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THE HOMOTOPY SOLVABILITY OF COMPACT LIE GROUPS AND HOMOGENOUS TOPOLOGICAL SPACES

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Abstract

We analyse the homotopy solvability of the classical Lie groups $O(n)$, $U(n)$, $Sp(n)$ and derive its heredity by closed subgroups. In particular, the homotopy solvability of compact Lie groups is shown.

Then, we study the homotopy solvability of the loop spaces $\Omega(G_{n,m}(\mathbb{F}))$, $\Omega(V_{n,m}(\mathbb{F}))$ and $\Omega(F_{n;n_1,\ldots,n_k}(\mathbb{F}))$ for Grassmann $G_{n,m}(\mathbb{F})$, Stiefel $V_{n,m}(\mathbb{F})$ and generalised flag $F_{n;n_1,\ldots,n_k}(\mathbb{F})$ manifolds for $\mathbb{F} = \mathbb{R}, \mathbb{C}$, the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$-algebra of quaternions. Furthermore, the homotopy solvability of the loop space $\Omega(\mathbb{O}P^2)$ for the Cayley plane $\mathbb{O}P^2$ is established as well.

Introduction

Given an associative H-space X, the functor $[-, X]$ sending a pointed space Y to the set of classes of maps $Y \to X$ takes its values in the category of groups. One may then ask when this functor takes its values in various subcategories of groups. For example X is homotopy commutative if and only if $[Y, X]$ is Abelian for all Y.

Berstein and Ganea [**4**] adapted the nilpotency and Zabrodsky [**25**, Chapter II] the solvability to H-spaces as follows. Given an associative H-space X, we write $\varphi_{X,1} = \psi_{X,0} = \iota_X$, $\varphi_{X,2} = \psi_{X,1} : X^2 \to X$ for the basic commutator map and define inductively:

(1) $\varphi_{X,n+1} = \varphi_{X,2}(\varphi_{X,1} \times \varphi_{X,n}) : X^{n+1} \to X$ for $n \geqslant 2$;

(2) $\psi_{X,n+1} = \psi_{X,1}(\psi_{X,n} \times \psi_{X,n}) : X^{2^{n+1}} \to X$ for $n \geqslant 1$.

The *nilpotency class* $\operatorname{nil} X$ of an associative H-space X is the least integer $n \geqslant 0$ for which the map $\varphi_{X,n+1} \simeq *$, is nullhomotopic. Similarly, the *solvability class* $\operatorname{sol} X$ of X is the least integer $n \geqslant 0$ for which the map $\psi_{X,n+1} \simeq *$, is nullhomotopic. In view of [**25**, Lemma 2.6.3], we have $\operatorname{sol} X \leqslant \operatorname{nil} X - 1$ provided $\operatorname{nil} X < \infty$. Berstein and Ganea [**4**] introduced a concept of the homotopy nilpotency of a pointed space X by

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means of its loop space $\Omega(X)$ which has been adjusted by Zabrodsky [**25**, Chapter II] to the homotopy solvability.

The homotopy nilpotency classes $\operatorname{nil} X$ of associative H-spaces X has been extensively studied as well as their homotopy commutativity. Hopkins [**14**] found cohomological criteria for a finite H-space to be homotopy nilpotent, and used it to prove that H-spaces with no torsion in homology are homotopy nilpotent. This result drew renewed attention to such problems by relating this classical nilpotency notion with the nilpotence theorem of Devinatz, Hopkins, and Smith [**8**].

It is well-known (see e.g., [**13**]) that for the loop space $\Omega(\mathbb{S}^m)$ of the m-sphere $\mathbb{S}^m$ we have $\operatorname{nil}\Omega(\mathbb{S}^m) = 1$ if and only if $m = 0, 1, 3, 7$ and

$$\operatorname{nil}\Omega(\mathbb{S}^m) = \begin{cases} 2 & \text{for odd } m \text{ and } m \neq 0,1,3,7 \text{ or } m = 2; \\ 3 & \text{for even } m \geqslant 4. \end{cases}$$

Write $\mathbb{F}P^m$ for the projective m-space for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$-algebra of quaternions. Then, the homotopy nilpotency of $\Omega(\mathbb{F}P^m)$ has been first studied by Ganea [**9**], Snaith [**21**] and then their p-localization $\Omega((\mathbb{F}P^m)_{(p)})$ by Meier [**17**]. The homotopy nilpotency of the loop spaces of Grassmann and Stiefel manifolds, and their p-localization have been extensively studied in [**11**].

Let $\mathbb{S}^{2m-1}_{(p)}$ be the p-localization of the sphere $\mathbb{S}^{2m-1}$ at a prime p. The main result of the paper [**12**] is the explicit determination of the homotopy nilpotency class of a wide range of homotopy associative multiplications on localized spheres $\mathbb{S}^{2m-1}_{(p)}$ for $p > 3$. Furthermore, the paper [**10**] develops techniques in the study of the homotopy nilpotency classes of Moore spaces $M(A, n)$ for $n \geqslant 1$.

Zabrodsky [**25**, Proposition 2.6.10] proved that the classical Lie groups $SU(n)$, $Sp(n)$ and $SO(2n+1)$ were homotopy solvable. But, there are no other works, in the literature known to the author, concerning the solvability of topological groups and spaces. This paper grew out of our desire to develop techniques in calculating the solvability classes of some classical Lie groups and loop spaces of associated homogenous spaces. Its results attempt to provide some insight into the differences between nilpotency and solvability classes.

Section 1 sets the stage for the developments to come. Subsection 1.1 examines known results on the homotopy solvability stated in [**25**, Chapter II] and establishes necessary notations on the homotopy solvability of H-spaces used in the rest of the paper. Subsection 1.2 relates the homotopy nilpotency and solvability of a space with its Postnikov sections. It is shown that the proof of [**4**, 4.11 Theorem] leads to the following generalised result.

Theorem 1.14. *Let X be a connected space with the solvable fundamental homotopy group $\pi_1(X)$ and its nilpotent actions on $\pi_m(X)$ for $m \geqslant 2$. If the k-invariants k_m of X are trivial for all but finitely many values of m then $\operatorname{nil}\Omega(X) < \infty$. In particular, $\operatorname{sol}\Omega(X) \leqslant \operatorname{nil}\Omega(X) - 1 < \infty$ provided $\operatorname{nil}\Omega(X) < \infty$.*

Section 2, based on [**15**] and [**25**, Chapter II], analyses the homotopy solvability of $\Omega(X)$ for some homogenous spaces X. Then, we take into account the homotopy solvability of Lie groups.

Subsection 2.1 makes use of a well-known result on the associated principal G-bundle $E(G) \to B(G)$ with a Lie group G, to derive

Corollary 2.2. *If G is a homotopy solvable Lie group and $K \leqslant G$ its closed subgroup then*

$$\operatorname{sol}\Omega(G/K) \leqslant \operatorname{sol} G + 1 \text{ and } \operatorname{sol} K \leqslant 2 + 1 \operatorname{sol} G.$$

Next, we take up the systematic study of the homotopy solvability of the spaces $\Omega(V_{n,m}(\mathbb{F}))$ for $V_{n,m}(\mathbb{F}) = SO(\mathbb{F},n)/SO(\mathbb{F},m)$ for $m < n$ with $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$-algebra. We present the following generalization of [**25**, 2.6.9 Proposition].

Proposition 2.5. *Let $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ be the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$-algebra of quaternions. Then,*

$$\operatorname{sol}\Omega(V_{n,m}(\mathbb{R})) \leqslant 2k_o(n,m) + k_e(n,m)$$

and

$$\operatorname{sol}\Omega(V_{n,m}(\mathbb{F})) \leqslant n - m \quad \text{for} \quad \mathbb{F} = \mathbb{C}, \mathbb{H},$$

where $k_o(n,m)$ and $k_e(n,m)$ are the numbers of all odd and even elements of the set $\{m+1, \ldots, n+m\}$.

Subsection 2.2 is concentrated with the homotopy solvability of classical Lie groups. First, we present the following generalization of [**25**, 2.6.10 Proposition].

Proposition 2.7.

(1) *The groups $SO(n)$ are homotopy solvable with*

$$\operatorname{sol} SO(n) \leqslant 2k_o(2n,n) + k_e(2n,n)$$

and the groups $SU(n)$, $Sp(n)$, $U(n)$ are homotopy solvable with

$$\operatorname{sol} SU(n), \operatorname{sol} Sp(n), \operatorname{sol} U(n) \leqslant n.$$

(2) *For any topological space Y (not necessarily of the homotopy type of a CW-complex),*

$$\operatorname{sol}\,[Y, SO(n)] \leqslant 2k_o(2n,n) + k_e(2n,n) \quad \text{and}$$

$$\operatorname{sol}\,[Y, SU(n)], \operatorname{sol}\,[Y, Sp(n)], \operatorname{sol}\,[Y, U(n)] \leqslant n.$$

Furthermore, we derive that:

(1) $\operatorname{sol} O(n) = \operatorname{sol} SO(n) \leqslant 2k_o(2n,n) + k_e(2n,n)$ and $\operatorname{sol} U(n) = \operatorname{sol} SU(n) \leqslant n$;

(2) $\operatorname{sol} Pin(n), \operatorname{sol} Spin(n) = \operatorname{sol} SO(n) \leqslant 2k_o(2n,n) + k_e(2n,n)$

for the universal covering groups $Pin(n) \to O(n)$ and $Spin(n) \to SO(n)$.

