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CLASSIFYING SPACE VIA HOMOTOPY COHERENT NERVE

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(*communicated by Emily Riehl*)

Abstract

We prove that the classifying space of a simplicial group is modeled by its homotopy coherent nerve. We will also show that the claim remains valid for simplicial groupoids.

1. Introduction

Classically, the classifying space of a topological group G is defined to be the base space of a principal G-bundle with weakly contractible total space. There is a parallel construction for simplicial groups: If G denotes a simplicial group, then a **principal G-fibration** is a map $p\colon E \to B$ of simplicial sets, such that E is a right G-simplicial set, each E_n is G_n-free, and $E/G \cong B$. The **classifying space** BG is defined to be the base of a principal G-fibration whose total space is a contractible Kan complex.

Just as the topological classifying space can be constructed by regarding a group as a category with one object, taking its nerve, and then applying the realization functor, it is expected that a similar result holds for simplicial groups. That is, given a simplicial group G, we might want to guess that the homotopy coherent nerve NG, where G is regarded as a simplicial category with one object, has the homotopy type of BG. The purpose of this note is to prove this result and a further generalization to the case where G is a simplicial groupoid.

We note that the statement of the main result of this note (Theorem 3.6) appeared in [**Hin07**] previously; however, Hinich's argument does not seem to hold: He computes the homotopy groups of NG and BG, and constructs a comparison map $BG \to NG$, but does not explain why the map induces isomorphisms in the homotopy groups. Fortunately, this gap was recently filled by Minichiello, Rivera, and Zeinalian in [**MRZ22**]. We will follow an alternative path: Instead of comparing BG and NG, we will compare the corresponding left adjoints. This key insight, which was communicated to us by Dmitri Pavlov in [**Mat**], leads to a more direct and concise argument.

Notation and terminology

By a **simplicial category**, we mean a simplicially enriched category. If $\mathcal{C}$ is a simplicial category and x, y are its objects, we will write $\mathcal{C}(x, y)$ for the simplicial set of maps from x to y and call its n-simplex an **n-arrow** from x to y. The ordinary

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category consisting of the n-arrows is denoted by $\mathcal{C}_n$. A **simplicial groupoid** is a simplicial category whose n-arrows are all invertible, for any $n \geqslant 0$. A **simplicial group** G, i.e., a simplicial object in the category of groups, will be identified with a simplicial groupoid with a single object whose endo-simplicial set is G.

A simplicial set is said to be **reduced** if it has only one vertex.

We will write Cat_Δ, Grpd_Δ, sGrp, sSet, sSet_0, $\boldsymbol{\Delta}$ for the categories of small simplicial categories, its full subcategory of simplicial groupoids, simplicial groups, simplicial sets, reduced simplicial sets, and the simplex category (the category of nonempty finite ordinals and poset map) respectively. Unless stated otherwise, Cat_Δ and sSet carry the Bergner and Kan-Quillen model structures, respectively.

The **homotopy coherent nerve** $N\colon \mathsf{Cat}_\Delta \to \mathsf{sSet}_{\text{Joyal}}$, originally due to Cordier [**Cor82**], is a right Quillen equivalence from the Bergner model structure to the Joyal model structure arising from a simplicial object in Cat_Δ. There are two conventions for the choice of a simplicial object. The first one uses the cosimplicial simplicial category $\mathfrak{C}[\Delta^\bullet]$. The simplicial category $\mathfrak{C}[\Delta^n]$ has as its objects the integers $0, \dots, n$, and its mapping simplicial sets are given by

$$\mathfrak{C}[\Delta^n](i,j) = N(P_{i,j}),$$

where $P_{i,j}$ is the poset of subset of subsets $I \subset [n]$ with minimal element i and maximal element j, ordered by inclusion. The other one uses the cosimplicial simplicial category $\widetilde{\mathfrak{C}}[\Delta^\bullet]$, where $\widetilde{\mathfrak{C}}[\Delta^n]$ is obtained from $\mathfrak{C}[\Delta^n]$ by taking the opposites of the mapping simplicial sets. We will opt for the latter convention because it makes our exposition more concise[1]. For a comprehensive account of the homotopy coherent nerve functor and the model structures of Bergner and Joyal, we refer the reader to [**Lur09**, §1.1.5, §2.2.5, §A.3.2]. But beware that in [**Lur09**], Lurie adopts the first convention for the homotopy coherent nerve functor.

