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UNSTABLE ALGEBRAS OVER AN OPERAD II

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Abstract

We work over the finite field $\mathbb{F}_q$. We introduce a notion of unstable $\mathcal{P}$-algebra over an operad $\mathcal{P}$. We show that the unstable $\mathcal{P}$-algebra freely generated by an unstable module is itself a free $\mathcal{P}$-algebra under suitable conditions. We introduce a family of 'q-level' operads which allows us to identify unstable modules studied by Brown–Gitler, Miller and Carlsson in terms of free unstable q-level algebras.

1. Introduction

In this article, we define unstable $\mathcal{P}$-algebras over an operad $\mathcal{P}$ in positive characteristic, and we characterise free unstable $\mathcal{P}$-algebras. The mod p Steenrod algebra $\mathcal{A}$ was introduced by Steenrod and Epstein [**22**] to study the stable operations of the usual cohomology functors with coefficients in the finite field $\mathbb{F}_p$ of order p. Unstable modules are a class of (graded) $\mathcal{A}$-modules which satisfy a property called instability. A detailed survey of unstable modules and their properties can be found in Schwartz's book [**21**]. The principal source of examples of unstable modules is precisely the mod p cohomology of topological spaces. These unstable modules coming from topology are endowed with an additional internal multiplication - the *cup-product* - which is associative and commutative. The category $\mathcal{U}$ of unstable modules is understood fairly well. It has injective cogenerators, the Brown–Gitler modules, which can be obtained as the 'Spanier–Whitehead dual' to the cohomology of a spectrum - the Brown–Gitler spectrum [**1, 19**]. Other injective objects of interest include the Carlsson modules [**3, 16**], which are obtained as a limit of Brown–Gitler modules. The Carlsson modules were introduced in [**3**] to prove the Segal conjecture for Burnside rings of elementary abelian groups, and were used by Miller [**19**] to prove the Sullivan conjecture on maps from classifying spaces. Both the Brown–Gitler and the Carlsson modules are endowed with a multiplication which naturally arises when studying their structure. It has been shown that certain of these modules, with their respective multiplication, are in fact free objects in a specified category of (non-associative) algebras [**5, 11**].

Here, we use the notion of operads to study algebraic structures in the category of unstable modules. As an example, the cup product equips the mod p cohomology of any topological space with the structure of a uCom-algebra in unstable modules,

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where uCom is the operad of unital associative and commutative algebras. This resulting mod p cohomology ring A satisfies an additional relation, called instability, and which reads: $\forall x \in A, P_0 x = x^p$, where P_0 is an operation obtained from the action of the mod p Steenrod algebra. The multiplications of the Brown–Gitler and Carlsson modules do not satisfy this instability relation as uCom-algebras in unstable modules. However, they can be equipped with the structure of algebras in unstable modules over a certain operad $\mathcal{P}$, and as such, they satisfy a similar relation $P_0 x = \star(x, \dots, x)$ for a certain p-ary operation $\star \in \mathcal{P}(p)$ (see [**11**] for the case $p = 2$).

In this article, we work in unstable modules over the Steenrod algebra of reduced powers over a field of order q. We define the notion of an unstable $\mathcal{P}$-algebra for a q-ary operation $\star \in \mathcal{P}(q)$. These are $\mathcal{P}$-algebras A in unstable modules such that for each $x \in A, P_0 x = \star(x, \dots, x)$. In the case where q is a prime, and $\star$ is the q-ary multiplication in uCom, we recover the usual definition of an unstable algebra. It is important to note that, in the instability relation, the action of P_0 is compatible with the $\mathcal{P}$-algebra structure. Therefore, for the instability relation to make sense, we need the operation $x \mapsto \star(x, \dots, x)$ to induce a $\mathcal{P}$-algebra endomorphism. To ensure this is the case, we introduce the notion of $\mathcal{P}$-centrality for operations $\star \in \mathcal{P}(q)$.

The main result of this article is a characterisation for certain free unstable $\mathcal{P}$-algebras. We build a functor $K^{\star}_{\mathcal{P}}$, which sends an unstable module M to the free unstable $\mathcal{P}$-algebra generated by M, and we obtain the following:

Theorem (Theorem 11.6)**.** *Let $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$ be a $\mathcal{P}$-central operation. For all connected reduced unstable module M, the underlying $\mathcal{P}$-algebra of $K^{\star}_{\mathcal{P}}(M)$ is the free $\mathcal{P}$-algebra over the underlying vector space of $\Sigma\Omega M$.*

Here, the conditions of being reduced and connected for an unstable module are defined in 9.2. The functor Σ is the one defined in 9.2, and Ω is a left adjoint for Σ. We warn the reader that, since we have regraded our modules using Kuhn's convention, following [**13**], the functor Σ differs from the usual suspension functor in the case where the base field is a prime field of odd characteristic.

We then apply this result to identify some algebraic structures on analogues of the Brown–Gitler modules, which are injective cogenerators of the category of unstable modules, and on analogues of the Brown–Gitler algebra, the Carlsson modules and the Carlsson algebra. We consider a q-ary operation on the Brown–Gitler modules and on the Carlsson modules, which corresponds to the multiplication studied by Carlsson [**3**] in the case where the base field is $\mathbb{F}_2$, and by Miller [**19**] in the case where the base field is $\mathbb{F}_p$ for p an odd prime. This multiplication of arity q is fixed by the action of $\mathfrak{S}_q$, and satisfies an additional relation of strong commutativity, but it is not associative. In order to study this operation, we introduce an operad Lev_n whose algebras are called n-level algebras, and which generalises the operad Lev of level algebras of Chataur and Livernet [**4**].

We first define the set operad $\mathfrak{Lev}_n$ as a suboperad of a certain operad of ordered partitions. We then obtain a presentation of this operad which yields the following characterisation:

Theorem (Theorem 5.3)**.** *The operad $\mathfrak{Lev}_n$ is generated by one element $\star \in \mathrm{Lev}_n(n)$, fixed under the action of $\mathfrak{S}_n$, and satisfying the following relation:*

$$(\star \circ_2 \star) \circ_1 \star = (\star \circ_2 \star) \circ_1 \star \cdot \sigma,$$

where $\sigma \in \mathfrak{S}_{3n-2}$ is the transposition $(n\ n+1)$, and $\circ_i$ is the i-th partial operadic composition (see 2.2).

Identifying the structure of free unstable n-level algebras for $n = q$ yields, for instance, the following result, which is analogous to a result of Davis [**5**] in the case where the base field is $\mathbb{F}_2$, and gives the explicit algebraic structure for the Carlsson module of weight 1:

Theorem (Theorem 12.4)**.** *The Carlsson module of weight one $K(1)$ over the field $\mathbb{F}_q$, with its Lev_q operation, is isomorphic to the free $\star$-unstable Lev_q-algebra generated by $F(1)$, where Lev_q denotes the linearisation of $\mathfrak{Lev}$.*

The results obtained in this article are based on the author's previous work [**11**], which they generalise. In contrast with this previous work, the approach we chose here is somewhat more straightforward, and relies on the algebraic and categorical framework which we study in more detail.

The notions introduced in this article give an opportunity to revisit classical results on unstable modules and algebras from the point of view of operads. For instance, we have obtained results concerning free $\mathcal{P}$-algebras generated by unstable modules equipped with an unstable 'twisted' action which generalise results of Campbell–Selick [**2**] and Duflot–Kuhn–Winstead [**6**]. These results are available in former versions of the present article, and we hope to publish them in future works. Other possible applications of these result stay open, including the link between unstable $\mathcal{P}$-algebras and the analytic functors viewpoint of Henn–Lannes–Schwartz [**10**], or with the generic representations viewpoint of [**13, 14**].

Notation.

- *Our base field will be denoted by $\mathbb{F}$. Unless clearly defined otherwise, it has characteristic p and order $q = p^\alpha$.*
- *The category of $\mathbb{F}$-vector space will be denoted by $\mathsf{Vect}_{\mathbb{F}}$.*
- *The set $\{1, \ldots, n\}$ will be denoted by $[n]$.*
- *The symmetric group on n letters (or permutation group of $[n]$) will be denoted by $\mathfrak{S}_n$.*

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Part 1. Algebraic background

In this first part of the article, we review the algebraic structures necessary to study unstable algebras over an operad. We introduce the notion of $\mathcal{P}$-central operation for an operad $\mathcal{P}$, and we define a family of operads Lev_n, for which we obtain a presentation in Theorem 5.3.

2. Recollections about operads

In this section we present the notions of symmetric sequences, operads, and their algebras in a category $\mathcal{C}$. We assume that the reader has some basic familiarity with these notions. Our main references on the subject are the two books [**17, 7**], as well as the article [**9**]. Later in this article, we will be interested in operads and algebras in the category of sets, in the category $\mathsf{Vect}_{\mathbb{F}}$ of $\mathbb{F}$-vector spaces, in the category of left modules over a (cocommutative) $\mathbb{F}$-bialgebra, and in the category $\mathcal{U}(q)$ of unstable modules.

For this section, fix a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ with all small colimits and all small limits. Assume that $\otimes$ preserve colimits. In particular, if G is a group, we can define the notion of a G-object in $\mathcal{C}$ as objects of $\mathcal{C}$ with an action of G. For a G-object X, we can define the object of G-orbits X_G, and if $H < G$ is a subgroup, we have a restriction functor Res_H^G from G-objects to H-objects, and this functor has a left adjoint Ind_H^G called induction, which can be obtained as a Kan extension. For more details on this, we refer the reader to [**7**, Chapter 2]. For clarity, we will assume that it is suitable to talk about elements for the objects of the category $\mathcal{C}$.

Definition 2.1 (Symmetric sequences [**7**, Section 2.1])**.** A **symmetric sequence** $\mathcal{M}$ is a sequence $(\mathcal{M}(n))_{n\in\mathbb{N}}$ of objects in $\mathcal{C}$ such that, for all $n \in \mathbb{N}$, the symmetric group $\mathfrak{S}_n$ acts on $\mathcal{M}(n)$ on the right. $\mathcal{M}(n)$ is said to be the object of **arity** n. Symmetric sequences form a category Sym. This category is endowed with a tensor product $\otimes_{\mathsf{Sym}}$ such that, if $\mathcal{M}$ and $\mathcal{N}$ are two symmetric sequences, then

$$\left(\mathcal{M} \otimes_{\mathsf{Sym}} \mathcal{N}\right)(n) = \coprod_{i+j=n} \mathrm{Ind}_{\mathfrak{S}_i\times\mathfrak{S}_j}^{\mathfrak{S}_n} \mathcal{M}(i) \otimes \mathcal{N}(j),$$

where $\mathrm{Ind}_{\mathfrak{S}_i\times\mathfrak{S}_j}^{\mathfrak{S}_n}$ will denote the induced representation from the Young subgroup $\mathfrak{S}_i \times \mathfrak{S}_j$ of the group $\mathfrak{S}_n$.

The category of symmetric sequences is endowed with another monoidal product $\circ$, called the **composition** of symmetric sequences, given by:

$$(\mathcal{M} \circ \mathcal{N})(n) = \coprod_{k\geqslant 0} \left(\mathcal{M}(k) \otimes (\mathcal{N}^{\otimes_{\mathsf{Sym}} k}(n))\right)_{\mathfrak{S}_k},$$

with unit the object I of $\mathcal{C}$ concentrated in arity 1. By a slight abuse of notation, I will both denote the unit of $\otimes$ and the unit of $\otimes_{\mathsf{Sym}}$. If $\mu \in \mathcal{M}(k)$, and for all $i \in [k]$ $\nu_i \in \mathcal{N}(n_i)$, let $(\mu; \nu_1, \ldots, \nu_k)$ denote the element

$$[\mu \otimes \nu_1 \otimes \cdots \otimes \nu_k]_{\mathfrak{S}_k} \in \mathcal{M} \circ \mathcal{N}(n_1 + \cdots + n_k).$$

Definition 2.2 (Operads [**7**, Section 3.1])**.** An **operad** is a monoid object in the monoidal category of symmetric sequences $(\mathsf{Sym}, \circ, I)$. For an operad $\mathcal{P}$, we denote its composition morphism $\gamma_{\mathcal{P}}\colon \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$, and $\mathbf{1}_{\mathcal{P}}\colon I \to \mathcal{P}$ its unit.

For $\mu \in \mathcal{P}(k)$, $\nu_1, \dots, \nu_k \in \mathcal{P}$, let $\mu(\nu_1, \dots, \nu_k)$ denote the element

$$\gamma_{\mathcal{P}}(\mu; \nu_1, \dots, \nu_k) \in \mathcal{P}.$$

The **partial compositions** in an operad $\mathcal{P}$ are maps

$$- \circ_i - : \mathcal{P}(k) \otimes \mathcal{P}(n) \to \mathcal{P}(k+n-1)$$

defined by:

$$\mathcal{P}(k) \otimes \mathcal{P}(n) \cong \mathcal{P}(k) \otimes \left(I \otimes \cdots \otimes \underbrace{\mathcal{P}(n)}_{i\text{-th input}} \otimes \cdots \otimes I \right)$$
$$\to \mathcal{P}(k) \otimes \big(\mathcal{P}(1) \otimes \cdots \otimes \mathcal{P}(n) \otimes \cdots \otimes \mathcal{P}(1)\big) \to \mathcal{P} \circ \mathcal{P}(n+k-1) \to \mathcal{P}(n+k-1),$$

for all $n \in \mathbb{N}, i \in \{1, \dots, k\}$, where the first arrow uses the unit $\mathbf{1}_{\mathcal{P}}$ of the operad and the third arrow uses the compositions $\gamma_{\mathcal{P}}$.

For any operad $\mathcal{P}$, let $S(\mathcal{P}, -)$ denote the associated functor $\mathcal{C} \to \mathcal{C}$, defined on objects by:

$$S(\mathcal{P}, X) = \coprod_{n \geqslant 0} \mathcal{P}(n) \otimes_{\mathfrak{S}_n} X^{\otimes n}.$$

It can also be defined as $S(\mathcal{P}, X) = \mathcal{P} \circ X$ where X is considered as a symmetric sequence concentrated in arity 0.

The unit and compositions in $\mathcal{P}$ induce a monad structure on $S(\mathcal{P}, -)$.

Definition 2.3 (Algebras over an operad [7, Section 3.2])**.** For an operad $\mathcal{P}$, a **$\mathcal{P}$-algebra** is an algebra over the monad $S(\mathcal{P}, -)$. In other words, a $\mathcal{P}$-algebra is a pair (A, θ) where A is an object of $\mathcal{C}$ and $\theta \colon S(\mathcal{P}, A) \to A$ is compatible with the composition and unit of $\mathcal{P}$.