Since any compact Lie group G is isomorphic to a closed subgroup of the orthogonal group $O(m)$ or the unitary group $U(n)$ for some positive integers m and n, respectively, we conclude

Corollary 2.10. *If G is a compact Lie group then G is homotopy solvable and*

$$\operatorname{sol} G \leqslant 2 \operatorname{sol} O(n) + 1 < \infty$$

for some positive integer n.

In particular, all exceptional compact Lie groups $G_2 \subseteq F_4 \subseteq E_6 \subseteq E_7 \subseteq E_8$ are homotopy solvable. We estimate that

$$\operatorname{sol} G_2 \leqslant \operatorname{sol} Spin(7) \leqslant 10 \ \text{ and } \ \operatorname{sol} F_4 \leqslant \operatorname{sol} O(27) \leqslant 81.$$

Next, we show that the diffeomorphism $\mathbb{O}P^2 \approx F_4/Spin(9)$ for the Cayley plane $\mathbb{O}P^2$ yields that

$$\operatorname{sol} \Omega(\mathbb{O}P^2) \leqslant \operatorname{sol} Spin(9) \leqslant 13.$$

Furthermore, Proposition 2.11 implies that

$$\operatorname{sol} GL(n,\mathbb{F}) = \operatorname{sol} SL(n,\mathbb{F}) = \operatorname{sol} O(n,\mathbb{F})$$

and

$$\operatorname{sol} \Omega(V_{n,m}(\mathbb{F})) = \operatorname{sol} \Omega(GL(n,\mathbb{F})/GL(m,\mathbb{F})) = \operatorname{sol} \Omega(SL(n,\mathbb{F})/SL(m,\mathbb{F})).$$

Then, Subsection 2.3 takes up the systematic study of the homotopy solvability of the spaces $V_{n,m}(\mathbb{F})$ for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$-algebra. some topological groups and loop spaces $\Omega(G_{n,m}(\mathbb{F}))$ and $\Omega(V_{n,m}(\mathbb{F}))$ of Grassmann $G_{n,m}(\mathbb{F})$ and Stiefel $V_{n,m}(\mathbb{F}))$ manifolds. We make use of results from Subsections 2.1 and 2.2 the state the main result.

Theorem 2.13. *If* $1 \leqslant m < n < \infty$ *then*

$$\operatorname{sol} \Omega(F_{n;n_1,\dots,n_k}(\mathbb{F})) < \infty \ \text{ for } \ \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$$

and

$$\operatorname{sol} \Omega(F^{+}_{n;n_1,\dots,n_k}(\mathbb{F})) \leqslant \operatorname{sol} \Omega((F_{n;n_1,\dots,n_k}(\mathbb{F}))) < \infty \ \text{ for } \ \mathbb{F} = \mathbb{R}, \mathbb{C}.$$

At the end, in Subsection 2.4 we follow [**22**] to consider the X*-projective* n*-space* $XP(n)$ for $n \leqslant m$, associated with an A_m-space X. We make use of [**11**, Theorem 1.5] on $\operatorname{nil} \mathbb{S}^{2m-1}_{(p)} < \infty$ to close the paper with the following conclusion.

Corollary 2.16. *If* $n \leqslant p-1$ *and* $p > 3$ *is a prime then*

$$\operatorname{sol} \Omega(\mathbb{S}^{2m-1}_{(p)} P(n-1)) \leqslant \operatorname{sol} \mathbb{S}^{2m-1}_{(p)} < \infty.$$

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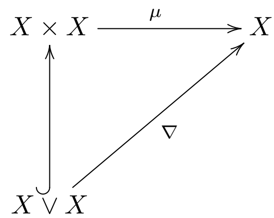
1. Prerequisites

All spaces in this note are assumed to be connected and based with the homotopy type of CW-complexes unless we assume otherwise. We also do not distinguish notationally between a continuous based map and its homotopy class. We write $\Omega(X)$ (resp. $\Sigma(X)$) for the loop (resp. suspension) space on a space X and $[X,Y]$ for the set of homotopy classes of maps $X \to Y$.

Given a space X, we use the customary notations $X \vee X$ and $X \wedge X$ for the *wedge* and the *smash square* of X, respectively.

Recall that an *H-space* is a pair (X, μ), where X is a space and $\mu : X \times X \to X$ is a map such that the diagram



commutes up to homotopy, where $\nabla : X \vee X \to X$ is the folding map. We call μ a *multiplication* or an *H-structure* for X. Two examples of H-spaces come to mind: topological groups and the loop spaces $\Omega(X)$. In the sequel, we identify an H-space (X, μ) with the space X.

An H-space X is called a *group-like space* if X satisfies all the axioms of groups up to homotopy. Recall that a homotopy associative H-CW-complex always has a homotopy inverse. More precisely, according to [**25**, 1.3.2 Corollary] (see also [**2**, Proposition 8.4.4]), we have

Proposition 1.1. *If X is a homotopy associative H-CW-complex then X is a group-like space.*

From now on, we assume that any H-space X is group-like.

Given spaces $X_1, \ldots, X_n$, we use the customary notations $X_1 \times \cdots \times X_n$ for their Cartesian product and $T_m(X_1, \ldots, X_n)$ for the subspace of $X_1 \times \cdots \times X_n$ consisting of those points with at least m coordinates at base points with $m = 0, 1, \ldots, n$. Then,

$T_0(X_1, \ldots, X_n) = X_1 \times \cdots \times X_n$,

$T_1(X_1, \ldots, X_n)$ is the so called the *fat wedge* of spaces $X_1, \ldots, X_n$ and

$T_{n-1}(X_1, \ldots, X_n) = X_1 \vee \cdots \vee X_n$, the wedge sum of spaces $X_1, \ldots, X_n$.

We write $j_m(X_1, \ldots, X_n) : T_m(X_1, \ldots, X_n) \to X_1 \times \cdots \times X_n$ for the inclusion map with $m = 0, 1, \ldots, n$ and

$$X_1 \wedge \cdots \wedge X_n = X_1 \times \cdots \times X_n / T_1(X_1, \ldots, X_n)$$

for the *smash product* of spaces $X_1, \ldots, X_n$.

Let $f_m : (X_m, \star_m) \to (Y_m, *_m)$ be continuous maps of pointed topological spaces for $m = 1, \ldots, n$. The map

$$f_1 \times \cdots \times f_n : (X_1 \times \cdots \times X_n, (\star_1, \ldots, \star_n)) \to (Y_1 \times \cdots \times Y_n, (*_1, \ldots, *_n))$$

sends the point $(x_1, \ldots, x_n)$ into

$$(f_1(x_1), \ldots, f_n(x_n)) \text{ for } (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$$

and restricts to maps

$$T_m(f_1, \ldots, f_n) : T_m(X_1, \ldots, X_n) \to T_m(Y_1, \ldots, Y_n)$$

with $m = 0, 1, \dots, n$. If $X_m = X$ and $f_m = f$ for $m = 1, \dots, n$ then we write

$$X^n = X_1 \times \cdots \times X_n, \quad X^{\wedge n} = X_1 \wedge \cdots \wedge X_n,$$
$$f^n = f_1 \times \cdots \times f_n \text{ and } f^{\wedge n} = f_1 \wedge \cdots \wedge f_n.$$

The identity map of a space X involved is consistently denoted by ι_X.

Given an H-group X, the functor $[-, X]$ takes its values in the category of groups. One may then ask when those functors take their values in various subcategories of groups. For example, X is homotopy commutative if and only if $[Y, X]$ is Abelian for all Y.

Furthermore, we write $\varphi_{X,1} = \psi_{X,0} = \iota_X$, $\varphi_{X,2} = \psi_{X,1} : X^2 \to X$ for the basic commutator map and define inductively:

(1) $\varphi_{X,n+1} = \varphi_{X,2}(\varphi_{X,1} \times \varphi_{X,n}) : X^{n+1} \to X$ for $n \geqslant 2$;

(2) $\psi_{X,n+1} = \psi_{X,1}(\psi_{X,n} \times \psi_{X,n}) : X^{2^{n+1}} \to X$ for $n \geqslant 1$.

