

ON STRICT POLYNOMIAL FUNCTORS WITH BOUNDED DOMAIN

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Abstract

We introduce a new functor category: the category $\mathcal{P}_{d,n}$ of strict polynomial functors of degree d with domain of dimension bounded by n . It is equivalent to the category of finite dimensional modules over the Schur algebra $S(n, d)$, hence it allows one to apply the tools available in functor categories to representations of the algebraic group GL_n . We investigate in detail the homological algebra in $\mathcal{P}_{d,n}$ for $d = p$, where $p > 0$ is the characteristic of a ground field. We also establish equivalences between certain subcategories of $\mathcal{P}_{d,n}$'s which resemble the Spanier-Whitehead duality in stable homotopy theory.

1. Introduction

It was demonstrated in a breakthrough work [10] that the category of functors from the category of finite dimensional vector spaces over a finite field \mathbf{k} to the category of all vector spaces over \mathbf{k} is a valuable tool for investigating representations of $\mathrm{GL}_n(\mathbf{k})$ for $n \gg 0$. This approach has turned out to be particularly effective in a study of homological problems for $\mathrm{GL}_n(\mathbf{k})$ -modules, like computing Ext-groups between important objects. Even more impressive results were obtained by using the category \mathcal{P}_d of strict polynomial functors of degree d introduced by Friedlander and Suslin in [11], which may be thought of as an algebro-geometric version of the afore-mentioned category of functors. This variant allows one to enjoy both flexibility offered by a functor category and various rich algebraic structures like the structure of a highest weight category, block theory etc. What is interesting here is that it can be shown that \mathcal{P}_d is equivalent to the category of finite dimensional modules over the Schur algebra $S(n, d)$ for $n \geq d$ [11, Thm. 3.2], which is a classically studied object. However, some crucial technical tools available in \mathcal{P}_d , like the sum-diagonal adjunction, are completely invisible from the level of modules over the Schur algebra.

In the present article we introduce the functor category $\mathcal{P}_{d,n}$, which is a modification of \mathcal{P}_d dedicated to the study of representations of $\mathrm{GL}_n(\mathbf{k})$ also for small values of n . Technically, our definition is quite straightforward: our definition differs from that of \mathcal{P}_d by the fact we restrict the domain of functors to the spaces of dimension at most n .

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Thus our category is closely related to \mathcal{P}_d and shares its basic properties, however, one quickly realizes that it is much more complicated than \mathcal{P}_d . In particular, we do not have the sum-diagonal adjunction anymore, which has some interesting consequences (see Section 2). Nevertheless, the functorial approach is still beneficial in our situation as we demonstrate in Section 2 by providing some non-trivial Ext-computations. We also introduce in Section 3 a certain duality, which is closely related to the linear duality for GL_n -modules, though it seems not to have any counterpart for \mathcal{P}_d . We think of our work as the starting point for a systematic study of functor categories in “the unstable case” and we hope to develop our investigations in future works.

Now we briefly describe the contents of the article. In Section 1 we introduce the category $\mathcal{P}_{d,n}$ and establish its basic properties. We show that it is equivalent to the category of finite dimensional modules over $S(n, d)$ (Thm. 1.2) and we recall the known relationship of the latter with the category of representations of the algebraic group GL_n . We also study the relationship between $\mathcal{P}_{d,n}$ and \mathcal{P}_d , which is best understood in terms of recollement diagrams (Thm. 1.4). It will be an important tool in the computations performed in Section 2.

In Section 2 we study in detail the homological algebra in $\mathcal{P}_{p,n}$, where p is the characteristic of a ground field. We observe many similarities to the case of \mathcal{P}_p , which was studied in [14] (e.g. the formality of the Yoneda algebra of simples, see Thm. 2.2), but we also notice important differences. In particular, we compute the groups $\mathrm{Ext}_{\mathcal{P}_{p,n}}^*(I^{\otimes p}, F)$ for various F 's, which are specific to “the unstable case” and we relate it to the classical Schur functor (Cor. 2.6). At last we establish quite surprising connection between $\mathcal{P}_{p,p-1}$ and the category of representations of the symmetric group Σ_p at the level of K -theory (Cor. 2.7).

