{cases} c'(\xi_2) = 2\xi_2', \quad c'(\xi_4) = \xi_4' & \text{if } n \text{ is even}, \\ c'(\xi_2) = \xi_2', \quad c'(\xi_4) = 2\xi_4' & \text{if } n \text{ is odd}. \end{cases}$$

Proof. First, the cofibration sequences

$$S^{4n-4} \to \Sigma^{4n-6}\mathbb{C}P^2 \to S^{4n-2}, \quad S^{4n} \to \Sigma^{4n-2}\mathbb{C}P^2 \to S^{4n+2},$$

induce the following commutative diagrams of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{KSp}^{-2}(S^{4n-2}) & \longrightarrow & \widetilde{KSp}^{-2}(\Sigma^{4n-6}\mathbb{C}P^2) & \longrightarrow & \widetilde{KSp}^{-2}(S^{4n-4}) & \longrightarrow & 0 \\ & & \downarrow c'={c'}_2 & & \downarrow c' & & \downarrow c'={c'}_1 & & \\ 0 & \longrightarrow & \tilde{K}^{-2}(S^{4n-2}) & \longrightarrow & \tilde{K}^{-2}(\Sigma^{4n-6}\mathbb{C}P^2) & \longrightarrow & \tilde{K}^{-2}(S^{4n-4}) & \longrightarrow & 0, \end{array} \tag{6}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{KSp}^{-2}(S^{4n+2}) & \longrightarrow & \widetilde{KSp}^{-2}(\Sigma^{4n-2}\mathbb{C}P^2) & \longrightarrow & \widetilde{KSp}^{-2}(S^{4n}) & \longrightarrow & 0 \\ & & \downarrow c'={c'}_4 & & \downarrow c' & & \downarrow c'={c'}_3 & & \\ 0 & \longrightarrow & \tilde{K}^{-2}(S^{4n+2}) & \longrightarrow & \tilde{K}^{-2}(\Sigma^{4n-2}\mathbb{C}P^2) & \longrightarrow & \tilde{K}^{-2}(S^{4n}) & \longrightarrow & 0, \end{array} \tag{7}$$

respectively, where the zeroes that appear on the left and right of the two $\widetilde{KSp}$ sequences are due to the fact that $\widetilde{KSp}^{-2}(S^{4m-3}) = \widetilde{KSp}^{-1}(S^{4m-2}) = 0$, for every $m \geqslant 1$. Since

$$\tilde{K}^{-2}(S^{2i}) \cong \mathbb{Z}, \quad \widetilde{KSp}^{-2}(S^{4i+2}) \cong \mathbb{Z}, \quad \widetilde{KSp}^{-2}(S^{4n}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

we have two cases. If n is even then $\widetilde{KSp}^{-2}(\Sigma^{4n-6}\mathbb{C}P^2) = \mathbb{Z}\{\xi_2\}$ and ξ_4 generates a subgroup of $\widetilde{KSp}^{-2}(\Sigma^{4n-2}\mathbb{C}P^2)$, where ξ_2 and ξ_4 are generators of

$$\widetilde{KSp}^{-2}(S^{4n-2}) \cong \mathbb{Z} \text{ and } \widetilde{KSp}^{-2}(S^{4n+2}) \cong \mathbb{Z},$$

respectively. We have ${c'}_1 = {c'}_3 = 0$ and as in [**10**], ${c'}_2 = 2, {c'}_4 = 1$. If n is odd then $\widetilde{KSp}^{-2}(\Sigma^{4n-6}\mathbb{C}P^2)$ includes the subgroup generated by ξ_2 and $\widetilde{KSp}^{-2}(\Sigma^{4n-2}\mathbb{C}P^2) = \mathbb{Z}\{\xi_4\}$. Also, in this case we have ${c'}_1 = {c'}_3 = 0, {c'}_2 = 1, {c'}_4 = 2$. In the two cases we have

$$\tilde{K}^{-2}(\Sigma^{4n-6}\mathbb{C}P^2) = \mathbb{Z}\{\xi'_1, \xi'_2\}, \quad \tilde{K}^{-2}(\Sigma^{4n-2}\mathbb{C}P^2) = \mathbb{Z}\{\xi'_3, \xi'_4\},$$

where $\xi'_1 \in \tilde{K}^{-2}(S^{4n-4})$, $\xi'_2 \in \tilde{K}^{-2}(S^{4n-2})$, $\xi'_3 \in \tilde{K}^{-2}(S^{4n})$ and $\xi'_4 \in \tilde{K}^{-2}(S^{4n+2})$.

