Homology, Homotopy and Applications, vol. 26(1), 2024, pp.15–27

INDEPENDENCE COMPLEXES OF $(n \times 6)$-GRID GRAPHS

TAKAHIRO MATSUSHITA AND SHUN WAKATSUKI

(*communicated by J. Daniel Christensen*)

Abstract

We determine the homotopy types of the independence complexes of the $(n \times 6)$-square grid graphs. In fact, we show that these complexes are homotopy equivalent to wedges of spheres.

1. Introduction

For a finite simple graph $G = (V, E)$, a subset σ of the vertex set V of G is called *independent* if no two elements in σ are adjacent. The family of independent sets of G forms a simplicial complex $I(G)$, which is called the *independence complex of* G. Homotopy types of independence complexes and their connection with combinatorial properties of graphs have been extensively studied in the last two decades [**Ko2**].

The independence complex $I(G)$ of G is the simplicial complex whose vertex set is V and whose set of minimal non-faces is the set E of edges in G. Hence the independence complex is another formulation of the clique complex, which has appeared in several branches in mathematics. The Vietoris–Rips complex, which appears in topological data analysis and geometric group theory [**PRSZ**], and the order complex of a poset [**Ko2**] are typical examples of clique complexes. In particular, the barycentric subdivision of every simplicial complex is a clique complex, and hence a very wide class of geometric objects can be obtained from the independence complex. This means that the independence complex is a fundamental construction of simplicial complexes.

However, it is in general quite difficult to determine the homotopy type of independence complex $I(G)$ even if the graph G can be easily described. It has been studied by several authors to determine the homotopy types or homotopy invariants of particular classes of graphs ([**Ba**], [**Br**], [**BH**], [**EH**], [**En**], [**GSS**], [**Ir**], [**Ka**], [**Ko1**], [**MT**], [**Ma1**], [**Ma2**]). In this paper, we treat the independence complexes of certain square grid graphs $\Gamma_{n,k}$ defined as follows: Let n and k be positive integers. We define the graph $\Gamma_{n,k}$ by

$$V(\Gamma_{n,k}) = \{(x, y) \in \mathbb{Z}^2 \mid 1 \leqslant x \leqslant n,\ 1 \leqslant y \leqslant k\},$$

$$E(\Gamma_{n,k}) = \{\{(x, y), (x', y')\} \mid (x, y), (x', y') \in V(\Gamma_{n,k}),\ |x' - x| + |y' - y| = 1\}.$$

The goal in this paper is to determine the homotopy type of $I(\Gamma_{n,6})$, for all n.

Received July 31, 2022, revised January 27, 2023; published on January 24, 2024.
2020 Mathematics Subject Classification: 55P10, 05C69.
Key words and phrases: independence complex, square grid graph.
Article available at `http://dx.doi.org/10.4310/HHA.2024.v26.n1.a2`

k	0	1	2	3	4	5	6
ν	0	0	0	2	2	4	4
μ	2	2	4	4	-	-	-

Table 1: Definitions of ν and μ

Before giving the precise statement of our main result, we review the background and known results in the independence complex of square grid graphs. The graph $\Gamma_{n,1}$ is the path graph P_n of n vertices, and the homotopy type of $I(\Gamma_{n,1})$ was determined by Kozlov [**Ko1**]. The homotopy types of $I(\Gamma_{n,2})$ and $I(\Gamma_{n,3})$ were determined by Adamaszek [**Ad2**]. The Euler characteristic of $I(\Gamma_{n,4})$ was determined by Okura [**Ok**], and the homotopy types of $I(\Gamma_{n,4})$ and $I(\Gamma_{n,5})$ were recently determined by the authors [**MW**].

Our grid graph $\Gamma_{n,k}$ is the cartesian product $P_n \times P_k$ of two path graphs P_n and P_k. As related graphs, the independence complexes of $C_n \times C_k$ and $P_n \times C_k$ have also been studied. Here C_n denotes the cycle graph with n vertices. Fendley, Schoutens, and van Eersten [**FSv**] suggested several conjectures related to the Euler characteristic of $I(C_n \times C_k)$ from a viewpoint of statistical physics. Jonsson [**Jo1**] solved one of their conjectures, which states that the reduced Euler characteristic of $I(C_n \times C_k)$ is 1 when n and k are coprime. After that, Bousquet-Mélou, Linusson, and Nevo [**BLN**] studied the homotopy types of several families of independence complexes of grid graphs, and since then the independence complexes of $C_n \times C_k$, $P_n \times C_k$, and $P_n \times P_k$ have been studied by several authors ([**Ad2**], [**Ir**], [**Jo2**], [**Jo3**], [**MW**], [**Ok**], [**Th**]).

Now we state our main result in this paper:

Theorem 1.1. *For $n \geqslant 1$, the homotopy type of $I(\Gamma_{n,6})$ is given as follows.*

(1) *Assume that n is odd and write $n = 14m + 2k + 1$ with $m \geqslant 0$ and $0 \leqslant k \leqslant 6$. Then there is a homotopy equivalence:*

$$I(\Gamma_{n,6}) \simeq S^{n'} \vee \Big(\bigvee_{n'-m \leqslant i \leqslant n'-1} \bigvee_{6} S^i \Big) \vee \bigvee_{\nu} S^{n'-m-1},$$

where $n' = 21m + 3k + 1$ and ν is a number defined by Table 1.