The term "**∞-category**" is a synonym for "quasi-category" in the sense of Joyal [**Joy02**]. The term "**∞-groupoid**" will be used as a synonym for "Kan complex."

Acknowledgments

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[1]The choice of the convention is insignificant. Indeed, since the geometric realization of a simplicial set is naturally homeomorphic to that of the opposite, if $\mathcal{C}$ is a topological category (i.e., a category enriched over the category of compactly generated weak Hausdorff spaces), then we have a natural bijection

$$\mathsf{Cat}_\Delta(\mathfrak{C}[\Delta^\bullet], \operatorname{Sing}\mathcal{C}) \cong \mathsf{Cat}_\Delta\left(\widetilde{\mathfrak{C}}[\Delta^\bullet], \operatorname{Sing}\mathcal{C}\right).$$

Combining this with the fact that the functor $\operatorname{Sing}|-| : \mathsf{Cat}_\Delta \to \mathsf{Cat}_\Delta$ admits a natural weak equivalence from the identity, we find that the two homotopy coherent nerve functors arising from different conventions can be joined by a zig-zag of natural transformations whose components at fibrant simplicial categories are weak categorical equivalences.

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2. Review of simplicial classifying spaces

In this section, we review some basic results and constructions on simplicial groups which we use freely in the next section.

One of the guiding principles in higher category theory is Grothendieck's homotopy hypothesis, which states that "spaces" and "higher groupoids" should be the same. The $\overline{W}$-construction, which we now introduce, provides an incarnation of this principle.

Construction 2.1. Let $\mathcal{G}$ be a simplicial groupoid. We define a simplicial set $\overline{W}\mathcal{G}$ as follows: An n-simplex of $\overline{W}\mathcal{G}$ is a sequence

$$x_n \xleftarrow{g_0} x_{n-1} \xleftarrow{g_1} \cdots \xleftarrow{g_{n-2}} x_1 \xleftarrow{g_{n-1}} x_0,$$

where $x_0, \dots, x_n$ are objects of $\mathcal{G}$ and $g_{n-i} \colon x_{i-1} \to x_i$ is an $(n-i)$-arrow in $\mathcal{G}$. For $n \geqslant 1$, the face map $d_i \colon \left(\overline{W}\mathcal{G}\right)_n \to \left(\overline{W}\mathcal{G}\right)_{n-1}$ is given by

$$d_i\left(g_0, \dots, g_{n-1}\right) = \begin{cases} (g_0, \dots, g_{n-2}) & \text{if } i = 0, \\ (g_0, \dots, g_{n-i-2}, & \\ \quad g_{n-i-1} \circ d_0 g_{n-i}, \dots, d_{i-2} g_{n-2}, d_{i-1} g_{n-1}) & \text{if } 0 < i < n, \\ (d_1 g_1, \dots, d_{n-1} g_{n-1}) & \text{if } i = n, \end{cases}$$

while for $n \geqslant 0$, the degeneracy map $s_i \colon \left(\overline{W}\mathcal{G}\right)_n \to \left(\overline{W}\mathcal{G}\right)_{n+1}$ is given by

$$s_i\left(g_0, \dots, g_{n-1}\right) = \begin{cases} (g_0, \dots, g_{n-1}, \mathrm{id}) & \text{if } i = 0, \\ (g_0, \dots, g_{n-i-1}, \mathrm{id}, s_0 g_{n-i}, \dots, s_{i-1} g_{n-1}) & \text{if } 0 < i < n, \\ (\mathrm{id}, s_0 g_0, \cdots, s_{n-1} g_{n-1}) & \text{if } i = n. \end{cases}$$

Remark 2.2. Our definition of the functor $\overline{W} \colon \mathsf{Grpd}_\Delta \to \mathsf{sSet}$ is the opposite of that defined in [**GJ09**, Chapter V, §7], in the sense that they instead consider the functor $\mathsf{Grpd}_\Delta \xrightarrow{(-)^{\mathrm{op}}} \mathsf{Grpd}_\Delta \xrightarrow{\overline{W}} \mathsf{sSet}$. The discrepancy arose from the fact that we consider *right* actions of simplicial groups, whereas in [**GJ09**] simplicial groups act from the *left*.