For (A, θ) a $\mathcal{P}$-algebra, $\mu \in \mathcal{P}(n)$ and $a_1, \dots, a_n \in A$, let $\mu(a_1, \dots, a_n)$ denote the element $\theta(\mu; a_1, \dots, a_n) \in A$. In the case where $a = a_1 = \cdots = a_n$, we use the notation $a^{\mu n}$ for the elements $(\mu; a, \dots, a) \in S(\mathcal{P}, A)$ and $\theta(\mu; a, \dots, a) \in A$ depending on the context.

Let $\mathcal{P}_{\mathsf{alg}}$ denote the category of $\mathcal{P}$-algebras.

Example 2.4. Here are basic example in the category Set of sets, with tensor product given by the Cartesian product:

- There is an operad $\mathfrak{uCom}$ such that, for all n, $\mathfrak{uCom}(n)$ is a singleton $\{*_n\}$ equipped with a trivial action of $\mathfrak{S}_n$, with unit $*_1 \in \mathfrak{uCom}(1)$ and with composition given by identities of singletons. $\mathfrak{uCom}$-algebras are exactly unital, commutative, associative monoids.
- If $\mathfrak{A}$ is a unital, associative monoid, then $\mathfrak{A}$ can be considered as an operad concentrated in arity 1. $\mathfrak{A}$-algebras correspond to sets with a (left) $\mathfrak{A}$-action.

Extending the examples of operads in set by linearity induce examples of operads in $\mathbb{F}$-vector spaces. Here are some basic examples of operads in $\mathbb{F}$-vector spaces:

- Denote by $\mathrm{uCom} = \mathbb{F}\mathfrak{uCom}$, the linearisation of $\mathfrak{uCom}$. One has $\mathrm{uCom}(n) = \mathbb{F}$ for all $n \in \mathbb{N}$, the unit of this operad is $1 \in \mathbb{F} = \mathrm{uCom}(1)$, compositions are given by identities of $\mathbb{F}$, and uCom-algebras are unital, associative, commutative $\mathbb{F}$-algebras. Let $X_n \in \mathrm{uCom}(n)$ denote the generator of the vector space $\mathrm{uCom}(n)$.

- If A is a unital, associative $\mathbb{F}$-algebra, then A can be considered as an operad concentrated in arity 1. A-algebras correspond to the usual notion of A-(left)-modules.

We now recall the notion of a distributive law for composition products of operads [**17**, Section 8.6.1].

Definition 2.5 (Distributive Law)**.** Let $\mathcal{P}$, $\mathcal{Q}$ be two operads. A **distributive law** is a morphism $\lambda\colon \mathcal{Q}\circ\mathcal{P}\to\mathcal{P}\circ\mathcal{Q}$ in Sym such that the symmetric sequence $\mathcal{P}\circ\mathcal{Q}$, with unit $(\mathbf{1}_{\mathcal{P}};\mathbf{1}_{\mathcal{Q}})$, and with composition map given by:

$$\mathcal{P}\circ\mathcal{Q}\circ\mathcal{P}\circ\mathcal{Q}\xrightarrow{\mathcal{P}\circ\lambda\circ\mathcal{Q}}\mathcal{P}\circ\mathcal{P}\circ\mathcal{Q}\circ\mathcal{Q}\xrightarrow{\gamma_P\circ\gamma_Q}\mathcal{P}\circ\mathcal{Q}\ ,$$

is an operad.

Examples of distributive laws on operads are given in 3.5 and 4.6.

3. Modules over a bialgebra

In this section, we study the category of (left) modules over an $\mathbb{F}$-bialgebra. This category is classically equipped with a tensor product which is symmetric as soon as the bialgebra is cocommutative. This will allow us to study operads in this category, and their algebras. Our main reference for the notion of bialgebras and their modules is the book [**12**]. Note that here, our bialgebras are not required to come equipped with a structure of Hopf algebra: they are not necessarily equipped with an antipode.

Definition 3.1 (Bialgebra [**12**, Section III.2])**.** An $\mathbb{F}$-**bialgebra** (or simply, bialgebra) is a quintuple $(B,\mu,\eta,\Delta,\epsilon)$ where (B,μ,η) is a unital associative $\mathbb{F}$-algebra, (B,Δ,ϵ) is a counital, coassociative $\mathbb{F}$-coalgebra, such that ϵ and Δ satisfy compatibility conditions that make them into algebra morphisms.

The category of (left) modules over a bialgebra B will be denoted by B_{mod}.

Definition 3.2 (Tensor product of B-modules [**12**, III.5])**.** Let M and N be two B-modules. We define a B-module structure on $M\otimes N$ by:

$$\begin{array}{ccc} B\otimes M\otimes N & \xrightarrow{\Delta\otimes M\otimes N} & B\otimes B\otimes M\otimes N \\ & & \downarrow{\scriptstyle B\otimes\tau\otimes N} \\ & & B\otimes M\otimes B\otimes N \longrightarrow M\otimes N, \end{array}$$

where $\tau\colon B\otimes M\to M\otimes B$ is the symmetry isomorphism of the tensor product in $\mathsf{Vect}_{\mathbb{F}}$ and the last map is given by the B-module structures on M and N.

This provides a tensor product that we still denote by $\otimes$ on B-modules.

Lemma 3.3 ([**12**, Proposition 111.5.1])**.** *If B is cocommutative, then $(B_{\mathsf{mod}},\otimes,\mathbb{F})$ is a symmetric monoidal category.*

Definition 3.4 ($\mathcal{P}$-algebras in B-modules)**.** Let B be a cocommutative bialgebra, and $\mathcal{P}$ be an operad in $\mathbb{F}$-vector spaces. Consider $\mathcal{P}$ as an operad in the symmetric monoidal category B_{mod} using the trivial action $\varepsilon \otimes \mathcal{P} \colon B \otimes \mathcal{P} \to \mathcal{P}$. A **$\mathcal{P}$-algebra in B-modules** is an algebra over the resulting operad in B_{mod}

$\mathcal{P}$-algebras in B-modules form a category denoted by $\mathcal{P}^B_{\mathsf{alg}}$.

Remark 3.5. $\mathcal{P}$-algebras in B-modules can also be defined as $\mathcal{P} \circ B$ algebras with a certain distributive law $\lambda \colon B \circ \mathcal{P} \to \mathcal{P} \circ B$.

Denote by $\Delta^{n-1} = (B^{\otimes n-2} \otimes \Delta) \circ \cdots \circ (B \otimes \Delta) \circ \Delta \colon B \to B^{\otimes n}$ the $n-1$-th iterated coproduct. In arity n, the map $\lambda \colon (B \circ \mathcal{P})(n) \to (\mathcal{P} \circ B)(n)$ is the composite:

$$\begin{array}{ccc} (B \circ \mathcal{P})(n) = B \otimes \mathcal{P}(n) & \longrightarrow & \mathcal{P}(n) \otimes B \\ & & \big\downarrow {\scriptstyle \mathcal{P} \otimes \Delta^{n-1}} \\ & & \mathcal{P}(n) \otimes B^{\otimes n} \longrightarrow \mathcal{P}(n) \otimes_{\mathfrak{S}_n} B^{\otimes n} = (\mathcal{P} \circ B)(n), \end{array}$$

where the first arrow is the symmetry of the tensor product, and the third map is the projection on orbits by the action of $\mathfrak{S}_n$.

We now present a basic example of bialgebra and identify the associated modules and $\mathcal{P}$-algebras. This example and all the variations introduced here are fundamental for the rest of this article. In particular, the bialgebras $T_s\,\mathrm{D}$, $Q_s\,\mathrm{D}$ and $\mathrm{D}^{\pm}$ will appear in Section 12 to compare certain unstable algebras to known unstable modules. Here, s designates any positive integer. A more involved example of bialgebra is given by the Steenrod algebra. This example is discussed in Section 9.

Example 3.6. Let $\mathrm{D} = \mathbb{F}[d]$ denote the polynomial algebra in one indeterminate d. Following [**12**, III.2.Example 2], we can equip D with a bialgebra structure such that d is grouplike (that is, $\Delta(d) = d \otimes d$). The D-modules are vector spaces V equipped with a linear map $d \colon V \to V$. The $\mathcal{P}$-algebras in D-modules are the $\mathcal{P}$-algebras A equipped with a $\mathcal{P}$-algebra morphism $d \colon A \to A$.

We can construct many variations of the previous example. For instance:

- Let $T_s D$ denote the quotient of D by the ideal generated by d^{s+1}. The bialgebra structure of D induces a bialgebra structure on $T_s\,\mathrm{D}$. $T_s\,\mathrm{D}$-modules (resp. $\mathcal{P}$-algebras in T_sD-modules) are D-modules (resp. $\mathcal{P}$-algebras in D-modules) such that d is nilpotent of degree $s+1$.
- Let $Q_s D$ denote the quotient of D by the ideal generated by $d^s - 1$. The bialgebra structure of D induces a bialgebra structure on $Q_s\,\mathrm{D}$. $Q_s\,\mathrm{D}$-modules (resp. $\mathcal{P}$-algebras in Q_sD-modules) are D-modules (resp. $\mathcal{P}$-algebras in D-modules) such that d is cyclic of degree $s+1$.
- Let $\mathrm{D}^{\pm} = \mathbb{F}[d, d^{-1}]$, the algebra of Laurent polynomials in one indeterminate. The bialgebra structure of D extends into a bialgebra structure on $\mathrm{D}^{\pm}$. $\mathrm{D}^{\pm}$-modules (resp. $\mathcal{P}$-algebras in $\mathrm{D}^{\pm}$-modules) are D-modules (resp. $\mathcal{P}$-algebras in D-modules) on which d acts bijectively.

4. Operad of partitions

In this section, we introduce and study a particular example of set operad, the operad of (ordered) partitions Π. We show that this operad is isomorphic to a composition product $\mathfrak{uCom} \circ \mathfrak{D}$ of two operads equipped with a distributive law, for which Π provides a useful combinatoric description. This operad Π will be crucial in order to define the operad of q-level algebras in Section 5.

In this section, the base category is the category Set of sets, equipped with the symmetric tensor product provided by the Cartesian product.

Definition 4.1. An **ordered partition** J of the set $[n] = \{1, \dots, n\}$ is a sequence $J = (J_i)_{i\in\mathbb{N}}$ of piecewise disjoint subsets of $[n]$ such that $\bigcup_{i\in\mathbb{N}} J_i = [n]$.

Since $[n]$ is finite, there is an integer s such that $J_s \neq \emptyset$ and $J_{s'} = \emptyset$ for all $s' > s$. We will often write $J = (J_0, \dots, J_s)$, omitting the empty sets that follow.

Let j, k, l be three positive natural number such that $l \leqslant j$. Denote by

$$\lambda^j_{l,k} \colon [j] \to [j-1+k]$$

the set map defined by

$$\lambda^j_{l,k}(m) = \begin{cases} m, & \text{if } m \leqslant l, \\ m-1+k & \text{if } m > l. \end{cases}$$

In other words, $\lambda^j_{l,k}$ "skips" all numbers between $l+1$ and $l-1+k$.

Definition 4.2 (The operad of partitions Π)**.** There is an operad Π in sets, called the operad of partitions, such that, for all $n \in \mathbb{N}$, $\Pi(n)$ is the set of ordered partitions $J = (J_i)_{0\leqslant i\leqslant s}$ of the set $[n]$. The unit of the operad is the partition $(\{1\}) \in \Pi(1)$ of $[1]$, and the partial compositions are induced by:

$$(J \circ_l K)_i = \begin{cases} \lambda^j_{l,k}(J_i), & \text{if } l \in J_{i'} \text{ with } i' > i, \\ \lambda^j_{l,k}(J_i) \setminus \{l\} \cup (K_0 + l - 1) & \text{if } l \in J_i, \\ \lambda^j_{l,k}(J_i) \cup (K_{i-i'} + l - 1), & \text{if } l \in J_{i'} \text{ with } i' < i, \end{cases}$$

where K is a partition of $[k]$, J is a partition of $[j]$, $l \in [n]$ and for a subset S of $\mathbb{N}$ and an integer $m \in \mathbb{N}$, let $S + m$ be the set $\{x + m \colon x \in S\}$.

Remark 4.3. One can alternatively define $\Pi(n)$ as the set of functions $f \colon [n] \to \mathbb{N}$. Then, The action of the symmetric group is given by $(\sigma f)(i) = f(\sigma^{-1} i)$. The unit $1_\Pi \colon [1] \to \mathbb{N} \in \Pi(1)$ satisfies $1_\Pi(1) = 0$, and the partial compositions are given by:

$$f \circ_l g(i) = \begin{cases} f(i), & \text{if } i < l, \\ f(l) + g(i-l+1), & \text{if } l \leqslant i \leqslant l+k-1, \\ f(i-k+1), & \text{if } l+k \leqslant i \leqslant l+k-1, \end{cases}$$

where k is the arity of g.

We give two example of composition of ordered partitions. The second example will be of key importance later in this section.