(2) *Assume that n is even and write $n = 14m + 2k$ with $m \geqslant 0$ and $0 \leqslant k \leqslant 6$. If $n \leqslant 6$, we have*

$$I(\Gamma_{2,6}) \simeq S^2,\ I(\Gamma_{4,6}) \simeq \bigvee_{3} S^5,\ I(\Gamma_{6,6}) \simeq \bigvee_{3} S^8.$$

If $n > 6$, there is a homotopy equivalence:

$$I(\Gamma_{n,6}) \simeq \begin{cases} \bigvee_{5} S^{n'} \vee \Big(\bigvee_{n'-m < i \leqslant n'-1} \bigvee_{6} S^i \Big) \vee \bigvee_{\mu} S^{n'-m} & (0 \leqslant k \leqslant 3, m \geqslant 1), \\ \bigvee_{5} S^{n'} \vee \Big(\bigvee_{n'-m \leqslant i \leqslant n'-1} \bigvee_{6} S^i \Big) & (4 \leqslant k \leqslant 6), \end{cases}$$

where $n' = 21m + 3k - 1$ and μ is a number defined by Table 1.

Finally, we discuss the Euler characteristic of $I(\Gamma_{n,k})$. For a positive integer k, let f_k denote the function assigning $\chi(I(\Gamma_{n,k}))$ to n. The known results ([**Ko1**],

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$f_6(n)$	0	2	2	-2	0	4	0	-4	2	6	-2	-4	4	4

n	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$f_6(n)$	-4	-2	6	2	-4	0	4	0	-2	2	2	0	0	0

Table 2: Periodicity of $f_6(n) = \chi(I(\Gamma_{n,6}))$

[**Ad2**], [**MW**]) showed that f_1, f_2, f_3, f_5 are periodic functions with periods 6, 4, 8, and 40, respectively. On the other hand, by the computation of Okura [**Ok**], f_4 is not a bounded function and $|f_4(n)|$ tends to infinity. Our main result implies the following:

Corollary 1.2. *The function f_6 assigning $\chi(I(\Gamma_{n,6}))$ to n is a periodic function with period* 28*, whose values are shown in Table 2.*

The rest of this paper is organized as follows: In Section 2, we review several definitions and facts related to independence complex. Section 3 is devoted to the proof of Theorem 1.1.

Acknowledgments

The authors are grateful to the anonymous referee for useful comments. The first author and second author were partially supported by JSPS KAKENHI Grant Number JP19K14536 and JP20J00404, respectively.

2. Preliminaries

In this section, we review definitions and facts related to independence complexes which we need in this paper. Our main tools are the cofiber sequence (Theorem 2.2) and the fold lemma (Corollary 2.3). For a comprehensive introduction to simplicial complexes, we refer to [**Ko2**].

A *(finite simple) graph* is a pair (V, E) such that V is a finite set and E is a subset of the family $\binom{V}{2}$ of 2-element subsets of V. A subset σ of V is *independent* or *stable* if there are no $v, w \in \sigma$ such that $\{v, w\} \in E$. The *independence complex of a graph G* is the simplicial complex whose vertex set is V and whose simplices are independent sets in G, and is denoted by $I(G)$.

A *subgraph of* $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subset V$ and $E' \subset E$. The subgraph G' of G is said to be *induced* if $E' = \binom{V'}{2} \cap E$. For a subset S of V, we write $G - S$ to mean the induced subgraph of G whose vertex set is $V - S$. For a vertex v of G, we write $G - v$ instead of $G - \{v\}$. Note that a subgraph G' of G is induced if and only if $I(G')$ is a subcomplex of $I(G)$.

If G is a disjoint union $G_1 \sqcup G_2$ of two subgraphs G_1 and G_2, then $I(G)$ coincides with the join $I(G_1) * I(G_2)$. In particular, if G has an isolated vertex, then $I(G)$ is contractible. Let K_2 denote the complete graph with 2 vertices, *i.e.* K_2 is the graph consisting of two vertices and one edge. Since $I(K_2) = S^0$, we have an equivalence $I(K_2 \sqcup G) \simeq \Sigma I(G)$. Here, Σ denotes a suspension.

Let K be an (abstract) simplicial complex and S a subset of $V(K)$. Let $K - S$ denote the simplicial complex consisting of the simplices of K disjoint from S. For a face σ of K, define the *star* $\mathrm{star}(\sigma)$ and the *link* $\mathrm{link}(\sigma)$ by

$$\mathrm{star}(\sigma) = \{\tau \in K \mid \sigma \cup \tau \in K\}, \ \mathrm{link}(\sigma) = \mathrm{star}(\sigma) - \sigma.$$

For $v \in V$, we write $\mathrm{star}(v)$ and $\mathrm{link}(v)$ instead of $\mathrm{star}(\{v\})$ and $\mathrm{link}(\{v\})$, respectively.

There is a functor called geometric realization from the category of simplicial complexes to the category of topological spaces [**Ko2**]. We often identify the simplicial complex K with its geometric realization $|K|$, and apply topological terms to simplicial complexes.