Proposition 2.3 ([**GJ09**, Theorems 7.6, 7.8])**.** *The following specifications determine a model structure on* sGrpd*:*

- *A map $f \colon \mathcal{G} \to \mathcal{H}$ is a weak equivalence if and only if the following conditions are satisfied:*
  - *The functor $f \colon \mathcal{G}_0 \to \mathcal{H}_0$ induces a bijection between the sets of components of $\mathcal{G}_0$ and $\mathcal{H}_0$.*
  - *For each object $x \in \mathcal{G}$, the map $\mathcal{G}(x, x) \to \mathcal{H}(fx, fx)$ is a weak homotopy equivalence.*
- *A map $f \colon \mathcal{G} \to \mathcal{H}$ is a fibration if and only if the following conditions are satisfied:*

- *Given a morphism $v\colon y \to y'$ in $\mathcal{H}_0$ and an object $x \in \mathcal{G}$ such that $fx = y$, there is a morphism $u\colon x \to x'$ in $\mathcal{G}_0$ such that $fu = v$.*
- *For each object $x \in \mathcal{G}$, the map $\mathcal{G}(x, x) \to \mathcal{H}(fx, fx)$ is a Kan fibration.*

Moreover, the $\overline{W}$-construction defines a right Quillen equivalence $\overline{W}\colon \mathsf{Grpd}_\Delta \to \mathsf{sSet}$.

If simplicial sets model "spaces" and hence "higher groupoids," then reduced simplicial sets should model "higher groups." It turns out that the $\overline{W}$-construction also substantiates this intuition.

Proposition 2.4 ([**Qui67**, Chapter II, §3, Theorem 2])**.** *The category* sGrp *admits a model structure whose fibrations and weak equivalences are created by the forgetful functor* $\mathsf{sGrp} \to \mathsf{sSet}$.

Proposition 2.5 ([**GJ09**, Proposition 6.2])**.** *The category* sSet_0 *admits a model structure whose cofibrations and weak equivalences are created by the forgetful functor* $\mathsf{sSet}_0 \to \mathsf{sSet}$.

Theorem 2.6 ([**GJ09**, Proposition 6.3])**.** *The $\overline{W}$-construction defines a right Quillen equivalence functor $\overline{W}\colon \mathsf{sGrp} \to \mathsf{sSet}_0$.*

As a corollary of the above theorem, we find that for any simplicial group G, the simplicial set $\overline{W}G$ is a Kan complex. As explained in [**GJ09**, Chapter V, §4], the simplicial set $\overline{W}G$ is the base of a certain principal G-fibration $WG \to \overline{W}G$, and the simplicial set WG is contractible [**GJ09**, Chapter V, Lemma 4.6]. Combining this with the fact that every principal G-fibration is a Kan fibration [**GJ09**, Corollary 2.7], we obtain:

Corollary 2.7. *For any simplicial group G, the simplicial set $\overline{W}G$ is a BG.*

3. Main result

The goal of this section is to prove the main result of this note: If $\mathcal{G}$ is a simplicial groupoid, there is a natural homotopy equivalence

$$\overline{W}\mathcal{G} \xrightarrow{\simeq} N\mathcal{G}.$$

Remark 3.1. Note that $N\mathcal{G}$ is a Kan complex. Indeed, since simplicial groups are Kan complexes, $\mathcal{G}$ is a fibrant simplicial category, and so its homotopy coherent nerve $N\mathcal{G}$ is an ∞-category. Moreover, its homotopy category $\mathrm{ho}(N\mathcal{G}) \cong \pi_0(\mathcal{G})$ is a groupoid. Thus $N\mathcal{G}$ is an ∞-groupoid and hence is a Kan complex.