Note that there is a cofibration sequence $\Sigma^{4n-6}\mathbb{C}P^2 \to X \to \Sigma^{4n-2}\mathbb{C}P^2$, which induces an exact sequence

$$\begin{aligned} \widetilde{KSp}^{-2}(\Sigma^{4n-5}\mathbb{C}P^2) \to & \widetilde{KSp}^{-2}(\Sigma^{4n-2}\mathbb{C}P^2) \to \widetilde{KSp}^{-2}(X) \to \\ & \widetilde{KSp}^{-2}(\Sigma^{4n-6}\mathbb{C}P^2) \to \widetilde{KSp}^{-2}(\Sigma^{4n-3}\mathbb{C}P^2). \end{aligned}$$

We need to study groups $\widetilde{KSp}^{-2}(\Sigma^{4n-5}\mathbb{C}P^2)$ and $\widetilde{KSp}^{-2}(\Sigma^{4n-3}\mathbb{C}P^2)$. Consider the following cofibration sequence

$$S^{4n} \xrightarrow{\Sigma^{4n-3}\eta} S^{4n-1} \to \Sigma^{4n-3}\mathbb{C}P^2 \to S^{4n+1}.$$

This sequence induces an exact sequence

$$\widetilde{KSp}^{-2}(S^{4n+1}) \to \widetilde{KSp}^{-2}(\Sigma^{4n-3}\mathbb{C}P^2) \to \widetilde{KSp}^{-2}(S^{4n-1}) \xrightarrow{(\Sigma^{4n-3}\eta)^*} \widetilde{KSp}^{-2}(S^{4n}).$$

Since

$$\widetilde{KSp}^{-2}(S^{4n+1}) \cong 0, \quad \widetilde{KSp}^{-2}(S^{4n-1}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

we have two cases. If n is even then we get the following exact sequence

$$0 \to \widetilde{KSp}^{-2}(\Sigma^{4n-3}\mathbb{C}P^2) \to \mathbb{Z}_2 \xrightarrow{(\Sigma^{4n-3}\eta)^*} \mathbb{Z}_2.$$

Since $Sp(\infty)$ is homotopy equivalent to $\Omega^4 O(\infty)$, we can determine the map

$$\widetilde{KSp}^{-2}(S^{4n-1}) \xrightarrow{(\Sigma^{4n-3}\eta)^*} \widetilde{KSp}^{-2}(S^{4n})$$

by

$$(\Sigma^{4n-3}\eta)^* \colon \pi_{4n+4}(SO(\infty)) \to \pi_{4n+5}(SO(\infty)).$$

Let l_1 be a generator of $\pi_{4n+4}(SO(\infty)) \cong \mathbb{Z}_2$. Then the composition

$$l_2 \colon S^{4n+5} \xrightarrow{\Sigma^{4n+2}\eta} S^{4n+4} \xrightarrow{l_1} SO(\infty)$$

generates $\pi_{4n+5}(SO(\infty))$. Since the map $(\Sigma^{4n-3}\eta)^*$ sends l_1 to l_2 so $(\Sigma^{4n-3}\eta)^*$ is injective. Therefore we can conclude that $\widetilde{KSp}^{-2}(\Sigma^{4n-3}\mathbb{C}P^2)$ is zero. If n is even then we have $\widetilde{KSp}^{-2}(\Sigma^{4n-3}\mathbb{C}P^2) \cong 0$.

Similarly, to calculate the group $\widetilde{KSp}^{-2}(\Sigma^{4n-5}\mathbb{C}P^2)$, we have the following exact sequence

$$\widetilde{KSp}^{-2}(S^{4n-2}) \xrightarrow{(\Sigma^{4n-4}\eta)^*} \widetilde{KSp}^{-2}(S^{4n-1}) \to \widetilde{KSp}^{-2}(\Sigma^{4n-5}\mathbb{C}P^2) \to \widetilde{KSp}^{-2}(S^{4n-3}).$$

If n is even then we get the following exact sequence

$$\mathbb{Z} \xrightarrow{(\Sigma^{4n-4}\eta)^*} \mathbb{Z}_2 \to \widetilde{KSp}^{-2}(\Sigma^{4n-5}\mathbb{C}P^2) \to 0.$$

Let l'_1 be a generator of $\pi_{4n+3}(SO(\infty)) \cong \mathbb{Z}$. Then the composition

$$l'_2 \colon S^{4n+4} \xrightarrow{\Sigma^{4n+1}\eta} S^{4n+3} \xrightarrow{l'_1} SO(\infty) \quad \text{generates} \quad \pi_{4n+4}(SO(\infty)).$$

Since the map $(\Sigma^{4n-4}\eta)^*$ sends l'_1 to l'_2 so $(\Sigma^{4n-4}\eta)^*$ is surjective. Therefore we can conclude that $\widetilde{KSp}^{-2}(\Sigma^{4n-5}\mathbb{C}P^2)$ is zero. If n is even, we have $\widetilde{KSp}^{-2}(\Sigma^{4n-5}\mathbb{C}P^2) \cong 0$.