Lemma 2.1. *Let K be a simplicial complex and v a vertex of K. Then K is homeomorphic to the mapping cone of the inclusion $\mathrm{link}(v) \hookrightarrow K - v$, and hence there is a cofiber sequence*

$$\mathrm{link}(v) \to K - v \to K.$$

In particular, if the inclusion $\mathrm{link}(v) \hookrightarrow K - v$ is null-homotopic, then

$$K \simeq (K - v) \vee \Sigma \mathrm{link}(v).$$

Let G be a graph and let $v \in V(G)$. Let $N(v)$ denote the set of vertices of G adjacent to v, and set $N[v] = N(v) \cup \{v\}$. It is straightforward to see $I(G) - v = I(G - v)$ and $\mathrm{link}(v) = I(G - N[v])$. Applying the above lemma to $K = I(G)$, we have the following:

Theorem 2.2 (see [**Ad1**])**.** *Let G be a graph and v a vertex of G. Then there is a cofiber sequence*

$$I(G - N[v]) \to I(G - v) \to I(G).$$

In particular, if the inclusion $I(G - N[v]) \to I(G - v)$ is null-homotopic, then

$$I(G) \simeq I(G - v) \vee \Sigma I(G - N[v]).$$

This theorem implies the following simple argument, which will be frequently used in the subsequent section.

Corollary 2.3 (fold lemma [**En**])**.** *Let G be a graph, and v, w vertices of G. Assume that $v \neq w$ and $N(v) \subset N(w)$. Then the inclusion $I(G - w) \hookrightarrow I(G)$ is a homotopy equivalence.*

3. Proofs

In this section, we write Γ_n instead of $\Gamma_{n,6}$. The goal of this section is to prove Theorem 1.1, which determine the homotopy type of $I(\Gamma_n)$.

3.1. Subgraphs X_n, Y_n, A_n and B_n

In this subsection, we determine the homotopy types of independence complexes of several induced subgraphs X_n, Y_n, A_n and B_n of Γ_n, and relates them to the homotopy type of the independence complex $I(\Gamma_n)$ of Γ_n.

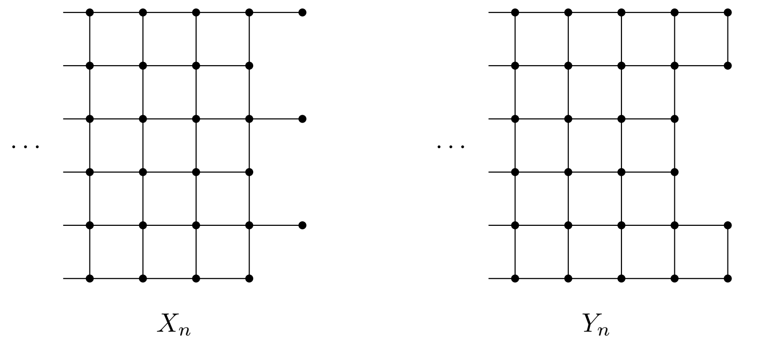

Figure 1: Graphs X_n and Y_n

Definition 3.1. For $n \geqslant 1$, define the induced subgraphs X_n, Y_n, A_n and B_n of Γ_n by

$$X_n = \Gamma_n - \{(n,1),(n,3),(n,5)\},$$

$$Y_n = \Gamma_n - \{(n,3),(n,4)\},$$

$$A_n = \Gamma_n - \{(n,1),(n,5)\},$$

$$B_n = \Gamma_n - \{(n,4)\}.$$

See Figures 1 and 2.

The homotopy type of $I(\Gamma_n)$ is determined by Lemmas 3.2 and 3.3, Corollaries 3.7 and 3.8, and Proposition 3.9. Lemmas 3.2 and 3.3 are easy consequences of the fold lemma (Corollary 2.3). The others are mainly deduced by Proposition 3.6. In this subsection, we prove these propositions, and the determination of the homotopy type of $I(\Gamma_n)$ is postponed to the next subsection.

