

ON FINITE DOMINATION AND POINCARÉ DUALITY

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Abstract

The object of this paper is to show that non-homotopy finite Poincaré duality spaces are plentiful. Let π be a finitely presented group. Assuming that the reduced Grothendieck group $\tilde{K}_0(\mathbb{Z}[\pi])$ has a non-trivial 2-divisible element, we construct a finitely dominated Poincaré space X with fundamental group π such that X is not homotopy finite. The dimension of X can be made arbitrarily large. Our proof relies on a result which says that every finitely dominated space possesses a stable Poincaré duality thickening.

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1. Introduction

For a group π , let $K_0(\mathbb{Z}[\pi])$ be the Grothendieck group of the category of finitely generated projective (left) $\mathbb{Z}[\pi]$ -modules. According to [6], a connected finitely dominated space X with fundamental group π determines an element

$$w(X) \in K_0(\mathbb{Z}[\pi]),$$

called the *Wall finiteness obstruction*. If we assume that π is finitely presented, then X has the homotopy type of a finite CW complex if and only if $\tilde{w}(X) = 0$, where $\tilde{w}(X) \in \tilde{K}_0(\mathbb{Z}[\pi])$ is the image $w(X)$ in the reduced Grothendieck group.

Recall that $K_0(\mathbb{Z}[\pi])$ comes equipped with an involution σ that is induced by mapping a finitely generated projective $\mathbb{Z}[\pi]$ -module P to its linear dual

$$P^* := \text{hom}_{\mathbb{Z}[\pi]}(P, \mathbb{Z}[\pi]),$$

where the latter is converted into a left module using the involution of the group ring induced by mapping a group element to its inverse.

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The main result of this paper is the following.

Theorem A. *Let π be a finitely presented group. Assume that $\tilde{K}_0(\mathbb{Z}[\pi])$ has an element κ such that $2\kappa \neq 0$. Then there is a connected, finitely dominated Poincaré duality space X such that*

- X is not homotopy finite,
- $\pi_1(X) = \pi$, and
- $\tilde{w}(X) = \kappa + (-1)^d \sigma(\kappa)$, where $d = \dim X$ may be chosen to be arbitrarily large.

Remarks 1.1.

- (1). Our method of proof shows that for any such κ , there is a positive integer D such that there is an X satisfying Theorem A for every even $d > D$ or every odd $d > D$, but not both.
- (2). Lizhen Qin pointed out to me that Theorem A is similar to Wall's [8, Thm. 1.5]. However, the approaches are quite different. Wall's method is to start with a finite complex K with fundamental group π . He then thickens K to a compact d -manifold with boundary $(N, \partial N)$. Lastly, Wall modifies ∂N by attaching certain cells while at the same time taking care to retain Poincaré duality. The resulting Poincaré space is designed so as to have finiteness obstruction $\kappa + (-1)^d \sigma(\kappa)$. On the other hand, our approach only requires Theorem B below, which asserts that any finitely dominated space L admits a *Poincaré thickening*, i.e., there is a Poincaré pair $(K, \partial K)$ with K having the homotopy type of L . This is proved with the help of the *dualizing spectrum* of [3].
- (3). Let $\pi = \mathbb{Z}/p\mathbb{Z}$ denote a cyclic group of odd prime order. The abelian group $\tilde{K}_0(\mathbb{Z}[\pi])$ has a nontrivial 2-divisible element whenever the relative class number $h_1(p)$ is not a power of 2 (cf. [4, pp. 30-31]). The smallest prime satisfying this condition is $p = 23$ (cf. [9] for many other examples). Moreover, $h_1(p)$ is not a power of 2 whenever p is an irregular prime, so there are infinitely many such p (recall that a prime p is irregular if and only if it divides the class number $h(p)$ of the cyclotomic field generated by $e^{2\pi i/p}$). Making use of the relative class number, Wall constructed a non-homotopy finite Poincaré duality space X of dimension 4 [6, Cor. 5.4.2], where $\pi_1(X) = \pi = \mathbb{Z}/p\mathbb{Z}$. Wall's approach is number theoretic.
- (4). A geometric framework to constructing non-homotopy finite Poincaré duality spaces is outlined in the work of Pedersen and Ranicki [5], who use Siebenmann's theory of tame ends. However, in the latter paper no examples are provided.

