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POLYNOMIAL GENERATORS OF $\mathbf{MSU}^*[1/2]$ RELATED TO CLASSIFYING MAPS OF CERTAIN FORMAL GROUP LAWS

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Abstract

This paper presents a commutative complex oriented cohomology theory that realizes the Buchstaber formal group law F_B localized away from 2. It is shown that the restriction of the classifying map of F_B on the special unitary cobordism ring localized away from 2 defines a four parameter genus, studied by Hoehn and Totaro.

1. Introduction

The ring of complex cobordism $\mathbf{MU}_*$ and the ring $\mathbf{MSU}_*$ of special unitary cobordism has been studied by many authors. We refer the reader to [20], [17] for details. In particular the ring $\mathbf{MSU}_*$, localized away from 2, is torsion free

$$\mathbf{MSU}_*[1/2] = \mathbb{Z}[1/2][x_2, x_3, \ldots], \quad |x_i| = 2i$$

and SU-structure forgetful homomorphism is the inclusion in complex cobordism ring

$$\mathbf{MSU}_*[1/2] \subset \mathbf{MU}_*[1/2] = \mathbb{Z}[1/2][x_1, x_2, x_3, \ldots].$$

In this paper we construct a commutative complex oriented cohomology theory (Theorem 5.1) such that the coefficient ring is the scalar ring of the Buchstaber formal group law F_B with inverted 2, and show (Proposition 5.1) that after restricted to $\mathbf{MSU}_*[1/2]$, the classifying map of F_B can become a genus

$$\mathbf{MSU}_*[1/2] \to \mathbb{Z}[1/2][x_2, x_3, x_4], \quad |x_i| = 2i, \tag{1}$$

studied by Hoehn [13] and Totaro [21].

Since $\mathbf{MSU}^*$ is not complex oriented, it is difficult to compute the genus (1) on specific explicit elements. Using the polynomial generators of the spherical cobordism ring $W^*[1/2]$ given by Chernykh and Panov [12], we derive certain polynomial generators of $\mathbf{MSU}^*[1/2]$ in terms of the universal formal group law. This gives a new understanding of the genus (1).

In particular, the classifying map f_B of F_B is a surjection on some infinitely generated ring Λ_B, with kernel generated by some explicit elements (Proposition 2.1). After it is tensored with rationals it is identical (Proposition 2.2) to the complex

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elliptic genus

$$\mathbf{MU}_* \otimes \mathbb{Q} \to \mathbb{Q}[x_1, x_2, x_3, x_4], \tag{2}$$

defined in Hoehn's thesis [**13**, Section 2.5]. Here x_1 is the image of complex projective plane $\mathbf{CP}_1$ and x_2, x_3, x_4 are the images of any first three generators of the polynomial ring $\mathbf{MSU}_* \otimes \mathbb{Q} = \mathbb{Q}[x_2, x_3, x_4, \ldots]$.

By Hoehn [**13**], for X an SU-manifold of complex dimension n, the exponential characteristic class $\phi(X)$ is in fact a Jacobi form of weight n. Jacobi forms are generalizations of modular forms. See details in [**21**]. Hoehn showed that the Jacobi forms x_2, x_3, x_4 arise as the elliptic genera of certain explicit SU-manifolds, of complex dimensions 2, 3, 4, so that the homomorphism

$$\mathbf{MSU}_* \to (\text{Jacobi forms over } \mathbb{Z})$$

becomes surjective after it is tensored with $\mathbb{Z}[1/2]$.

In [**21**, Theorem 4.1] Totaro proved that the Krichever–Hoehn complex elliptic genus on complex cobordism viewed as a homomorphism (2) is surjective and the kernel is equal to the ideal of complex flops.

Then Totaro proved (Theorem 6.1) that the kernel of the complex elliptic genus on $\mathbf{MSU}_* \otimes \mathbb{Z}[1/2]$ is equal to the ideal I of SU-flops. Also, the quotient ring is a polynomial ring:

$$\mathbf{MSU}_*[1/2]/I = \mathbb{Z}[1/2][x_2, x_3, x_4]. \tag{3}$$

Unfortunately $\mathbf{MSU}^*$ is not complex oriented. It would be nice to develop a method for calculating (3) explicitly in terms of the universal formal group law using some generators of $\mathbf{MSU}_*[1/2]$ treated as explicit elements in $\mathbf{MU}_*[1/2]$.

This goal can be achieved as follows: in Section 5 we replace the ideal of complex flops with a more explicit ideal by considering the integral Buchstaber genus which is identical to the Krichever–Hoehn complex elliptic genus over $\mathbf{MU}_* \otimes \mathbb{Q}$.

In Section 6 we use the polynomial generators of the spherical cobordism ring $W_*[1/2]$ constructed in [**12**] and define certain polynomial generators of $\mathbf{MSU}_*[1/2]$. In particular, we use fact that $W_*[1/2]$ is generated by the coefficients of the corresponding formal group law. Given Novikov's criteria and that $W_*[1/2]$ is a free $\mathbf{MSU}_*[1/2]$ module generated by 1 and $\mathbf{CP}^1$, we define some generators in $\mathbf{MSU}_*[1/2]$ (Proposition 6.3) in terms of the universal formal group law. Finally the explicit quotient map of the Buchstaber formal group law (Proposition 2.1) gives the decompositions of constructed generators $\mathbf{MSU}_*$ of dimensions $\geqslant 10$ in polynomial ring $\mathbb{Z}[1/2][x_2, x_3, x_4]$. It is a way to calculate the genus (3) in terms of the universal formal group law.

In Section 7 we consider the restriction of classifying map of the universal abelian formal group law on $\mathbf{MSU}_*[1/2]$ to define a genus with one parameter.