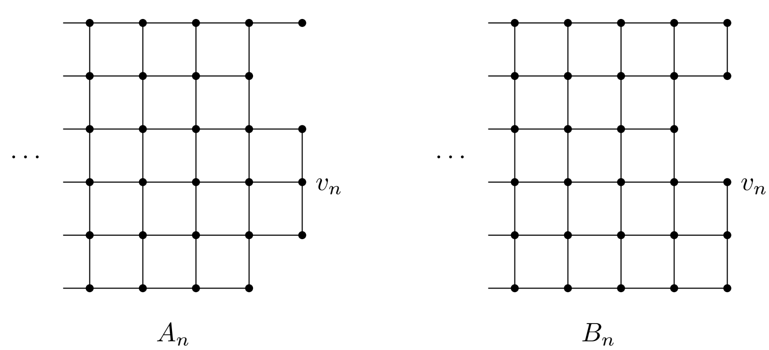

Figure 2: Graphs A_n and B_n

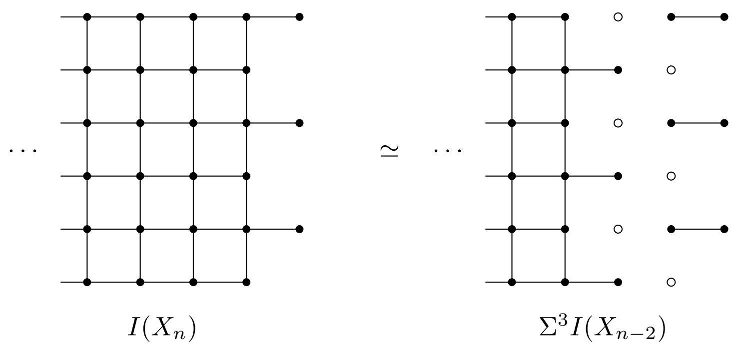

Figure 3: $I(X_n) \simeq \Sigma^3 I(X_{n-2})$

We first prove Lemmas 3.2 and 3.3, which determine the homotopy types of $I(X_n)$ and $I(Y_n)$. Here we write pt to mean the topological space consisting of one point.

Lemma 3.2. *For $n \geqslant 1$, there is a following homotopy equivalence:*

$$I(X_n) \simeq \begin{cases} \mathrm{pt} & (n = 2k+1), \\ S^{3k-1} & (n = 2k). \end{cases}$$

Proof. The fold lemma implies

$$I(X_n) \simeq \Sigma^3 I(X_{n-2})$$

for $n \geqslant 3$ (see Figure 3). The fold lemma again implies

$$I(X_1) \simeq \mathrm{pt} \quad \text{and} \quad I(X_2) \simeq S^2.$$

This completes the proof. □

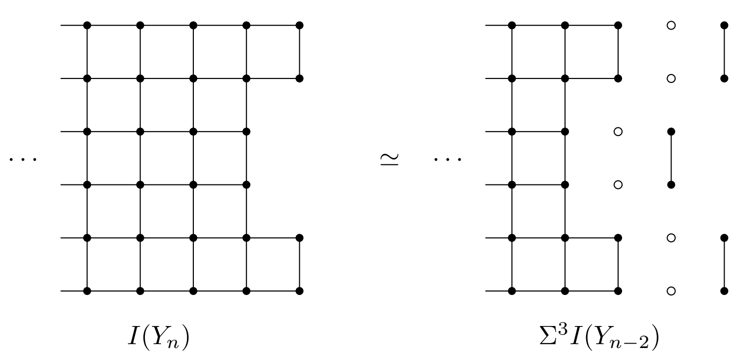

Figure 4: $I(Y_n) \simeq \Sigma^3 I(Y_{n-2})$

Lemma 3.3. *For $n \geqslant 1$, there is a following homotopy equivalence:*

$$I(Y_n) \simeq \begin{cases} S^{3k+1} & (n = 2k+1), \\ S^{3k-1} & (n = 2k). \end{cases}$$

Proof. The fold lemma implies

$$I(Y_n) \simeq \Sigma^3 I(Y_{n-2})$$

for $n \geqslant 3$ (see Figure 4). The fold lemma again implies

$$I(Y_1) \simeq S^1 \quad \text{and} \quad I(Y_2) \simeq S^2.$$

This completes the proof. □

Set $v_n = (n, 3)$. Note that $A_n - v_n = X_n$ and $B_n - v_n = Y_n$ (see Definition 3.1).

Lemma 3.4. *There are following homotopy equivalences:*

(1) *For $n \geqslant 4$, there is a following homotopy equivalence:*

$$I(A_n - N[v_n]) \simeq \Sigma^3 I(B_{n-3}).$$

(2) *For $n \geqslant 5$, there is a following homotopy equivalence:*

$$I(B_n - N[v_n]) \simeq \Sigma^5 I(A_{n-4}).$$

Proof. These homotopy equivalences are immediately deduced by the fold lemma. See Figures 5 and 6. □

Lemma 3.5. *The following hold:*

$$I(A_1) \simeq \mathrm{pt}, I(A_2) \simeq S^2, I(A_3) \simeq S^3,$$

$$I(B_1) \simeq S^1, I(B_2) \simeq S^2, I(B_3) \simeq S^4, I(B_4) \simeq S^5 \vee S^5.$$

Proof. These homotopy equivalences clearly follow from the fold lemma except for $I(B_4) \simeq S^5 \vee S^5$. This last homotopy equivalence is deduced from Theorem 2.2,

$$I(B_4 - v_4) \simeq S^5, \quad \text{and} \quad I(B_4 - N[v_4]) \simeq S^4.$$

These two homotopy equivalences are deduced from the fold lemma. □

The following proposition is a key to the whole proof. This proposition allows us to determine the homotopy types of $I(A_n)$ and $I(B_n)$ inductively.