Therefore in both cases, we get the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{KSp}^{-2}(\Sigma^{4n-2}\mathbb{C}P^2) & \longrightarrow & \widetilde{KSp}^{-2}(X) & \longrightarrow & \widetilde{KSp}^{-2}(\Sigma^{4n-6}\mathbb{C}P^2) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle c'} & & \\ 0 & \longrightarrow & \tilde{K}^{-2}(\Sigma^{4n-2}\mathbb{C}P^2) & \longrightarrow & \tilde{K}^{-2}(X) & \longrightarrow & \tilde{K}^{-2}(\Sigma^{4n-6}\mathbb{C}P^2) & \longrightarrow & 0. \end{array} \tag{8}$$

Thus, we can conclude that $\widetilde{KSp}^{-2}(X)$ is a free abelian group that includes the subgroup generated by ξ_2 and ξ_4 and also $\widetilde{K}^{-2}(X)$ is a free abelian group generated by ξ'_1, ξ'_2, ξ'_3 and ξ'_4. Now according to the definition of the maps $c' = {c'}_i$, for $1 \leqslant i \leqslant 4$, we can choose ξ_1, ξ'_1, ξ_2, ξ'_2, ξ_3, ξ'_3, ξ_4 and ξ'_4 such that

$$\begin{cases} c'(\xi_2) = 2\xi'_2, \qquad c'(\xi_4) = \xi'_4 & \text{if } n \text{ is even,} \\ \\ c'(\xi_2) = \xi'_2, \qquad c'(\xi_4) = 2\xi'_4 & \text{if } n \text{ is odd.} \end{cases}$$

□

Consider the map $c' : Sp(2) \to SU(4)$. The composite

$$Q_2 \to Sp(2) \xrightarrow{c'} SU(4)$$

factors through the 7-skeleton of $SU(4)$, which is $\Sigma\mathbb{C}P^3$. Thus, we obtain the map $\bar{c}' : Q_2 \to \Sigma\mathbb{C}P^3$. The cohomologies of Q_2 and $\Sigma\mathbb{C}P^3$ are given by

$$H^*(Q_2) = \mathbb{Z}\{\bar{y}_3, \bar{y}_7\}, \qquad H^*(\Sigma\mathbb{C}P^3) = \mathbb{Z}\{\bar{x}_3, \bar{x}_5, \bar{x}_7\},$$

such that $\bar{c}'(\bar{x}_3) = \bar{y}_3$, $\bar{c}'(\bar{x}_5) = 0$ and $\bar{c}'(\bar{x}_7) = \bar{y}_7$. Denote by ζ_n a generator of $\tilde{K}(S^{2n})$, recall that

$$H^*(\mathbb{C}P^3) = \mathbb{Z}[t]/(t^4), \qquad K(\mathbb{C}P^3) = \mathbb{Z}[x]/(x^4),$$

where $|t| = 2$. Note that $\tilde{K}^{-2}(\mathbb{C}P^2 \wedge \Sigma^{4n-8}\mathbb{C}P^3) \cong \tilde{K}^0(\mathbb{C}P^2 \wedge \Sigma^{4n-6}\mathbb{C}P^3)$ is a free abelian group generated by $\zeta_{2n-3} \otimes x^i \otimes x^j$, where $1 \leqslant i \leqslant 2$ and $1 \leqslant j \leqslant 3$, with the following Chern characters

$$ch_{2n+1}(\zeta_{2n-4} \otimes x \otimes x) = ch_{2n-4}\zeta_{2n-4} \otimes ch_2 x \otimes ch_3 x = \frac{1}{12}\sigma^{4n-8}t^2 \otimes t^3,$$

similarly

$$\begin{aligned} ch_{2n+1}(\zeta_{2n-4} \otimes x \otimes x^2) &= \frac{1}{4}\sigma^{4n-8}t^2 \otimes t^3, \\ ch_{2n+1}(\zeta_{2n-4} \otimes x \otimes x^3) &= \sigma^{4n-8}t^2 \otimes t^3, \\ ch_{2n+1}(\zeta_{2n-4} \otimes x^2 \otimes x) &= \frac{1}{6}\sigma^{4n-8}t^2 \otimes t^3, \\ ch_{2n+1}(\zeta_{2n-4} \otimes x^2 \otimes x^2) &= \frac{1}{2}\sigma^{4n-8}t^2 \otimes t^3, \\ ch_{2n+1}(\zeta_{2n-4} \otimes x^2 \otimes x^3) &= 2\sigma^{4n-8}t^2 \otimes t^3. \end{aligned}$$