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2. Preliminaries

The theory W_* of c_1-spherical bordism is defined geometrically in [20, Chapter VIII]. The closed manifolds M with a c_1-spherical structure, consist of
– a stably complex structure on the tangent bundle TM;
– a $\mathbf{CP}^1$-reduction of the determinant bundle, that is, a map $f\colon M \to \mathbf{CP}^1$ and an equivalence $f^*(\eta) \simeq \det TM$, where η is the tautological bundle over $\mathbf{CP}^1$.

This is a natural generalization of an SU-structure, which can be thought of as a trivialization of the determinant bundle. The corresponding bordism theory is called c_1 -spherical bordism and is denoted W_*. The unitary and special unitary bordism rings are denoted by $\mathbf{MU}_*$ and $\mathbf{MSU}_*$ respectively. We refer to [17] and [12] for details on $\mathbf{MSU}_*$ and W_* that will be used throughout the paper.

Motivated by string theory in [14], [13], [21] the universal Krichever–Hoehn complex elliptic genus ϕ_{KH} is defined as the ring homomorphism

$$\phi_{KH}\colon \mathbf{MU}_* \to \mathbb{Q}[q_1, q_2, q_3, q_4] \tag{4}$$

associated to the Hirzebruch characteristic power series $Q(x) = \frac{x}{f(x)}$, where

$$h(x) = \frac{f'(x)}{f(x)}$$

is the solution of the differential equation in $\mathbf{MU}_* \otimes \mathbb{Q}$

$$(h')^2 = S(h), \tag{5}$$

where

$$S(x) = x^4 + q_1x^3 + q_2x^2 + q_3x + q_4,$$

for some formal parameters $|q_i| = 2i$.

One consequence of Krichever–Hoehn's rigidity theorem [14], [13], [21], is that ([13, Kor 2.2.3]) if $F \to E \to B$ is a fiber bundle of closed connected weakly complex manifolds, with structure group a compact connected Lie group G, and if F is an SU-manifold, then the elliptic genus ϕ_{KH} satisfies $\phi_{KH}(E) = \phi_{KH}(F)\phi_{KH}(B)$. In fact, the elliptic genus is the universal genus with the above multiplicative property.

In ([21, Theorem 6.1]) Totaro gave a geometric description of the kernel ideal I of the complex elliptic genus restricted to $\mathbf{MSU}_*[1/2]$, the ideal of SU flops. This kernel is equal to the ideal in $\mathbf{MSU}_*[1/2]$ generated by twisted projective bundles $\tilde{\mathbf{CP}}(A \oplus B)$ over weakly complex manifolds Z such that the complex vector bundles A and B over Z have rank 2 and $c_1Z + c_1A + c_1B = 0$; in this case, the total space is an SU-manifold. Then Totaros's result says that the $I \in \mathbf{MU}_*[1/2]$ contains a polynomial generator of $\mathbf{MSU}_*[1/2]$ in real dimension $2n$ for all $n \geqslant 5$ and

$$\mathbf{MSU}_*[1/2]/I \simeq \mathbb{Z}[1/2][x_2, x_3, x_4]. \tag{6}$$

In [13] by using of characteristic classes, Hoehn constructed a base sequence $W_1, W_2, W_3, \ldots$ of the rational cobordism ring $\mathbf{MU}_* \otimes \mathbb{Q}$ on which ϕ_{KH} has the values A, B, C, D, and 0 for W_i with $i > 4$.

As another generalization of Oshanin's elliptic genus Schreieder in [19] studied a genus ψ with logarithmic series

$$\log_\psi(x) = \int_0^x \frac{dt}{R(t)}, \quad R(t) = \sqrt{1 + q_1t + q_2t^2 + q_3t^3 + q_4t^4}.$$

The genus ψ is easily calculable on cobordism classes of complex projective spaces $\mathbf{CP}_i$, the generators of the domain

$$\mathbf{MU}_* \otimes \mathbb{Q} = \mathbb{Q}[\mathbf{CP}_1, \mathbf{CP}_2, \dots].$$

This is because of the equation $(\log_\psi(x)')^2 = 1/R^2(x) = \sum_{i \geqslant 1} \psi(\mathbf{CP}_i)x^i$ we need only the Taylor expansion of $(1+y)^{-1/2}$.

It is natural to ask whether one can calculate ϕ_{KH} in an elementary manner, different from that relying on the formulas in [13] and [8].

Viewed as a classifying map ψ is strongly isomorphic to genus ϕ by the series $\mu(x) = \sum_{i \geqslant 0} \mathbf{CP}_i x^{i+1}$ [4]. This gives a method for explicit calculation of ϕ_{KH}.

In [2, 3] we introduced the formal power series

$$A(x,y) = \sum A_{ij}x^i y^j = F(x,y)(x\omega(y) - y\omega(x)) \in \mathbf{MU}^*[[x,y]], \tag{7}$$

where

$$F = F(x,y) = \sum \alpha_{ij}x^i y^j$$

is the universal formal group law over complex cobordism ring $\mathbf{MU}^*$ and

$$\omega(x) = \frac{\partial F(x,y)}{\partial y}(x,0) = 1 + \sum_{i \geqslant 1} w_i x^i$$

is the invariant differential of F.

The series $A(x,y)$ has proven to be interesting for the following reasons.

Proposition 2.1 ([3]). *(i) The obvious quotient map*

$$f_B \colon \mathbf{MU}^* \to \mathbf{MU}^*/(A_{ij},\, i,j \geqslant 3)$$

classifies a formal group law which is identical to the universal Buchstaber formal group law F_B, the universal formal group law of the form

$$\frac{x^2A(y) - y^2A(x)}{xB(y) - yB(x)},$$

where $A(0) = B(0) = 1$.