In Section 3 we establish certain equivalences between certain subcategories of $\mathcal{P}_{d,n}$'s, which have formal similarities with the Spanier-Whitehead duality in stable homotopy theory. Our construction is somewhat surprising, since it provides a functorial interpretation of some classical computations of duals of representations which seemed not to fit into a categorical context.

2. The category $\mathcal{P}_{d,n}$

Throughout the article we work over a fixed ground field \mathbf{k} .

In this section we define the category of homogeneous strict polynomial functors of degree d with bounded domain in the spirit of the Pirashvili and Kuhn reformulation of the definition of strict polynomial functors [11] (c.f. [16], [20]) and we discuss properties of that category. The category of strict polynomial functors of degree d is denoted by \mathcal{P}_d .

Let $\mathcal{Vect}_{\mathbf{k}}^{\leq n}$ be the category of \mathbf{k} -linear spaces with dimension at most n . We define the category $\Gamma^d \mathcal{Vect}_{\mathbf{k}}^{\leq n}$ as follows. The objects of that category are the same as the objects of $\mathcal{Vect}_{\mathbf{k}}^{\leq n}$ and for two linear spaces V, W we set

$$\mathrm{Hom}_{\Gamma^d \mathcal{Vect}_{\mathbf{k}}^{\leq n}}(V, W) = \Gamma^d \mathrm{Hom}_{\mathbf{k}}(V, W),$$

where $\mathrm{Hom}_{\mathbf{k}}(V, W)$ is the linear space of \mathbf{k} -linear maps between V and W . The composition of morphisms is determined by the natural identification

$$\Gamma^d \text{Hom}_{\mathbf{k}}(V, W) \simeq \text{Hom}_{\mathbf{k}}(V^{\otimes d}, W^{\otimes d})^{\Sigma_d}.$$

Finally, the category $\mathcal{P}_{d,n}$ of homogeneous strict polynomial functors of degree d with bounded domain $\Gamma^d \mathcal{V}ect_{\mathbf{k}}^{\leq n}$ is defined as the category of \mathbf{k} -linear functors from $\Gamma^d \mathcal{V}ect_{\mathbf{k}}^{\leq n}$ to the category of finite dimensional linear spaces $\mathcal{V}ect_{\mathbf{k}}$. $\mathcal{P}_{d,n}$ is an abelian category. The Kuhn duality $(-)^{\#}$ in $\mathcal{P}_{d,n}$ is defined as the restriction of the Kuhn duality in \mathcal{P}_d to the category $\Gamma^d \mathcal{V}ect_{\mathbf{k}}^{\leq n}$ and we define the tensor product in $\mathcal{P}_{d,n}$ in the same manner. Let $\Gamma^{d,n} = \Gamma^d \circ \text{Hom}(\mathbf{k}^n, -)$ and $S^{d,n} = (\Gamma^{d,n})^{\#}$.

Proposition 2.1. $\Gamma^{d,n}$ is a compact projective generator of $\mathcal{P}_{d,n}$ and $S^{d,n}$ is an injective cogenerator of $\mathcal{P}_{d,n}$. In particular, $\mathcal{P}_{d,n}$ has enough injective and projective objects.