Remark 1.2. I am not aware of a general criterion for determining when $\tilde{K}_0(\mathbb{Z}[\pi])$ has non-trivial 2-divisible elements. However, if π has a factor H such that $\tilde{K}_0(\mathbb{Z}[H])$ has non-trivial 2-divisible elements, then so will $\tilde{K}_0(\mathbb{Z}[\pi])$.

The proof of Theorem A will rely on another result which may be of independent interest:

Theorem B (Stable Poincaré Thickening). *Given a finitely dominated space L , there is a Poincaré pair*

$$(K, \partial K)$$

such that

- ∂K connected and finitely dominated,
- K homotopy equivalent to L , and
- $\pi_1(\partial K) \rightarrow \pi_1(K)$ is an isomorphism.

Remark 1.3. The proof of Theorem B is relatively easy when L is a finite complex, since one may appeal to induction on the number of cells to embed L up to homotopy in a high dimensional Euclidean space. This results in a smooth compact manifold thickening M of L as in [7]. Then $(M, \partial M)$ will fulfill the assertion of Theorem B.

However, in the finitely dominated case, the inductive approach is no longer available as the number of cells of L may be infinite. Instead, our approach in this case will be homotopy theoretic: we will follow the proof of ‘3 \Rightarrow 1’ of [3, Thm. A].

Remark 1.4. Finitely dominated Poincaré duality spaces fall under the rubric of projective surgery theory [5]. If $(f, b): M \rightarrow X$ is a normal map from a compact d -manifold M to a finitely dominated Poincaré duality space X of dimension d , then it has a projective surgery obstruction $\sigma^p(f, b) \in L_d^p(\pi_1(X))$. If $d \geq 4$, then $(f, b) \times 1: M \times S^1 \rightarrow X \times S^1$ will be normally cobordant to a homotopy equivalence if and only if $\sigma^p(f, b) = 0$.

Outline. §2 is mostly language. In §3 we show how Theorem B implies Theorem A. In §4 we prove Theorem B.

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2. Conventions

2.1. Spaces

Let Top be the Quillen model category of compactly generated weak Hausdorff spaces [1]. The weak equivalences of Top are the weak homotopy equivalences, and the fibrations are the Serre fibrations. The cofibrations are defined using the right lifting property with respect to the trivial fibrations. In particular, every object in Top is fibrant. An object is cofibrant whenever it is a retract of a cell complex. We let Top_* denote the category of based spaces. Then Top_* inherits a Quillen model structure from Top by means of the forgetful functor $\text{Top}_* \rightarrow \text{Top}$.

An object in Top or Top_* is *finite* if it is a finite cell complex. It is *homotopy finite* if it weakly equivalent to a finite object. An object X is *finitely dominated* if it is a retract of a homotopy finite object.

If X is an unbased space, we write X_+ for the based space $X \amalg *$ given by taking the disjoint union with a basepoint.

2.2. Poincaré duality spaces

Recall that an object $X \in \text{Top}$ is a *Poincaré duality space* of (formal) dimension d if there exists a pair

$$(\mathcal{L}, [X])$$

in which \mathcal{L} is a rank one local coefficient system and

$$[X] \in H_d(X; \mathcal{L})$$

is a *fundamental class* such that for all local coefficient systems \mathcal{B} , the cap product homomorphism

$$\cap[X]: H^*(X; \mathcal{B}) \rightarrow H_{d-*}(X; \mathcal{L} \otimes \mathcal{B})$$

is an isomorphism in all degrees (cf. [8], [2]). The Poincaré spaces considered in this paper cofibrant. If the pair $(\mathcal{L}, [X])$ exists it is determined up to unique isomorphism. Also note that closed manifolds are homotopy finite Poincaré duality spaces.