1.1. Homotopy nilpotency and solvability

Let G be an abstract group. Recall that G is called *nilpotent* if it has the lower central series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_n = E,$$

where $G_{m+1} = [G_m, G]$ for $m \geqslant 0$ terminating in the trivial subgroup E after finitely many steps. We write $\operatorname{nil} G$ for the *nilpotency class* of G.

A group G is called *solvable* or *soluble* if its derived series, the descending normal series

$$G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq G^{(2)} \trianglerighteq \cdots \trianglerighteq G^{(n)} = E,$$

where every subgroup $G^{(m+1)}$ for $m \geqslant 0$ is the commutator subgroup $(G^{(m)})'$ of the previous one $G^{(m)}$, eventually reaches the trivial subgroup E of G. We write $\operatorname{sol} G$ for the *solvability class* of G. Since $G^{(m)} \leqslant G_m$ for $m \geqslant 0$, we derive that the nilpotency of G implies its solvability.

Recall that by [**20**, Theorems 5.15–5.17], the solvability is closed under subgroups, quotient groups and group extensions. Furthermore,

$$\operatorname{sol} G' \leqslant \operatorname{sol} G \tag{1}$$

for any subgroup $G' \leqslant G$ and

$$\operatorname{sol} G \leqslant \operatorname{sol} G' + \operatorname{sol} G'' \tag{2}$$

for any group extension $E \to G' \to G \to G'' \to E$.

The *nilpotency class* $\operatorname{nil}(X, \mu)$ of an H-space (X, μ) is the least integer $n \geqslant 0$ for which the map $\varphi_{X,n+1} \simeq *$, is nullhomotopic and we call the homotopy associative H-space X *homotopy nilpotent*. If no such integer exists, we put $\operatorname{nil}(X, \mu) = \infty$. In the sequel, we simply write $\operatorname{nil} X$ for the nilpotency class of an H-space X.

The *solvability class* $\operatorname{sol}(X, \mu)$ of an H-space (X, μ) is the least integer $n \geqslant 0$ for which the map $\psi_{X,n+1} \simeq *$, is nullhomotopic and we call the homotopy associative H-space X *homotopy solvable*. If no such integer exists, we put $\operatorname{sol}(X, \mu) = \infty$. In the sequel, we also simply write $\operatorname{sol} X$ for the solvability class of an H-space X.

Notice that X is homotopy commutative if and only if $\operatorname{nil} X = 1$ or equivalently $\operatorname{sol} X = 0$.

Given a space X, the numbers $\operatorname{nil}\Omega(X)$ and $\operatorname{sol}\Omega(X)$ (if any) are called the *nilpotency class* and the *solvability class* of X, respectively. A space X with $\operatorname{nil}\Omega(X) < \infty$ (resp. $\operatorname{sol}\Omega(X) < \infty$) is called *homotopy nilpotent* (resp. *homotopy solvable*).

Notice that $\operatorname{nil}\Omega(K(G,1)) = \operatorname{nil} G$ and $\operatorname{sol}\Omega(K(G,1)) = \operatorname{sol} G$ for the Eilenberg-MacLane space $K(G,1)$ and an abstract group G.

It is obvious that for homotopy nilpotent group-like spaces $X_1, \ldots, X_m$, we have

$$\operatorname{nil}(X_1 \times \cdots \times X_m) = \max\{\operatorname{nil} X_1, \ldots, \operatorname{nil} X_m\} \tag{3}$$

and

$$\operatorname{sol}(X_1 \times \cdots \times X_m) = \max\{\operatorname{sol} X_1, \ldots, \operatorname{sol} X_m\}. \tag{4}$$

The set $\pi_0(X)$ of all path-components of an H-space X is known to be a group. The following result is easy to prove

Lemma 1.2. *If X is an H-space and the path component of the base-point $\star \in X$ is contractible then* $\operatorname{nil}\ \pi_0(X) = \operatorname{nil} X$ *and* $\operatorname{sol}\ \pi_0(X) = \operatorname{sol} X$.

The definition of the nilpotency and solvability classes may be extended to maps. The *nilpotency class* $\operatorname{nil} f$ of an H-map $f : X \to Y$ is the least integer $n \geqslant 0$ for which the map $f \circ \varphi_{X,n+1} : X^{n+1} \to Y$ is nullhomotopic; if no such integer exists, we put $\operatorname{nil} f = \infty$.

The *solvability class* $\operatorname{sol} f$ of an H-map $f : X \to Y$ is the least integer $n \geqslant 0$ for which the map $f \circ \psi_{X,n} : X^{2^n} \to Y$ is nullhomotopic; if no such integer exists, we put $\operatorname{sol} f = \infty$. Since $f\varphi_{X,n} \simeq \varphi_{Y,n} f^n$ and $f\psi_{X,n} \simeq \varphi_{Y,n} f^{2^n}$, we derive

$$\operatorname{sol} f \leqslant \min\{\operatorname{sol} X, \operatorname{sol} Y\} \ \text{ and } \ \operatorname{nil} f \leqslant \min\{\operatorname{nil} X, \operatorname{nil} Y\}.$$

In the sequel, we need

Lemma 1.3. *Let X be an H-space. Then, the composite map:*

(1)

$$T_1(X, \ldots, X) \xrightarrow{j_1(X,\ldots,X)} X^n \xrightarrow{\varphi_{X,n}} X$$

is nullhomotopic;

(2)

$$T_1(X, \ldots, X) \xrightarrow{j_1(X,\ldots,X)} X^{2^n} \xrightarrow{\psi_{X,n}} X$$

is nullhomotopic.

Since the space $X^{\wedge n}$, the n-th smash power of X is the homotopy cofiber of the map $j_1(X, \ldots, X) : T_1(X, \ldots, X) \to X^n$, the result above implies the existence of maps

$$\overline{\varphi}_{X,n} : X^{\wedge n} \to X \text{ for } n \geqslant 1 \text{ and } \overline{\psi}_{X,n} : X^{\wedge 2^n} \to X \text{ for } n \geqslant 0$$

with

$$\overline{\varphi}_{1,X} = \overline{\psi}_{0,X} = \iota_X, \quad \overline{\varphi}_{X,2} = \overline{\psi}_{1,X} = \varphi_{X,2} : X^{\wedge 2} \to X$$

for the basic commutator map and define inductively:

(1) $\overline{\varphi}_{X,n+1} = \overline{\varphi}_{X,2}(\overline{\varphi}_{X,n} \wedge \iota_X) : X^{\wedge(n+1)} \to X$ for $n \geqslant 2$;

(2) $\overline{\psi}_{n+1,X} = \overline{\psi}_{2,X}(\overline{\psi}_{n,X} \wedge \overline{\psi}_{n,X}) : X^{\wedge 2^{n+1}} \to X$ for $n \geqslant 1$.

It is well known that the quotient map $X^n \to X^{\wedge n}$ has a right homotopy inverse after suspending for $n \geqslant 1$, and the fact that X is an H-space means that the suspension map $[Y, X] \to [\Sigma Y, \Sigma X]$ is a monomorphism for any space Y. Thus, we may state

Proposition 1.4. *Let X be an H-space. Then:*

(1) $\varphi_{X,n} \simeq *$ *if and only if* $\overline{\varphi}_{X,n} \simeq *$ *for* $n \geqslant 1$*;*

(2) $\psi_{X,n} \simeq *$ *if and only if* $\overline{\psi}_{X,n} \simeq *$ *for* $n \geqslant 1$*.*

Then, [4, 2.7 Theorem] and Proposition 1.4 lead to

Theorem 1.5. *If X is an H-space then:*

(1)

$$\operatorname{nil} X = \sup_m \operatorname{nil}[X^m, X] = \sup_m \operatorname{nil}[X^{\wedge m}, X] = \sup_Y \operatorname{nil}[Y, X],$$

where m ranges over all integers and Y over all topological spaces;

(2)

$$\operatorname{sol} X = \sup_m \operatorname{sol}[X^{2^m}, X] = \sup_m \operatorname{sol}[X^{\wedge 2^m}, X] = \sup_Y \operatorname{sol}[Y, X],$$

where m ranges over all integers and Y over all topological spaces.

Recall that a *nilpotent space*, first defined by Dror [7] is a based topological space X such that:

- the fundamental group $\pi_1(X)$ is a nilpotent group;
- $\pi_1(X)$ acts nilpotently on the higher homotopy groups $\pi_k(X)$ for $k \geqslant 2$.

A basic theorem about nilpotent spaces [7] states that any map that induces an integral homology isomorphism between two nilpotent space is a weak homotopy equivalence.

Remark 1.6. If X is a connected nilpotent space then following proofs of:

(1) [17, Proposition 1.2], one can show

$$\operatorname{sol} \Omega(X) = \sup_Y \operatorname{sol}\, [Y, \Omega(X)],$$

where Y ranges over all CW-spaces provided X is a CW-space;

(2) [17, Theorem 3.6], one can show

$$\operatorname{sol} \Omega(X) \leqslant \sup_p \operatorname{sol} \Omega(X_{(p)}) + \operatorname{sol} \Omega(X_{(0)})$$

for the p-localization $X_{(p)}$ at a prime p and the rationalization $X_{(0)}$ of the space X.