Example 4.4. Consider $J = (\{2\},\{1,3\}) \in \Pi(3)$ and $K = (\emptyset,\{1,2\}) \in \Pi(2)$. Let us compute $J \circ_1 K$. For this, notice that $\lambda^3_{1,2}\colon [3] \to [4]$ sends 1 to 1, 2 to 3 and 3 to 4. So, $\lambda^3_{1,2}(J_0) = \{3\}$, $\lambda^3_{1,2}(J_1) = \{1,4\}$, $K_0 = K_0 + 1 - 1 = \emptyset$, $\lambda^3_{1,2}(J_2) = \emptyset$, as well as $K_1 + l - 1 = K_1 = \{1,2\}$, so finally,

$$J \circ_1 K = (\{3\},\{4\},\{1,2\}).$$

Example 4.5. Consider $J = (\emptyset,[q]) \in \Pi(q)$ for a certain integer $q \geqslant 2$. We want to compute $(J \circ_2 J) \circ_1 J$. Note first that $\lambda^q_{2,q}$ sends 1 to 1, 2 to 2, and all $n \in \{3,\dots,q\}$ to $n+q-1$. So, $\lambda^q_{2,q}(J_0) = \emptyset$, $\lambda^q_{2,q}(J_1) = \{1,2,q+2,q+3,\dots,2q-1\}$, $J_0 + 2 - 1 = \emptyset$, and $J_1 + 2 - 1 = J_1 + 1 = \{2,3,\dots,q+1\}$. So,

$$J \circ_2 J = (\emptyset,\{1,q+2,q+3,\dots,2q-1\},\{2,3,\dots,q+1\}).$$

Now, note that $\lambda^{2q-1}_{1,q}\colon [2q-1] \to [3q-2]$ send 1 to 1 and all $n \in \{2,3,\dots,2q-1\}$ to $n+q-1$. So, $\lambda^{2q-1}_{1,q}((J \circ_2 J)_0) = \emptyset$,

$$\lambda^{2q-1}_{1,q}((J \circ_2 J)_1) = \{1,2q+1,2q+2,\dots,3q-2\},$$

$\lambda^{2q-1}_{1,q}((J \circ_2 J)_2) = \{q+1,q+2\dots,2q\}$. Since $J_0 + 1 - 1 = \emptyset$ and $J_1 + 1 - 1 = J_1 = \{1,\dots,q)$, we conclude:

$$(J \circ_2 J) \circ_1 J = (\emptyset,\{2q+1,2q+2,\dots,3q-2\},\{1,2,\dots,2q\}).$$

Alternatively, using the approach of Remark 4.3, this J corresponds to the function $f\colon [q] \to \mathbb{N}$ which is constant with $f(i) = 1$ for all $i \in [q]$. Then

$$\begin{aligned} f \circ_2 f\colon [2q-1] &\to \mathbb{N} \\ i &\mapsto \begin{cases} 1, & \text{if } i = 1 \text{ or } i > q+1, \\ 2, & \text{if } i \in \{2,3,\dots,q\} \end{cases} \end{aligned}$$

and

$$\begin{aligned} (f \circ_2 f) \circ_1 f\colon [3q-2] &\to \mathbb{N} \\ i &\mapsto \begin{cases} 1, & \text{if } i > 2q, \\ 2, & \text{if } i \leqslant 2q. \end{cases} \end{aligned}$$

The operad Π can be identified to a composition product of operads with distributive law (see Definition 2.5). Recall from Example 2.4 that there is an operad $\mathfrak{uCom}$ in sets whose algebras are unital, associative, commutative monoids. Denote by $\mathfrak{D}$ the unital, associative monoid whose underlying set is $\{d^i\}_{i\in\mathbb{N}}$ with multiplication $d^i \cdot d^j = d^{i+j}$. According to Example 2.4, $\mathfrak{D}$ can be seen as an operad in Set concentrated in arity 1. We prove the following:

Proposition 4.6. *The composition product $\mathfrak{uCom} \circ \mathfrak{D}$ is endowed with an operad structure with the distributive law $\lambda\colon \mathfrak{D} \circ \mathfrak{uCom} \to \mathfrak{uCom} \circ \mathfrak{D}$ such that:*

$$\lambda(d^j;*_n) = (*_n,\underbrace{d^j,\dots,d^j}_{n}).$$

There is an morphism of operads: $\varphi\colon \mathfrak{uCom} \circ \mathfrak{D} \to \Pi$ sending $(*_n;d^{i_1},\dots,d^{i_n})$ to the unique partition $J \in \Pi(n)$ satisfying $j \in J_{i_j}$ for all $j \in [n]$.

The morphism φ is an isomorphism of operads.

Proof. Verifying that λ above satisfies the definition of a distributive law is straightforward, and left to the reader.

Note that $\mathfrak{uCom}$ is generated as an operad by $*_2$, and $\mathfrak{D}$ is generated by d^1. This implies that $\mathfrak{uCom} \circ \mathfrak{D}$ is generated by $(*_2; d^0, d^0)$ and $(*_1; d^1)$. Let us define a morphism of operads $\varphi\colon \mathfrak{uCom} \circ \mathfrak{D} \to \Pi$, induced by

$$\varphi(*_2; d^0, d^0) = (\{1,2\}) \in \Pi(2), \quad \varphi(*_1; d^1) = (\emptyset, \{1\}) \in \Pi(1).$$

Since φ is a morphism of operads, it is compatible to operadic composition and actions of the symmetric group. It is then easy to check that $\varphi(*_n; d^{i_1}, \dots, d^{i_n})$ is the unique partition $J \in \Pi(n)$ satisfying $j \in J_{i_j}$ for all $j \in [n]$, and that we have described a bijection between $\mathfrak{uCom} \circ \mathfrak{D}$ and Π compatible with composition. □

From this proposition we obtain the straightforward corollary:

Corollary 4.7. *A* Π*-algebra in* Set *is a commutative monoid equipped with a monoid endomorphism.*

Remark 4.8. We can transfer the preceding construction to the category of $\mathbb{F}$-vector spaces using linearisation. Recall from Example 2.4 that the linearisation of $\mathfrak{uCom}$ is the operad uCom of unital, associative, commutative $\mathbb{F}$-algebras. Note that the linearisation of $\mathfrak{D}$ is D, the polynomial algebra in one indeterminate d. The previous result yields an isomorphism of operads between $\mathrm{uCom} \circ \mathrm{D}$, equipped with the distributive law of 3.5 (with $B = D$ seen as a bialgebra using Example 3.6), and the linearisation $\mathbb{F}\Pi$ of Π.

5. The operad of n-level algebras

In this section, we introduce the set operad $\mathfrak{Lev}_n$ of n-level monoids and its linearisation Lev_n. This is a new construction which generalises the operad Lev of level algebras defined by Chataur and Livernet [4]. Intuitively, a n-level algebra is a vector space endowed with a n-ary operation which is strongly commutative in a certain sense, but which is not associative.

Definition 5.1 (The operads $\mathfrak{Lev}_n$, Lev_n)**.** For all integers $n > 1$, **the operad of n-level monoids** is the sub-operad $\mathfrak{Lev}_n \subseteq \Pi$ generated by the element $(\emptyset, [n]) \in \Pi(n)$.

The operad of n-level algebras is the linearisation $\mathrm{Lev}_n = \mathbb{F}\mathfrak{Lev}_n$ of the operad of n-level monoids.

As suggested in the name of this operad, we will call **n-level algebras** the algebras over the operad Lev_n.

Remark 5.2. Since $\mathfrak{Lev}_n$ is generated by an element of arity n, $\mathfrak{Lev}_n(k)$ is empty whenever $k \not\equiv 1 \mod n-1$.

Recall that operads can be described by generators and relations [**17**, 5.4.5]. We now present one of our main new results, which gives a presentation for the operad $\mathfrak{Lev}_n$, and by linearisation, for the operad Lev_n:

Theorem 5.3. *The operad* $\mathfrak{Lev}_n$ *is generated by one element* $\star \in \mathfrak{Lev}_n(n)$*, fixed under the action of* $\mathfrak{S}_n$*, and satisfying the following relation:*

$$(\star \circ_2 \star) \circ_1 \star = (\star \circ_2 \star) \circ_1 \star \cdot \sigma,$$

where $\sigma \in \mathfrak{S}_{3n-2}$ *is the transposition* $(n\ n+1)$*.*

The proof of this result, which is somewhat long and technical, is postponed to Section 6.

Let's illustrate the notion of n-level algebra through examples:

Example 5.4. Consider the set $\mathbb{Z}^{\mathbb{N}}$ of sequences of relative integers. Fix a positive integer $n > 0$. Then $\mathbb{Z}^{\mathbb{N}}$ is endowed with a structure of n-level monoid such that, if $u_i = (u_{ij})_{j\in\mathbb{N}}$ for all $i \in [n]$,

$$\star(u_1, \dots, u_n) = \big(u_{1j} + \cdots + u_{nj} - 1\big)_{j\in\mathbb{N}}.$$

There are several possible variants of this example. Note that the linearisation of this particular example gives an n-level algebra similar to the n-level algebra structure of the Carlsson algebra described in Theorem 12.4.

Remark 5.5. The operad Lev_2 is the operad Lev of level algebras of [**4**].

We now construct a cofiltration of Lev_n using a notion of truncation. This new family of operads will be used in Section 12 to identify certain unstable modules.

For all $s > 0$, $k \in \mathbb{N}$, denote by $\mathfrak{Lev}_n^{>s}(k)$ the subset of $\mathfrak{Lev}_n(k)$ of partitions J such that there exists $i > s$ such that $J_i \neq \emptyset$. It is easy to check that $\mathfrak{Lev}_n^{>s} \subset \mathfrak{Lev}_n$ forms an operadic ideal (see [**17**, 5.2.16]), which allows us to define an operad structure on the quotient $\mathfrak{Lev}_n/\mathfrak{Lev}_n^{>s}$.

Definition 5.6. The **s-truncation** of $\mathfrak{Lev}$ is the quotient operad

$$T_s\mathfrak{Lev}_n = \mathfrak{Lev}_n/\mathfrak{Lev}_n^{>s}.$$

The **s-truncation** of Lev is the quotient operad $T_s \mathrm{Lev}_n = \mathrm{Lev}_n / \mathrm{Lev}_n^{>s}$, where $\mathrm{Lev}_n^{>s} = \mathbb{F}\mathfrak{Lev}_n^{>s}$.

Remark 5.7. Since we have a filtration $\mathfrak{Lev}_n^{>1} \supset \mathfrak{Lev}_n^{>2} \supset \cdots$ whose limit is $\emptyset$, we obtain a cofiltration of Lev_n:

$$\mathrm{Lev}_n \longrightarrow \dots \longrightarrow T_{s+1}\mathrm{Lev}_n \longrightarrow T_s\mathrm{Lev}_n \longrightarrow \dots \qquad (*)$$

6. Proof of Theorem 5.3

This section is devoted to the proof of Theorem 5.3, which gives a presentation of the operad Lev_n of Definition 5.1, and gives us insights on its associated algebras. This proof uses the description of free operads using trees (see, for example, [**17**], section 5.4). Before proving this theorem, we need one preliminary result:

Lemma 6.1. *As a subset of* $\Pi(k)$*,* $\mathfrak{Lev}_n(k)$ *is the set of all ordered partitions* $J = (J_i)_{0\leqslant i\leqslant s}$ *of* $[k]$ *such that:*

$$\sum_{i=0}^{s} \frac{|J_i|}{n^i} = 1. \qquad (**)$$

Proof. By induction on k. Since $\mathfrak{Lev}_n(k)$ is generated by a single element $(\emptyset, [n])$ of arity n, $\mathfrak{Lev}_n(0) = \emptyset$, $\mathfrak{Lev}_n(1)$ is equal to $\{1_\Pi\} = \{([1])\}$, $\mathfrak{Lev}_n(n)$ is equal to $\{(\emptyset, [n])\}$, and $\mathfrak{Lev}_n(i) = \emptyset$ for all $1 < i < n$. It is easy to see that $([1])$ is the only partition of $[1]$ satisfying equation $(**)$, that $(\emptyset, [n])$ is the only partition of $[n]$ satisfying equation $(**)$, and that no partition of $\emptyset = [0]$, and no partition of $[2], [3], \dots, [n-1]$ can satisfy equation $(**)$.

Suppose now that $k > n$ and that we have proven that $\mathfrak{Lev}_n(k-n+1)$ is the set of all ordered partitions $J = (J_i)_{0 \leqslant i \leqslant s}$ of $[k-n+1]$ satisfying equation $(**)$.

On one hand, let $J = (J_i)_{0 \leqslant i \leqslant s}$ a partition of $[k]$, and suppose that $J \in \mathfrak{Lev}_n(k)$. Then, since $k > n$ and $\mathfrak{Lev}_n$ is spanned by $(\emptyset, [n])$, there exist a partition $\hat{J} = (\hat{J}_i)_{0 \leqslant i \leqslant s}$ of $[k-n+1]$, an integer $l \in [k-n+1]$, and a permutation $\sigma \in \mathfrak{S}_k$ such that $J = \hat{J} \circ_l (\emptyset, [n]) \cdot \sigma$. By the induction hypothesis, $\hat{J}$ satisfies equation $(**)$. Denote by i_0 the unique integer such that $l \in \hat{J}_{i_0}$. Then,

$$\sum_{i=0}^{s} \frac{|J_i|}{n^i} - \sum_{i=0}^{s} \frac{|\hat{J}_i|}{n^i} = \frac{1}{n^{i_0}} - \frac{n}{n^{i_0+1}} = 0,$$

so J satisfies equation $(**)$.

On the other hand, let $J = (J_i)_{0 \leqslant i \leqslant s}$ be a partition of $[k]$, and suppose that J satisfies equation $(**)$. One can assume that J_s is non-empty without loss of generality. Then, since $\sum_{i=0}^{s} \frac{|J_i|}{n^i} = 1$ and $k > n$, this implies that $s > 1$, and

$$|J_s| = n^s \left(1 - \sum_{i=0}^{s-1} \frac{|J_i|}{n^i}\right) = n \left(n^{s-1} - \sum_{i=0}^{s-1} |J_i| n^{s-1-i}\right).$$

So $|J_s|$ is a multiple of n. In particular, since $J_s \neq \emptyset$, $|J_s| \geqslant n$. There exists a permutation $\sigma \in \mathfrak{S}_k$ such that $\{k-n+1, \dots, k\} \in (J \cdot \sigma)_s$. Let $\bar{J} \in \Pi(k-n+1)$ be the partition of $[k-n+1]$ such that $\bar{J}_i = (J \cdot \sigma)_i$ for all $i < s-1$, $\bar{J}_{s-1} = \{k-n+1\} \cup (J \cdot \sigma)_{s-1}$, and $\bar{J}_s = \emptyset$. Then, one has:

$$\sum_{i=0}^{s} \frac{|\bar{J}_i|}{n^i} = \left(\sum_{i=0}^{s} \frac{|J_i|}{n^i}\right) - \frac{n}{n^s} + \frac{1}{n^{s-1}} = 1,$$

so, by the induction hypothesis, $\bar{J} \in \mathfrak{Lev}_n(k-n+1)$, and, since

$$J = (\bar{J} \circ_{k-n+1} (\emptyset, [n])) \cdot \sigma^{-1},$$

we conclude that $J \in \mathfrak{Lev}_n(k)$. □

Proof of Theorem 5.3. By definition, as a suboperad of Π, $\mathfrak{Lev}_n$ is generated by an operation $\star$ of arity n fixed by the action of $\mathfrak{S}_n$. As a partition, recall from Example 4.5 that $(\star \circ_2 \star) \circ_1 \star = (\emptyset, \{2n+1, \dots, 3n-2\}, \{1, \dots, 2n\})$, so $\star$ satisfies the relation

$$(\star \circ_2 \star) \circ_1 \star = (\star \circ_2 \star) \circ_1 \star \cdot (n\ n+1). \qquad (***)$$

Let $\mathfrak{MagCom}_n$ be the free operad generated by an arity n operation μ fixed by the action of $\mathfrak{S}_n$. Let $\psi\colon \mathfrak{MagCom}_n \to \mathfrak{Lev}_n$ be the unique operad morphism sending μ to $\star$. Denote by $\mathfrak{Mag}_n$ the free operad generated by an arity n operation ν, this time with $\mathfrak{S}_n$ acting freely on ν. There is a unique operad morphism $\mathfrak{Mag}_n \twoheadrightarrow \mathfrak{MagCom}_n$

sending $\nu \in \mathfrak{Mag}_n$ to $\mu \in \mathfrak{MagCom}_n$. The operad $\mathfrak{Mag}_n$ admits a characterisation using planar n-ary trees (see, for example, [**8**, Section 6.1.2], for the case $n = 2$). Using this characterisation, $\mathfrak{Mag}_n(k)$ is the set of planar n-ary trees (all vertices are either the root, leaves, or have n incoming edges) with k leaves labelled by the set $[k]$. The generator $\mu \in \mathfrak{Mag}_n(n)$ is identified with the corolla with n leaves labelled 1 to n. The unit $1_{\mathfrak{Mag}_n}$ is identified with the tree which consists of just the root and no leaves. Using the surjection $\mathfrak{Mag}_n \twoheadrightarrow \mathfrak{MagCom}_n$, we will consider the elements of $\mathfrak{MagCom}_n$ as (classes of) trees.