Consider the map $\bar{c}' : \tilde{K}^{-2}(\mathbb{C}P^2 \wedge \Sigma^{4n-8}\mathbb{C}P^3) \to \tilde{K}^{-2}(\mathbb{C}P^2 \wedge \Sigma^{4n-9}Q_2)$, we can put

ξ'_1, ξ'_2, ξ'_3 and ξ'_4 such that

$$\xi'_1 = \bar{c}'(\zeta_{2n-4} \otimes x \otimes x), \quad \xi'_2 = \bar{c}'(\zeta_{2n-4} \otimes x^2 \otimes x),$$
$$\xi'_3 = \bar{c}'(\zeta_{2n-4} \otimes x \otimes x^3) \text{ and } \xi'_4 = \bar{c}'(\zeta_{2n-4} \otimes x^2 \otimes x^3).$$

We have the following proposition.

Proposition 3.3. *The Image of ψ is generated by*

$$\begin{cases} \frac{1}{3}(2n+1)!\sigma^{4n-9}t^2 \otimes \bar{y}_7 & \text{if } n \text{ is even,} \\ \frac{1}{6}(2n+1)!\sigma^{4n-9}t^2 \otimes \bar{y}_7 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \tilde{K}^{-2}(\mathbb{C}P^2 \wedge \Sigma^{4n-8}\mathbb{C}P^3) & \xrightarrow{\varphi'} & H^{4n+2}(\mathbb{C}P^2 \wedge \Sigma^{4n-8}\mathbb{C}P^3) \\ {\scriptstyle \bar{c}'}\downarrow & & \downarrow\cong \\ \tilde{K}^{-2}(\mathbb{C}P^2 \wedge \Sigma^{4n-9}Q_2) & \xrightarrow{\varphi} & H^{4n+2}(\mathbb{C}P^2 \wedge \Sigma^{4n-9}Q_2) \end{array} \tag{9}$$

where the map φ' is defined similarly to the map φ. That is, when $1 \leqslant i \leqslant 2$ and $1 \leqslant j \leqslant 3$, we have

$$\varphi'(\zeta_{2n-4} \otimes x^i \otimes x^j) = (2n+1)!ch_{2n+1}(\zeta_{2n-4} \otimes x^i \otimes x^j).$$

By definition of the map of φ' we have

$$\varphi'(\zeta_{2n-4} \otimes x \otimes x) = \frac{1}{12}(2n+1)!\sigma^{4n-8}t^2 \otimes t^3,$$
$$\varphi'(\zeta_{2n-4} \otimes x \otimes x^2) = \frac{1}{4}(2n+1)!\sigma^{4n-8}t^2 \otimes t^3,$$
$$\varphi'(\zeta_{2n-4} \otimes x \otimes x^3) = (2n+1)!\sigma^{4n-8}t^2 \otimes t^3,$$
$$\varphi'(\zeta_{2n-4} \otimes x^2 \otimes x) = \frac{1}{6}(2n+1)!\sigma^{4n-8}t^2 \otimes t^3,$$
$$\varphi'(\zeta_{2n-4} \otimes x^2 \otimes x^2) = \frac{1}{2}(2n+1)!\sigma^{4n-8}t^2 \otimes t^3,$$
$$\varphi'(\zeta_{2n-4} \otimes x^2 \otimes x^3) = 2(2n+1)!\sigma^{4n-8}t^2 \otimes t^3.$$

Therefore according to the commutativity of diagram (9) we get

$$\varphi(\xi'_1) = \varphi'(\zeta_{2n-4} \otimes x \otimes x) = \frac{1}{12}(2n+1)!\sigma^{4n-9}t^2 \otimes \bar{y}_7,$$
$$\varphi(\xi'_2) = \varphi'(\zeta_{2n-4} \otimes x^2 \otimes x) = \frac{1}{6}(2n+1)!\sigma^{4n-9}t^2 \otimes \bar{y}_7,$$
$$\varphi(\xi'_3) = \varphi'(\zeta_{2n-4} \otimes x \otimes x^3) = (2n+1)!\sigma^{4n-9}t^2 \otimes \bar{y}_7,$$
$$\varphi(\xi'_4) = \varphi'(\zeta_{2n-4} \otimes x^2 \otimes x^3) = 2(2n+1)!\sigma^{4n-9}t^2 \otimes \bar{y}_7.$$