In particular, given a topological space X, we deduce from Theorem 1.5 that

$$\operatorname{nil} \pi_1(X) \leqslant \operatorname{nil} \Omega(X) \quad \text{and} \quad \operatorname{sol} \pi_1(X) \leqslant \operatorname{sol} \Omega(X).$$

Furthermore, in view of [25, Lemma 2.6.1], we may state

Corollary 1.7. *A connected H-space X is:*

(1) *homotopy nilpotent if and only if the functor* $[-, X]$ *on the category of all spaces is nilpotent group valued;*

(2) *homotopy solvable if and only if the functor* $[-, X]$ *on the category of all spaces is solvable group valued.*

Proof. (1) Certainly, the homotopy nilpotency of a connected associative H-space X implies that the functor $[-, X]$ on the category of all pointed spaces is nilpotent group valued.

Now, suppose that the functor $[-, X]$ is nilpotent group valued and also that $\operatorname{nil}[\prod_1^\infty X, X] \leqslant n$. Then, for the projection map $\prod_1^\infty X \to X^n$ on the first n factors, the composite map

$$\prod_1^\infty X \to X^n \xrightarrow{\varphi_{X,n}} X$$

is null-homotopic. Since, the projection $\prod_1^\infty X \to X^n$ has a retraction, we deduce that the map $\varphi_{X,n} : X^n \to X$ is also null-homotopic.

(2) We follow *mutatis mutandis* the arguments above for (1) and the proof is complete. □

1.2. Homotopy solvability of loop spaces

First, notice that [**25**, Lemma 2.6.3, Lemma 2.6.6] and [**4**, 3.3 Theorem] yield

Proposition 1.8.

(1) ([**25**, Lemma 2.6.3]) $\overline{\psi}_{X,n} = \overline{\varphi}_{X,n+1}(\iota_X \wedge \overline{\psi}_{X,0} \wedge \overline{\psi}_{X,1} \wedge \cdots \wedge \overline{\psi}_{X,n-1})$ *for* $n \geqslant 1$.

(2) ([**25**, Lemma 2.6.6]) *Let* $F \xrightarrow{j} E \xrightarrow{p} B$ *be an* H*-fibration, i.e., it is a fibration and all spaces and maps are* H*-spaces and* H*-maps.*

$$\textit{If}\ \ \operatorname{sol} j, \operatorname{sol} p < \infty \ \ \textit{then}\ \ \operatorname{sol} E \leqslant \operatorname{sol} j + \operatorname{sol} p.$$

Corollary 1.9.

(1) *If* $F \xrightarrow{j} E \xrightarrow{p} B$ *is an* H*-fibration with* $\operatorname{sol} F < \infty$ *and* $\operatorname{sol} B < \infty$ *then*

$$\operatorname{sol} E \leqslant \operatorname{sol} F + \operatorname{sol} B.$$

(2) *Let* $F \to E \xrightarrow{p} B$ *be a fibration with* $\operatorname{sol}\Omega(E) < \infty$*. Then*

$$\operatorname{sol}\Omega(F) \leqslant \operatorname{sol}\Omega(E) < \infty.$$

Proof. (1) Is a direct consequence of Proposition 1.8(2).

(2) The fibration $F \to E \xrightarrow{p} B$ leads to the H-fibration $\Omega^2(B) \to \Omega(F) \to \Omega(E)$ of H-spaces. Since $\Omega^2(B)$ is an Abelian H-space and $\operatorname{sol}\Omega(E) < \infty$, part (1) implies that $\operatorname{sol}\Omega(F) \leqslant \operatorname{sol}\Omega(E) < \infty$ and the proof follows. □

Now, let X be an H-space with $\operatorname{nil} X < \infty$. Then, by Proposition 1.8(1), we get that

$$\operatorname{sol} X \leqslant \operatorname{nil} X - 1$$

provided $\operatorname{nil} X < \infty$.

Recall that for the m-th sphere $\mathbb{S}^m$ with $m \geqslant 0$, in view of (see e.g., [**13**]), we have $\operatorname{nil}\Omega(\mathbb{S}^m) = 1$ if and only if $m = 0, 1, 3, 7$ and

$$\operatorname{nil}\Omega(\mathbb{S}^m) = \begin{cases} 2 & \text{for odd } m \text{ and } m \neq 1, 3, 7 \text{ or } m = 2; \\ 3 & \text{for even } m \geqslant 4. \end{cases}$$

Consequently, by Proposition 1.8(1), we have $\operatorname{sol}\Omega(\mathbb{S}^m) = 0$ if and only if $m = 0, 1, 3, 7$ and

$$\operatorname{sol}\Omega(\mathbb{S}^m) \leqslant \begin{cases} 1 & \text{for odd } m \text{ and } m \neq 1, 3, 7 \text{ or } m = 2; \\ 2 & \text{for even } m \geqslant 4. \end{cases}$$

Recall that the *Postnikov decomposition* (Postnikov tower or Moore-Postnikov tower/system) of a space X is a system $(P_n(X), p_n, f_n)_{n\geqslant 0}$, where:
$P_n(X)$ are spaces called *nth-Postnikov sections* of X with $P_0(X) = *$;
$p_n : P_n(X) \to P_{n-1}(X)$ are fibration with fiber $K(\pi_n(X), n)$;
$f_n : X \to P_n(X)$ are $(n+1)$-connected maps.

Write $\widetilde{\pi_n(X)}$ for the $\pi_1(P_{n-1}(X)) = \pi_1(X)$-module $\pi_n(X)$ with $n \geqslant 2$. Then, the characteristic class

$$k_n = \overline{c}(p_n) \in H^{n+1}(P_{n-1}(X), \widetilde{\pi_n(X)}) \quad \text{of the fibration} \quad p_n : P_n(X) \to P_{n-1}(X)$$

is called the n-th *Postnikov invariant*, or n-th k-invariant of X. It can be shown that the k-invariants k_n are just the characteristic classes of $f_n : X \to P_{n-1}(X)$ made into a fibration.

Lemma 1.10. *If X is a homotopy associative H-space and Y a finite dimensional CW-complex with* $\dim Y = 2^n$ *then the group $[Y, X]$ is solvable with the solvability class at most n.*

Proof. First, recall that given a homotopy associative H-space X, in view of [**16**], all its m-th Postnikov stages P_mX are also a homotopy associative H-space and the canonical map $X \to P_mX$ is an H-map. Hence, for a CW-complex Y with $\dim Y = 2^n$, there is an isomorphism $[Y, X] \approx [Y, P_{2^n}X]$ determined by the canonical map $X \to P_{2^n}X$. Then, the map $\overline{\psi}_{P_{2^n}X, n+1}(f_1 \wedge \cdots \wedge f_{2^{n+1}}) : Y^{\wedge 2^{n+1}} \to P_{2^n}X$ is homotopy trivial for any maps $f_1, \ldots, f_{2^{n+1}} : Y \to P_nX$ since the space $Y^{\wedge 2^{n+1}}$ is 2^n-connected. Consequently, $\operatorname{nil}[Y, X] = \operatorname{nil}[Y, P_{2^n}X] \leqslant n$ and the proof follows. □

Next, any CW-complex Y can be expressed as

$$Y = \lim_{\rightarrow} Y_\alpha$$

with Y_α finite. This leads to the short exact sequence

$$E \to \lim_{\leftarrow}{}^1[\Sigma Y_\alpha, X] \longrightarrow [Y, X] \longrightarrow \lim_{\leftarrow}[Y_\alpha, X] \to E$$

for any connected homotopy associative H-space X.

In view of [**14**, Proposition 1.2], we have that the intersection

$$([Y, X]_d) \cap (\lim_{\leftarrow}{}^1[\Sigma Y_\alpha, X]) = 0$$

for any finite connected, homotopy associative H-space X with $\dim X = d < \infty$, where $[Y, X]_d$ stands for the d-th member of the lower central series of the group

$[Y, X]$. Since the d-th derived subgroup $[Y, X]^{(d)} \subseteq [Y, X]_d$, Lemma 1.10 and [**14**, Proposition 1.1] yield:

Proposition 1.11. *If X is a finite connected, homotopy associative H-space then the group $[Y, X]$ is pro-solvable for any CW-complex Y.*

Since $p_n : P_n(X) \to P_{n-1}(X)$ is a fibration with the fiber $K(\pi_n(X), n)$ starting with $P_1(X) = K(\pi_1(X), 1)$, it follows by an inductive argument and Proposition 1.8 that Postnikov sections $P_n(X)$ are solvable spaces for $n \geqslant 0$ provided X is nilpotent. Furthermore, in view of [**17**, Proposition 1.3], the homotopy nilpotency of a space X implies its nilpotency. But, not every space X is homotopy solvable if X is nilpotent or even simply connected.