(ii) If $A'(0) = B'(0)$ then $B(x)$ is identical to the image of $\omega(x)$ under the classifying map f_B.

Proposition 2.2 ([4]). *After it is tensored with rationals the classifying map f_B of the Buchstaber formal group law is identical to the Krichever–Hoehn complex elliptic genus*

$$\phi \colon \mathbf{MU}_* \otimes \mathbb{Q} \to \Lambda_B \otimes \mathbb{Q} = \mathbb{Q}[\mathbf{CP}_1, \mathbf{CP}_2, \mathbf{CP}_3, \mathbf{CP}_4],$$

where $\mathbf{CP}_i$ are complex projective spaces.

For explicit calculation of the Krichever–Hoehn genus the following observation is helpful.

Proposition 2.3 ([4]). *Over the ring $\mathbf{MU}^* \otimes \mathbb{Q}$ the series $\frac{x}{\omega(x)}$ is the strong isomorphism from the formal group law with logarithm series*

$$\int_0^x \frac{dt}{\sqrt{1+p_1t+p_2t^2+p_3t^3+p_4}}$$

in [19] to the formal group law classified by f.

3. Some auxiliary combinatorial definitions

By Euclid's algorithm for the natural numbers $m_1, m_2, \dots, m_k$ one can find integers $\lambda_1, \lambda_2, \dots \lambda_k$, such that

$$\lambda_1 m_1 + \lambda_2 m_2 + \dots + \lambda_k m_k = gcd(m_1, m_2, \dots, m_k). \tag{8}$$

Let

$$d(m) = gcd\Big\{ \binom{m+1}{1}, \binom{m+1}{2}, \dots, \binom{m+1}{m-1} \Big| \ m \geqslant 1 \Big\}. \tag{9}$$

By [15] one has

$$d(m) = \begin{cases} p, & \text{if } m+1 = p^s \text{ for some prime } p, \\ 1, & \text{otherwise.} \end{cases}$$

For the coefficients of universal formal group law $F(x,y) = \sum \alpha_{i\,j} x^i y^j$ the elements

$$e_m = \lambda_1 \alpha_{1\,m} + \lambda_2 \alpha_{2\,m-1} + \dots + \lambda_m \alpha_{m\,1} \tag{10}$$

are multiplicative generators in $\mathbf{MU}_*$.

By [9, Theorem 9.9], or [22]

$$\frac{D(m)}{d(m)} = \begin{cases} d(m-1) & \text{if } m \neq 2^k - 2, \\ 2 & \text{if } m = 2^k - 2, \end{cases} \tag{11}$$

where

$$D(m) = gcd\Big\{ \binom{m+1}{i} - \binom{m+1}{i-1} \big| \ 2 < i \leqslant m-1, m \geqslant 5 \Big\}. \tag{12}$$

Let $m \geqslant 4$ and let $\lambda_2, \dots, \lambda_{m-2}$ are such integers that

$$d_2(m) := \sum_{i=2}^{m-2} \lambda_i \binom{m+1}{i} = gcd\Big\{ \binom{m+1}{2}, \dots, \binom{m+1}{m-2} \Big\}. \tag{13}$$

Then by [9, Lemma 9.7] one has for $m \geqslant 3$

$$d_2(m) = d(m)d(m-1). \tag{14}$$

Note $d(n)$ are the Chern numbers of the generators in complex cobordism of dimension $2n$.

For the generators

$$A_{ij}, \ i,j \geqslant 3, \ i+j-2 = m$$

of the quotient ideal corresponding to Λ_B, the scalar ring of the universal Buchstaber formal group law in Proposition 2.1 and the integers $\lambda_3, \dots, \lambda_{m-1}$ corresponding to (12) consider the linear combinations

$$T_m = \lambda_3 A_{3\,m-1} + \lambda_4 A_{4\,m-2} + \dots + \lambda_{m-1} A_{m-1\,3}. \tag{15}$$

The elements T_{p^s}, where p is a prime number, and e_i in (10) for $i \neq p^s$ will play a major role in Section 4.

4. Realization of the universal Buchstaber formal group law localized away from 2.

Let e_i and T_i be as in (10) and (15) respectively.

Let J_B be the ideal of $\mathbf{MU}_* = \mathbb{Z}[e_1, e_2, \ldots]$

$$J_B = \{A_{ij}, i, j \geqslant 3\},$$

the quotient ideal of the universal Buchstaber formal group law F_B classified by

$$f_B \colon \mathbf{MU}_* \to \mathbf{MU}_*/J_B = \Lambda_B.$$

The ideal J_B is not prime as the quotient ring Λ_B is not an integral domain: it has 2-torsion element of degree 12 [**3**]. Here we use the results of [**9**], that Λ_B is generated by $f_B(e_j)$, $j = 1, 2, 3, 4$ and $j = p^r$, $r \geqslant 1$, p is prime, and $j = 2^k - 2$, $k \geqslant 3$. Then the ideal $Tor(\Lambda_B)$ is generated by the elements of order 2, namely $f_B(e_j)$, $j = 2^k - 2$, $k \geqslant 3$.

The ideal $Tor(\Lambda_B)$ is prime as $\Lambda_{\mathcal{B}}$ in

$$f_{\mathcal{B}} \colon \mathbf{MU}_* \to \Lambda_B \to \Lambda_B/Tor(\Lambda_B) = \Lambda_{\mathcal{B}} \tag{16}$$

is an integral domain and so is $J_{\mathcal{B}} = f_B^{-1}(Tor(\Lambda_B))$, the preimage ideal in $\mathbf{MU}_*$. Then the ideal

$$J_{\mathcal{B}} = J_B + (e_{2^k-2}), k \geqslant 3 \tag{17}$$

is the kernel of the composition (16). Denote by $F_{\mathcal{B}}$ the formal group law classified by $f_{\mathcal{B}}$.