Thus by the commutativity of diagram (5) when n is even then we get

$$\psi(\xi_2) = \varphi(c'(\xi_2)) = -\frac{1}{3}(2n+1)!\sigma^{4n-9}t^2 \otimes \bar{y}_7,$$
$$\psi(\xi_4) = \varphi(c'(\xi_4)) = -2(2n+1)!\sigma^{4n-9}t^2 \otimes \bar{y}_7,$$

and when n is odd then we get

$$\psi(\xi_2) = \frac{1}{6}(2n+1)!\sigma^{4n-9}t^2 \otimes \bar{y}_7,$$
$$\psi(\xi_4) = 4t'(2n+1)!\sigma^{4n-9}t^2 \otimes \bar{y}_7.$$

Thus we can conclude that

$$\operatorname{Im}\psi \cong \begin{cases} \mathbb{Z}\{\frac{1}{3}(2n+1)!\sigma^{4n-9}t^2 \otimes \bar{y}_7\} & \text{if } n \text{ is even,} \\ \\ \mathbb{Z}\{\frac{1}{6}(2n+1)!\sigma^{4n-9}t^2 \otimes \bar{y}_7\} & \text{if } n \text{ is odd.} \end{cases} \qquad \square$$

Therefore by Lemma 3.1 and Proposition 3.3 we get the following theorem.

Theorem 3.4. *There is an isomorphism*

$$[X, Sp(n)] \cong \begin{cases} \mathbb{Z}_{\frac{1}{3}(2n+1)!} & \text{if } n \text{ is even,} \\ \\ \mathbb{Z}_{\frac{1}{6}(2n+1)!} & \text{if } n \text{ is odd.} \end{cases} \qquad \square$$

4. Proof of Theorem 1.1

In this section we will prove Theorem 1.1. Recall $A = \Sigma^{4n-9}Q_2$. Since the dimension of A is equal to $4n-2$, we have

$$[\Sigma A, BSp(n)] \cong [\Sigma A, BSp(\infty)] \cong \widetilde{KSp}(\Sigma A).$$

The cofibration sequence $S^{4n-5} \to \Sigma A \to S^{4n-1}$ induces the following exact sequence

$$\to \widetilde{KSp}(S^{4n-1}) \to \widetilde{KSp}(\Sigma A) \to \widetilde{KSp}(S^{4n-5}) \to \cdots .$$

Since $\widetilde{KSp}(S^{4i-1}) = 0$ for all $i \geqslant 1$, this implies that $\widetilde{KSp}(\Sigma A) = 0$. Thus we get the following lemma.

Lemma 4.1. *There is an isomorphism* $[\Sigma A, BSp(n)] \cong 0$. $\square$

Apply the functor $[\Sigma A, -]$ to fibration (2) to obtain the following exact sequence

$$[\Sigma A, Sp(n)] \xrightarrow{(\beta_k)_*} [\Sigma A, Map_0^*(\mathbb{C}P^2, BSp(n))] \to [\Sigma A, B\mathcal{G}_k(\mathbb{C}P^2)] \to [\Sigma A, BSp(n)], \quad (10)$$

where by Lemma 4.1, $[\Sigma A, BSp(n)] \cong 0$. Note that

$$[\Sigma A, Sp(n)] \cong [\Sigma^2 A, BSp(n)] \cong \widetilde{KSp}(\Sigma^2 A).$$

Also by adjunction,

$$[\Sigma A, Map_0^*(\mathbb{C}P^2, BSp(n))] \cong [\Sigma A \wedge \mathbb{C}P^2, BSp(n)] \cong [\mathbb{C}P^2 \wedge A, Sp(n)].$$

Thus the exact sequence becomes

$$\widetilde{KSp}(\Sigma^2 A) \xrightarrow{(\beta_k)_*} [\mathbb{C}P^2 \wedge A, Sp(n)] \to [\Sigma A, B\mathcal{G}_k(\mathbb{C}P^2)] \to 0, \quad (11)$$

therefore we get the following lemma.

Lemma 4.2. *There is an isomorphism*

$$[\Sigma A, B\mathcal{G}_k(\mathbb{C}P^2)] \cong [A, \mathcal{G}_k(\mathbb{C}P^2)] \cong \operatorname{Coker}(\beta_k)_*. \qquad \square$$

In what follows we will calculate $\operatorname{Im}(\beta_k)_*$. For this, we need the following lemmas.