Proposition 3.6. *The following hold:*

(1) *If $n = 2k+1$, then $I(A_n)$ is homotopy equivalent to a wedge of spheres whose dimension is at most $3k$. If $n = 2k$, then $I(A_n)$ is homotopy equivalent to a wedge of spheres whose dimension is at most $3k-1$.*

(2) *For $n \geqslant 4$, the inclusion $I(A_n - N[v_n]) \hookrightarrow I(A_n - v_n)$ is null-homotopic. In particular, we have*

$$I(A_n) \simeq I(X_n) \vee \Sigma^4 I(B_{n-3}).$$

(3) *If $n = 2k+1$, then $I(B_n)$ is homotopy equivalent to a wedge of spheres whose dimension is at most $3k+1$. If $n = 2k$, then $I(B_n)$ is homotopy equivalent to a wedge of spheres whose dimension is at most $3k-1$.*

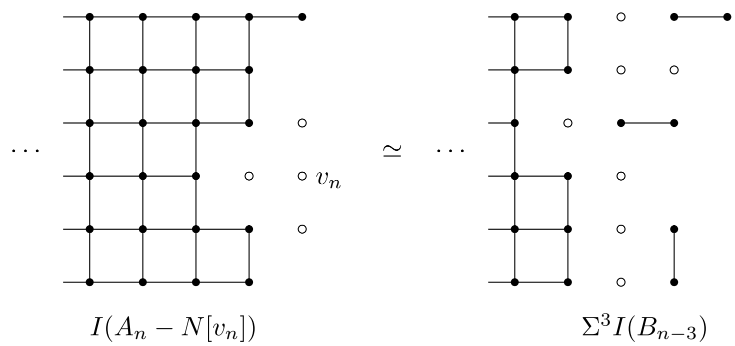

Figure 5: $I(A_n - N[v_n]) \simeq \Sigma^3 I(B_{n-3})$

(4) *For $n \geqslant 5$, the inclusion $I(B_n - N[v_n]) \hookrightarrow I(B_n - v_n)$ is null-homotopic. In particular, we have*

$$I(B_n) \simeq I(Y_n) \vee \Sigma^6 I(A_{n-4}).$$

Proof. We simultaneously show these four statements by induction on n. The case $n \leqslant 3$ immediately follows from Lemma 3.5. Suppose $n \geqslant 4$. We note that in the statement of (2) (or (4)), Theorem 2.2 and Lemma 3.4 imply that the former assertion implies the latter.

We now show (1) and (2). Suppose that n is odd, and set $n = 2k + 1$. Then Lemma 3.4 implies that

$$I(A_{2k+1} - v_{2k+1}) = I(X_{2k+1}) \simeq \mathrm{pt}.$$

Hence (2) is clear and $I(A_{2k+1}) \simeq \Sigma I(A_{2k+1} - N[v_{2k+1}]) \simeq \Sigma^4 I(B_{2k-2})$. Then, by the inductive hypothesis, $I(B_{2k-2})$ is homotopy equivalent to a wedge of spheres whose dimension is at most $3k - 4$. This implies (1) in this case.

Next suppose that n is even, and set $n = 2k$. Lemmas 3.2 and 3.4 imply

$$I(A_{2k} - v_{2k}) = I(X_{2k}) \simeq S^{3k-1} \quad \text{and} \quad I(A_{2k} - N[v_{2k}]) \simeq \Sigma^3 I(B_{2k-3}).$$

By the inductive hypothesis, $I(B_{2k-3})$ is homotopy equivalent to a wedge of spheres whose dimension is at most $3(k-2)+1 = 3k-5$. Hence the inclusion

$$\Sigma^3 I(B_{2k-3}) \to I(X_{2k})$$

is null-homotopic. This implies (2) and

$$I(A_{2k}) \simeq I(X_{2k}) \vee \Sigma^4 I(B_{2k-3}).$$

This means $I(A_{2k})$ is homotopy equivalent to a wedge of spheres whose dimension is at most $3k - 1$. This completes the proof of (1).

Next we show (3) and (4). Suppose that n is odd, and set $n = 2k + 1$. Then Lemmas 3.3 and 3.4 imply

$$I(B_{2k+1} - v_{2k+1}) = I(Y_{2k+1}) \simeq S^{3k+1} \quad \text{and} \quad I(B_{2k+1} - N[v_{2k+1}]) \simeq \Sigma^5 I(A_{2k-3}).$$

By the inductive hypothesis, $I(A_{2k-3}) = I(A_{2(k-2)+1})$ is homotopy equivalent to a wedge of spheres whose dimension is at most $3(k-2) = 3k - 6$. Thus the inclusion

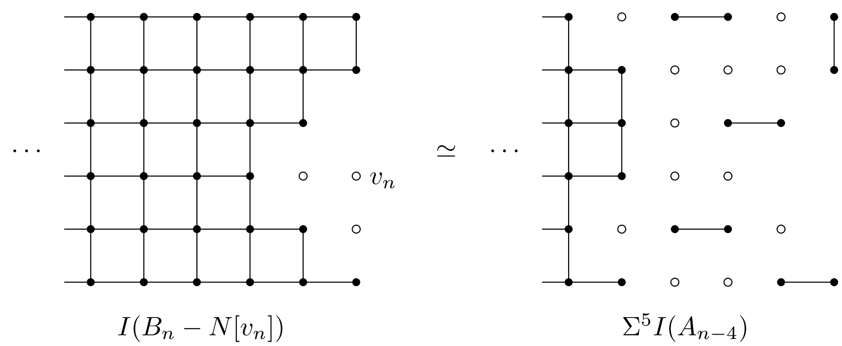

Figure 6: $I(B_n - N[v_n]) \simeq \Sigma^5 I(A_{n-4})$

$\Sigma^5 I(A_{2k-3}) \to I(X_{2k+1})$ is null-homotopic, which implies (4), and

$$I(B_{2k+1}) \simeq I(Y_{2k+1}) \vee \Sigma^6 I(A_{2k-3}).$$

This implies that $I(B_{2k+1})$ is homotopy equivalent to a wedge of spheres whose dimension is at most $3k+1$, which implies (3).