Remark 2.1. If X is a connected Poincaré space, then X is finitely dominated if and only if $\pi_1(X)$ is finitely presented (cf. [3, Thm. D]).

3. The proof Theorem A

In this section we show how Theorem B implies Theorem A. The proof of Theorem B will appear in the next section.

As mentioned in the introduction, the group ring $\mathbb{Z}[\pi]$ comes equipped with a canonical involution which on group elements is defined by $g \mapsto g^{-1}$. One may use this involution to convert right modules to left modules and *vice versa*. The Grothendieck group $K_0(\mathbb{Z}[\pi])$ is, in turn, equipped with an involution that is induced by the operation $P \mapsto P^*$, in which P is a finitely generated projective left $\mathbb{Z}[\pi]$ -module and $P^* = \text{hom}_{\mathbb{Z}[\pi]}(P, \mathbb{Z}[\pi])$ is its linear dual. The latter is a finitely generated projective right $\mathbb{Z}[\pi]$ -module which we identify as a finitely generated projective right $\mathbb{Z}[\pi]$ -module. Denote the involution on $K_0(\mathbb{Z}[\pi])$ by σ . Then σ restricts to an involution on the reduced Grothendieck group $\tilde{K}_0(\mathbb{Z}[\pi])$.

Let $(K, \partial K)$ be a finitely dominated Poincaré pair of dimension d . Assume that both K and ∂K are connected and $\pi_1(\partial K) \rightarrow \pi_1(K) = \pi$ is an isomorphism. Then we have

Lemma 3.1 (Wall [6, Thm. 1.4]). *Let $\kappa = \tilde{w}(K)$. Then*

$$\tilde{w}(\partial K) = \kappa + (-1)^{d-1} \sigma(\kappa).$$

With $(K, \partial K)$ as above, the double

$$X := K \cup_{\partial K} K$$

is a finitely dominated Poincaré duality space of dimension d . Assume that ∂K is connected and homotopy finite. Then by additivity of the finiteness obstruction, we infer that

$$\tilde{w}(X) = \tilde{w}(K) + \tilde{w}(K) - \tilde{w}(\partial K) = \kappa + (-1)^d \sigma(\kappa).$$

Corollary 3.2. *If $2\kappa \neq 0$, then either ∂K is not homotopy finite or X is not homotopy finite.*

Proof. If ∂K is homotopy finite, then $\kappa + (-1)^{d-1}\sigma(\kappa) = 0$. If X is homotopy finite, then $\kappa + (-1)^d\sigma(\kappa) = 0$. Consequently, if both ∂K and X are homotopy finite, we infer that $\sigma(\kappa) = -\sigma(\kappa)$ and this implies that $\kappa = -\kappa$ or $2\kappa = 0$, a contradiction. \square

Recall that for a map of spaces $Y \rightarrow B$, the associated j -fold unreduced *fiberwise suspension* is

$$S_B^j Y := Y \times D^j \cup_{Y \times S^{j-1}} B \times S^{j-1}.$$

Applying this construction to $\partial K \rightarrow K$, we obtain a finitely dominated Poincaré duality space

$$X_j := S_K^j \partial K$$

having formal dimension $d_{j-1} := d + j - 1$. Note that $X_0 = \partial K$ and X_1 is weakly equivalent to the double of $(K, \partial K)$.

Corollary 3.3. *Assume $2\kappa \neq 0$. Then for every single $j \geq 0$, at least one of the Poincaré duality spaces X_{2j}, X_{2j+1} is not homotopy finite. Furthermore, $\tilde{w}(X_j) = \kappa + (-1)^{d_{j-1}}\sigma(\kappa)$.*

Proof of Theorem A. Let $\kappa \in \tilde{K}_0(\mathbb{Z}[\pi])$ be such that $2\kappa \neq 0$. Then by [6, Thm. F] there is a finitely dominated space L with fundamental group π such that $\tilde{w}(L) = \kappa$. By Theorem B, there is a finitely dominated Poincaré pair $(K, \partial K)$ and a homotopy equivalence $K \simeq L$. The proof is completed by applying Corollary 3.3. \square

4. The proof of Theorem B

As mentioned in the introduction, we will follow the proof of ‘3 \Rightarrow 1’ of [3, Thm. A]. However, some details in that proof were omitted in the finitely dominated case. We provide those details here.