Our natural transformation $\overline{W} \to N$ is constructed from a morphism of cosimplicial objects in the category Cat_Δ. We thus construct a cosimplicial object corresponding to $\overline{W}$:

Construction 3.2 ([**Hin07**])**.** For each $n \geqslant 0$, define $\Delta^n_{\overline{W}}$ to be the simplicial category freely generated by an $(n-i)$-arrow $g_{n,i}\colon i-1 \to i$, for each $1 \leqslant i \leqslant n$. In other words, the objects of $\Delta^n_{\overline{W}}$ are the integers $0, \dots, n$, and the hom-simplicial sets are given by

$$\Delta^n_{\overline{W}}(i,j) = \begin{cases} \prod_{n-j \leqslant s < n-i} \Delta^s & \text{if } i \leqslant j, \\ \emptyset & \text{if } i > j. \end{cases}$$

We interpret the empty product as Δ^0. The map $\Delta^n_{\overline{W}}(j,k) \times \Delta^n_{\overline{W}}(i,j) \to \Delta^n_{\overline{W}}(i,k)$ defining the simplicial composition is the identity map.

If $\mathcal{C}$ is a simplicial category, a simplicial functor $f\colon \Delta^n_{\overline{W}} \to \mathcal{C}$ can be identified with a sequence $x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n$, where f_i is the image of the morphism $g_{n,i}$ under f. We make $\{\Delta^n_{\overline{W}}\}_{n\geqslant 0}$ into a cosimplicial object in Cat_Δ as follows: For $n \geqslant 1$ and $0 \leqslant i \leqslant n$, the map $\partial_i\colon \Delta^{n-1}_{\overline{W}} \to \Delta^n_{\overline{W}}$ is given by

$$\partial_i(g_{n-1,j}) = \begin{cases} d_{i-j} g_{n,j} & \text{if } j < i \text{ or } i = n, \\ g_{n,i-1} \circ d_0 g_{n,i} & \text{if } j = i < n, \\ g_{n,j+1} & \text{if } j > i. \end{cases}$$

For $n \geqslant 0$ and $0 \leqslant i \leqslant n$, the map $\sigma_i\colon \Delta^{n+1}_{\overline{W}} \to \Delta^n_{\overline{W}}$ is given by

$$\sigma_i(g_{n+1,j}) = \begin{cases} s_{i-j} g_{n,j} & \text{if } j \leqslant i, \\ \mathrm{id}_i & \text{if } j = i+1, \\ g_{n,j-1} & \text{if } j > i+1. \end{cases}$$

With this definition, the functor $\overline{W}\colon \mathsf{Grpd}_\Delta \to \mathsf{sSet}$ is the restriction of the functor $\mathsf{Cat}_\Delta\left(\Delta^\bullet_{\overline{W}}, -\right)\colon \mathsf{Cat}_\Delta \to \mathsf{sSet}$.

Proposition 3.3. *There is a unique morphism*

$$\widetilde{\mathfrak{C}}[\Delta^\bullet] \to \Delta^\bullet_{\overline{W}}$$

of cosimplicial objects in Cat_Δ *such that each simplicial functor* $\widetilde{\mathfrak{C}}[\Delta^n] \to \Delta^n_{\overline{W}}$ *is the identity on objects.*

Remark 3.4. If one wants to stick to the cosimplicial simplicial category $\mathfrak{C}[\Delta^\bullet]$, one needs to modify the $\overline{W}$-construction by replacing the mapping simplicial sets of $\Delta^\bullet_{\overline{W}}$ by their opposites to obtain a corresponding claim for Proposition 3.3.

Proof of Proposition 3.3. We begin by showing uniqueness. If there is a morphism $\varphi\colon \widetilde{\mathfrak{C}}[\Delta^\bullet] \to \Delta^\bullet_{\overline{W}}$ of cosimplicial objects as in the statement, then the first entry $\varphi_1\colon \widetilde{\mathfrak{C}}[\Delta^1] \to \Delta^1_{\overline{W}}$ must be the identity map since both $\widetilde{\mathfrak{C}}[\Delta^1]$ and $\Delta^1_{\overline{W}}$ are isomorphic to the poset [1]. Since φ commutes with the cosimplicial structure maps, it follows that for each $0 \leqslant i \leqslant j \leqslant n$, the image of $\{i,j\} \in \widetilde{\mathfrak{C}}[\Delta^n](i,j)$ is completely determined. Since φ commutes with compositions, we find that $\varphi_n\colon \widetilde{\mathfrak{C}}[\Delta^n](i,j) \to \Delta^n_{\overline{W}}(i,j)$ is completely determined on vertices. But now this is a map between the nerves of posets, and such a map is determined by its values on vertices. This proves the uniqueness part.