For a tree $t \in \mathfrak{MagCom}_n(k)$, $\psi(t) \in \mathfrak{Lev}_n(k) \subset \Pi(k)$ is a partition $\psi(t) = (\psi(t)_i)_{0 \leqslant i \leqslant s}$ of $[k]$ where $\psi(t)_i$ is the set of labels of leaves of t at height i. Since $\Pi(k)$ is the set of ordered partitions of $[k]$, this implies that two trees $t, t' \in \mathfrak{MagCom}_n(k)$ have the same image under ψ if and only if for all i, t and t' have the same set of labels for their leaves of height i, the height being the distance (number of edges) from the root. So, the stabiliser of $\psi(t)$ is generated by the transpositions that permute two leaves in t of same height.

Denote by $\approx$ the relation generated by $(\mu \circ_2 \mu) \circ_1 \mu \approx (\mu \circ_2 \mu) \circ_1 \mu \cdot (n\ \ n+1)$ under operadic composition and action of the symmetric groups.

Thus, in order to prove that $\mathfrak{Lev}_n$ admits the presentation suggested, we have to prove that ψ is surjective, and that $\approx$ allows permutations of leaves of same height, that is: if $\tau \in \mathfrak{S}_k$ is a transposition which permutes two leaves of same height in $t \in \mathfrak{MagCom}_n(k)$, then $\tau t \approx t$.

The morphism ψ is surjective: Let $J \in \mathfrak{Lev}_n(k)$. Like in 4.1, we will denote s the smallest integer such that $J_{s'} = \emptyset$ for all $s' > s$. We will prove, by induction on s, that J is in the image of ψ.

In the case where $s = 0$, then necessarily, $J = (\{0\}) = 1_{\mathfrak{Lev}_n} = \psi(1_{\mathfrak{MagCom}})$.

Suppose the result proven in the case where $s = l$ for a certain $l \in \mathbb{N}$ and suppose now that $s = l + 1$. The condition $**$ implies that $|J_s|$ is divisible by n, say $|J_s| = mn$. Since $J_s \neq \emptyset$, $m > 0$. There is a $\sigma \in \mathfrak{S}_k$ such that $\sigma J_s = \{k - mn + 1, \ldots, k\}$. Consider a partition $\bar{J}$ such that:

$$\bar{J}_i = \begin{cases} \sigma J_i, & \text{if } i < s-1, \\ \sigma J_{s-1} \cup \{k - mn + 1, \ldots, k - m(n-1)\}, & \text{if } i = s-1, \\ \emptyset, & \text{if } i \geqslant s. \end{cases}$$

One can easily show that $\bar{J}$ satisfies $**$, and $\bar{J}_{s'} = \emptyset$ for all $s' > s - 1$, so by induction hypothesis, there is a tree $t \in \mathfrak{MagCom}_n(k - m(n-1))$ such that $\psi(t) = \bar{J}$. One then has:

$$\psi(t(1_{\mathfrak{MagCom}_n}^{\times k-mn}, \mu^{\times m})) = \bar{J}(1_{\mathfrak{Lev}_n}^{\times k-mn}, *^{\times m}),$$

which is equal to σJ. Finally, this implies that:

$$\psi(\sigma^{-1} t(1_{\mathfrak{MagCom}_n}^{\times k-mn}, \mu^{\times m})) = J,$$

so we have proven that J is in the image of ψ.

We have proven by induction on s that all $J \in \mathfrak{Lev}_n(k)$ are in the image of ψ.

The relation $\approx$ allows permutation of leaves of the same height: Let $t \in \mathrm{MagCom}_n(k)$ be a tree with k leaves, and let τ be a permutation of leaves of the same height. Denote by s the height of t, that is, the maximum of the heights of leaves of t. For all $i \in \{0, \dots, s\}$ let $\alpha_i = |\psi(t)_i|$. There exists a $\sigma \in \mathfrak{S}_k$ such that

$$\psi(\sigma t)_s = \{1, \dots, \alpha_s\}, \quad \psi(\sigma t)_{s-1} = \{\alpha_s + 1, \dots, \alpha_s + \alpha_{s-1}\}, \dots,$$

$$\psi(\sigma t)_i = \{\alpha_s + \alpha_{s-1} + \dots + \alpha_{i+1} + 1, \dots, \alpha_s + \dots + \alpha_i\}.$$

Denote by ρ the permutation $\sigma\tau\sigma^{-1}$. Then ρ acts on σt by permuting two leaves of same height, so ρ is a transposition in $\mathfrak{S}_{\alpha_s} \times \mathfrak{S}_{\alpha_{s-1}} \times \dots \times \mathfrak{S}_{\alpha_0}$. Since t is of height s, $\alpha_s > 0$. Since all vertices of t have n incoming edges, this implies that $\alpha_s = ln$ for an integer $l > 0$. Without loss of generality, we can assume that we have chosen σ such that, for all $j \in [l]$, the leaves of σt labelled $\{jn+1, \dots, (j+1)n\}$ are connected to the same internal vertex of height $s-1$.

We will now prove that $\approx$ allows permutation of leaves of same height by recurrence on the height s. From what precedes, ρ admits a decomposition

$$\rho = \rho_s \times \rho_{s-1} \times \dots \times \rho_0$$

where $\rho_i \in \mathfrak{S}_{\alpha_i}$ is either the neutral permutation, or a transposition.

If $s = 0$, then $t = 1_{\mathfrak{MagCom}_n}$ and there is nothing to prove.

Suppose that we have proven that $\approx$ allows permutation of leaves of same height of all trees t of height $s = \beta$ for a certain integer β, and suppose now that $s = \beta + 1$.

Let $\bar{t}$ be the tree obtained from σt by removing the ingoing edges of all the inner vertices of height $s-1$ (there are l such vertices), relabelling these vertices with $\{1, \dots, l\}$, and relabelling the other leaves in t labelled i by $i - \alpha_s + l$ for all $i > \alpha_s$. We have described an element $\bar{t} \in \mathfrak{MagCom}_n(k - \alpha_s + l)$ of height $s-1$, such that $\sigma t = \bar{t}(\mu^{\times l}, 1_{\mathfrak{MagCom}_n}^{\times k - \alpha_s})$ (this is $\bar{t}$ composed with μ in the l first inputs). Denote by $\hat{t} = (\mathrm{id}_{[l]} \times \rho_{s-1} \times \dots \times \rho_0)\bar{t}$. By the induction hypothesis, we know that $\bar{t} \approx \hat{t}$, so, composing by μ in the l first input, $\sigma t \approx (\mathrm{id}_{[\alpha_s]} \times \rho_{s-1} \times \dots \times \rho_0)t$.

It remains to prove that $\rho\sigma t \approx (\mathrm{id}_{[\alpha_s]} \times \rho_{s-1} \times \dots \times \rho_0)t$, or equivalently, that $\rho_s \sigma t \approx \sigma t$, where we identified ρ_s with $\rho_s \times \mathrm{id}_{[\alpha_{s-1}]} \times \dots \times \mathrm{id}_{\alpha_0}$. From what precedes, ρ_s is either neutral or a transposition. If it is neutral, there is nothing to prove. Suppose now that ρ_s is a transposition $(i_1\ i_2)$.

If $i_1, i_2 \in \{jn+1, \dots, (j+1)n\}$ for a certain $j \in [l]$, then since $\mu \in \mathfrak{MagCom}_n(n)$ is fixed by $\mathfrak{S}_n$, $\rho_s t = t$. Suppose now that there exists $j_1, j_2 \in [l]$ with $j_1 \neq j_2$, such that $j_1 n + 1 \leqslant i_1 \leqslant (j_1+1)n$ and $j_2 n + 1 \leqslant i_2 \leqslant (j_2+1)n$. Since $\mu \in \mathfrak{MagCom}_n(n)$ is fixed by $\mathfrak{S}_n$, we can assume that $i_1 = (j_1+1)n$ and $i_2 = j_2 n + 1$

Since $\bar{t}$ is a tree of height $s-1$ then l is again a multiple of n, $l = nl'$, and since we have assumed that $l > 1$, $l' > 0$.

By the induction hypothesis, $(j_2\ 2)(j_1\ 1)\bar{t} \approx \bar{t}$. We can again suppose, without loss of generality, that we have chosen σ such that the leaves of $(j_2\ 2)(j_1\ 1)\bar{t}$ labelled $\{jn+1, \dots, (j+1)n\}$ are the leaves of the same internal vertex of height $s-2$ for all $j \in [l']$. Composing with μ in the l first inputs of $(j_2\ 2)(j_1\ 1)\bar{t}$ and $\bar{t}$ then yields $\sigma'\sigma t \approx \sigma t$, where σ' is a block permutation, where the blocks have size n, and σ' transpose the j_1-th block with the first block and the j_2-th block with the second block. Then, $\sigma'\rho_s(\sigma')^{-1}$ is the transposition $(n\ n+1)$. I pretend to be certain that $\sigma'\sigma t \approx (n\, n+1)\sigma'\sigma t = \sigma'\rho_s\sigma t$.

Let $\bar{u}$ be the tree obtained from $(j_2\ 2)(j_1\ 1)\bar{t}$ by collapsing the leaves labelled 1 to n, relabelling the new leaf of level $s-2$ by 1, and relabelling all other leaves labelled i in $(j_2\ 2)(j_1\ 1)\bar{t}$ by $i-n+1$. Then $\bar{u}$ is a new element in $\mathrm{MagCom}_n(k-\alpha_s+l)$ such that $(j_2\ 2)(j_1\ 1)\bar{t}=\bar{u}\circ_1\mu$. Denote by $u=\bar{s}(1_{\mathfrak{MagCom}_n},\mu^{\times l-1},1_{\mathfrak{MagCom}_n}^{\times k-\alpha_s})$ (that is $\bar{u}$ composed by μ in the inputs $2,3\dots,l$). We then have:

$$\sigma'\sigma t = u\circ_1\left(\dots\left((((\mu\circ_2\mu)\circ_1\mu)\circ_{2n+l-2}\mu)\circ_{2n+l-3}\mu\right)\cdots\circ_{2n+1}\mu\right),$$

and

$$(n\ n+1)\sigma'\sigma t$$
$$= u\circ_1\left(\dots\left(((((\mu\circ_2\mu)\circ_1\mu)(n\ n+1))\circ_{2n+l-2}\mu)\circ_{2n+l-3}\mu\right)\cdots\circ_{2n+1}\mu\right).$$

Since $\approx$ is generated by $(\mu\circ_2\mu)\circ_1\mu\approx((\mu\circ_2\mu)\circ_1\mu)(n\ n+1)$, this shows that

$$\sigma'\sigma t\approx(n\,n+1)\sigma'\sigma t=\sigma'\rho_s\sigma t.$$

So, applying $(\sigma')^{-1}$ on both sides, we get $\rho_s\sigma t\approx\sigma t$, which, from what precedes, implies that $\rho\sigma t\approx\sigma t$, and since $\rho=\sigma\tau\sigma^{-1}$, that implies $\sigma\tau t\approx\sigma t$. Finally, applying σ^{-1} to both sides, we get $\tau t\approx t$, which is what we wanted to prove. □

7. $\star$-powers in $\mathcal{P}$-algebras

In this section, we work over the finite field $\mathbb{F}_q$ of order q, and we study the $\star$-power operation $x\mapsto\star(x,\dots,x)$ in $\mathcal{P}$-algebras, where $\mathcal{P}$ is an operad equipped with an q-ary operation $\star$. We observe that, when the operation $\star$ is symmetric, the $\star$-power induces a linear map on all $\mathcal{P}$-algebras. We build a functor $\psi_!$, which to each D-module M, produces a $\mathcal{P}$-algebra containing M and in which $dx=\star(x,\dots,x)$ for all $x\in M$.

Recall that $D=\mathbb{F}_q[d]$ is the polynomial algebra in one indeterminate d (see Example 3.6).

Fix an operad $\mathcal{P}$ and a symmetric arity q operation $\star\in\mathcal{P}(q)^{\mathfrak{S}_q}$.

Notation 7.1. *Let $\star_k\in\mathcal{P}(q^k)$ denote the operation inductively defined by:*

$$\star_0=1_{\mathcal{P}}\quad\textit{and}\quad\star_{k+1}=\star(\underbrace{\star_k,\dots,\star_k}_{q}).$$

In particular, $\star_1=\star$. The associativity of the composition in $\mathcal{P}$ implies that

$$\star_i(\underbrace{\star_j,\dots,\star_j}_{q^i})=\star_j(\underbrace{\star_i,\dots,\star_i}_{q^j})=\star_{i+j}.$$

For all vector spaces V, we define a map $\psi_V\colon \mathrm{D}\otimes V\to S(\mathcal{P},V)$ by $\psi_V(d^k\otimes v)=v^{\star_k q^k}$, where the notation $a^{\mu n}$ was introduced in Definition 2.3, and extending by linearity on D.