We first review the various notions of finiteness in the equivariant setting. Let G be a topological group object of Top whose underlying space is cofibrant. Let $\text{Top}_*(G)$ denote the category of based G -spaces. An object $Z \in \text{Top}_*(G)$ is *finite* if it is built up from a point by finitely many free G -cell attachments, where by a free G -cell, we mean $D^k \times G$. Similarly, Z is *homotopy finite* if it is weakly equivalent to a G -finite object. Lastly, Z is *finitely dominated* if it is a retract of a homotopy finite object. One also has the corresponding notions of finiteness in the category of unbased G -spaces as well as in the category of (naive) G -spectra.

Proof of Theorem B. We may assume without loss in generality that $L = BG$ for a suitable cofibrant topological group G . The idea is to construct a finitely dominated based G -space Z and a G -equivariant weak equivalence

$$\Sigma^\infty Z \simeq \Sigma^n \mathcal{D}_G,$$

where $n \geq 0$ is some integer and $\mathcal{D}_G = S[G]^{hG}$ is the *dualizing spectrum* of G (i.e., the homotopy fixed points of G acting on $S[G]$, where the latter denotes the suspension spectrum of G_+).

Assuming this to be the case, then as in the proof of ‘3 \Rightarrow 1’ of [3, Thm. A] the desired Poincaré pair $(K, \partial K)$ is given by the pair of unreduced Borel constructions

$$(EG \times_G CZ, EG \times_G Z),$$

where CZ is the cone on Z .

We now proceed with the construction of Z . Choose a factorization of the identity map

$$L \rightarrow X \rightarrow L$$

in which the unbased space X is homotopy finite. Let $\tilde{L} = EG$ and $\tilde{X} = X \times^L EG$ be the fiber product of X and EG along L . Then one has a factorization of G -spaces

$$\tilde{L} \rightarrow \tilde{X} \rightarrow \tilde{L}$$

in which \tilde{X} is a homotopy finite unbased G -space.

Then the equivariant dual

$$\mathrm{hom}_G(\tilde{X}_+, S[G])$$

is a homotopy finite G -spectrum. This means that there is an $n > 0$, a homotopy finite based G -space Y and an equivariant weak equivalence

$$\Sigma^\infty Y \simeq \Sigma^n \mathrm{hom}_G(\tilde{X}_+, S[G]).$$

Moreover, the dualizing spectrum $\mathcal{D}_G = \mathrm{hom}_G(\tilde{L}_+, S[G])$ is an equivariant retract of $\mathrm{hom}_G(\tilde{X}_+, S[G])$. It follows that $\Sigma^n \mathcal{D}_G$ is an equivariant homotopy retract of the homotopy finite G -spectrum $\Sigma^\infty Y$. Consequently, there is a factorization

$$\Sigma^n \mathcal{D}_G \xrightarrow{s'} \Sigma^\infty Y \xrightarrow{r'} \Sigma^n \mathcal{D}_G$$

in which $r' \circ s'$ is equivariantly homotopic to the identity. Consider the self-map $s' \circ r': \Sigma^\infty Y \rightarrow \Sigma^\infty Y$. As Y is homotopy finite, there is an integer $k > 0$ such that $s' \circ r'$ equivariantly desuspends to a G -map $\Sigma^k Y \rightarrow \Sigma^k Y$. Let Z denote the mapping telescope of

$$\Sigma^k Y \xrightarrow{s' \circ r'} \Sigma^k Y \xrightarrow{s' \circ r'} \Sigma^k Y \xrightarrow{s' \circ r'} \dots$$

Then the based G -space Z is an equivariant retract of $\Sigma^k Y$ and $\Sigma^\infty Z \simeq \Sigma^n \mathcal{D}_G$, as was to be proved. \square

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