Let

$$J = \big(\mathcal{T}_l, l \geqslant 5\big), \text{ where } \mathcal{T}_l = \begin{cases} T_l, & n = p^s,\ p \text{ is a prime}, \\ e_l, & \text{otherwise}. \end{cases} \tag{18}$$

Let $J_{\mathcal{B}}(l) \subset J_{\mathcal{B}}$, generated by those elements whose degree is greater or equal $-2l$, and let $J(l) \subset J$ be generated by $\mathcal{T}_5, \ldots, \mathcal{T}_l$.

Remark 4.1. We note that those polynomials in $\mathbb{Z}[e_1, e_2, \ldots, e_l]$ that are in the kernel of $f_{\mathcal{B}}$ can be viewed as the elements of $J_{\mathcal{B}}(l)$.

Proposition 4.2. *$J(l) = J_{\mathcal{B}}(l)$ for any natural $l \geqslant 5$.*

Proof. It is clear that $J(l) \subset J_{\mathcal{B}}(l)$.

Let us prove $J_{\mathcal{B}}(l) \subset J(l)$ by induction on l. It is obvious for $l = 5$ as $T_5 = A_{34}$.

To prove $J_{\mathcal{B}}(l) = J_{\mathcal{B}}(l-1) + T_l$ note

$$s_{i+j-2}(A_{ij}) = \binom{i+j-1}{j-1} - \binom{i+j-1}{j}.$$

Indeed, modulo decomposable elements

$$A_{ij} = \alpha_{i-1j} - \alpha_{ij-1}$$

and $s_{i+j-1}(\alpha_{ij}) = -\binom{i+j}{i}$. Now apply Euclid's algorithm for $m_i = s_l(A_{i,l+2-i})$, fix the integers λ_i and consider the elements T_l in (15).

The combinatorial identities in Section 3 implies that

$$s_l(T_l) = D(l) \tag{19}$$

is the greatest common divisor of the integers $s_l(A_{ij})$ for $A_{ij} \in J_B$. $i+j-2=l$.

It follows that

$$A_{ij} = \frac{s_n(A_{ij})}{D(l)} T_l + P(e_1, e_2, \ldots, e_{l-1}),$$

for some polynomial P, i.e.,

$$A_{ij} = P(e_1, e_2, \ldots, e_{l-1}) \text{ modulo } T_l.$$

Therefore $P(e_1, e_2, \ldots, e_{l-1})$ is in the kernel of $f_{\mathcal{B}}$, i.e., is in $J_{\mathcal{B}}(l-1) = J(l-1)$ by above Remark 4.1. □

Let $A_l = \mathbb{Z}[e_1, e_2, \ldots, e_l]$ and $A^{l+1} = \mathbb{Z}[e_{l+1}, e_{l+2}, \ldots]$ i.e., $\mathbf{MU}_* = A_l \otimes A^{l+1}$. Let $J(l)$ as above be generated by $\mathcal{T}_1, \ldots, \mathcal{T}_l$. The preimage of $J(l)$ by obvious inclusion defines the ideal of A_l denoted by same symbol so that

$$\mathbf{MU}_*/J(l) = A_l/J(l) \otimes A^{l+1}.$$

Proposition 4.3. *i) The ideal J in (18) is regular;*
ii) $A_l/J(l)$ and $\mathbf{MU}_*/J(l)$ are integral domains, or equivalently $J(l)$ is prime.

It is clear that ii) implies i): If $\mathbf{MU}_*/J(l)$ is integral domain for any $l \geqslant 5$, i.e., it has no zero divisors, then multiplication by $\mathcal{T}_{l+1}$ is monorphism. Therefore the sequence $\mathcal{T}_5, \mathcal{T}_6, \ldots$ of generators of J is regular.

We will see that ii) follows from the proof of Proposition 6.5 in [9] and the following

Lemma 4.4. *For $p^r \leqslant l < p^{r+1}$ the ring $A_l/J(l) \otimes \mathbb{F}_p$ is additively generated by the following monomials*

For $p = 2$,

$$\alpha_1^{m_1} \beta_2^{m_2} \alpha_3^{m_3} \beta_2^{k_1} \beta_{2^2}^{k_2} \cdots \beta_{2^{r-1}}^{k_{r-1}} \beta_{2^r}^{m}, \quad k_1, k_2, k_{r-1} = 0, 1.$$

For $p = 3$,

$$\alpha_1^{m_1} \beta_2^{m_2} \beta_4^{m_4} \beta_3^{k_1} \beta_{3^2}^{k_2} \cdots \beta_{3^{r-1}}^{k_{r-1}} \beta_{3^r}^{m}, \quad k_1, k_2, k_{r-1} = 0, 1, 2.$$

For prime $p > 3$,

$$\alpha_1^{m_1} \beta_2^{m_2} \alpha_3^{m_3} \beta_4^{m_4} \beta_p^{k_1} \beta_{p^2}^{k_2} \cdots \beta_{p^{r-1}}^{k_{r-1}} \beta_{p^r}^{m}, \quad 0 \leqslant k_1, k_2, k_{r-1} \leqslant p-1,$$

not divisible by $\alpha_1^{j_1} \beta_2^{j_2} \alpha_3^{j_3} \beta_4^{j_4}$, where $(j_1 j_2 j_3 j_4)$ corresponds to leading lexicographical monomial ordering for which $\lambda_{j_1 j_2 j_3 j_4} \not\equiv 0 \mod p$ in

$$p\beta_p = \sum \lambda_{j_1 j_2 j_3 j_4} \alpha_1^{j_1} \beta_2^{j_2} \alpha_3^{j_3} \beta_4^{j_4}.$$