Lemma 4.3. *Let $\mu : \Sigma A \to Sp(n)$ be an element of $\widetilde{KSp}(\Sigma^2 A)$ and $\varepsilon' : S^3 \to Sp(n)$ be the inclusion of bottom cell. If $\mu' : \mathbb{C}P^2 \wedge A \to Sp(n)$ is the adjoint of the following composition*

$$\mathbb{C}P^2 \wedge \Sigma A \xrightarrow{q\wedge \mathbb{1}} \Sigma S^3 \wedge \Sigma A \xrightarrow{\Sigma\varepsilon'\wedge\mu} \Sigma Sp(n) \wedge Sp(n) \xrightarrow{[ev,ev]} BSp(n),$$

where q is the quotient map. Then there is a lift $\tilde{\mu}$ of μ'

$$\begin{array}{ccc} & & \Omega X_n \\ & \overset{\tilde{\mu}}{\nearrow} & \downarrow \\ \mathbb{C}P^2 \wedge A & \xrightarrow{\mu'} & Sp(n) \end{array}$$

such that $\tilde{\mu}^*(a_{4n+2}) = t^2 \otimes \Sigma^{-1}\mu^*(y_{4n-1})$.

Proof. Nagao in [**15**, Lemma 3.1] showed that there is a lift

$$\lambda : \Sigma Sp(n) \wedge Sp(n) \longrightarrow X_n$$

of $[ev, ev]$ such that

$$\lambda^*(\bar{y}_{4n+3}) = \sum_{i+j=n+1} \Sigma y_{4i-1} \otimes y_{4j-1}.$$

Now let $\tilde{\lambda}$ be the following composition

$$\tilde{\lambda} : \mathbb{C}P^2 \wedge \Sigma A \xrightarrow{q\wedge \mathbb{1}} \Sigma S^3 \wedge \Sigma A \xrightarrow{\Sigma\varepsilon'\wedge\mu} \Sigma Sp(n) \wedge Sp(n) \xrightarrow{\lambda} X_n.$$

We have

$$\begin{aligned}\tilde{\lambda}^*(\bar{y}_{4n+3}) &= (q\wedge \mathbb{1})^*(\Sigma\varepsilon' \wedge \mu)^*\lambda^*(\bar{y}_{4n+3}) \\ &= (q\wedge \mathbb{1})^*(\Sigma\varepsilon' \wedge \mu)^*\Big(\sum_{i+j=n+1} \Sigma y_{4i-1} \otimes y_{4j-1}\Big) \\ &= (q\wedge \mathbb{1})^*(\Sigma u_3 \otimes \mu^*(y_{4n-1})) = t^2 \otimes \mu^*(y_{4n-1}),\end{aligned}$$

where u_3 is the generator of $H^3(S^3)$. We take $\tilde{\mu} : \mathbb{C}P^2 \wedge A \longrightarrow \Omega X_n$ to be the adjoint of the following composition

$$\Sigma\mathbb{C}P^2 \wedge A \xrightarrow{S} \mathbb{C}P^2 \wedge \Sigma A \xrightarrow{\tilde{\lambda}} X_n,$$

that is $\tilde{\mu} : ad(\tilde{\lambda} \circ S)$, where the map $S : \Sigma\mathbb{C}P^2 \wedge A \longrightarrow \mathbb{C}P^2 \wedge \Sigma A$ is the swapping map and the map $ad : [\Sigma\mathbb{C}P^2 \wedge A, X_n] \longrightarrow [\mathbb{C}P^2 \wedge A, \Omega X_n]$ is the adjunction. Then $\tilde{\mu}$ is a lift of μ. Note that

$$(\tilde{\lambda} \circ S)^*(\bar{y}_{4n+3}) = S^* \circ \tilde{\lambda}^*(\bar{y}_{4n+3}) = S^*(t^2 \otimes \mu^*(y_{4n-1})) = \Sigma t^2 \otimes \Sigma^{-1}\mu^*(y_{4n-1}),$$

thus we get

$$\tilde{\mu}^*(a_{4n+2}) = t^2 \otimes \Sigma^{-1}\mu^*(y_{4n-1}).$$

□

Consider the map $\theta : \Sigma A \to Sp(n)$, then by Lemma 4.3 there is a lift $\tilde{\theta}$ of $(\beta_k)_*(\theta)$ such that