Finally suppose that n is even, and set $n = 2k$. We have already showed the case that $k = 2$ in the previous lemma and its proof. Suppose $k \geqslant 3$. Then Lemmas 3.3 and 3.4 imply

$$I(B_{2k} - v_{2k}) = I(Y_{2k}) \simeq S^{3k-1} \quad \text{and} \quad I(B_{2k} - N[v_{2k}]) \simeq \Sigma^5 I(A_{2k-4}).$$

Then $I(A_{2k-4}) = I(A_{2(k-2)})$ is homotopy equivalent to a wedge of spheres whose dimension is at most $3(k-2) - 1 = 3k - 7$. This implies that the inclusion

$$\Sigma^5 I(A_{2k-4}) \to I(Y_{2k})$$

is null-homotopic, which implies (4), and there is a homotopy equivalence

$$I(B_{2k}) \simeq I(X_{2k}) \vee \Sigma^6 I(A_{2k-4}).$$

This implies that $I(B_{2k})$ is homotopy equivalent to a wedge of spheres whose dimension is at most $3k-1$. This completes the proof. □

Combining (2) and (4) of Proposition 3.6, we have the following.

Corollary 3.7. *For $n \geqslant 8$, there is a following homotopy equivalence:*

$$I(A_n) \simeq I(X_n) \vee \Sigma^4 I(Y_{n-3}) \vee \Sigma^{10} I(A_{n-7}).$$

Combining Lemma 3.5 and (2) of Proposition 3.6, we have the following.

Corollary 3.8. *There are following homotopy equivalences:*

$$I(A_1) \simeq \mathrm{pt}, I(A_2) \simeq S^2, I(A_3) \simeq S^3, I(A_4) \simeq S^5 \vee S^5,$$

$$I(A_5) \simeq S^6, I(A_6) \simeq S^8 \vee S^8, I(A_7) \simeq S^9 \vee S^9.$$

In the rest of this subsection, we prove the following proposition.

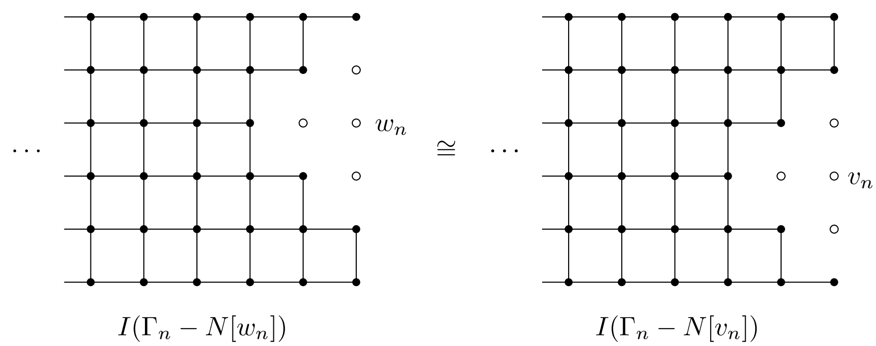

Figure 7: $I(\Gamma_n - N[w_n]) \cong I(\Gamma_n - N[v_n])$

Proposition 3.9. *For $n \geqslant 5$, we have the following homotopy equivalence:*

$$I(\Gamma_n) \simeq I(Y_n) \vee \bigvee_2 \Sigma^6 I(A_{n-4}).$$

Proof. Set $w_n = (n, 4)$. We first show that the inclusion

$$I(\Gamma_n - N[w_n]) \hookrightarrow I(\Gamma_n - w_n)$$

is null-homotopic. Note that the inclusion $I(\Gamma_n - N[w_n]) \hookrightarrow I(\Gamma_n - w_n)$ has the following factorization:

$$I(\Gamma_n - N[w_n]) \to I(Y_n) \to I(\Gamma_n - w_n).$$

Hence we show that the inclusion $I(\Gamma_n - N[w_n]) \hookrightarrow I(Y_n)$ is null-homotopic.

Let α be the automorphism of Γ_n defined by sending a vertex (i, j) to the vertex $(i, 7-j)$. Then α restricts to isomorphisms $B_n - N[w_n] \xrightarrow{\cong} B_n - N[v_n]$ (see Figure 7) and $Y_n \xrightarrow{\cong} Y_n$. Hence it suffices to see that the inclusion $I(\Gamma_n - N[v_n]) \hookrightarrow I(Y_n)$ is null-homotopic. Since

$$I(B_n - N[v_n]) = I(\Gamma_n - N[v_n]) \hookrightarrow I(Y_n) = I(B_n - v_n),$$

Proposition 3.6 implies that the inclusion $I(\Gamma_n - N[v_n]) \hookrightarrow I(Y_n)$ is null-homotopic. This concludes that the inclusion $I(\Gamma_n - N[w_n]) \hookrightarrow I(Y_n)$ is null-homotopic, as desired.