Proof. We follow the proof of Proposition 6.5 in [9]. To get the generating monomials in Lemma 4.4 we need only to modify the generating monomials of $\Lambda_{\mathcal{B}} \otimes \mathbb{F}_p$. In particular, in (6.17), (6.18) and (6.20) there are extra factors for $A_l/J(l)$, satisfying $p^r \leqslant l < p^{r+1}$, namely

$$\beta_{p^r+1}^{k_1} \beta_{p^r+2}^{k_2} \cdots \beta_{p^r+s}^{k_s} \cdots, \quad k_1, k_2, \ldots \leqslant p-1.$$

We have to replace these factors by

$$\beta_{p^r}^{pk_1+p^2k_2+\cdots+p^sk_s+\cdots}.$$

This is because of $\beta_{p^{r+1}} \notin A_l$. In this way for each m we keep the total number of generating monomials of $\Lambda^{-2m} \otimes \mathbb{F}_p$ since there is no relation (6.2) of [9] in our ring $A_l/J(l) \otimes \mathbb{F}_p$. □

Denote by $[\mathcal{T}_i]$ the cobordism class representing the generator $\mathcal{T}_i$ of the ideal J. Consider the sequence

$$\Sigma_T = ([\mathcal{T}_5], [\mathcal{T}_6], \ldots).$$

The Sullivan–Baas construction [1] of cobordism with singularities Σ_T gives a cohomology theory $\mathbf{MU}^*_{\Sigma_T}(-)$ which by regularity of the ideal J has a scalar ring

$$\mathbf{MU}^*_{\Sigma_T}(pt) = \mathbf{MU}_*/J = \Lambda_{\mathcal{B}}.$$

By Mironov [16, Theorem 4.3 and Theorem 4.5] $\mathbf{MU}^*_{\Sigma_T}(-)$ admits an associate multiplication and all obstructions to commutativity are in $\Lambda_{\mathcal{B}} \otimes \mathbb{F}_2$. Therefore after localization away from 2 all obstructions vanish and we get a commutative cohomology

$$h^*_{\mathcal{B}}(-) := \mathbf{MU}^*_{\Sigma_T}[1/2](-).$$

Here we recall that $\Lambda_B[1/2] = \Lambda_{\mathcal{B}}[1/2]$ by definition of $\Lambda_{\mathcal{B}}$.

It is clear that $h^*_{\mathcal{B}}(-)$ is complex oriented since the Atiyah–Hirzebruch spectral sequence $H^*(-, h^*_{\mathcal{B}}(pt)) \Rightarrow h^*_{\mathcal{B}}(-)$ collapses for $\mathrm{BU}(1) \times \mathrm{BU}(1)$.

Thus we can state

Theorem 4.5. *There exist a commutative complex oriented cohomology $h^*_{\mathcal{B}}(-)$ with scalar ring isomorphic to $\Lambda_{\mathcal{B}}[1/2]$, the ring of coefficients of the universal Buchstaber formal group law localized away from 2.*

This result without a complete proof, was announced in short communications of MMS [5].

5. The restriction of the Buchstaber genus on $\mathbf{MSU}_*[1/2]$

Taking into account [18] that after localized away from 2, the forgetful map from the special unitary cobordism $\mathbf{MSU}_*[1/2] = \mathbb{Z}[1/2][x_2, x_3, \ldots]$ to complex cobordism $\mathbf{MU}_*[1/2]$ is an injection, define the following ideal extensions in $\mathbf{MU}_*][1/2]$:

J^e_{SU}, generated by any polynomial generators x_n of $\mathbf{MSU}_*[1/2]$, $n \geqslant 5$ viewed as elements in $\mathbf{MU}_*[1/2]$ by forgetful injection map;

J^e_T, generated by SU-flops [21] of dimension $\geqslant 10$ again viewed as elements in $\mathbf{MU}_*[1/2]$;

J^e_B, the contraction ideal by the obvious inclusion $\mathbf{MU}_* \to \mathbf{MU}_*[1/2]$ of the ideal J_B of $\mathbf{MU}_*$ generated by the elements $\{A_{ij}, i, j \geqslant 3\}$, defined in Section 3.

Proposition 5.1. *i) $J^e_B = J^e_T$; ii) When restricted on $\mathbf{MSU}_*[1/2]$ the classifying map of the Buchstaber formal group law localized away from 2, gives a genus with the scalar ring $\mathbb{Z}[1/2][x_2, x_3, x_4]$, $|x_i| = 2i$.*

One motivation is the restricted Krichever–Hoehn complex elliptic genus below, studied in [**13**] and [**21**]. Another construction with the scalar ring $\mathbb{Z}_{(2)}[a,b]$, $|a|=2$, $|b|=6$ see in [**6**].