Example 1.12. Take $X = \mathbb{S}^k \vee \mathbb{S}^l$ with $k, l \geqslant 2$ and write $i_1 : \mathbb{S}^k \hookrightarrow \mathbb{S}^k \vee \mathbb{S}^l$, as well as $i_2 : \mathbb{S}^l \hookrightarrow \mathbb{S}^k \vee \mathbb{S}^l$ for the canonical inclusion maps. Then, for the 2^n-fold non-trivial Whitehead products $[i_1, \ldots, i_1]$ and $[i_2, \ldots, i_2]$ with $n \geqslant 1$, the non-trivial Whitehead product $[[i_1, \ldots, i_1], [i_2, \ldots, i_2]]$ yields (via the Samelson product) a non-trivial map

$$(\mathbb{S}^{k-1})^{\wedge 2^n} \wedge (\mathbb{S}^{l-1})^{\wedge 2^n} \longrightarrow \Omega(X)^{\wedge 2^{n+1}} \xrightarrow{\psi_{X,n+1}} \Omega(X).$$

Consequently, the space X is not homotopy solvable.

In view of [**3**, Corollary 5.3.10] or [**19**, Theorem 1.1], we have

Proposition 1.13. *Let $(P_n(X), p_n, f_n)_{n \geqslant 0}$ be the Postnikov decomposition of a nilpotent space X. Then, the Postnikov sections $P_n(X)$ are nilpotent spaces fro $n \geqslant 0$ and the fibrations $p_n : P_n(X) \to P_{n-1}(X)$ have finite principal refinements for $n \geqslant 1$. That is, for each $n \geqslant 1$ there are Abelian groups A_i and principal $K(A_i, n)$-fibrations q_i for $i = 1, \ldots, m_n$ such $p_n = q_1 \cdots q_{m_n}$.*

Then, the proof of [**4**, 4.11 Theorem] leads to the following generalised result.

Theorem 1.14. *Let X be a connected space with the solvable fundamental homotopy group $\pi_1(X)$ and its nilpotent actions on $\pi_m(X)$ for $m \geqslant 2$. If the k-invariants k_m of X are trivial for all but finitely many values of m, then $\operatorname{nil} \Omega(X) < \infty$. In particular,*

$$\operatorname{sol} \Omega(X) \leqslant \operatorname{nil} \Omega(X) - 1 < \infty.$$

Let $(P_n(X), p_n, f_n)_{n \geqslant 0}$ be the Postnikov decomposition of a space X. Recall that the homotopy fiber $\tilde{X}_n$ of the map $f_n : X \to P_n(X)$ is called the *n-connected covering* of X for $n \geqslant 0$. Then, the fibration

$$\Omega^2(P_n(X)) \to \Omega(\tilde{X}_n) \to \Omega(X),$$

in view of Corollary 1.9(2), leads to

Corollary 1.15. *If X is a nilpotent space and $(P_n(X), p_n, f_n)_{n \geqslant 0}$ its Postnikov decomposition then its n-connected coverings $\tilde{X}_n$ are homotopy solvable for $n \geqslant 0$.*

2. Homotopy solvability of some homogenous spaces and Lie groups

Basing on some results from [**15**] and [**25**], we first analyse homotopy solvability of homogenous spaces. Then, we take into account the homotopy solvability of some topological groups.

2.1. Homotopy solvability of some homogenous spaces

First, we recall (see e.g., [**6**, Lemma 1.33]) the following

Proposition 2.1. *Let G be a Lie group and $E(G) \to B(G)$ the associated principal G-bundle. If $K \leqslant G$ is a closed subgroup of G and $i : K \hookrightarrow G$ the embedding map then passage to orbits yields a G/K-bundle $G/K \xrightarrow{\eta} B(K) \xrightarrow{B(i)} B(G)$.*

Then, we derive

Corollary 2.2. *If G is a homotopy solvable Lie group and $K \leqslant G$ its closed subgroup then*

$$\operatorname{sol}\Omega(G/K) \leqslant \operatorname{sol} G + 1 \quad \textit{and} \quad \operatorname{sol} K \leqslant 2 \operatorname{sol} G + 1.$$

Proof. We mimic the proof of [**4**, Theorem 3.3]. Namely, given a homotopy solvable Lie group G and its closed subgroup $K < G$, the G/K-bundle

$$G/K \xrightarrow{\eta} B(K) \xrightarrow{B(i)} B(G)$$

from Proposition 2.1 and the associated Puppe sequence yield an exact sequence of groups and homomorphisms

$$\cdots \xrightarrow{i_*} [Y, \Omega(G)] \xrightarrow{\partial} [Y, \Omega(G/K)] \xrightarrow{\Omega(\eta)_*} [Y, K] \xrightarrow{i_*} [Y, G]$$

for any space Y.

Since the group $[Y, \Omega(G)]$ is Abelian, formulas (1), (2) and the definition of the homotopy solvability, in view of the above, lead to $\operatorname{sol}\Omega(G/K) \leqslant \operatorname{sol} G + 1$. Furthermore, Corollary 1.9(1) implies that $\operatorname{sol} K \leqslant 2 \operatorname{sol} G + 1$ and the proof is complete. □

To make use of the above, we need some further result. Given a map $f : X \to Y$ of pointed topological spaces, write F_f for its homotopy fiber.

Proposition 2.3 ([**18**, p. 96])**.** *If $X \xrightarrow{g} Y \xrightarrow{f} Z$ are maps of pointed topological spaces then the commutative diagram*

$$\begin{array}{ccccc} X & = & X & \xrightarrow{g} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle fg} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{f} & Z & = & Z \end{array}$$

leads to the fibration

$$F_f \longrightarrow F_{gf} \longrightarrow F_f.$$

Then, Propositions 1.8 and 2.3 lead to

Corollary 2.4. *If $G_1 \leqslant G_2 \leqslant \cdots \leqslant G_n$ is a sequence of topological groups such that $n \geqslant 3$ and $\Omega(G_{i+1}/G_i)$ are homotopy solvable for $i = 1, \ldots, n-1$ then $\Omega(G_{i+k}/G_i)$ are homotopy solvable for $k = 0, 1, \ldots, n-i$ with $i = 1, \ldots, n-1$.*

Let now $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ be the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$-algebra of quaternions. Then, we set for classical groups:

$$O(\mathbb{F}, n) = \begin{cases} O(n) \text{ if } \mathbb{F} = \mathbb{R}; \\ U(n) \text{ if } \mathbb{F} = \mathbb{C}; \\ Sp(n) \text{ if } \mathbb{F} = \mathbb{H}. \end{cases} \quad \text{and } SO(\mathbb{F}, n) = \begin{cases} SO(n) \text{ if } \mathbb{F} = \mathbb{R}; \\ SU(n) \text{ if } \mathbb{F} = \mathbb{C}; \\ Sp(n) \text{ if } \mathbb{F} = \mathbb{H}. \end{cases}$$

Notice that the groups $SO(\mathbb{F}, n)$ are each subject to the standard embeddings $SO(\mathbb{F}, m) \subseteq SO(\mathbb{F}, n)$ provided $1 \leqslant m \leqslant n$. We write $V_{n,m}(\mathbb{F}) = SO(\mathbb{F}, n)/SO(\mathbb{F}, m)$, the Stiefel manifold and aim to show that $\Omega(V_{n,m}(\mathbb{F}))$ are homotopy solvable provided $1 \leqslant m \leqslant n$.

First, given $m < n$ we write $k_o(n, m)$ and $k_e(n, m)$ for the numbers of all odd and even elements of the set $\{m+1, \ldots, n+m\}$. Certainly, $k_o(n, m) + k_e(n, m) = n - m$. Then, Corollary 2.4 yields the following generalization of [**25**, 2.6.9 Proposition].

Proposition 2.5. *Let $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ be the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$-algebra of quaternions. Then,*

$$\operatorname{sol} \Omega(V_{n,m}(\mathbb{R})) \leqslant 2k_o(n, m) + k_e(n, m)$$

and

$$\operatorname{sol} \Omega(V_{n,m}(\mathbb{F})) \leqslant n - m \quad \textit{for} \quad \mathbb{F} = \mathbb{C}, \mathbb{H}.$$

Proof. First, notice that $V_{n,n-1}(\mathbb{R}) \approx \mathbb{S}^{n-1}$ and $V_{n,n-1}(\mathbb{F}) \approx \mathbb{S}^{nd-1}$ ($d = 2$ if $\mathbb{F} = \mathbb{C}$ and $d = 4$ if $\mathbb{F} = \mathbb{H}$).