$$\tilde{\theta}^*(a_{4n+2}) = t^2 \otimes \Sigma^{-1}\theta^*(y_{4n-1}).$$

We define the map $\lambda_k : \widetilde{KSp}(\Sigma^2 A) \to H^{4n+2}(X)$ by $\lambda_k(\theta) = \tilde{\theta}^*(a_{4n+2}) = a_{4n+2} \circ \tilde{\theta}$. Now consider the following commutative diagram

$$\begin{array}{ccccccc} \longrightarrow & \widetilde{KSp}(\Sigma^2 A) & \xrightarrow{(\beta_k)_*} & [X, Sp(n)] & \longrightarrow & [A, \mathcal{G}_k] \\ & \downarrow{\scriptstyle \lambda_k} & & \| & & \\ \widetilde{KSp}^{-2}(X) \xrightarrow{\psi} & H^{4n+2}(X) & \longrightarrow & [X, Sp(n)] & \longrightarrow & 0, \end{array} \tag{12}$$

we have $\operatorname{Im}(\beta_k)_* = \operatorname{Im}\lambda_k / \operatorname{Im}\psi$. Note that in Proposition 3.3 we calculated $\operatorname{Im}\psi$, thus we need to calculate $\operatorname{Im}\lambda_k$.

Let $a : \Sigma Q_2 \to BSp(\infty)$ be the adjoint of the composition of the inclusions

$$Q_2 \to Sp(2) \to Sp(\infty)$$

and also $b : \Sigma Q_2 \to BSp(\infty)$ be the pinch map of the bottom cell $q : \Sigma Q_2 \to S^8$ followed by a generator of $\pi_8(BSp(\infty)) \cong \mathbb{Z}$. Note that $\widetilde{KSp}(\Sigma Q_2)$ is a free abelian group with a basis a, b. Then we have

$$ch(c'(a)) = \Sigma y_3 - \frac{1}{6}\Sigma y_7, \qquad ch(c'(b)) = -2\Sigma y_7.$$

Let $\theta_1 = \mathbf{q}(\zeta_{2n-4}c'(a)) \in \widetilde{KSp}(\Sigma^{4n-7}Q_2)$, where $\mathbf{q} : K \to KSp$ is the quaternionization.

Also let $\theta_2 : \Sigma^{4n-7}Q_2 \to BSp(\infty)$ be the composite of the pinch map to the top cell $\Sigma^{4n-7}Q_2 \to S^{4n}$ and a generator of $\pi_{4n}(BSp(\infty)) \cong \mathbb{Z}$. Then $\widetilde{KSp}(\Sigma^2 A)$ is a free abelian group with a basis θ_1, θ_2. We have

$$ch(c'(\theta_1)) = \begin{cases} \Sigma^{4n-7}y_3 - \frac{1}{6}\Sigma^{4n-7}y_7 & \text{if } n \text{ is even,} \\ 2\Sigma^{4n-7}y_3 + \frac{1}{3}\Sigma^{4n-7}y_7 & \text{if } n \text{ is odd,} \end{cases}$$

$$ch(c'(\theta_2)) = \begin{cases} -2\Sigma^{4n-7}y_7 & \text{if } n \text{ is even,} \\ \Sigma^{4n-7}y_7 & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 4.4. $\operatorname{Im}\lambda_k$ *is isomorphic to*

$$\begin{cases} \mathbb{Z}\{\frac{1}{6}(2n-1)!kt^2 \otimes \Sigma^{4n-9}y_7\} & \textit{if } n \textit{ is even,} \\ \mathbb{Z}\{\frac{1}{12}(2n-1)!kt^2 \otimes \Sigma^{4n-9}y_7\} & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. For $i = 1, 2$, by Lemma 4.3, $(\beta_k)_*(\theta_i)$ has a lift $\tilde{\theta}_{i,k} : \mathbb{C}P^2 \wedge A \to \Omega X_n$, such that

$$\tilde{\theta}^*_{i,k}(a_{4n+2}) = kt^2 \otimes \Sigma^{-1}{\theta_i}^*(y_{4n-1}).$$

Since $y_{4i-1} = \sigma(q_i)$ and $q_i = c'^*(c_{2i})$ so we get

$$\tilde{\theta}^*_{i,k}(a_{4n+2}) = kt^2 \otimes \Sigma^{-2}q_n(\theta_i) = kt^2 \otimes \Sigma^{-2}c_{2n}(c'(\theta_i)).$$