Before moving on to the proof of the existence part, we prove the following auxiliary assertion: Let $0 \leqslant i \leqslant j \leqslant n$ be integers, and let $\iota = \iota^{(n)}_{i,j}\colon [1] \to [n]$ denote the poset map defined by $\iota^{(n)}_{i,j}(0) = i$ and $\iota^{(n)}_{i,j}(1) = j$. Then the map

$$\iota_* \colon \Delta^1_{\overline{W}}(0,1) \to \Delta^n_{\overline{W}}(i,j) = \prod_{i<k\leqslant j} \Delta^{n-k}$$

maps the unique vertex of $\Delta^1_{\overline{W}}(0,1)$ to the vertex

$$(0,1,\dots,j-i-1) \in \left(\prod_{i<k\leqslant j} \Delta^{n-k}\right)_0 .$$

(When $i = j$, interpret the left hand side as the unique vertex $0 \in \Delta^0_0 = \Delta^n_{\overline{W}}(i,j)$.) The claim is proved by induction on n. The claim is trivial if $n = 0$. For the inductive step, suppose that the claim holds for $n-1$. There are three cases to consider:

(1) If $i = j$, there is nothing to prove.

(2) If $i < j < n$, then $\iota^{(n)}_{i,j} = \partial_n \iota^{(n-1)}_{i,j}$. By definition, the map

$$\partial_n \colon \Delta^{n-1}_{\overline{W}}(i,j) \to \Delta^n_{\overline{W}}(i,j)$$

is given by

$$\partial_{n-j} \times \cdots \times \partial_{n-i} \colon \Delta^{n-1-j} \times \cdots \times \Delta^{n-1-i} \to \Delta^{n-j} \times \cdots \times \Delta^{n-i}.$$

So it fixes the vertex $(0,1,\dots,j-i-1)$.

(3) If $i < j = n$, then $\iota^{(n)}_{i,j} = \partial_{n-1} \iota^{(n-1)}_{i,n-1}$. By definition, the map

$$\partial_{n-1} \colon \Delta^{n-1}_{\overline{W}}(i,n-1) \to \Delta^n_{\overline{W}}(i,n)$$

is given by

$$(\mathrm{id},\partial_0) \times \partial_1 \times \cdots \times \partial_{n-i} \colon \Delta^0 \times \cdots \times \Delta^{n-1-i} \to \Delta^0 \times \Delta^1 \times \cdots \times \Delta^{n-i}.$$

Thus it carries the vertex $(0,1,\dots,n-i-2)$ to the vertex $(0,1,\dots,n-i-1)$.

We now proceed to the proof of the existence part. Define a simplicial functor $\varphi_n \colon \widetilde{\mathfrak{C}}[\Delta^n] \to \Delta^n_{\overline{W}}$ as follows: Recall that $\widetilde{\mathfrak{C}}[\Delta^n](i,j)$ is the nerve of the *opposite* of the poset

$$P_{i,j} = \{I \subset [i,j] \mid \min I = i,\ \max I = j\},$$

with ordering given by inclusion. For $0 \leqslant i < j \leqslant n$, we define a map of simplicial sets

$$\varphi_n \colon \widetilde{\mathfrak{C}}[\Delta^n](i,j) = N\left(P^{\mathrm{op}}_{i,j}\right) \to \Delta^n_{\overline{W}}(i,j) = \Delta^{n-j} \times \cdots \times \Delta^{n-i-1}$$

on vertices to be the map induced by the poset map

$$\begin{gathered} P^{\mathrm{op}}_{i,j} \to [n-j] \times \cdots \times [n-i-1] \\ \{i = i_0 < \cdots < i_k = j\} \mapsto (0,\dots,i_k - i_{k-1} - 1,\dots,0,\dots,i_1 - i_0 - 1)\,. \end{gathered}$$