Since

$$\operatorname{sol} \Omega(\mathbb{S}^{nd-1}) \leqslant 1 \text{ for } d = 2, 4 \text{ and } \operatorname{sol} \Omega(\mathbb{S}^{n-1}) \leqslant \begin{cases} 2 & \text{if } n \text{ is odd}, \\ 1 & \text{if } n \text{ is even}, \end{cases}$$

we get by Corollary 2.4 that $\operatorname{sol} \Omega(V_{n,m}(\mathbb{F})) < \infty$ for $m < n$. But, in view of Proposition 2.3, one has the fibration

$$V_{k,m}(\mathbb{F}) \to V_{n,m}(\mathbb{F}) \to V_{n,k}(\mathbb{F})$$

for $m < k < n$ and by looping we obtain an H-fibration

$$\Omega(V_{k,m}(\mathbb{F})) \to \Omega(V_{n,m}(\mathbb{F})) \to \Omega(V_{n,k}(\mathbb{F})).$$

Therefore, Propositions 1.8 and 2.3 imply

$$\operatorname{sol} \Omega(V_{n,m}(\mathbb{F})) \leqslant \operatorname{sol} \Omega(V_{k,m}(\mathbb{F})) + \operatorname{sol} \Omega(V_{n,k}(\mathbb{F})).$$

Thus, by induction with respect to $n - m$ and Proposition 1.8, the proof is complete. □

Remark 2.6. Since

$$V_{n,m}(\mathbb{R}) \approx O(n)/O(m) \text{ and } V_{n,m}(\mathbb{C}) \approx U(n)/U(m),$$

Proposition 2.5 leads to

$$\operatorname{sol} \Omega(O(n)/O(m)) \leqslant 2k_o(n, m) + k_e(n, m) \quad \text{and} \quad \operatorname{sol} \Omega(U(n)/U(m)) \leqslant n - m.$$

2.2. Homotopy solvability of some Lie groups

Given a topological group G and its subgroup $H \leqslant G$, following [**15**], we describe H as *homotopy-Abelian* in G if maps $f, \overline{f} : H \times H \to G$ defined by

$$f(x,y) = xy = \overline{f}(y,x)$$

for $x, y \in H$ are homotopic.

This is the case, for example, when G is pathwise-connected and H is conjugate to, subgroup whose elements commute with those of H. If H is homotopy-Abelian in G, then any subgroup of H is homotopy-Abelian in any group which contains G as a subgroup.

Certainly, the groups $O(n)$ $(n \geqslant 2)$, $SO(n)$ $(n \geqslant 3)$, $U(n)$ $(n \geqslant 2)$, $Sp(n)$ $(n \geqslant 1)$ are non-Abelian, but elements of $SO(\mathbb{F}, n)$ and $O(\mathbb{F}, n)$ commute with those of appropriate conjugate subgroups. Since the groups $SO(\mathbb{F}, n)$ are pathwise-connected and the short exact sequence $1 \to SU(n) \to U(n) \to \mathbb{S}^1 \to 1$ implies ([**5**, 1.12 Proposition]) the pathwise-connectivity of $U(n)$, it follows that $SO(\mathbb{F}, n)$ and $U(n)$ are homotopy-Abelian in $SO(\mathbb{F}, 2n)$ and $U(2n)$, respectively. Thus, the standard embeddings $SO(\mathbb{F}, n) \hookrightarrow SO(\mathbb{F}, 2n)$ and $U(n) \hookrightarrow U(2n)$ are $\leqslant 1$ homotopy nilpotent (or $= 0$ homotopy solvable). (This property is stated as $SO(\mathbb{F}, n)$ and $U(n)$ homotopy commute in $SO(\mathbb{F}, 2n)$ and $U(2n)$, respectively.)

Since

$$\Omega(V_{2n,n}(\mathbb{F})) \to SO(\mathbb{F}, n) \hookrightarrow SO(\mathbb{F}, 2n), \quad \Omega(U(2n)/U(n)) \to U(n) \hookrightarrow U(2n)$$

are H-fibrations with the loops addition and Lie group multiplications, Proposition 1.8(2), Remark 2.6 and Proposition 2.5 imply the following generalization of [**25**, 2.6.10 Proposition].

Proposition 2.7.

(1) The groups $SO(n)$ are homotopy solvable with

$$\operatorname{sol} SO(n) \leqslant 2k_o(2n, n) + k_e(2n, n)$$

and the groups $SU(n)$, $Sp(n)$, $U(n)$ are homotopy solvable with

$$\operatorname{sol} SU(n), \operatorname{sol} Sp(n), \operatorname{sol} U(n) \leqslant n.$$

(2) For any topological space Y (not necessarily of the homotopy type of a CW-complex),

$$\operatorname{sol}\,[Y, SO(n)] \leqslant 2k_o(2n, n) + k_e(2n, n) \text{ and}$$

$$\operatorname{sol}\,[Y, SU(n)], \operatorname{sol}\,[Y, Sp(n)], \operatorname{sol}\,[Y, U(n)] \leqslant n.$$

Remark 2.8.

(1) The group $O(n)$ is disconnected. But, either the group isomorphism $O(n) \approx SO(n) \rtimes O(1)$ or the diffeomorphism $O(n) \approx SO(n) \times O(1)$ yields that $O(n)$ is homotopy-Abelian in $O(2n)$ via the standard embedding $O(n) \hookrightarrow O(2n)$.
Since $O(n+1)/O(n) \approx \mathbb{S}^n$, as in Proposition 2.7, we get that

$$\operatorname{sol} O(n) = \operatorname{sol} SO(n) \leqslant 2k_o(2n, n) + k_e(2n, n).$$

(2) Let $Pin(n) \to O(n)$ and $Spin(n) \to SO(n)$ be the universal covering maps. Then, fibrations

$$O(1) \to Spin(n) \to SO(n), \quad O(1) \to Pin(n) \to O(n),$$

homeomorphisms

$$Spin(n+1)/Spin(n),\ Pin(n+1)/Pin(n) \approx \mathbb{S}^n$$

and Propositions 1.8(2), 2.5 yield that

$$\operatorname{sol} Pin(n),\ \operatorname{sol} Spin(n) = \operatorname{sol} SO(n) \leqslant 2k_o(2n,n) + k_e(2n,n).$$

But, the homotopy solvability of a topological group does not imply its solvability.

Remark 2.9. Since the commutators

$$[SO(3), SO(3)] = SO(3) \text{ and } [SU(2), SU(2)] = SU(2),$$

and:

$$\begin{aligned} &SO(3) \subseteq SO(n) \subseteq O(n) \text{ for } n \geqslant 3,\\ &SU(2) \subseteq SU(n) \subseteq U(n) \text{ for } n \geqslant 2, \text{ and}\\ &SU(2) = Sp(1) \subseteq Sp(n) \text{ for } n \geqslant 1, \end{aligned}$$

we derive that the groups:

(1) $SO(n)$ and $O(n)$ are not solvable for $n \geqslant 3$;

(2) $U(n)$ and $SU(n)$ are not solvable for $n \geqslant 2$;

(3) $Sp(n)$ is not solvable for $n \geqslant 1$.

In view of [**5**, Theorem 5.1], any compact Lie group G is (isomorphic to) a matrix Lie group. (More precisely, any compact Lie group G is isomorphic to a closed subgroup of the orthogonal group $O(m)$ or the unitary group $U(n)$ for some positive integers m and n, respectively). Therefore, Corollary 2.2 and Proposition 2.7 yield

Corollary 2.10. *If G is a compact Lie group then G is homotopy solvable and*

$$\operatorname{sol} G \leqslant 2\operatorname{sol} O(m) + 1,\ 2\operatorname{sol} U(n) + 1 < \infty$$

for some positive integers m and n.

By [**1**, Chapter 8], we have the increasing sequence of exceptional compact Lie groups: $G_2 \subseteq F_4 \subseteq E_6 \subseteq E_7 \subseteq E_8$. In view of Corollary 2.10 all of them are homotopy solvable. We estimate the solvability class of G_2 and F_4.

First, consider the exceptional compact Lie group $G_2 = \operatorname{Aut}(\mathbb{O})$, the automorphism group of the octonions $\mathbb{O}$ as a normed algebra. Consequently, the canonical inclusion $G_2 \subseteq SO(7)$ leads to $G_2 \subseteq Spin(7)$. Furthermore, by e.g., [**24**, Theorem 3], the group G_2 is the stabilizer group of any point on the sphere $\mathbb{S}^7$ of the canonical transitive action of $Spin(7)$ on $\mathbb{S}^7$.