Proof. $J_T^e \subseteq J_B^e$: The homomorphism

$$\mathbf{MU}_* \xrightarrow{f_B} \Lambda_B \xrightarrow{\subset} \Lambda_B \otimes \mathbb{Q} = \mathbb{Q}[x_1,x_2,x_3,x_4]$$

is a specialization of the complex elliptic genus

$$\mathbf{MU}_* \xrightarrow{\subset} \mathbf{MU}_* \otimes \mathbb{Q} \to \mathbb{Q}[x_1,x_2,x_3,x_4]$$

by [**2**], therefore vanishes on the kernel of the complex elliptic genus which is the ideal I of complex flops by [**21**, Theorem 4.1]. On the other hand the ring $\Lambda_B[1/2]$ is torsion free and injected in $\mathbb{Q}[x_1,x_2,x_3,x_4]$ by [**9**]. So the ring homomorphism $\mathbf{MU}_*[1/2] \xrightarrow{f_B} \Lambda_B[1/2]$ vanishes on I, therefore it vanishes on SU-flops. Moreover by [**21**] the ideal of SU-flops J_T in $\mathbf{MSU}_*[1/2]$ contains the polynomial generators $x_n,\ n \geqslant 5$ constructed by using Euclid's algorithm and SU-flops. Therefore $J_T = (x_5,\ldots)$ and $J_T^e \subseteq J_B^e$.

To prove $J_B^e \subseteq J_T^e$ note by (19) T_n satisfies the criteria for the membership of the set of polynomial generators in

$$\mathbf{MSU}^*[1/2] = \mathbb{Z}[1/2][x_2,x_3,\ldots],\ |x_i| = 2i,$$

described by Novikov in [**18**]. In particular, an SU-manifold M of real dimension $2n$, $n \geqslant 2$ is a polynomial generator if and only if $s_n(M)$, the main Chern characteristic number is as follows

$$s_n(M) = \begin{cases} \pm 2^k p & \text{if } n = p^l,\ \ p \text{ is odd prime}, \\ \pm 2^k p & \text{if } n+1 = p^l,\ \ p \text{ is odd prime}, \\ \pm 2^k & \text{otherwise}. \end{cases} \tag{20}$$

It follows that the generators A_{ij} of J_B, with $i+j=n+2$ and the generators x_n of J_T^e are related as follows $A_{ij} \equiv \frac{s_n(A_{ij})}{D(n)} T_n,\ \ T_n \equiv \pm 2^{k(n)} x_n,\ $ mod decomposables. Therefore we can proceed as in the proof of Proposition 4.2. ☐

Recall also that $\Lambda_B[1/2] = \Lambda_{\mathcal{B}}[1/2]$ is an integral domain. Note that Proposition 4.3 implies

Proposition 5.2. *Let J_{SU}^e be the ideal as before. The sequence $\{x_n\}$, $n \geqslant 5$ of any polynomial generators in $\mathbf{MSU}_*[1/2]$ viewed as elements in $\mathbf{MU}_*[1/2]$ by forgetful map is regular. Moreover, the ideal J_{SU}^e is prime.*

Corollary 5.3. *After restriction on $\mathbf{MSU}_*[1/2]$ the Buchsteber genus $f_{\mathcal{B}}$ gives a cohomology theory $\mathbf{MSU}^*_\Sigma[1/2]$ with singularities $\Sigma = (x_5,\ldots)$, with the scalar ring*

$$\mathbb{Z}[1/2][x_2,x_3,x_4],\ \ |x_i| = 2i.$$

6. The ring $\mathbf{MSU}_*[1/2]$

Let $F_U(x,y) = \sum \alpha_{ij} u^i v^j$ be the universal formal group law. Recall the idempotent in [**7**], [**12**]

$$\pi_0 \colon \mathbf{MU}_* \to \mathbf{MU}_* : \quad \pi_0 = 1 + \sum_{k \geqslant 2} \alpha_{1k} \partial_k$$

and the projection $\pi_0 \colon \mathbf{MU}_* \to W_* = Im\pi_0$.

Then W_* is a ring with multiplication $*$ and $\pi_0(a \cdot b) = a * b$. By [12, Proposition 2.15] the multiplication $*$ is given by

$$a * b = ab + 2[V]\partial a \partial b,$$

where $[V] = \alpha_{12} \in \mathbf{MU}_4$ is the cobordism class $\mathbf{CP}_1^2 - \mathbf{CP}_2$.

W_* is complex oriented and by [12, Proposition 3.12] the ring $W_*[1/2]$ is generated by the coefficients of the formal group law

$$F_W = F_W(x, y) = \sum w_{ij} x^i y^j.$$

Following [7], [12] one can calculate w_{ij} in terms of α_{kl} as follows. Consider the multiplicative cohomology theory Γ with

$$\pi_*(\Gamma) = \mathbf{MU}_*[t]/(t^2 - \alpha_{11}t - 2\alpha_{21}),$$

the free $\mathbf{MU}_*$ module generated by 1 and t.

There is a natural multiplicative transformation $\phi \colon W \to \Gamma$ given by

$$\phi_*(x) = x + t\partial x,$$

for $x \in W_*$. The restriction of $\phi_*[1/2]$ on $\mathbf{MSU}_*[1/2]$, the subring in $W_*[1/2]$ of cycles of ∂, is the natural inclusion in $\mathbf{MU}_*[1/2]$.

Then

$$\phi_* F_W = u + v + \sum_{i,j \geqslant 1} (w_{ij} + t\partial w_{ij}) u^i v^j$$

is strongly isomorphic to F_U (considered as a formal group low over $\pi_*(\Gamma)$ via the natural inclusion) by a series γ^{-1}, i.e.,

$$u + v + \sum_{i,j \geqslant 1} (w_{ij} + t\partial w_{ij}) u^i v^j = \gamma F_U(\gamma^{-1}(u), \gamma^{-1}(v)). \tag{21}$$

Finally, we need to apply for γ in (21) Lemma 3 in [7], which says that any orientation $w \in W_2$ gives the following identity in $\pi_*(\Gamma)$

$$\gamma(u) = \phi_*(w) = \pi_0(u) + t\partial u = u + tu\bar{u} + \sum_{i \geqslant 2} \alpha_{i1} u \bar{u}^i. \tag{22}$$

By [12] one can specify an orientation of W such that

$$gcd(w_{ij},\, i + j - 1 = k) = d(k)d(k-1)$$

modulo a power of 2. This allows to construct the generators of $W_*[1/2]$.