Lemma 7.2. *Let $\star\in\mathcal{P}(q)^{\mathfrak{S}_q}$. The map $\psi_V\colon D\otimes V\to S(\mathcal{P},V)$ defined above is linear and natural in V. Moreover, the natural transformation $\psi\colon \mathrm{D}\otimes-\to S(\mathcal{P},-)$ is monadic.*

Proof. To show that ψ_V is linear, it suffices to show, by induction on k, that $v \mapsto v^{\star_k q^k}$ is linear in v. If $k = 0$, the linearity is evident, let us prove the case $k = 1$. Let $u, v \in V$ and $\lambda \in \mathbb{F}_q$. One has:

$$(u + \lambda v)^{\star q} = (\star; u + \lambda v, \dots, u + \lambda v) = \sum_{i+j=q} \sum_{\sigma \in \mathfrak{S}_q/\mathfrak{S}_i \times \mathfrak{S}_j} (\star; \sigma(u^{\times i}, (\lambda v)^{\times j})).$$

Since $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$, we deduce:

$$(u + \lambda v)^{\star q} = \sum_{i+j=q} \sum_{\sigma \in \mathfrak{S}_q/\mathfrak{S}_i \times \mathfrak{S}_j} (\star; u^{\times i}, (\lambda v)^{\times j}) = \sum_{i+j=q} \binom{q}{i} \lambda^j (\star; u^{\times i}, v^{\times j}).$$

Since $\binom{q}{i}$ is divisible by q for all $i \in \{1, \dots, q-1\}$, we then get:

$$(u + \lambda v)^{\star q} = (\star; u^{\times q}) + \lambda^q (\star; v^{\times q}).$$

Finally, since $\lambda^q = \lambda$ in $\mathbb{F}_q$,

$$(u + \lambda v)^{\star q} = u^{\star q} + \lambda v^{\star q}.$$

Suppose that we have proven that $v \mapsto v^{\star_k q^k}$ is linear in v. Then, by definition of $\star_q$,

$$\begin{aligned}(u + \lambda v)^{\star_{k+1} q^{k+1}} &= (\star_{k+1}; (u + \lambda v)^{\times q^{k+1}}) \\ &= (\star(\star_k, \dots, \star_k); (u + \lambda v)^{\times q^{k+1}}) = \star((u + \lambda v)^{\star_k q^k}, \dots, (u + \lambda v)^{\star_k q^k}),\end{aligned}$$

so, by induction hypothesis,

$$(u + \lambda v)^{\star_{k+1} q^{k+1}} = (\star; u^{\star_k q^k} + \lambda v^{\star_k q^k}, \dots, u^{\star_k q^k} + \lambda v^{\star_k q^k}) = \left(u^{\star_k q^k} + \lambda v^{\star_k q^k}\right)^{\star q}.$$

But, we have shown above that $v \mapsto v^{\star q}$ was linear in v. So,

$$(u + \lambda v)^{\star_{k+1} q^{k+1}} = \left(u^{\star_k q^k},\right)^{\star q} + \lambda \left(v^{\star_k q^k}\right)^{\star q} = u^{\star_{k+1} q^{k+1}} + \lambda v^{\star_{k+1} q^{k+1}}.$$

The map ψ_V is clearly natural in V. To show that $\psi \colon \mathrm{D} \otimes - \to S(\mathcal{P}, V)$ induces a monad morphism, we need to check that it is compatible with the unit and composition of the monads $D \otimes -$ and $S(\mathcal{P}, -)$. For the unit, note that $1_{\mathrm{D}} = d^0$, so $\psi_V(\eta_{\mathrm{D}}(v)) = \psi_V(1_{\mathrm{D}} \otimes v) = v^{\star_0 1} = (1_{\mathcal{P}}; v) = \eta_{\mathcal{P}}(v)$. Since the composition of $\mathrm{D} \otimes -$ is induced by $d^i \otimes d^j \otimes v \mapsto d^{i+j} \otimes v$, the fact that ψ_V is compatible with the composition is a straightforward consequence of $\left(v^{\star_j q^j}\right)^{\star_i q^i} = v^{\star_{i+j} q^{i+j}}$. □

The monad morphism ψ induces a functor $\psi^* \colon \mathcal{P}_{\mathsf{alg}} \to \mathrm{D}_{\mathsf{mod}}$ on the categories of algebras which restricts the structure along ψ.

Proposition 7.3. *The functor ψ^* defined above admits a left adjoint $\psi_! \colon D_{\mathsf{mod}} \to \mathcal{P}_{\mathsf{alg}}$.*

Proof. One can build the desired left adjoint using left Kan extensions (see [**18**, Chapter X]). Explicitly, for a D-module M, the $\mathcal{P}$-algebra $\psi_!(M)$ is obtained as a coequaliser between two $\mathcal{P}$-algebra morphisms $S(\mathcal{P}, \mathrm{D} \otimes M) \to S(\mathcal{P}, M)$. The first of these $\mathcal{P}$-algebra morphisms is given by the D-action $\mathrm{D} \otimes M \to M$. The second is given by the composition:

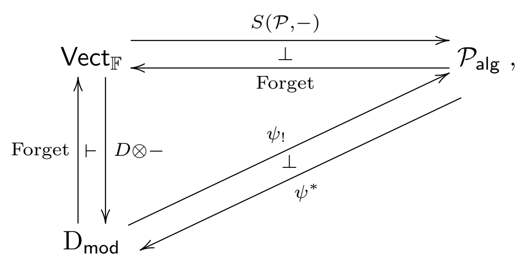
$$S(\mathcal{P}, \mathrm{D} \otimes M) \xrightarrow{S(\mathcal{P},\psi_M)} S(\mathcal{P}, S(\mathcal{P}, M)) \xrightarrow{(\gamma_{\mathcal{P}})_M} S(\mathcal{P}, M) \ ,$$

where the first map is ψ applied to the underlying vector space of M, and the second map is the composition of the monad $S(\mathcal{P}, -)$. In other words, $\psi_!(M)$ is the quotient of the free $\mathcal{P}$-algebra $S(\mathcal{P}, M)$ by the $\mathcal{P}$-ideal generated by the elements $d^n m - m^{\star_n q^n}$ for $m \in M$ and $n \in \mathbb{N}$ (see [**9**, 2.3.1] for the notion of ideal in a $\mathcal{P}$-algebra). The proof that this provides indeed a left adjoint to ψ^* is left as an exercise. □

Proposition 7.4. *Let $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$. There is a natural isomorphism:*

$$\psi_! \circ (D \otimes -) \cong S(\mathcal{P}, -).$$

Proof. Consider the following diagram of categories and adjunctions:



where Forget denotes all the right adjoints that extract underlying vector spaces. One can easily check that the composite Forget $\circ\psi^* \colon \mathcal{P}_{\mathsf{alg}} \to \mathsf{Vect}_{\mathbb{F}}$ is naturally isomorphic to Forget: $\mathcal{P}_{\mathsf{alg}} \to \mathsf{Vect}_{\mathbb{F}}$. Since $\psi_! \circ (\mathrm{D} \otimes -)$ is a left adjoint of Forget $\circ\psi^*$, it is also a left adjoint of Forget: $\mathcal{P}_{\mathsf{alg}} \to \mathsf{Vect}_{\mathbb{F}}$. Left adjoints being unique up to natural isomorphisms, we obtain the desired natural isomorphism $\psi_! \circ (D \otimes -) \cong S(\mathcal{P}, -)$. □

8. $\mathcal{P}$-central operations

In this section,, we work over the finite field $\mathbb{F}_q$ of order q, and we introduce the notion of *central operation* in an operad. A central operation is an operation which satisfies a certain interchange relation with respect to all other operations in the operad. The condition of centrality is the minimal condition on $\star$ which ensures that the $\star$-power is always a $\mathcal{P}$-algebra endomorphism. This notion of centrality generalises the notion of centrality for binary commutative operations defined in [**11**, Definition 5.2]. Since we want the $\star$-power to be linear over $\mathbb{F}_q$, we restrict ourselves to the case of symmetric operations $\star$ of arity q.

Given a central operation $\star$ of arity q, we construct a functor $\psi_!$ which takes a vector space V with a linear self-map d and produces a $\mathcal{P}$-algebra in D-modules over V satisfying $dv = \star(v, \dots, v)$. This construction allows us to produce free unstable algebras over an unstable module in Section 9. The results of the present section, in particular the commutative diagram D1 of 8.4, are crucial to the proof of Theorem 11.6 on the structure of free unstable algebras.

Recall the construction of the functors ψ^* and $\psi_!$ from Section 7. We equip the $\mathcal{P}$-algebra $\psi_!(M)$ with a D-module structure through ψ^*. This way, $\psi_!(M)$ is a $\mathcal{P}$-algebra and a D-module, but it is not necessarily a $\mathcal{P}$-algebra in D-modules as defined in 3.4. To extend $\psi_!$ into a functor $\mathrm{D}_{\mathsf{mod}} \to \mathcal{P}_{\mathsf{alg}}^{\mathrm{D}}$, we need $\star$ to satisfy an additional condition, which we call $\mathcal{P}$-centrality.

Definition 8.1 ($\mathcal{P}$-central operations)**.** An operation $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$ is said to be **$\mathcal{P}$-central** if, for all $\mu \in \mathcal{P}(m)$,

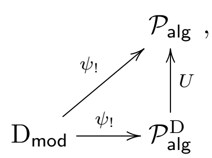
$$\star(\mu, \dots, \mu) = \mu(\star, \dots, \star) \cdot \sigma_{q,m}, \tag{I}$$

where $\sigma_{q,m} \in \mathfrak{S}_{qm}$ sends $(i-1)m + k$ to $(k-1)q + i$ for all $i \in [q]$, $k \in [m]$. Under the usual identifications $[qm] \cong [q] \times [m]$ and $[qm] \cong [m] \times [q]$ the permutation $\sigma_{q,m}$ corresponds to the transposition $[q] \times [m] \to [m] \times [q]$, $(i, k) \mapsto (k, i)$.

Example 8.2. Consider the operation $X_q \in \mathrm{uCom}(q)$, where uCom is the operad defined in 2.4. Note that any operation in $\mathrm{Com}(n)$ is of the type λX_m with $\lambda \in \mathbb{F}_q$. One has $X_q(\lambda X_n, \dots, \lambda X_n) = \lambda^q X_{qm} = \lambda X_{qm}$, and since X_{qm} is fixed under the action of $\mathfrak{S}_{qm}$, we then deduce that $X_q(\lambda X_m, \dots, \lambda X_m) = \lambda X_{qm} \cdot \sigma_{q,m} = \lambda X_m(X_q, \dots, X_q) \cdot \sigma_{q,m}$, and so, X_q is uCom-central.

We get the following result:

Lemma 8.3. *Let $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$. The functor $\psi_! \colon \mathrm{D}_{\mathsf{mod}} \to \mathcal{P}_{\mathsf{alg}}$ defined in 7.3 can be extended into a functor $\psi_! \colon \mathrm{D}_{\mathsf{mod}} \to \mathcal{P}_{\mathsf{alg}}^{\mathrm{D}}$ if and only if $\star$ is $\mathcal{P}$-central. In other words, the operation $\star$ is $\mathcal{P}$-central if and only if there is a diagram of functors that commutes up to natural isomorphism:*



were U is the forgetful functor that extracts the underlying $\mathcal{P}$-algebra.

Proof. Fix an element $\mu \in \mathcal{P}(m)$. Let V be an $\mathbb{F}$-vector space spanned by elements $x_1, \dots, x_m$. According to Proposition 7.3, the $\mathcal{P}$-algebra $\psi_!(\mathrm{D} \otimes V)$ is isomorphic to $S(\mathcal{P}, V)$. Using the definition of ψ^*, the D-module structure on this $\mathcal{P}$-algebra gives $d(\mu; x_1, \dots, x_m) = (\mu; x_1, \dots, x_m)^{\star q}$ and $dx_i = x_i^{\star q}$ for all i. Suppose that d induces a $\mathcal{P}$-algebra morphism on $\psi_!(\mathrm{D} \otimes V)$. Then, one has:

$$\begin{aligned}
(\star(\mu, \dots, \mu); x_1, \dots, x_m, x_1, \dots, x_m, \dots, x_m) &= (\mu; x_1, \dots, x_m)^{\star q} \\
&= d(\mu; x_1, \dots, x_m) = \mu(dx_1, \dots, dx_m) = \mu(x_1^{\star q}, \dots, x_m^{\star q}), \\
&= (\mu(\star, \dots, \star); x_1, \dots, x_1, x_2, \dots, x_2, \dots, x_m), \\
&= (\mu(\star, \dots, \star) \cdot \sigma_{q,m}; x_1, \dots, x_m, x_1, \dots, x_m, \dots, x_m).
\end{aligned}$$

Since $x_1, \dots, x_m$ are independent, this implies that $\star(\mu, \dots, \mu)$ and $\mu(\star, \dots, \star) \cdot \sigma_{q,m}$ are equal up to a permutation $\sigma \in \prod_{i=1}^m \mathfrak{S}_{\{i+(k-1)m : k \in [q]\}}$. But, since $\star$ is stable under the action of $\mathfrak{S}_q$, $\mu(\star, \dots, \star) \cdot \sigma_{q,m}$ is stable under the action of $\prod_{i=1}^m \mathfrak{S}_{\{i+(k-1)m : k \in [q]\}}$. So $\star(\mu, \dots, \mu) = \mu(\star, \dots, \star) \cdot \sigma_{q,m}$. This holds for all $\mu \in \mathcal{P}(m)$, and so, $\star$ is $\mathcal{P}$-central.

Inversely, if M is a D-module, recall that $\psi_!(M)$ is a quotient of $S(\mathcal{P}, M)$. Supposing that $\star$ is $\mathcal{P}$-central, then for all $\mu \in \mathcal{P}(m)$ and $x_1, \ldots, x_m \in M$, the action of d yields

$$d(\mu; x_1, \ldots, x_m) = (\mu; x_1, \ldots, x_m)^{\star q} = (\star(\mu, \ldots, \mu); x_1, \ldots, x_m, x_1, \ldots, x_m, \ldots, x_m),$$

and,

$$\begin{aligned} \mu(dx_1, \ldots, dx_m) = \mu(x_1^{\star q}, \ldots, x_m^{\star q}) &= (\mu(\star, \ldots, \star); x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_m) \\ &= (\mu(\star, \ldots, \star) \cdot \sigma_{q,m}; x_1, \ldots, x_m, x_1, \ldots, x_m, \ldots, x_m). \end{aligned}$$

Now, using the $\mathcal{P}$-centrality of $\star$, we conclude that d is compatible with μ. Since this is true for all μ, we then conclude that d is a $\mathcal{P}$-algebra morphism. □

We now prove the main result of this section, which will allow us to prove Theorem 11.6.

Combining Lemma 8.3 and Proposition 7.4, one gets the following:

Proposition 8.4. *Let $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$ be a $\mathcal{P}$-central operation. The following diagram of functors commutes up to natural isomorphism:*

$$\begin{array}{ccc} \mathsf{Vect}_{\mathbb{F}} & \xrightarrow{S(\mathcal{P},-)} & \mathcal{P}_{\mathsf{alg}} \\ {\scriptstyle \mathrm{D}\otimes -}\downarrow & & \uparrow{\scriptstyle U} \\ \mathrm{D}_{\mathsf{mod}} & \xrightarrow{\psi_!} & \mathcal{P}^{\mathrm{D}}_{\mathsf{alg}} \end{array} \qquad \text{(D1)}$$

The next proposition allows one to check, from a presentation of an operad $\mathcal{P}$, if an operation $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$ is $\mathcal{P}$-central.

Proposition 8.5. *Let $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$. Let F be a sub-symmetric sequence of $\mathcal{P}$. Suppose that F generates the operad $\mathcal{P}$. The operation $\star$ is $\mathcal{P}$-central if and only if it satisfies relation* (I) *of Definition 8.1 for all $\mu \in F$.*

Proof. This is a fairly straightforward generalisation of [**11**, Proposition 5.7]. □

Proposition 8.5 allows us to describe further example of operads equipped with a central operation.