Note that this map is indeed a poset map because the ordering of $P^{\mathrm{op}}_{i,j}$ is given by the reverse inclusion. This defines a simplicial functor $\varphi_n \colon \widetilde{\mathfrak{C}}[\Delta^n] \to \Delta^n_{\overline{W}}$. We claim that the simplicial functors $(\varphi_n)_{n\geqslant 0}$ define a morphism of cosimplicial objects. In other words, we claim that for any poset map $\alpha\colon [p] \to [q]$ and $0 \leqslant i \leqslant j \leqslant p$, the diagram

$$\begin{array}{ccc} \widetilde{\mathfrak{C}}[\Delta^p](i,j) & \xrightarrow{\varphi_p} & \Delta^p_{\overline{W}}(i,j) \\ {\scriptstyle\alpha_*}\downarrow & & \downarrow{\scriptstyle\alpha_*} \\ \widetilde{\mathfrak{C}}[\Delta^q](\alpha(i),\alpha(j)) & \xrightarrow[\varphi_q]{} & \Delta^q_{\overline{W}}(\alpha(i),\alpha(j)) \end{array}$$

commutes. Since the simplicial sets in the diagrams are nerves of posets, it suffices to show that the diagram commutes on the level of vertices. Also, since φ_n, α_* commutes with compositions in $\widetilde{\mathfrak{C}}[\Delta^n]$ and $\Delta^n_{\overline{W}}$, it suffices to establish the identity

$$\alpha_*\varphi_p(\{i,j\}) = \varphi_q\alpha_*(\{i,j\}).$$

This is clear, because by construction, both sides are equal to $\iota^{(q)}_{\alpha(i),\alpha(j)}(g_{1,1})$. The proof is now complete. □

We wish to show that the induced natural transformation $\overline{W} \to N\colon \mathsf{Grpd}_\Delta \to \mathsf{sSet}$ is a natural weak equivalence. For this, the following proposition comes in handy.

Proposition 3.5. *The functor* $N\colon \mathsf{Grpd}_\Delta \to \mathsf{sSet}$ *is right Quillen.*

Proof. We must show that N is a right adjoint and preserves fibrations and trivial fibrations.

Let us begin by showing that N is a right adjoint. According to the adjoint functor theorem [**AR94**, Theorem 1.66], we only need to show that N preserves limits and filtered colimits, and that both sSet and Grpd_Δ are locally presentable. The preservation of limits of follows from the fact that the inclusion $\mathsf{Grpd}_\Delta \hookrightarrow \mathsf{Cat}_\Delta$ preserves limits and $N\colon \mathsf{Cat}_\Delta \to \mathsf{sSet}$ is a right adjoint. The preservation of filtered colimits follows from the facts that the inclusion $\mathsf{Grpd}_\Delta \hookrightarrow \mathsf{Cat}_\Delta$ preserves filtered colimits, and that the simplicial categories $\widetilde{\mathfrak{C}}[\Delta^n]$ are compact objects of Cat_Δ. For the local presentability, let us recall the following facts:

(1) Every functor category of a locally presentable category is locally presentable [**AR94**, Corollary 1.54].
(2) A full subcategory of a locally presentable category closed under limits and filtered colimits is again locally presentable [**AR94**, Theorem 2.48].

It follows from (1) that sSet is locally presentable, and combining this with (2) shows that Cat and $\mathsf{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$ are locally presentable. Another application of (2) to the inclusion $\mathsf{Grpd}_\Delta \hookrightarrow \mathsf{Cat}^{\mathbf{\Delta}^{\mathrm{op}}}$ shows that Grpd_Δ is locally presentable.

To see that N preserves weak equivalences, note that the inclusion $\mathsf{Grpd}_\Delta \hookrightarrow \mathsf{Cat}_\Delta$ maps weak equivalences in Grpd_Δ to weak equivalences in Cat_Δ between fibrant objects, because any simplicial group is a Kan complex. Since weak categorical equivalences are weak homotopy equivalences, it follows that N preserves weak equivalences.