In particular, one can calculate main Chern numbers $s_k(w_{ij})$ in terms of main Chern numbers of α_{ij} as follows. By [12, Lemma 3.5] any orientation $w \in W_2$ gives $w_i \in W_{2i}$ and $\lambda \in \mathbf{MU}_2 = W_2$ such that one has in $\pi_*(\Gamma)$

$$\gamma(u) = u - (\lambda + (2l+1)t)u^2 + \sum_{i \geqslant 2} \gamma_{i+1} u^{i+1} mod J^2 + tJ,$$

where $2l = \partial\lambda$, $l \in \mathbb{Z}$, $\gamma_{i+1} = (-1)^i \alpha_{1i} + w_i$, J is the ideal in $\mathbf{MU}_*$ of elements of positive degree.

Lemma 6.1. *Let $k = i + j - 1 \geqslant 3$ is not of the form $k = 2^l = p^s - 1$ for some odd prime p. There is a choice of complex orientation w for the theory W such that*

$$s_k(w_{ij}) = \begin{cases} ps_k(\alpha_{ij}) & \text{if } k = p^s,\ p \text{ is any prime, } s > 0, \\ s_k(\alpha_{ij}) & \text{otherwise.} \end{cases}$$

Proof. By using (22) it is proved in [**12**, Lemma 3.9] that for $k \geqslant 3$ one has modulo decomposable elements

$$\phi_*(w_{1k}) = \alpha_{1k} + (k+1)((-1)^k\alpha_{1k} + w_k); \tag{23}$$

$$\phi_*(w_{ij}) = \alpha_{ij} + (-1)^k\binom{k+1}{i}\alpha_{1k} + \binom{k+1}{i}w_k. \tag{24}$$

Choose a complex orientation w for the theory W such that elements w_k satisfy the following conditions

$$1 + (-1)^k(k+1) - s_k(w_k) = d(k-1). \tag{25}$$

The it is easily checked that w_k indeed belongs to W_*, that is $s_k(w_k)$ is divisible by $d(k-1)d(k)$.

Then by (23) and (24) we have for $k = p^s$

$$\begin{aligned} w_{1k} =& -k\alpha_{1k} + (k+1)w_k = -k\alpha_{1k} + \alpha_{1k}(p^s + p) = p\alpha_{1k}; \\ w_{ij} =& \alpha_{ij} - \binom{k+1}{i}\alpha_{1k} + \binom{k+1}{i}w_k \\ =& \alpha_{ij} - \alpha_{ij}(k+1) + \alpha_{ij}(k+p) = p\alpha_{ij}. \end{aligned}$$

For $k = 2^l$ one has

$$\begin{aligned} w_{1k} =& (2+k)\alpha_{1k} + (k+1)w_k = (2+k)\alpha_{1k} - \alpha_{1k}k = 2\alpha_{1k}; \\ w_{ij} =& \alpha_{ij} + \binom{k+1}{i}\alpha_{1k} - \binom{k+1}{i}w_k = \alpha_{ij} + \alpha_{ij}(k+1) - k\alpha_{ij} = 2\alpha_{ij}. \end{aligned}$$

Similarly for other cases. □

If $k = 2^l = p^s - 1 \geqslant 3$ for some odd prime p, then by [**12**, Lemma 3.15] one has $k = 8$ or $k = 2^{2^n}$. We have to replace $d(k-1) = 2$ in (25) by 4 , if $k = 8$ to get $\phi_*(w_{i\,9-i}) = 4\alpha_{i\,9-i}$ and by -2^{2^n}, if $k = 2^{2^n}$ to get $\phi_*(w_{i\,2^{2^n}+1-i}) = -2^{2^n}\alpha_{i\,2^{2^n}+1-i}$.

Together with Lemma 6.1, this implies

Corollary 6.2. *Let $k \geqslant 3$. By (8) and (9) let λ_i be such that the linear combination $\mathbf{a}_k = \sum_{i=1}^k \lambda_i\alpha_{ik+1-i}$ is a polynomial generator in $\mathbf{MU}_{2k}$. Then*

$$\mathbf{b}_k = \sum_{i=1}^k \lambda_i w_{ik+1-i}.$$

is a polynomial generator in $W_{2k}[1/2]$.

Then $\mathbf{MSU}_*[1/2]$ is a subring of cycles of the boundary operation ∂ in $W_*[1/2]$ with multiplication $*$

$$\partial(a * b) = a * \partial b + \partial a * b - \mathbf{CP}_1\partial a * \partial b.$$

One has $\partial\mathbf{CP}_1 = 2$ and $a * b = a \cdot b$ whenever $a \in Im\partial$ or $b \in Im\partial$. Therefore

$$\partial(\mathbf{CP}_1 * \alpha) = 2\alpha - \mathbf{CP}_1 \cdot \partial\alpha, \forall\alpha \in W.$$

As mentioned in [17] this implies that

$$\alpha = 1/2\partial(\mathbf{CP}_1 * \alpha) + 1/2\mathbf{CP}_1 \cdot \partial\alpha, \tag{26}$$

and therefore $W[1/2]$ is generated by 1 and $\mathbf{CP}_1$ as a $\mathbf{MSU}_*[1/2]$ module. It is easily checked that this module is free.