Proposition 8.6. *The generator $\star \in \mathrm{Lev}_q(q)^{\mathfrak{S}_q}$ of the operad Lev_q defined in Definition 5.1 is Lev_q-central.*

Proof. In Π, note that $\star(\star, \ldots, \star)$ is the partition $(\emptyset, \emptyset, [q^2])$ of $[q^2]$, which is fixed under the action of $\mathfrak{S}_{q^2}$. In particular, $\star \in \mathrm{Lev}_q(q)$ satisfies the relation I with $\mu = \star$. According to Proposition 8.5, this implies that $\star$ is Lev_q-central. □

We describe one more example which we will refer to in Section 12:

Example 8.7. Denote by $T_s \mathrm{Lev}_q$ the image of the composite

$$\mathrm{Lev}_q \hookrightarrow \Pi \cong \mathrm{uCom} \circ \mathrm{D} \to \mathrm{uCom} \circ T_s \mathrm{D}\,.$$

The operadic generator $\star \in \mathrm{Lev}_q(q)$ yields a $T_s \mathrm{Lev}_q$-central operation (one can show this using Proposition 8.5). Similarly, the operation $(X_q; d^{\times q})$, where X_q is the generator of the arity module $\mathrm{uCom}(q)$, is a $\mathrm{uCom} \circ \mathrm{D}$-central operation, a $\mathrm{uCom} \circ \mathrm{D}^{\pm}$-central operation, and a $\mathrm{uCom} \circ Q_s \mathrm{D}$-central operation. More generally, if $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$ is $\mathcal{P}$-central, then $(d^i)^{\star q}$ is $\mathcal{P} \circ \mathrm{D}$, $\mathcal{P} \circ \mathrm{D}^{\pm}$, and $\mathcal{P} \circ Q_s \mathrm{D}$-central, for all $i \in \mathbb{N}$.

Part 2. Unstable modules, unstable $\mathcal{P}$-algebras

We now turn to unstable modules and algebras over the Steenrod algebra. The main result of this part of the article is Theorem 11.6, which characterise certain free unstable algebras over operads.

9. The Steenrod algebra, unstable modules, $\mathcal{P}$-algebras in unstable modules

The Steenrod algebra is a central object in homotopy theory and algebraic topology. It was built to represent the natural operations in cohomology that are stable under the suspension operation. As such, the cohomology of any topological space can be seen as a module over the Steenrod algebra. Unstable modules are modules over the Steenrod algebra satisfying an additional property which models the behaviour of these cohomology modules.

In this section, we describe the Steenrod algebra $\mathcal{A}(q)$ of reduced q-th powers (without the Bockstein operator). In the case where the base field is a prime field $\mathbb{F}_p$, this algebra is a sub-bialgebra of the usual Steenrod algebra. This algebra can be presented in various equivalent ways, including the presentation given by [**20**], and more recently [**13**]. For the sake of clarity of our argument, we will give our own presentation, which is again equivalent. The statements for $\mathcal{A}(q)$ and the category $\mathcal{U}(q)$ of unstable modules are well known to easily lead to results on the Steenrod algebra and its unstable modules (see for example [**16**, appendix A.1]). At the end of this section, we study algebras over an operad in the category of $\mathcal{A}(q)$-modules, and in the category $\mathcal{U}(q)$, using the notions introduced in Section 3.

We recall that, throughout this paper, $\mathbb{F}$ is a field of characteristic p and order $q = p^\alpha$. For the definitions of the category of unstable modules over this algebra, we will rely on the very detailed book [**21**]. The definitions in [**21**] concern unstable modules over the Steenrod algebra, but extend readily to the unstable modules over the sub-bialgebra of reduced powers.

Definition 9.1 (The Steenrod algebra of reduced powers [**13**, Definition 6.5])**.** The **Steenrod algebra of q-th reduced powers** (without Bockstein operator, with grading divided by 2 if $p > 2$), is the bialgebra $\mathcal{A}(q)$ generated, as a unital associative algebra, by elements P^i for all $i > 0$ of degree $(q-1)i$, satisfying relations called **Adem relations**. The coproduct in $\mathcal{A}(q)$ is given by:

$$\Delta(P^i) = \sum_{j+k=i} P^j \otimes P^k,$$

where P^0 is understood to be the unit of $\mathcal{A}(q)$.

Definition 9.2 (Unstable modules over the Steenrod algebra, reduced modules, connected modules, the suspension)**.** A graded $\mathcal{A}(q)$-module $M = (M^i)_{i\in\mathbb{N}}$ is **unstable** if for all $x \in M^i$, $P^j x = 0$ whenever $j > i$. Let $\mathcal{U}(q)$ denote the full subcategory of $\mathcal{A}(q)_{\mathsf{mod}}$ with objects the unstable $\mathcal{A}(q)$-modules.

For any $\mathcal{A}(q)$-module M, and any element $x \in M^i$, let $P_0 x = P^i x$. This induces an endomorphism $P_0 \colon M \to M$ which multiplies the degree by q.

An unstable module M is said to be **reduced** if P_0 is injective.

An unstable module M is said to be **connected** if $M^0 = \mathbf{0}$.

Let M be an $\mathcal{A}(q)$-module. The **suspension** of M is the $\mathcal{A}(q)$-module ΣM such that $(\Sigma M)^0 = 0$ and $(\Sigma M)^i = M^{i-1}$ for all $i > 0$, and such that $P^i(\sigma x) = \sigma(P^i x)$, where $\sigma x \in (\Sigma M)^{j+1}$ denotes the element representing $x \in M^j$.

Remark 9.3. Our definition of a reduced unstable module differs from the usual notion of reduced unstable module over the Steenrod algebra [**21**, p. 47]. However, [**21**, Lemma 2.6.4] shows that the two notions are equivalent.

If M is unstable, ΣM is also unstable. This induces a functor $\Sigma\colon \mathcal{U}(q) \to \mathcal{U}(q)$.

Proposition 9.4 ([**21**, p. 28])**.** *The suspension functor $\Sigma\colon \mathcal{U}(q) \to \mathcal{U}(q)$ admits a left adjoint $\Omega\colon \mathcal{U}(q) \to \mathcal{U}(q)$. Moreover, for all unstable modules M,*

$$\Sigma\Omega M = M/P_0 M,$$

where $P_0 M \subseteq M$ denotes the unstable submodule which is the image of the top operation P_0.

Definition 9.5 (Free unstable modules, see [**21**, pp. 19, 23, 25])**.** For $n \in \mathbb{N}$, let $F(n)$ denote the **free unstable module** generated by one element ι_n of degree n. One has $\mathsf{Hom}_{\mathcal{U}(q)}(F(n), M) \cong M^n$.

The following assertions are all consequences of the definition of $F(n)$: As a consequence of Proposition 9.4, for all $n > 0$, there is an isomorphism in $\mathcal{U}(q)$:

$$\Omega F(n) \cong F(n-1).$$

Indeed, for M an unstable module, there is a one-to-one correspondence (natural in M):

$$\begin{aligned}\mathsf{Hom}_{\mathcal{U}(q)}(\Omega F(n), M) &\cong \mathsf{Hom}_{\mathcal{U}(q)}(F(n), \Sigma M)\\ &\cong (\Sigma M)^n\\ &\cong M^{n-1}\\ &\cong \mathsf{Hom}_{\mathcal{U}(q)}(F(n-1), M).\end{aligned}$$

We now study $\mathcal{P}$-algebras in $\mathcal{A}(q)$-modules and $\mathcal{P}$-algebras in $\mathcal{U}(q)$.

Proposition 9.6. *A $\mathcal{P}$-algebra in $\mathcal{A}(q)$-module (see Definition 3.4) is a $\mathcal{P}$-algebra M endowed with an action of $\mathcal{A}(q)$ that satisfies the (generalised) Cartan formula, that is, for all $\mu \in \mathcal{P}(n)$, $(x_i)_{1\leqslant i\leqslant n} \in M^{\times n}$,*

$$P^i \mu(x_1, \dots, x_n) = \sum_{i_1+\dots+i_n=i} \mu(P^{i_1}x_1, \dots, P^{i_n}x_n).$$

Proof. This is an easy verification, noting that the $n-1$-th iterated coproduct $\Delta^{n-1}\colon \mathcal{A}(q) \to \mathcal{A}(q)^{\otimes n}$ sends P^i to $\sum_{i_1+\dots+i_n=i} P^{i_1} \otimes \dots \otimes P^{i_n}$. □

Definition 9.7. A $\mathcal{P}$-algebra in $\mathcal{A}(q)$-modules which is unstable as an $\mathcal{A}(q)$-module is called a **$\mathcal{P}$-algebra in unstable module**.

Let $\mathcal{P}_{\mathsf{alg}}^{\mathcal{U}(q)}$ denote the full subcategory of $\mathcal{P}_{\mathsf{alg}}^{\mathcal{A}(q)}$ with objects the $\mathcal{P}$-algebras in unstable modules.

We then obtain the following straightforward result:

Proposition 9.8. *If M is an unstable module, then $S(\mathcal{P}, M)$ is a $\mathcal{P}$-algebra in unstable modules. The forgetful functor $\mathcal{P}_{\mathsf{alg}}^{\mathcal{U}(q)} \to \mathcal{U}(q)$ admits as left adjoint the functor $S(\mathcal{P}, -): \mathcal{U}(q) \to \mathcal{P}_{\mathsf{alg}}^{\mathcal{U}(q)}$.*

10. Unstable algebras over the Steenrod algebra

The classical notion of an unstable algebra over the Steenrod algebra is that of a unital, commutative, associative algebra A, which is also an unstable module over the Steenrod algebra, and satisfies the Cartan formula and the additional relation $P_0 x = x^q$, also called instability. In our terminology, this is an object of $\mathrm{uCom}_{\mathsf{alg}}^{\mathcal{U}(q)}$ satisfying the instability relation. A variant of this notion is the notion of unstable level algebra due to [**4**]. Here we generalise the notion of unstable algebra to algebras over any operad $\mathcal{P}$ equipped with a q-ary operation $\star$.

In this section, we introduce the Brown–Gitler and Carlsson modules and algebra. We choose to define these objects using their algebraic structures. They are identified with injective objects, as is shown in [**14**], see also [**1, 19, 3**], and [**21**]. These objects come equipped with a q-ary operation, but are not unstable algebras in the usual sense: in the Brown–Gitler algebra and Carlsson algebra for example, $P_0 x$ is not equal to x^q, but instead, $P_0 x = \phi(x)^q$ where ϕ is a certain endomorphism. These provide examples of unstable q-level algebras, where this notion has been introduced in Section 5. In Section 12, we will see that some of those objects are in fact free, as unstable algebras over certain operads.

Definition 10.1 ($\star$-Unstable $\mathcal{P}$-algebras)**.** Let $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$. A **$\star$-unstable $\mathcal{P}$-algebra** is a $\mathcal{P}$-algebra A in $\mathcal{U}(q)$ such that $P_0 a = \star(a, \dots, a)$ for all $a \in A$.

The $\star$-unstable $\mathcal{P}$-algebras form a full subcategory of $\mathcal{P}_{\mathsf{alg}}^{\mathcal{U}(q)}$ which is denoted by $\mathcal{K}_{\mathcal{P}}^{\star}$.

As an example, an **unstable q-level algebra** is a $\star$-unstable Lev_q-algebra, where Lev_q is defined in 5.1 and $\star \in \mathrm{Lev}_q$ is the operadic generator.

When $\mathcal{P} = \mathrm{uCom}$ and $\star = X_q$ (see Example 2.4), we recover the classical notion of unstable algebras from, for example, [**14**].

Let us give some examples of $\star$-unstable $\mathcal{P}$-algebras that appear in literature for different operads $\mathcal{P}$ and operations $\star$. The classic example is the following:

Example 10.2. Let X be a topological space. Taking $q = p$, the cohomology of X with coefficients in $\mathbb{F}_p$ inherits an unstable $\mathcal{A}(p)$-action (see, for example, [**21**, Theorem 1.1.1.]). The cup-product endows $H^*(X, \mathbb{F}_p)$ with the structure of a unital, commutative, associative algebra in $\mathcal{U}(p)$, with the additional relation $P_0 x = x^p$. In other words, $H^*(X, \mathbb{F}_p)$ is an X_p-unstable uCom-algebra (see Example 2.4 for the definition of uCom and X_p).

Definition 10.3 (Brown–Gitler modules and algebra, Carlsson modules and algebra)**.** The **Brown–Gitler algebra** J is the uCom-algebra in $\mathcal{U}(q)$ whose underlying uCom-algebra is the polynomial algebra $\mathbb{F}[x_i, i \in \mathbb{N}]$, with $|x_i| = 1$, endowed with the (unstable) action of $\mathcal{A}(q)$ induced on generators by:

$$P^j x_i = \begin{cases} x_i, & \text{if } j = 0, \\ x_{i-1}^q, & \text{if } j = 1, \\ 0, & \text{otherwise,} \end{cases} \tag{1}$$

where we set $x_{-1} = 0$.

The Brown–Gitler algebra is equipped with a second grading w called the **weight**, which is additive with respect to multiplication, and such that $w(x_i) = q^i$.

The **Brown–Gitler module of weight** n is the submodule $J(n) \subseteq J$ of homogeneous elements of weight n.

The **Carlsson algebra** K is the uCom-algebra in $\mathcal{U}(q)$ whose underlying uCom-algebra is the polynomial algebra $\mathbb{F}[x_i, i \in \mathbb{Z}]$, with $|x_i| = 1$, endowed with the (unstable) action of $\mathcal{A}(q)$ induced on generators by:

$$P^j x_i = \begin{cases} x_i, & \text{if } j = 0, \\ x_{i-1}^q, & \text{if } j = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

There is a canonical algebra morphism $K \to J$ sending $x_i \in K$ to $x_i \in J$ if $i \geqslant 0$, and sending $x_i \in K$ to 0 if $i < 0$. This morphism is clearly compatible with the $\mathcal{A}(q)$-action.

The Carlsson algebra is equipped with a second grading w called the **weight**, additive with respect to multiplication, with $w(x_i) = q^i$. Note that this weight has range $\mathbb{N}[\frac{1}{q}]$.

The **Carlsson module of weight** n is the submodule $K(n) \subseteq K$ of homogeneous elements of weight n.

Remark 10.4. Consider the algebra endomorphism $\phi\colon J \to J$ sending x_i to x_{i-1}. This induces morphisms of unstable modules $J(qn) \to J(n)$. For $n \in \mathbb{N}$, the Carlsson module $K(n)$ is isomorphic to the limit in $\mathcal{U}(q)$ of the diagram:

$$J(n) \xleftarrow{\phi} J(qn) \xleftarrow{\phi} J(q^2 n) \longleftarrow \cdots .$$

This implies that there are isomorphisms $K(n) \cong K(qn)$. In fact, these isomorphisms are induced by the algebra isomorphism $\phi\colon K \to K$ sending x_i to x_{i-1}.