Then, the fibration

$$\mathbb{S}^7 = Spin(7)/G_2 \to B(G_2) \to B(Spin(7))$$

yields the H-fibration

$$\Omega(\mathbb{S}^7) \to G_2 \to Spin(7).$$

Since $\Omega(\mathbb{S}^7)$ is an Abelian H-space and $\operatorname{sol} Spin(7) \leqslant 10$, Proposition 1.8(2) implies

$$\operatorname{sol} G_2 \leqslant \operatorname{sol} Spin(7) \leqslant 10.$$

The group F_4 is known as $\operatorname{Aut}(\operatorname{Herm}_3(\mathbb{O}), \circ)$, the automorphism group of the Jordan algebra $(\operatorname{Herm}_3(\mathbb{O}), \circ)$ with $\operatorname{Herm}_3(\mathbb{O})$ as Hermitian 3×3-matrices over the octonions $\mathbb{O}$. Since, $F_4 \subseteq O(27)$, the fibration

$$O(27)/F_4 \to B(F_4) \to B(O(27))$$

leads to the H-fibration

$$\Omega(O(27)/F_4) \to F_4 \to O(27).$$

Hence, the relation $\operatorname{sol} O(27) \leqslant 40$, in view of Corollary 2.10, yields

$$\operatorname{sol} F_4 \leqslant 2 \operatorname{sol} O(27) + 1 \leqslant 81.$$

It is well-known a transitive action of the exceptional group F_4 on the Cayley plane $\mathbb{O}P^2$ which yields a diffeomorphism

$$F_4/Spin(9) \approx \mathbb{O}P^2.$$

Since, $\operatorname{sol} Spin(9) \leqslant 13$, Corollary 1.9(2) implies

$$\operatorname{sol} \Omega(\mathbb{O}P^2) \leqslant \operatorname{sol} Spin(9) \leqslant 13.$$

The result below has been stated in [**23**, 11.44 Proposition]. Nevertheless, we present its proof with other details.

Proposition 2.11. *If $n \geqslant 1$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ then the group $O(n, \mathbb{F})$ is a strong deformation retract of the group $GL(n, \mathbb{F})$.*

Proof. Let $T(n, \mathbb{F}) \subseteq GL(n, \mathbb{F})$ be the subgroup of upper-triangular matrices with positive diagonal entries. An immediate consequence of the Gram-Schmidt orthonormalization theorem is that every matrix $A \in GL(n, \mathbb{F})$ has a unique representation in the form $A = OT$, where $O \in O(n, \mathbb{F})$ and $T \in T(n, \mathbb{F})$. Thus, the map

$$\theta : O(n, \mathbb{F}) \times T(n, \mathbb{F}) \to GL(n, \mathbb{F})$$

given by $\theta(O, T) = OT$ for $(O, T) \in O(n, \mathbb{F}) \times T(n, \mathbb{F})$ is a continuous bijection. In fact, θ is a homeomorphism.

Now, the group $T(n, \mathbb{F})$ is contractible: we define a map

$$G : T(n, \mathbb{F}) \times I \to T(n, \mathbb{F})$$

given by $G(T,t) = G\left(\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & \cdots \\ & & \cdots & \cdots \\ & & a_{n-1n-1} & a_{n-1n} \\ & & & a_{nn} \end{pmatrix}, t\right) =$

$$\begin{pmatrix} t + a_{11}(1-t) & (1-t)a_{12} & \cdots & (1-t)a_{1n} \\ & t + a_{22}(1-t) & \cdots & \cdots \\ & & \cdots & \cdots \\ & & t + a_{n-1n-1}(1-t) & (1-t)a_{n-1n} \\ & & & t + a_{nn}(1-t) \end{pmatrix}$$

for $I = [0,1]$ and $(T,t) \in T(n,\mathbb{F}) \times I$. Then,

$$GL(n,\mathbb{F}) \times I \xrightarrow{\theta^{-1}\times I} O(n,\mathbb{F}) \times T(n,\mathbb{F}) \times I \xrightarrow{\mathrm{id}_{O(n,\mathbb{F})}\times G} O(n,\mathbb{F}) \times T(n,\mathbb{F})$$

is a deformation of $GL(n,\mathbb{F})$ on $O(n,\mathbb{F}) \times \{I_n\}$ and the proof is complete. □

Proposition 2.11 leads to a map $f : GL(n,\mathbb{F}) \to O(n,\mathbb{F})$ being a homotopy inverse of the canonical inclusion map $O(n,\mathbb{F}) \hookrightarrow GL(n,\mathbb{F})$. Since, $O(n,\mathbb{F}) \hookrightarrow GL(n,\mathbb{F})$ is an H-map, we deduce that $f : GL(n,\mathbb{F}) \to O(n,\mathbb{F})$ is also an H-map. Consequently, we get

$$\operatorname{sol} GL(n,\mathbb{F}) = \operatorname{sol} O(n,\mathbb{F}).$$

Furthermore, this proposition yields a homotopy equivalence

$$V_{n,m}(\mathbb{F}) \simeq GL(n,\mathbb{F})/GL(m,\mathbb{F})$$

provided $m \leqslant n$. In particular, there is a homotopy equivalence

$$GL(n,\mathbb{F})/GL(n-1,\mathbb{F}) \simeq \mathbb{S}^{dn-1},$$

where $d = \dim_{\mathbb{R}} \mathbb{F}$ and

$$\operatorname{sol}\Omega(V_{n,m}(\mathbb{F})) = \operatorname{sol}\Omega(GL(n,\mathbb{F})/GL(m,\mathbb{F})).$$

As a byproduct of the proof of Proposition 2.11, we get that there is a deformation of $SL(n,\mathbb{F})$ on $SO(n,\mathbb{F}) \times \{I_n\}$. Hence, we get

$$\operatorname{sol} SL(n,\mathbb{F}) = \operatorname{sol} SO(n,\mathbb{F}) = \operatorname{sol} O(n,\mathbb{F}) \text{ and}$$
$$\operatorname{sol}\Omega(V_{n,m}(\mathbb{F})) = \operatorname{sol}\Omega(SL(n,\mathbb{F})/SL(m,\mathbb{F})).$$

2.3. Grassmannians and generalised flag manifolds

Let $\mathbb{F} = \mathbb{R}, \mathbb{C}$ be the field of reals or complex numbers and $\mathbb{H}$, the skew $\mathbb{R}$-algebra of quaternions. Write $G_{n,m}(\mathbb{F})$ (resp. $G^+_{n,m}(\mathbb{F})$) for the (resp. oriented with $\mathbb{F} = \mathbb{R}, \mathbb{C}$) Grassmannian of m-dimensional subspaces in the n-dimensional $\mathbb{F}$-vector space. For example, the set of lines $G_{n+1,1}(\mathbb{F}) = \mathbb{F}P^n$, the projective n-space over $\mathbb{F}$.

The homotopy nilpotency of $\mathbb{F}P^n$ has been studied by Ganea [**9**, Propositions 1.3–1.5], Meier [**17**, Theorem 5.4] and Snaith [**21**, Corollary 3.3]. Then, results presented in [**11**] extend this investigation onto Grassmannians and generalised flag manifolds. The aim of this subsection is to present some results on the homotopy nilpotency of those manifolds.

It is well known that $G_{n,m}(\mathbb{F})$ (resp. $G^+_{n,m}(\mathbb{F})$) are smooth manifolds with diffeomorphisms

$$G_{n,m}(\mathbb{F}) \approx O(\mathbb{F},n)/(O(\mathbb{F},m) \times O(\mathbb{F},n-m)) \text{ and}$$
$$G^+_{n,m}(\mathbb{F}) \approx SO(\mathbb{F},n)/SO(\mathbb{F},m) \times SO(\mathbb{F},n-m))$$

which lead to fibrations

$$\Omega(O(\mathbb{F},n)) \to \Omega(G_{n,m}(\mathbb{F})) \to O(\mathbb{F},m) \times O(\mathbb{F},n-m)$$

and

$$\Omega(SO(\mathbb{F},n)) \to \Omega(G^+_{n,m}(\mathbb{F})) \to SO(\mathbb{F},m) \times SO(\mathbb{F},n-m).$$

Next, there is the universal covering map

$$\mathbb{S}^0 \longrightarrow G^+_{n,m}(\mathbb{R}) \longrightarrow G_{n,m}(\mathbb{R})$$

and a fibre bundle

$$\mathbb{S}^1 \longrightarrow G^+_{n,m}(\mathbb{C}) \longrightarrow G_{n,m}(\mathbb{C}).$$

Furthermore, recall that the classifying space

$$BO(\mathbb{F},m) = \lim_{n\to\infty} G_{n,m}(\mathbb{F}) = G_{\infty,m}(\mathbb{F}).$$

Then, Propositions 1.8 and 2.7 yield

Theorem 2.12. *If* $1 \leqslant m < n \leqslant \infty$ *then*

$$\operatorname{sol}\Omega(G_{n,m}(\mathbb{F})) < \infty \quad \textit{for} \quad \mathbb{F} = \mathbb{R},\, \mathbb{C},\, \mathbb{H}$$

and

$$\operatorname{sol}\Omega(G^+_{n,m}(\mathbb{F})) \leqslant \operatorname{sol}\Omega(G_{n,m}(\mathbb{F})) < \infty \quad \textit{for} \quad \mathbb{F} = \mathbb{R},\, \mathbb{C}.$$