Proposition 6.3. *Let* $\mathbf{b}_k$ *be as in Corollary 6.2. Then*

$$\mathbf{MSU}_*[1/2] = \mathbb{Z}[1/2][x_2, x_k : k \geqslant 3],$$

where

$$x_2 = \mathbf{CP}_2 - 9/8\mathbf{CP}_1^2,$$
$$x_k := \partial(\mathbf{CP}_1 * \mathbf{b}_k) = 2\mathbf{b}_k - \mathbf{CP}_1 \cdot \partial\mathbf{b}_k.$$

Proof. One has for the values of the Chern numbers

$$c_1c_1[\mathbf{CP}_1^2] = 8, \qquad c_1c_1[\mathbf{CP}_2] = 9, \qquad c_2[\mathbf{CP}_1^2] = 4, \qquad c_2[\mathbf{CP}_2] = 3.$$

This imply that $c_1c_1[\mathbf{b}_2] = 0$. There are no more Chern numbers having c_1 as a factor and $s_2[\mathbf{b}_2] = s_2[\mathbf{CP}^2] = 3$. Therefore $\mathbf{b}_2$ forms a generator of $\mathbf{MSU}_4[1/2]$.

Apply (26). The main Chern number vanishes on the second (decomposable) component of

$$\mathbf{b}_k = 1/2\partial(\mathbf{CP}_1 * \mathbf{b}_k) + 1/2\mathbf{CP}_1 \cdot \partial\mathbf{b}_k,$$

i.e., the first component x_k has the main Chern number $2s_k(\mathbf{b}_k)$. □

7. The restriction of the classifying map of F_{Ab} on $\mathbf{MSU}^*[1/2]$.

As above let $F_U = \sum \alpha_{ij}x^iy^j$ be the universal formal group law. By definition the coefficient ring of the universal abelian formal group law F_{Ab} is the quotient ring

$$\Lambda_{Ab} = \mathbf{MU}_*/I_{Ab}, \text{ where } I_{Ab} = (\alpha_{ij}, i, j > 1). \tag{27}$$

Let us apply Euclid's algorithm for the Chern numbers $s_{m-1}(\alpha_{i,m-i})$ in (13)

Let

$$z_k = \sum_{i=2}^{k-1} \lambda_i \alpha_{i\,k+1-i}, \;\; k \geqslant 3.$$

By [10], [11] one has $I_{Ab} = I_{AB} = (z_k, \; k \geqslant 3)$.

Consider the composition

$$r_{Ab} \colon \mathbf{MSU}_*[1/2] \xrightarrow{\subset} \mathbf{MU}_*[1/2] \xrightarrow{f_{Ab}} \Lambda_{Ab}[1/2], \tag{28}$$

where $\subset$ is forgetful map.

Proposition 7.1. *One has the following polynomial generators in* $\mathbf{MSU}_*[1/2]$ *viewed as the elements in* $\mathbf{MU}_*[1/2]$

$$x_2 = \mathbf{CP}_2 - \frac{9}{8}\mathbf{CP}_1^2, \qquad x_3 = -\alpha_{22}, \qquad x_4 = -\alpha_{23} - \frac{3}{2}x_3\mathbf{CP}_1.$$

To prove this we have to check that all Chern numbers of x_i having factor c_1 are zero. Then we have to check the main Chern number $s_i(x_i)$ for Novikov's criteria.

We already did this for x_2 in the proof of Proposition 6.3. Then by definition x_3 is the coefficient $-\alpha_{22} \in I_{AB}$ of the universal formal group law. In $\mathbf{MU}_*$ one has

$$2\alpha_{22} = -3\mathbf{CP}_3 + 8\mathbf{CP}_1\mathbf{CP}_2 - 5\mathbf{CP}_1^3,$$
$$\alpha_{23} = 2\mathbf{CP}_1^4 - 7\mathbf{CP}_1^2 * \mathbf{CP}_2 + 3\mathbf{CP}_2^2 + 4\mathbf{CP}_1\mathbf{CP}_3 - 2\mathbf{CP}_4.$$

Let us compute the Chern numbers of $x_3 = -\alpha_{22}$. One has

X	$c_3(X)$	$c_1c_2(X)$	$c_1c_1c_1(X)$
$\mathbf{CP}_3$	4	24	64
$\mathbf{CP}_1\mathbf{CP}_2$	6	24	54
$\mathbf{CP}_1^3$	8	24	48.

It follows all Chern numbers of α_{22} having factor c_1 are zero. Then α_{22} forms a generator in $\mathbf{MSU}_6[1/2]$ as $s_3[-2\alpha_{22}] = 4 \cdot 3$.

Similarly for x_4: the main Chern number $s_4(x_4) = 2 \cdot 5$ fits for Novikov's criteria and one has

X	$c_1c_1c_1c_1(X)$	$c_1c_1c_2(X)$	$c_1c_3(X)$
$\mathbf{CP}_1^4$	384	192	64
$\mathbf{CP}_1^2\mathbf{CP}_2$	432	204	60
$\mathbf{CP}_2^2$	486	216	54
$\mathbf{CP}_1\mathbf{CP}_3$	512	224	56
$\mathbf{CP}_4$	625	250	50.

Note F_{Ab} is a specialization of the Buchstaber formal group law F_B. In particular one can put $A(x) = B(x)^2$ in Proposition (2.1) to specify F_B to F_{Ab} over torsion free ring $\Lambda_{Ab}[1/2]$. Then Proposition 5.2 implies

Proposition 7.2. *After restriction on* $\mathbf{MSU}_*[1/2]$ *the classifying map of the universal abelian formal group law becomes the one-parameter genus*

$$r_{Ab} \colon \mathbf{MSU}_*[1/2] \to \mathbb{Z}[1/2][x_2].$$

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