From the very definition of these objects, it is clear that neither the Brown–Gitler algebra nor the Carlsson algebra are X_q-unstable uCom-algebras. However, we have the following:

Lemma 10.5. *The Brown–Gitler algebra and the Carlsson algebra are equipped with a $\star$-unstable* Lev_q*-algebra structure, where $\star$ and* Lev_q *are defined in 5.1.*

Proof. Note that $\phi\colon J \to J$ and $\phi\colon K \to K$ defined above are endomorphisms of uCom-algebras in $\mathcal{U}(q)$. So, J and K, endowed with ϕ, form two uCom $\circ$ D-algebras, where uCom $\circ$ D is defined in 4.8 and D acts by $d = \phi$. Since uCom $\circ$ D is isomorphic to $\mathbb{F}\Pi$ (see 4.8), and Lev_q is defined as a suboperad of $\mathbb{F}\Pi$ (see 5.1), it ensues that J and K can be equipped with the structure of a Lev_q-algebra in $\mathcal{U}(q)$ by restriction of structure. Since Lev_q is generated by the element $\star = (\emptyset, [q]) \in \mathbb{F}\Pi(q)$, which corresponds to the operation $(X_q; d, \dots, d)$ under the isomorphism $\mathbb{F}\Pi \cong \mathrm{uCom} \circ \mathrm{D}$, these Lev_q-algebra structures on J and K are defined by:

$$\star(a_1, \dots, a_q) = \phi(a_1) \cdots \phi(a_q),$$

for all elements $a_1, \dots, a_q$ of J, or of K. To show that J and K are $\star$-unstable, it suffices to check that $P_0 x_i = \star(x_i, \dots, x_i)$ on the generators x_i. But $P_0 x_i = x_{i-1}^q$, and $\star(x_i, \dots, x_i) = \phi(x_i)^q = x_{i-1}^q$. □

Lemma 10.6. *The Lev_q-operation $\star$ defined above on J and K preserve the weight.*

Proof. Note that ϕ divides the weight by q, so, if $a_1, \dots, a_q$ all have weight n, $\star(a_1, \dots, a_q) = \phi(a_1) \cdots \phi(a_q)$ is a product of q elements of weight n/q, which is then of weight n. □

11. Free unstable $\mathcal{P}$-algebras

The aim of this section is to prove our main result, Theorem 11.6, which identifies free unstable algebras generated by certain unstable modules to free algebras. More precisely, given an operad $\mathcal{P}$ equipped with a $\mathcal{P}$-central operation $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$ (see Definition 8.1), we will build a diagram of categories:

$$\begin{array}{ccc}
\mathsf{Vect}_{\mathbb{F}} & \xrightarrow{S(\mathcal{P},-)} & \mathcal{P}_{\mathsf{alg}} \\
{\scriptstyle \mathrm{D}\otimes -}\downarrow & & \uparrow \\
\mathrm{D}_{\mathsf{mod}} & \xrightarrow{\psi_!} & \mathcal{P}^{\mathrm{D}}_{\mathsf{alg}} \\
{\scriptstyle \varphi}\uparrow & & \uparrow{\scriptstyle \varphi_{\mathcal{P}}} \\
\mathcal{U}(q) & \xrightarrow{K^\star_{\mathcal{P}}} & \mathcal{K}^\star_{\mathcal{P}}
\end{array} \quad , \tag{D2}$$

where the top square is the diagram D1 from Proposition 8.4, and where the bottom functor, $K^\star_{\mathcal{P}} \colon \mathcal{U}(q) \to \mathcal{K}^\star_{\mathcal{P}}$, is left adjoint to the forgetful functor $\mathcal{K}^\star_{\mathcal{P}} \to \mathcal{U}(q)$. Here, the functor $\varphi \colon \mathcal{U}(q) \to \mathrm{D}_{\mathsf{mod}}$ sends an unstable module to its underlying vector space equipped with a linear self-map provided by the top operation P_0. We show that the $\mathcal{P}$-centrality of $\star$ implies that φ extends to a functor $\varphi_{\mathcal{P}} \colon \mathcal{K}^\star_{\mathcal{P}} \to \mathcal{P}^{\mathrm{D}}_{\mathsf{alg}}$. We show that the bottom square of this diagram commutes up to natural isomorphism.

We will then show that the image of certain unstable modules (namely, the connected, reduced unstable modules) under $\varphi \colon \mathcal{U}(q) \to \mathrm{D}_{\mathsf{mod}}$, are in the essential image of $D \otimes - \colon \mathsf{Vect}_{\mathbb{F}} \to \mathrm{D}_{\mathsf{mod}}$.

For any unstable module M, denote by φM the D-module whose underlying vector space is M and such that $dx = P_0 M$. This induces a functor $\varphi \colon \mathcal{U}(q) \to \mathrm{D}_{\mathsf{mod}}$.

Lemma 11.1. *Let A be a $\mathcal{P}$-algebra in $\mathcal{U}(q)$. Then, P_0 is a $\mathcal{P}$-algebra endomorphism of A. In other words, the functor $\varphi \colon \mathcal{U}(q) \to \mathrm{D}_{\mathsf{mod}}$ extends to a functor $\varphi_{\mathcal{P}} \colon \mathcal{P}^{\mathcal{U}(q)}_{\mathsf{alg}} \to \mathcal{P}^{\mathrm{D}}_{\mathsf{alg}}$. For all $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$, this restricts to a functor $\bar{\varphi}_{\mathcal{P}} \colon \mathcal{K}^\star_{\mathcal{P}} \to \mathcal{P}^{\mathrm{D}}_{\mathsf{alg}}$.*

Proof. This is a straightforward generalisation of [**11**, Corollary 2.7], which covers the case $q = 2$. □

Recall that, for any $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$, we built, in Section 7, a functor $\psi_!\colon \mathrm{D}_{\mathsf{mod}} \to \mathcal{P}_{\mathsf{alg}}$ that sends a D-module M to the free $\mathcal{P}$-algebra over M modulo the relation $dx = \star(x,\dots,x)$ for all $x \in M$.

Proposition 11.2. *The functor $\psi_! \circ \varphi\colon \mathcal{U}(q) \to \mathcal{P}_{\mathsf{alg}}$, sending M to*

$$S(\mathcal{P}, M)/(P_0^k x - \star_k(x,\dots,x), x \in M, k \in \mathbb{N}),$$

extends into a functor $K_{\mathcal{P}}^{\star}\colon \mathcal{U}(q) \to \mathcal{P}_{\mathsf{alg}}^{\mathcal{U}(q)}$.

Additionally, if $\star$ is $\mathcal{P}$-central (see Definition 8.1), then $K_{\mathcal{P}}^{\star}$ restricts into a functor $K_{\mathcal{P}}^{\star}\colon \mathcal{U}(q) \to \mathcal{K}_{\mathcal{P}}^{\star}$.

Proof. Note that $S(\mathcal{P}, M)$ is also the free $\mathcal{P}$-algebra in $\mathcal{U}(q)$ generated by M. In other words, $S(\mathcal{P}, M)$ can be equipped with an unstable $\mathcal{A}(q)$-action by:

$$P^i \cdot (\mu; x_1, \dots, x_n) = \sum_{j_1+\dots+j_n = i} (\mu; P^{j_1}x_1, \dots, P^{j_n}x_n).$$

To make sure that this induces an unstable $\mathcal{A}(q)$-action on:

$$S(\mathcal{P}, M)/(P_0^k x - \star_k(x,\dots,x), x \in M, k \in \mathbb{N}),$$

it suffices to check that the set $X = \{P_0^k x - \star_k(x,\dots,x), x \in M, k \in \mathbb{N}\}$ is stable under the action of $\mathcal{A}(q)$. For all $k, i \in \mathbb{N}$ and $x \in M$, one can show that

$$P^i P_0^k x = \begin{cases} P_0^k P^{\frac{i}{q^k}} x, & \text{if } q^k | i,\\ 0, & \text{otherwise.}\end{cases}$$

See for example, [**21**, Section 1.7].

We will show, by induction on k, that,

$$P^i x^{\star_k q^k} = \begin{cases} \left(P^{\frac{i}{q^k}} x\right)^{\star_k q^k}, & \text{if } q^k | i,\\ 0, \ \text{otherwise.}\end{cases} \tag{Q_k}$$

For the case $k = 0$, there is nothing to prove. Suppose Q_k proven. Denote by $\mathsf{t} = x^{\star_k q^k}$ Then,

$$\begin{aligned} P^i x^{\star_{k+1} q^{k+1}} &= P^i \star (\mathsf{t}, \dots \mathsf{t})\\ &= \sum_{j_1+\dots+j_q} \star(P^{j_1}\mathsf{t}, \dots, P^{j_q}\mathsf{t}).\end{aligned}$$

Fix $j'_1 \leqslant \dots \leqslant j'_q \in \mathbb{N}$ such that $j'_1 + \dots + j'_{q^k} = i$. Suppose that we have:

$$j'_1 = j'_2 = \dots = j'_{r_1} < j'_{r_1+1} = \dots = j'_{r_1+r_2} < \dots < j'_{r_1+\dots+r_{s-1}+1} = \dots = j'_{r_1+\dots+r_s},$$

with $r_1 + \dots + r_s = q$, that is, $r_1 + \dots + r_s$ is the coarsest partition of the integer q such that $j'_l = j'_{l'}$ if and only if there exists $m \in [s]$ such that

$$r_1 + \dots + r_{m-1} < l, l' < r_1 + \dots + r_m.$$

Since $\star$ is fixed under the action of $\mathfrak{S}_q$, the term $\star(P^{j'_1}\mathfrak{s},\dots,P^{j'_q}\mathfrak{s})$ appears in the above sum exactly $\frac{q!}{r_1!\dots r_s!}$ times. This integer is divisible by q unless $s=1$, in which case, $r_1=q$, and the condition $j'_1+\dots+j'_q=i$ implies that $i\equiv 0\ [q]$ and $j'_1=\dots=j'_q=\frac{n}{q}$.

This implies that

$$P^i\star(\mathfrak{t},\dots,\mathfrak{t})=\begin{cases}\star\left(P^{i/q}\mathfrak{t},\dots,P^{i/q}\mathfrak{t}\right), & \text{if } i\equiv 0\ [q]\\ 0, & \text{otherwise.}\end{cases}$$

Now, the induction hypothesis gives:

$$P^{i/q}\mathfrak{t}=P^{i/q}(\star_k;x,\dots,x)=\begin{cases}\left(\star_k;P^{i/q^{k+1}}x,\dots,P^{i/q^{+1}}x\right), & \text{if } q^{k+1}|i\\ 0, & \text{otherwise,}\end{cases}$$

and so:

$$P^ix^{\star_{k+1}q^{k+1}}=\begin{cases}\left(\star_{k+1};P^{i/q^{k+1}}x,\dots,P^{i/q^{+1}}x\right), & \text{if } q^{k+1}|i\\ 0, & \text{otherwise.}\end{cases}$$

We have proven Q_k for all $k\in\mathbb{N}$. Since both 0 and $P_0P^{\frac{i}{q^k}}x-\left(\star_k;P^{i/q^k}x,\dots,P^{i/q^k}x\right)$ belong to X, we conclude that X is stable under the action of $\mathcal{A}(q)$. So, we defined an unstable $\mathcal{A}(q)$-action on $\psi_!\circ\varphi(M)$ which is compatible with the $\mathcal{P}$-algebra structure.

To prove the second assertion of our proposition, suppose now that $\star$ is $\mathcal{P}$-central. We want to show that $\psi_!\circ\varphi(M)$, with the above $\mathcal{A}(q)$-action, is $\star$-unstable. In other words, we want to show that, for all $\mu\in\mathcal{P}(n)$, $x_1,\dots,x_n\in M$, one has:

$$P_0(\mu;x_1,\dots,x_n)\sim_X\star((\mu;x_1,\dots,x_n)^{\times q})=(\star(\mu^{\times q});x_1,\dots,x_n,\dots,x_1,\dots,x_n),$$

where $\sim_X$ denotes the equivalence relation defined by the quotient by the $\mathcal{P}$-ideal generated by X. From Lemma 11.1, we deduce that:

$$P_0(\mu;x_1,\dots,x_n)=(\mu;P_0x_1,\dots,P_0x_n).$$

But, one has:

$$(\mu;P_0x_1,\dots,P_0x_n)\sim_X(\mu,\star(x_1^{\times q}),\dots,\star(x_n^{\times q})).$$

This last element is equal to:

$$(\mu(\star^{\times n});x_1^{\times q},\dots,x_n^{\times q}).$$

Using the fact that $\star$ is $\mathcal{P}$-central, $\mu(\star^{\times n})=\star(\mu^{\times q})\cdot\sigma_{q,n}^{-1}$, so,

$$(\mu(\star^{\times n});x_1^{\times q},\dots,x_n^{\times q})=(\star(\mu^{\times q});x_1,\dots,x_n,\dots,x_1,\dots,x_n).$$

This concludes our proof. □

Proposition 11.3. *Let $\star\in\mathcal{P}(q)^{\mathfrak{S}_q}$ be a $\mathcal{P}$-central operation. Then $K_{\mathcal{P}}^{\star}(M)$ is the free $\star$-unstable $\mathcal{P}$-algebra over M. In other words, $K_{\mathcal{P}}^{\star}\colon\mathcal{U}(q)\to\mathcal{K}_{\mathcal{P}}^{\star}$ is a left adjoint to the forgetful functor $\mathcal{K}_{\mathcal{P}}^{\star}\to\mathcal{U}(q)$.*

Proof. This is a somewhat straightforward generalisation of [**11**, Proposition 6.7]. Since we are using different constructions in this article, let us give the detailed proof.

Let M be an object in $\mathcal{U}(q)$, A an object in $\mathcal{K}^\star_\mathcal{P}$. We show that there is a bijection $\mathsf{Hom}_{\mathcal{U}(q)}(M, A) \cong \mathsf{Hom}_{\mathcal{K}^\star_\mathcal{P}}(K^\star_\mathcal{P}(M), A)$.