Proof. By the Yoneda lemma, we have:

$$\text{Hom}_{\mathcal{P}_{d,n}}(\Gamma^{d,n}, F) \simeq F(\mathbf{k}^n), \quad (1)$$

for any $F \in \mathcal{P}_{d,n}$, hence $\Gamma^{d,n}$ is a projective functor. We also see from (1) that $\Gamma^{d,n}$ is compact. It remains to prove that $\Gamma^{d,n}$ is a generator of $\mathcal{P}_{d,n}$, i.e. $\text{Hom}_{\mathcal{P}_{d,n}}(\Gamma^{d,n}, -)$ is a faithful functor. By the Yoneda lemma it is sufficient to show that for any $F, G \in \mathcal{P}_{d,n}$, if $\eta: F \rightarrow G$ is a natural transformation such that $\eta(\mathbf{k}^n) = 0$ then $\eta = 0$, but it follows from the fact that for any $m < n$, \mathbf{k}^m is a retract of \mathbf{k}^n in $\mathcal{V}ect_{\mathbf{k}}^{\leq n}$, hence also in $\Gamma^d \mathcal{V}ect_{\mathbf{k}}^{\leq n}$.

By the Kuhn duality $S^{d,n}$ is an injective cogenerator of $\mathcal{P}_{d,n}$. The last statement is evident from the first part of the proposition. \square

In the sequel $\text{GL}_n = \text{GL}_n(\mathbf{k})$ is regarded as an algebraic group over \mathbf{k} . Let GL_n^{rat} -mod be the category of rational left GL_n -modules. We denote by $\text{GL}_n^{\text{Pol},d}$ -mod the category of finite dimensional homogeneous polynomial GL_n -modules of degree d . Let $S(n, d)$ -mod be the category of finite dimensional left modules over the Schur algebra $S(n, d)$.

Theorem 2.2. *There are equivalences of abelian categories $\mathcal{P}_{d,n} \simeq S(n, d)$ -mod and $\mathcal{P}_{d,n} \simeq \text{GL}_n^{\text{Pol},d}$ -mod. In particular, $\mathcal{P}_{d,n} \simeq \mathcal{P}_d$ for $n \geq d$.*

Proof. By Proposition 2.1 and the Gabriel theorem (c.f. [3, II Thm. 1.3]) we obtain the equivalence

$$\mathcal{P}_{d,n} \simeq \text{End}_{\mathcal{P}_{d,n}}(\Gamma^{d,n}) - \text{mod}.$$

It follows from (1) that

$$\text{End}_{\mathcal{P}_{d,n}}(\Gamma^{d,n}) \simeq \Gamma^{d,n}(\mathbf{k}^n) = \Gamma^d(\text{End}(\mathbf{k}^n)) \simeq S(n, d),$$

hence

$$\mathcal{P}_{d,n} \simeq \text{End}_{\mathcal{P}_{d,n}}(\Gamma^{d,n}) - \text{mod} \simeq S(n, d) - \text{mod}. \quad (2)$$

It immediately implies the last statement, because $\mathcal{P}_d \simeq S(n, d)$ -mod for $n \geq d$ (c.f. [11, Thm. 3.2]). The second equivalence in Theorem follows from the first one and the well-known equivalence $S(n, d) - \text{mod} \simeq \text{GL}_n^{\text{Pol},d} - \text{mod}$ (c.f. [17, Thm. 2.2.7]). \square

Let us recall that \mathcal{P}_d is a highest weight category in the sense of [7] with poset $\Lambda(d)$ of Young diagrams of weight d with the reversed dominance order (c.f. [15,

Thm. 8.6.6]). In this category the costandard (resp. standard) object corresponding to $\lambda \in \Lambda(d)$ is the Schur functor S_λ (resp. Weyl functor W_λ) in the sense of [2] (c.f. [15, Prop. 8.5.12]). The simple functor corresponding to λ will be denoted by F_λ . In the following corollary we show that $\mathcal{P}_{d,n}$ is a highest weight category.

Corollary 2.3. *$\mathcal{P}_{d,n}$ is a highest weight category with poset $\Lambda(d, n)$ of Young diagrams of weight d and with at most n columns, with the partial order on $\Lambda(d, n)$ being the reversed dominance order.*

Proof. The assertion follows immediately from Theorem 2.2 and the well-known fact that $S(n, d)\text{-mod}$ is a highest weight category (c.f. [18, Thm. 4.1]). \square

Remark 2.4. An unfortunate fact that the order is the reversed dominance order is caused by the fact that we label the family of Schur functors as in [2], [22] and simultaneously follow the general conventions concerning highest weight categories (see e.g [7]).