In particular,

$$\operatorname{sol}\Omega(\mathbb{F}P^n) < \infty \quad \textit{for} \quad \mathbb{F} = \mathbb{R},\, \mathbb{C},\, \mathbb{H}.$$

The (resp. oriented) generalised flag manifold

$F_{n;n_1,\ldots,n_k}(\mathbb{F})$ (resp. $F^+_{n;n_1,\ldots,n_k}(\mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}$) with $1 \leqslant n_1 < \cdots < n_k \leqslant n-1$

in the n-dimensional $\mathbb{F}$-vector space is smooth with a diffeomorphism

$$F_{n;n_1,\ldots,n_k}(\mathbb{F}) \approx$$
$$(O(\mathbb{F},n)/(O(\mathbb{F},n_1) \times O(\mathbb{F},n_1-n_2) \times \cdots \times O(\mathbb{F},n_{k-1}-n_k) \times O(\mathbb{F},n-n_k))$$

and

$$F^+_{n;n_1,\ldots,n_k}(\mathbb{F}) \approx$$
$$(SO(\mathbb{F},n)/SO(\mathbb{F},n_1) \times SO(\mathbb{F},n_1-n_2) \times \cdots \times SO(\mathbb{F},n_{k-1}-n_k) \times SO(\mathbb{F},n-n_k)).$$

Then, the universal covering map,

$$(\mathbb{S}^0)^k \to F^+_{n;n_1,\ldots,n_k}(\mathbb{R}) \to F_{n;n_1,\ldots,n_k}(\mathbb{R})$$

a fibre bundle

$$(\mathbb{S}^1)^k \to F^+_{n;n_1,\ldots,n_k}(\mathbb{C}) \to F_{n;n_1,\ldots,n_k}(\mathbb{C})$$

and the fibrations

$$\Omega(SO(\mathbb{F},n)) \longrightarrow \Omega(F_{n;n_1,\dots,n_k}(\mathbb{F})) \longrightarrow \\ SO(\mathbb{F},n_1) \times SO(\mathbb{F},n_1-n_2) \times \cdots \times SO(\mathbb{F},n_{k-1}-n_k) \times SO(\mathbb{F},n-n_k),$$

and

$$\Omega(SO(\mathbb{F},n)) \longrightarrow \Omega(F^{+}_{n;n_1,\dots,n_k}(\mathbb{F})) \longrightarrow \\ SO(\mathbb{F},n_1) \times SO(\mathbb{F},n_1-n_2) \times \cdots \times SO(\mathbb{F},n_{k-1}-n_k) \times SO(\mathbb{F},n-n_k)$$

lead to the following generalization of Theorem 2.12.

Theorem 2.13. *If* $1 \leqslant m < n < \infty$ *then*

$$\operatorname{sol}\Omega(F_{n;n_1,\dots,n_k}(\mathbb{F})) < \infty \quad \textit{for} \quad \mathbb{F} = \mathbb{R},\, \mathbb{C},\, \mathbb{H}$$

and

$$\operatorname{sol}\Omega(F^{+}_{n;n_1,\dots,n_k}(\mathbb{F})) \leqslant \operatorname{sol}\Omega((F_{n;n_1,\dots,n_k}(\mathbb{F}))) < \infty \quad \textit{for} \quad \mathbb{F} = \mathbb{R},\, \mathbb{C}.$$

2.4. $\mathbf{A_m}$-spaces

Recall that by Stasheff [**22**], an A_m*-structure* on a space X consists of m-tuples

$$\begin{array}{ccccccccc} X & = & \mathcal{E}_1(X) & \hookrightarrow \cdots \hookrightarrow & \mathcal{E}_n(X) & \hookrightarrow \cdots \hookrightarrow & \mathcal{E}_m(X) \\ & & \downarrow{\scriptstyle \mathfrak{q}_1} & & \downarrow{\scriptstyle \mathfrak{q}_n} & & \downarrow{\scriptstyle \mathfrak{q}_m} \\ * & = & \mathcal{B}_1(X) & \hookrightarrow \cdots \hookrightarrow & \mathcal{B}_n(X) & \hookrightarrow \cdots \hookrightarrow & \mathcal{B}_m(X) \end{array}$$

such that $\mathfrak{q}_{n*} : \pi_k(\mathcal{E}_n(X), X) \to \pi_k(\mathcal{B}_n(X))$ is an isomorphism for all $k \geqslant 1$, together with a contracting homotopy

$$h : C\mathcal{E}_{n-1}(X) \to \mathcal{E}_n(X) \ \text{ such that } \ h(C\mathcal{E}_{n-1}(X)) \subseteq \mathcal{E}_n(X)$$

for the cone space $C\mathcal{E}_{n-1}(X)$ on $\mathcal{E}_{n-1}(X)$ with $n = 2, \dots, m$. For the purposes of homotopy theory, in the light of [**22**, Proposition 2], we can think of

$$X \longrightarrow \mathcal{E}_n(X) \xrightarrow{\mathfrak{q}_n} \mathcal{B}_n(X),$$

as a fibration.

An A_m*-space* for $m = 0, 1 \dots, \infty$ is a space X with a multiplication $\mu : X \times X \to X$ that is associative up to higher homotopies involving up to n variables. Further, an A_∞-space has all coherent higher associativity homotopies and is equivalent to a loop space $\Omega(Y)$ for a space Y called the *classifying space* of X.

By [**22**, Theorem 5], classes of spaces with A_m-structures and A_m-spaces coincide.

Proposition 2.14. *A space* X *admits an* A_m*-structure if and only if* X *is an* A_m*-space.*

The X*-projective* n*-space* $XP(n)$ for $n \leqslant m$, associated with an A_m-space is the base space $\mathcal{B}_{n+1}(X)$ of the derived A_m-structure. The space $\mathcal{B}_1(X)$ is a point and $\mathcal{B}_2(X)$ can be recognized as the suspension $\Sigma(X)$. Notice that $\mathcal{B}_{m+1}(X)$ can be defined even when $\mathfrak{p}_{m+1}$ cannot; it has the homotopy type of the mapping cone $C\mathcal{E}_m(X) \cup_{\mathfrak{q}_m} \mathcal{B}_m(X)$. By means of [**22**, Theorem 11, Theorem 12], the spaces $\mathcal{E}_n(X)$ and $\mathcal{B}_{n+1}(X)$ have the homotopy types of the n-th join X^{*^n} and $C\mathcal{E}_n(X) \cup_{\mathfrak{p}_n} \mathcal{B}_n(X)$ for $n \leqslant m$, respectively provided X is path-connected. Because of a homotopy equivalence $X^{*^n} \simeq \Sigma^{n-1}(X^{\wedge n})$ for the $(n-1)$-th suspension Σ^{n-1}, we deduce that the fibration

$$X \longrightarrow \mathcal{E}_n(X) \xrightarrow{\mathfrak{q}_n} \mathcal{B}_n(X)$$

is homotopy equivalent to

$$X \to \Sigma^{n-1} X^{\wedge n} \xrightarrow{q_n} XP(n-1).$$

Now, let $\mathbb{S}^{2m-1}_{(p)}$ be the p-localization of the sphere $\mathbb{S}^{2m-1}$ at a prime p. Then, [**11**, Theorem 1.5] yields

Proposition 2.15. *If $m \geqslant 2$ and $p > 3$ is a prime then*

$$\operatorname{nil} \mathbb{S}^{2m-1}_{(p)} < \infty$$

with respect to any homotopy associative H-structure on $\mathbb{S}^{2m-1}_{(p)}$.

But, $\mathbb{S}^{2m-1}_{(p)}$ admits an A_{p-1}-structure for $p > 3$. Hence, the fibration

$$\mathbb{S}^{2m-1}_{(p)} \longrightarrow \mathbb{S}^{2mn-1} \longrightarrow \mathbb{S}^{2m-1}_{(p)} P(n-1)$$

yields the H-fibration

$$\Omega(\mathbb{S}^{2mn-1}_{(p)}) \longrightarrow \Omega(\mathbb{S}^{2m-1}_{(p)} P(n-1)) \longrightarrow \mathbb{S}^{2m-1}_{(p)}$$

with an Abelian H-space $\Omega(\mathbb{S}^{2mn-1}_{(p)})$ and $\operatorname{sol} \mathbb{S}^{2m-1}_{(p)} \leqslant \operatorname{nil} \mathbb{S}^{2m-1}_{(p)} - 1 < \infty$. Consequently, in view of Proposition 1.8, we derive

Corollary 2.16. *If $n \leqslant p-1$ and $p > 3$ is a prime then*

$$\operatorname{sol} \Omega(\mathbb{S}^{2m-1}_{(p)} P(n-1)) \leqslant \operatorname{sol} \mathbb{S}^{2m-1}_{(p)} < \infty.$$

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