Theorem 2.2 implies that $I(\Gamma_n) \simeq I(\Gamma_n - w_n) \vee \Sigma I(\Gamma_n - N[w_n])$. Lemma 3.4 implies that $I(\Gamma_n - N[w_n]) \cong I(\Gamma_n - N[v_n]) \simeq \Sigma^5 I(A_{n-4})$. Since $\Gamma_n - w_n = B_n$, (4) of Proposition 3.6 implies

$$I(\Gamma_n) \simeq I(\Gamma_n - w_n) \vee \Sigma I(\Gamma_n - N[w_n]) \simeq I(Y_n) \vee \bigvee_2 \Sigma^6 I(A_{n-4}).$$

This completes the proof. □

3.2. Proof of Theorem 1.1

The goal of this subsection is to complete the proof of Theorem 1.1, which determines the homotopy type of $I(\Gamma_n)$. By Proposition 3.9, the homotopy type of $I(\Gamma_n)$ is determined by the homotopy types of $I(X_n)$, $I(Y_n)$, and $I(A_n)$. The homotopy types of $I(X_n)$ and $I(Y_n)$ have been already determined in Lemmas 3.2 and 3.3. Thus the next task is to determine the homotopy type of $I(A_n)$.

Lemma 3.10. *The following hold:*

(1) *If $2k+1 \geqslant 15$, there is a following homotopy equivalence:*

$$I(A_{2k+1}) \simeq \Big(\bigvee_{3} S^{3k}\Big) \vee \Sigma^{20} I(A_{2k+1-14}).$$

(2) *There is a following homotopy equivalence:*

$$I(A_{14m+2k+1}) \simeq \bigvee_{3} S^{21m+3k} \vee \cdots \vee \bigvee_{3} S^{20m+3k+1} \vee \Sigma^{20m} I(A_{2k+1}).$$

Proof. It follows from Lemmas 3.2,3.3 and Corollary 3.7 that

$$\begin{aligned} I(A_{2k+1}) &\simeq I(X_{2k+1}) \vee \Sigma^4 I(Y_{2k-2}) \vee \Sigma^{10} I(A_{2k-6}) \\ &\simeq \Sigma^4 I(Y_{2k-2}) \vee \Sigma^{10} I(X_{2k-6}) \vee \Sigma^{14} I(Y_{2k-9}) \vee \Sigma^{20} I(A_{2k-13}) \\ &\simeq S^{3k} \vee S^{3k} \vee S^{3k} \vee \Sigma^{20} I(A_{2k+1-14}) \end{aligned}$$

if $2k+1 \geqslant 15$. We deduce (2) by iterating (1). □

We first determine the homotopy type of $I(A_n)$ for odd n.

Proposition 3.11. *For $m \geqslant 0$ and $0 \leqslant k \leqslant 6$, there is a following homotopy equivalence:*

$$I(A_{14m+2k+1}) \simeq \Big(\bigvee_{i=20m+3k+1}^{21m+3k} \bigvee_{3} S^{i}\Big) \vee \bigvee_{a} S^{20m+3k},$$

where a is a number defined by Table 3.

Proof. The case that $14m+2k+1 \leqslant 7$ has been already proved in Corollary 3.8. If $14m+2k+1 > 7$, Corollary 3.7 implies

$$I(A_{2k+1}) \simeq I(X_{2k+1}) \vee \Sigma^4 I(Y_{2k-2}) \vee \Sigma^{10} I(A_{2k-6}) \simeq S^{3k} \vee \Sigma^{10} I(A_{2k-6}).$$

Hence Corollary 3.8 implies

$$I(A_9) \simeq S^{12} \vee \Sigma^{10} I(A_2) \simeq S^{12} \vee S^{12},$$

$$I(A_{11}) \simeq S^{15} \vee \Sigma^{10} I(A_4) \simeq S^{15} \vee S^{15} \vee S^{15},$$

$$I(A_{13}) \simeq S^{18} \vee \Sigma^{10} I(A_6) \simeq S^{18} \vee S^{18} \vee S^{18}.$$

Then Lemma 3.10 completes the proof. □

Next we determine the homotopy type of $I(A_n)$ for even n.

n	0	1	2	3	4	5	6
a	0	1	1	2	2	3	3
b	-	-	0	0	0	1	1

Table 3: Definitions of a and b

Proposition 3.12. *Assume $m \geqslant 0$ and $0 \leqslant k \leqslant 6$. Except for $I(A_2) \simeq S^2$, there is a following homotopy equivalence:*

$$I(A_{14m+2k}) \simeq \begin{cases} \bigvee_2 S^{21m+3k-1} \vee \Big(\bigvee_{i=20m+3k}^{21m+3k-2} \bigvee_3 S^i \Big) \vee \bigvee_2 S^{20m+3k-1} & (k=0,1), \\ \bigvee_2 S^{21m+3k-1} \vee \Big(\bigvee_{i=20m+3k-1}^{21m+3k-2} \bigvee_3 S^i \Big) \vee \bigvee_b S^{20m+3k-2} & (2 \leqslant k \leqslant 6), \end{cases}$$

where b is a number defined by Table 3.