On the other hand we have

$$c_{2n}(c'(\theta_1)) = \begin{cases} \frac{1}{6}(2n-1)!\Sigma^{4n-7}y_7 & \text{if } n \text{ is even,} \\ \frac{1}{3}(2n-1)!\Sigma^{4n-7}y_7 & \text{if } n \text{ is odd,} \end{cases}$$

$$c_{2n}(c'(\theta_2)) = \begin{cases} 2(2n-1)!\Sigma^{4n-7}y_7 & \text{if } n \text{ is even,} \\ (2n-1)!\Sigma^{4n-7}y_7 & \text{if } n \text{ is odd.} \end{cases}$$

Therefore we get

$$\tilde{\theta}^*_{1,k}(a_{4n+2}) = \begin{cases} kt^2 \otimes \frac{1}{6}(2n-1)!\Sigma^{4n-9}y_7 & \text{if } n \text{ is even,} \\ kt^2 \otimes \frac{1}{3}(2n-1)!\Sigma^{4n-9}y_7 & \text{if } n \text{ is odd,} \end{cases}$$

and

$$\tilde{\theta}^*_{2,k}(a_{4n+2}) = \begin{cases} kt^2 \otimes 2(2n-1)!\Sigma^{4n-9}y_7 & \text{if } n \text{ is even,} \\ kt^2 \otimes (2n-1)!\Sigma^{4n-9}y_7 & \text{if } n \text{ is odd.} \end{cases}$$

If n is even, then $\operatorname{Im}\lambda_k \cong \mathbb{Z}\{\alpha,\beta\}$, where

$$\alpha = \frac{1}{6}(2n-1)!kt^2 \otimes \Sigma^{4n-9}y_7 \quad \text{and} \quad \beta = 2(2n-1)!kt^2 \otimes \Sigma^{4n-9}y_7.$$

Also $\operatorname{Im}\lambda_k$ is generated by $2\alpha - \frac{1}{12}\beta = \frac{1}{6}(2n-1)!kt^2 \otimes \Sigma^{4n-9}y_7$. If n is odd, then $\operatorname{Im}\lambda_k \cong \mathbb{Z}\{\alpha,\beta\}$, where

$$\alpha = \frac{1}{3}(2n-1)!kt^2 \otimes \Sigma^{4n-9}y_7 \quad \text{and} \quad \beta = (2n-1)!kt^2 \otimes \Sigma^{4n-9}y_7.$$

Also $\operatorname{Im}\lambda_k$ is generated by $\alpha - \frac{1}{4}\beta = \frac{1}{12}(2n-1)!kt^2 \otimes \Sigma^{4n-9}y_7$. Therefore we can conclude

$$\operatorname{Im}\lambda_k \cong \begin{cases} \mathbb{Z}\{\frac{1}{6}(2n-1)!kt^2 \otimes \Sigma^{4n-9}y_7\} & \text{if } n \text{ is even,} \\ \mathbb{Z}\{\frac{1}{12}(2n-1)!kt^2 \otimes \Sigma^{4n-9}y_7\} & \text{if } n \text{ is odd.} \end{cases} \qquad \square$$

Therefore by Proposition 3.3 and Lemma 4.4 we obtain the following proposition.

Proposition 4.5. *There is an isomorphism*

$$\operatorname{Im}(\beta_k)_* \cong \begin{cases} \mathbb{Z}/((2n+1)!/(3(k,4n(2n+1)))) & \textit{if } n \textit{ is even,} \\ \mathbb{Z}/((2n+1)!/(6(k,4n(2n+1)))) & \textit{if } n \textit{ is odd.} \end{cases} \qquad \square$$

Therefore by Theorem 3.4, Lemma 4.2 and Proposition 4.5 we get the following theorem.

Theorem 4.6. *There is an isomorphism* $[\Sigma A, B\mathcal{G}_k(\mathbb{C}P^2)] \cong \mathbb{Z}_{(k,4n(2n+1))}$. $\square$

Now we prove Theorem 1.1.

Proof of Theorem 1.1. Consider the exact sequence (11). Suppose that $\mathcal{G}_k(\mathbb{C}P^2) \simeq \mathcal{G}_{k'}(\mathbb{C}P^2)$, then there is an isomorphism of groups

$$[\Sigma A, B\mathcal{G}_k(\mathbb{C}P^2)] \cong [\Sigma A, B\mathcal{G}_{k'}(\mathbb{C}P^2)].$$

Thus the order of $[\Sigma A, B\mathcal{G}_k(\mathbb{C}P^2)]$ is equal to the order of $[\Sigma A, B\mathcal{G}_{k'}(\mathbb{C}P^2)]$. Therefore by Theorem 4.6 we can conclude Theorem 1.1. □

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