It remains to verify that N preserves fibrations. Since the homotopy coherent nerve functor is a right Quillen functor from Cat_Δ to $\mathsf{sSet}_{\mathrm{Joyal}}$, and since the inclusion $\mathsf{Grpd}_\Delta \hookrightarrow \mathsf{Cat}_\Delta$ preserves fibrations, the functor N maps fibrations in Grpd_Δ to fibrations in the Joyal model structure. Now recall that $N\mathcal{G}$ is a Kan complex for any simplicial groupoid $\mathcal{G}$. By Joyal's lifting theorem [**Lan21**, Theorem 2.1.8], every Joyal fibration between Kan complexes is a Kan fibration. Thus N preserves fibrations, as claimed. □

We now arrive at the main result.

Theorem 3.6. *Let $\mathcal{G}$ be a simplicial groupoid. The morphism $\widetilde{\mathfrak{C}}[\Delta^\bullet] \to \Delta^\bullet_{\overline{W}}$ of cosimplicial objects induces a homotopy equivalence*

$$\overline{W}\mathcal{G} \xrightarrow{\simeq} N\mathcal{G}$$

of Kan complexes.

Proof. The natural transformation $\overline{W} \to N\colon \mathsf{Grpd}_\Delta \to \mathsf{sSet}$ induces a natural transformation $\mathbb{R}\overline{W} \to \mathbb{R}N$ between the total right derived functors. We must show that the latter natural transformation is a natural isomorphism. By the uniqueness of adjoints, it suffices to show that the induced natural transformation $\mathbb{L}L_N \to \mathbb{L}L_{\overline{W}}$ is a natural isomorphism, where $L_{\overline{W}}, L_N\colon \mathsf{sSet} \to \mathsf{Grpd}_\Delta$ are the left adjoints of $\overline{W}$ and N.

We begin by showing that $L_N(\Delta^n) \to L_{\overline{W}}(\Delta^n)$ is a weak equivalence for every $n \geqslant 0$. Since the map $\Delta^n \to \Delta^0$ is a weak homotopy equivalence, it suffices to prove this for $n = 0$. But now we have natural bijections

$$\mathsf{Grpd}_\Delta\left(L_N\left(\Delta^0\right), \mathcal{G}\right) \cong \mathsf{sSet}\left(\Delta^0, N\mathcal{G}\right) \cong \operatorname{ob}\mathcal{G}$$

and

$$\mathsf{Grpd}_\Delta\left(L_{\overline{W}}\left(\Delta^0\right), \mathcal{G}\right) \cong \mathsf{sSet}_0\left(\Delta^0, \overline{W}\mathcal{G}\right) \cong \operatorname{ob}\mathcal{G},$$

which shows that $L_N\left(\Delta^0\right)$ and $L_{\overline{W}}\left(\Delta^0\right)$ are the terminal simplicial groupoids. Thus the claim holds trivially.

Now let X be an arbitrary simplicial set. We show that $L_N(X) \to L_{\overline{W}}(X)$ is a weak equivalence of simplicial groups. Since X is a colimit of the sequence of cofibrations between cofibrant objects

$$\operatorname{sk}_0 X \to \operatorname{sk}_1 X \to \cdots,$$

it suffices to consider the case where X is isomorphic to its n-skeleton. We prove the claim by induction on n. The base case $n = 0$ follows from the result in the previous paragraph. For the inductive step, assume the claim holds for n. We have a pushout diagram of the form

$$\begin{array}{ccc} \coprod_\alpha \partial\Delta^{n+1} & \longrightarrow & \operatorname{sk}_n X \\ \downarrow & & \downarrow \\ \coprod_\alpha \Delta^{n+1} & \longrightarrow & X, \end{array}$$

and by the induction hypothesis, the claim holds for all the corners except for X. Now observe that the image of the above square under any left Quillen functor is a homotopy pushout, because all the relevant objects are cofibrant and the left vertical arrow is a cofibration. Hence $L_N X \to L_{\overline{W}} X$ is a weak equivalence, as required. □

As a corollary, we find that the homotopy coherent nerve models the classifying space:

Corollary 3.7. *For any simplicial group G, the map*

$$\overline{W}G \to NG$$

is a homotopy equivalence of Kan complexes.

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