Proof. The case $14m+2k \leqslant 7$ follows from Corollary 3.8. Thus, we suppose that $14m+2k > 7$. Then Corollary 3.7 implies

$$\begin{aligned} I(A_{14m+2k}) &\simeq I(X_{14m+2k}) \vee \Sigma^4 I(Y_{14m+2k-3}) \vee \Sigma^{10} I(A_{14m+2k-7}) \\ &\simeq \bigvee_2 S^{21m+3k-1} \vee \Sigma^{10} I(A_{14m+2k-7}). \end{aligned}$$

Then Proposition 3.11 completes the proof. □

Now we complete the proof of our main theorem.

Proof of Theorem 1.1. Suppose $n \geqslant 5$. Then Proposition 3.9 asserts

$$I(\Gamma_n) \simeq I(Y_n) \vee \bigvee_2 \Sigma^6 I(A_{n-4}).$$

Then the proof is completed by the combination of Propositions 3.11 and 3.12, and Lemma 3.3. The case $n < 5$ follows from the known results (see [**Ko1**], [**Ad2**] and [**MW**]). □

References

[**Ad1**] M. Adamaszek, *Splittings of independence complexes and the powers of cycles*, J. Combin. Theory Ser. A 119 (2012) 1031–1047.

[**Ad2**] M. Adamaszek, *Hard squares on cylinders revisited*, arXiv:1202.1655.

[**Ba**] J. A. Barmak, *Star clusters in independence complexes of graphs*, Adv. Math. 241 (2013) 33–57.

[**Br**] B. Braun, *Independence complexes of stable Kneser graphs*, Electron. J. Combin. 18(1) (2011).

[**BH**] B. Braun, W. K. Hough. *Matching and independence complexes related to small grids*, Electron. J. Combin. 24(4) (2017).

[**BLN**] M. Bousquet-Mélou, S. Linusson, E. Nevo, *On the independence complex of square grids*, J. Algebraic Combin. 27 (2008) 423–450.

[**EH**] R. Ehrenborg, G. Hetyei, *The topology of independence complex*, European J. Combin. 27 (2006) 906–923.

[**En**] A. Engström, *Complexes of directed trees and independence complexes*, Discrete Math. 309 (2009) 3299–3309.

[**FSv**] P. Fendley, K. Schoutens, H. van Eerten, *Hard squares with negative activity*, J. Phys. A: Math. Gen. 38(2) (2005) 315–322.

[GSS] S. Goyal, S. Shukla, A. Singh, *Homotopy Type of Independence Complexes of Certain Families of Graphs*, Contrib. Discrete Math. 16(3) (2021) 74–92.

[Ir] K. Iriye, *On the homotopy types of the independence complexes of grid graphs with cylindrical identification*, Kyoto J. Math. 52(3) (2012) 479–501.

[Jo1] J. Jonsson, *Hard squares with negative activity and rhombus tilings of the plane*, Electron. J. Combin. 13(1) (2006), #R67.

[Jo2] J. Jonsson, *Hard squares with negative activity on cylinders with odd circumference*, Electron. J. Combin. 16(2) (2009), #R5.

[Jo3] J. Jonsson, *Certain homology cycles of the independence complex of grids*, Discrete Comput. Geom. 43(4) (2010) 927–950.

[Ka] K. Kawamura, *Independence complexes of chordal graphs*, Discrete Math. 310 (2010) 2204–2211.

[Ko1] D. N. Kozlov, *Complexes of directed trees*, J. Combin. Theory Ser. A 88 (1999) 112–122.

[Ko2] D. N. Kozlov, Combinatorial algebraic topology, Algorithms Comput. Math., Vol. 21, Springer, Berlin, 2008.

[MT] M. Marietti, D. Testa, *A uniform approach to complexes arising from forests*, Electron. J. Combin. 15 (2008).

[Ma1] T. Matsushita, *Matching complexes of small grids*, Electron. J. Combin. 26(3) (July, 2019)

[Ma2] T. Matsushita, *Matching complexes of polygonal line tilings*, Hokkaido Math. J. 51 (2022) 339–359.

[MW] T. Matsushita, S. Wakatsuki, *Independence complexes of $(n \times 4)$ and $(n \times 5)$-grid graphs*, Topology Appl., 334 (2023), Paper No. 108541, 18 pp.

[Ok] K. Okura, *Simple homotopy types of independence complexes of graphs involving grid graphs*, arXiv:1908.09356.

[PRSZ] L. Polterovich, D. Rosen, K. Samvelyan, J. Zhang, Topological Persistence in Geometry and Analysis, Univ. Lecture Ser., American Mathematical Society, 2020.

[Th] J. Thapper, *Independence complexes of cylinders constructed from square and hexagonal grid graphs*, arXiv:0812.1165.

Takahiro Matsushita `matsushita@shinshu-u.ac.jp`

Department of Mathematical Sciences, Faculty of Science, Shinshu University, Matsumoto, Nagano 390-8621, Japan

Shun Wakatsuki `shun.wakatsuki@math.nagoya-u.ac.jp`

Graduate School of Mathematics, Nagoya University, Furocho, Chikusaku, Nagoya, 464-8602, Japan