Recall from the construction of $K^\star_\mathcal{P}$ that there is a surjective morphism

$$S(\mathcal{P}, M) \to K^\star_\mathcal{P}(M)$$

in $\mathcal{U}(q)$ (see the proof of 11.2). The unit of the monad $S(\mathcal{P}, -)$ provides a morphism of unstable modules $M \to S(\mathcal{P}, M)$. To any morphism $f\colon K^\star_\mathcal{P}(M) \to A$ in $\mathcal{K}^\star_\mathcal{P}$, we associate the morphism in $\mathcal{U}(q)$ which is the following composition:

$$\bar{f}\colon\ M \longrightarrow S(\mathcal{P}, M) \longrightarrow K^\star_\mathcal{P}(M) \xrightarrow{f} A.$$

Now suppose $g\colon M \to A$ is a morphism in $\mathcal{U}(q)$. Then there is a unique morphism $g\colon S(P, M) \to A$ in $\mathcal{P}^{\mathcal{U}(q)}_{\mathsf{alg}}$. For any $x \in M$ and $k \in \mathbb{N}$, since g' is compatible with the action of $\mathcal{A}(q)$, $g'(P_0^k x) = P_0^k g'(x)$. Since g' is compatible with the $\mathcal{P}$-algebra structures, one has $g'(\star_k(x, \dots, x)) = \star_k(g'(x), \dots, g'(x))$. Finally, since A is $\star$-unstable, $P_0^k g'(x) = \star_k(g'(x), \dots, g'(x))$. So, for any $x \in M$ and $k \in \mathbb{N}$, $g'(P_0^k x - \star_k(x, \dots, x)) = 0$, which implies that g' passes to the quotient into a morphism

$$\hat{g}\colon S(P, M)/(P_0^k x - \star_k(x, \dots, x), x \in M, k \in \mathbb{N}) \to A$$

in $\mathcal{P}^{\mathcal{U}(q)}_{\mathsf{alg}}$, which can be seen as a morphism $\hat{g}\colon K^\star_\mathcal{P}(M) \to A$. Showing that the associations $f \mapsto \bar{f}$ and $g \mapsto \hat{g}$ provide inverse bijections between $\mathsf{Hom}_{\mathcal{U}(q)}(M, A)$ and $\mathsf{Hom}_{\mathcal{K}^\star_\mathcal{P}}(K^\star_\mathcal{P}(M), A)$ is a straightforward verification that is left to the reader. □

Remark 11.4. The construction of the categories $\mathcal{P}^{\mathcal{U}(q)}_{\mathsf{alg}}$, $\mathcal{K}^\star_\mathcal{P}$ and of the functor $K^\star_\mathcal{P}$ are natural with respect to $\mathcal{P}$, and to $\star$, in the following sense: let $f\colon \mathcal{P} \to \mathcal{Q}$ be an operad morphism. Then, f induces a restriction functor $f^*\colon \mathcal{Q}^{\mathcal{U}(q)}_{\mathsf{alg}} \to \mathcal{P}^{\mathcal{U}(q)}_{\mathsf{alg}}$. One can readily check that, for all $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$, this restricts into a functor: $f^*\colon \mathcal{K}^{f(\star)}_\mathcal{Q} \to \mathcal{K}^\star_\mathcal{P}$.

In this same setting, $f\colon \mathcal{P} \to \mathcal{Q}$ also induces a morphism

$$f_*\colon K^\star_\mathcal{P}(M) \to f^*\left(K^{f(\star)}_Q(M)\right),$$

natural in M.

Lemma 11.5. *For any connected reduced unstable module M, $\varphi(M)$ is isomorphic to $D \otimes \mathrm{Forget}(\Sigma\Omega M)$, where $\mathrm{Forget}\colon \mathcal{U}(q) \to \mathsf{Vect}_\mathbb{F}$ is the forgetful functor that extracts the underlying vector space of an unstable module. This isomorphism is not unique in general.*

Proof. The unit of the adjunction $\Sigma \dashv \Omega$ from Proposition 9.4 now provides a map $M \to \Sigma\Omega M = M/P_0M$. Pick any linear section $s\colon M/P_0M \to M$ (this is not a morphism in $\mathcal{U}(q)$). The fact that M is reduced connected and that P_0 is injective and multiplies the degree by q then implies that M is freely generated by $s(\Sigma\Omega M)$ under the action of P_0. □

We now obtain our main result as an easy consequence of the preceding results:

Theorem 11.6. *Let $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$ be a $\mathcal{P}$-central operation. For all connected reduced M, the underlying $\mathcal{P}$-algebra of $K^\star_\mathcal{P}(M) \in \mathcal{K}^\star_\mathcal{P}(M)$ is isomorphic to $S(\mathcal{P}, \mathrm{Forget}(\Sigma\Omega M))$. This isomorphism is not unique in general.*

Proof. Consider the diagram of categories D2 from the beginning of the present section. Proposition 8.4 shows that the top square commutes up to natural isomorphism. The functor $K_{\mathcal{P}}^{\star}$ has been constructed in Proposition 11.2 so that the bottom square commutes up to natural isomorphism.

Let M be a connected, reduced unstable module. Lemma 11.5 shows that $\varphi(M) = D \otimes \mathrm{Forget}(\Sigma\Omega M)$. So, the $\mathcal{P}$-algebra in D-modules $\varphi \circ K_{\mathcal{P}}^{\star}(M)$ is isomorphic to $\psi_!(D \otimes \mathrm{Forget}(\Sigma\Omega M))$. Its underlying $\mathcal{P}$-algebra is then isomorphic to the free $\mathcal{P}$-algebra $S(\mathcal{P}, \mathrm{Forget}(\Sigma\Omega M))$. Since the underlying $\mathcal{P}$-algebra of $\varphi \circ K_{\mathcal{P}}^{\star}(M)$ is also the underlying $\mathcal{P}$-algebra of $K_{\mathcal{P}}^{\star}(M)$, we obtain the result. □

12. Applications

In this section, we apply our result to free unstable modules to obtain a description of the free $\star$-unstable $\mathcal{P}$-algebra generated by one element. This allows us identify certain Brown–Gitler modules and Carlsson modules from Section 10 to free unstable algebras over certain operads.

Theorem 11.6 shows that, for $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$ a $\mathcal{P}$-central operation, the free $\star$-unstable $\mathcal{P}$-algebra generated by a connected, reduced unstable module M is itself free as a $\mathcal{P}$-algebra, generated by the underlying vector space of $\Sigma\Omega M$. In particular, if $\Sigma\Omega M$ comes with a natural choice of linear basis, and when the operad $\mathcal{P}$ comes itself with a basis, we can give a basis of the $\mathcal{P}$-algebra $K_{\mathcal{P}}^{\star}(M)$.

Recall from Section 9 that the free unstable module $F(n)$ over an element of degree n satisfies $\Sigma\Omega F(n) \cong \Sigma F(n-1)$. In particular, the underlying vector space of $\Sigma\Omega F(1) \cong \Sigma F(0)$ is one dimensional, concentrated in degree 1. We then get:

Proposition 12.1. *Let $\star \in \mathcal{P}(q)^{\mathfrak{S}_q}$ be a $\mathcal{P}$-central operation. Then, $K_{\mathcal{P}}^{\star}(F(1))$ is the $\mathcal{P}$-algebra in $\mathcal{U}(q)$ whose underlying $\mathcal{P}$-algebra is freely generated by one element ι_1 of degree 1, endowed with the unstable action of $\mathcal{A}(q)$ induced by:*

$$P^j \iota_1 = \begin{cases} \iota_1, & \text{if } j = 0, \\ \star(\iota_1, \ldots, \iota_1), & \text{if } j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This is a straightforward generalisation of [**11**, Proposition 8.4], which treats the case $q = 2$. In particular, since $\Sigma\Omega F(1) \cong \Sigma F(0)$, and applying Theorem 11.6, we know that $K_{\mathcal{P}}^{\star}(F(1))$ is a $\star$-unstable $\mathcal{P}$-algebra whose underlying $\mathcal{P}$-algebra is generated by one element ι_1 of degree 1. Since $K_{\mathcal{P}}^{\star}(F(1))$ is a $\mathcal{P}$-algebra in $\mathcal{U}(q)$, it suffices to inspect the action of $\mathcal{A}(q)$ on ι_1, and the instability relation reads $P^i \iota_1 = 0$ for all $i > 1$. Since $K_{\mathcal{P}}^{\star}(F(1))$ is $\star$-unstable, we necessarily have $P^1 \iota_1 = P_0 \iota_1 = \star(\iota_1, \ldots, \iota_1)$. □

Example 12.2. When $\mathcal{P} = \mathrm{uCom}$ and $\star = X_q$, the functor $K_{X_q}^{\mathrm{uCom}}$ corresponds to the functor denoted by U in [**15**]. In this case, Theorem 11.6 corresponds to a remark of Kuhn [**15**, p. 4223]. In [**15**, Theorem 1.6], Kuhn identifies a large family of free unstable algebras generated by unstable modules defined as representations of symmetric powers. This contains, for example, the computation of the mod 2 cohomology of Eilenberg–MacLane spaces of finite elementary abelian 2-groups [**15**, Remark 1.8 (2)].

Theorem 12.3 (Compare with [**11**, Proposition 9.10])**.** *The Carlsson algebra K (see Definition 10.3) with the multiplication of monomials and the operator ϕ, is isomorphic to the free $(X_q; d, \dots, d)$-unstable* $\mathrm{uCom} \circ \mathrm{D}^{\pm}$*-algebra generated by $F(1)$.*

Proof. This is a straightforward generalisation of the proof of the first assertion of [**11**, Proposition 9.10], which treats the case $q = 2$. □

From Theorem 12.3, we deduce:

Theorem 12.4. *The Carlsson module of weight one $K(1)$ (see Definition 10.3) with its* Lev_q *operation from Lemma 10.6, is isomorphic to the free $\star$-unstable* Lev_q*-algebra generated by $F(1)$.*

Proof. We use the naturality of $K^{\star}_{\mathcal{P}}$ (see remark 11.4). In particular, recall from 5.1 the definition of the operad Lev_q as a suboperad of $\mathbb{F}\Pi$. Proposition 4.8 then implies that we have an injective operad morphism $\mathrm{Lev}_q \to \mathrm{uCom} \circ \mathrm{D}$, and $\mathrm{uCom} \circ \mathrm{D}$ embeds in $\mathrm{uCom} \circ \mathrm{D}^{\pm}$. Hence, following remark 11.4, the resulting morphism of operads $f\colon \mathrm{Lev}_q \to \mathrm{uCom} \circ \mathrm{D}^{\pm}$ induces an injective morphism of $\star$-unstable Lev_q-algebras $K^{\star}_{\mathrm{Lev}_q}(F(1)) \to f^* \left(K^{(X_q; d, \dots, d)}_{\mathrm{uCom} \circ \mathrm{D}^{\pm}}(F(1)) \right)$. Using Theorem 12.3, we can therefore view $f^* \left(K^{(X_q; d, \dots, d)}_{\mathrm{uCom} \circ \mathrm{D}^{\pm}}(F(1)) \right)$ as the $\star$-unstable Lev_q-algebra given by the Carlsson algebra K and with q-level multiplication as in Lemma 10.5. We want to identify the image of $K^{\star}_{\mathrm{Lev}_q}(F(1))$ as a sub-q-level algebra of K. For this, recall (see Lemma 6.1) that Lev_q is spanned, as a suboperad of $\mathrm{uCom} \circ \mathrm{D}^{\pm}$, by elements

$$(X_m; d^{i_1}, \dots, d^{i_m}) \in \mathrm{uCom} \circ \mathrm{D}^{\pm}(m)$$

such that $\sum_{j=1}^{m} \frac{1}{q^{i_j}} = 1$. According to Proposition 12.1, $K^{\star}_{\mathrm{Lev}_q}(F(1))$ is then spanned by the elements $((X_m; d^{i_1}, \dots, d^{i_m}); \iota_1, \dots, \iota_1)$. Using the operad morphism f, these are identified with the monomials $x_{i_1} \cdots x_{i_m}$ in K such that $\sum_{j=1}^{m} \frac{1}{q^{i_j}} = 1$, that is, all the monomials of weight 1. □

Proposition 12.5. *The Carlsson module $K(1)$ is the limit of the Brown–Gitler module $J(q^s)$ in the category of $\star$-unstable* Lev_q*-algebras.*

Proof. Recall from Remark 10.4 that $K(1)$ is isomorphic to the limit of the following diagram in $\mathcal{U}(q)$:

$$J(n) \xleftarrow{\phi} J(qn) \xleftarrow{\phi} J(q^2 n) \longleftarrow \cdots .$$

Following Lemmas 10.5 and 10.6, $J(q^s)$ is equipped with a $\star$-unstable Lev_q-algebra structure for all s. Since the map $\phi\colon J(q^{s+1}) \to J(q^s)$ sends x_{s+1} to x_s, and is compatible with the Lev_q-algebra structures, the diagram above is a diagram in $\star$-unstable Lev_q-algebras. Hence, $K(1)$ is the limit of this diagram in the category of $\star$-unstable Lev_q-algebras. □

Theorem 12.6. *For all $s \geqslant 1$, the Brown–Gitler module $J(q^s)$, with its* Lev_q *operation from Lemma 10.6, is isomorphic to the free $\star$-unstable T_s* Lev_q*-algebra (see 5.6) generated by $F(1)$.*

Proof. Consider the diagram of operads $(*)$ of 5.7. Using the naturality of our constructions (see remark 11.4), we get, for all $s \geqslant 1$, a surjective morphism of $\star$-unstable q-level algebras $K^{\star}_{\mathrm{Lev}_q}(F(1)) \to K^{\star}_{T_s\,\mathrm{Lev}_q}(F(1))$. By the identification $K^{\star}_{\mathrm{Lev}_q}(F(1)) \cong K(1)$ as above, one can check that $K^{\star}_{T_s\,\mathrm{Lev}_q}(F(1))$ is, as a q-level algebra, the quotient of $K(1)$ by all x_i such that $i > s$. It follows that $K^{\star}_{T_s\,\mathrm{Lev}_q}(F(1))$ is isomorphic, as a q-level algebra (and so, as a $T_s\,$Lev-algebra), to the Brown–Gitler module $J(q^s)$. In fact, the diagram of $\star$-unstable q-level algebras:

$$J(n) \xleftarrow{\phi} J(qn) \xleftarrow{\phi} J(q^2n) \longleftarrow \cdots,$$

whose limit is $K(1)$, is exactly the diagram of $\star$-unstable q-level algebras obtained from the diagram $(*)$ of 5.7 by naturality of our constructions. □

Remark 12.7. Theorem 12.4 shows that the Carlsson algebra K can be identified with the free d^{X_2}-unstable $\mathrm{uCom} \circ \mathrm{D}^{\pm}$-algebra generated by $F(1)$. Restriction along the inclusion of operads $\Pi = \mathrm{uCom} \circ \mathrm{D} \to \mathrm{uCom} \circ \mathrm{D}^{\pm}$ makes K a (non-free) d^{X_2}-unstable Π-algebra, and the quotient of K by the ideal generated by x_i for all $i < 0$ yields the Brown–Gitler algebra J, seen as a Π-algebra. As a Π-ideal, this ideal is generated by the unique element $x_{-1} = d\iota_1$.

Similarly, when $q = 2$ considering $K^{d^{X_2}}_{\mathrm{uCom} \circ \mathrm{D}^{\pm}}(F(n))$ as a d^{X_2}-unstable Π-algebra, and quotienting by the ideal generated by $d\iota_n$, yields a description of the algebras denoted by $H^*(T(n,*), \mathbb{F}_2)$ in [**15**]. As the notation suggests, these algebras are obtained as the cohomology of a spectrum, which is related to the Eilenberg–MacLane spaces $K(V, n)$.

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