In the following theorem we describe the categories $\mathcal{P}_{d,n}$ and $\mathcal{D}^b\mathcal{P}_{d,n}$ in terms of recollements of abelian and triangulated categories and we characterize costandard and standard objects in $\mathcal{P}_{d,n}$.

Theorem 2.5. *Denote the full subcategory of \mathcal{P}_d consisting of strict polynomial functors of degree d , which assign the zero space to linear spaces with dimension at most n by $\mathcal{P}_d^{>n}$. Let $i_*: \mathcal{P}_d^{>n} \rightarrow \mathcal{P}_d$ be the inclusion functor and let $j^*: \mathcal{P}_d \rightarrow \mathcal{P}_{d,n}$ be the exact functor, which restricts a given strict polynomial functor of degree d to the category $\Gamma^d \text{Vect}_{\mathbf{k}}^{\leq n}$.*

1. *There is a recollement of abelian categories*

$$\begin{array}{ccccc} & \overset{i^*}{\curvearrowright} & & \overset{j_!}{\curvearrowright} & \\ \mathcal{P}_d^{>n} & \xrightarrow{i_*} & \mathcal{P}_d & \xrightarrow{j^*} & \mathcal{P}_{d,n} \\ & \underset{i^!}{\curvearrowleft} & & \underset{j_*}{\curvearrowleft} & \end{array}$$

and the recollement of triangulated categories induced by the recollement given above on the level of bounded derived categories

$$\begin{array}{ccccc} & \overset{Li^*}{\curvearrowright} & & \overset{Lj_!}{\curvearrowright} & \\ \mathcal{D}\mathcal{P}_d^{>n} & \xrightarrow{i_*} & \mathcal{D}\mathcal{P}_d & \xrightarrow{j^*} & \mathcal{D}\mathcal{P}_{d,n} \\ & \underset{Ri^!}{\curvearrowleft} & & \underset{Rj_*}{\curvearrowleft} & \end{array}$$

2. *Let $\lambda \in \Lambda(d, n)$. The costandard (resp. standard, simple) object corresponding to λ in $\mathcal{P}_{d,n}$ is the functor $j^*(S_\lambda)$ (resp. $j^*(W_\lambda)$, $j^*(F_\lambda)$). Thus, costandard and standard functors in $\mathcal{P}_{d,n}$ will be also called, respectively, Schur and Weyl functors and will be denoted by S_λ and W_λ . The simple functor in $\mathcal{P}_{d,n}$ corresponding to $\lambda \in \Lambda(d, n)$ will be denoted by F_λ .*
3. *Let $\lambda \in \Lambda(d, n)$. Then $j_*(S_\lambda) \simeq Rj_*(S_\lambda) \simeq S_\lambda$ and $j_!(W_\lambda) \simeq Lj_!(W_\lambda) \simeq W_\lambda$. The functor j_* (resp. $j_!$) preserves injective envelopes (resp. projective covers).*

Proof. We observe that $J = \Lambda(d) \setminus \Lambda(d, n)$ is an order ideal of $\Lambda(d)$ and $\mathcal{P}_d^{>n}$ consists of strict polynomial functors of degree d with composition factors F_λ , where $\lambda \in J$. Then

the part (1) follows from [7, Thm. 3.9(b)]. (In [7, Thm. 3.9(b)] only the triangulated recollement was established, however, the abelian case is proved in the similar manner by using the recollement $(R/ReR\text{-mod}, R\text{-mod}, eRe\text{-mod})$ where R is a ring and $e \in R$ is an idempotent (see e.g. [21, Ex. 2.7]).) The part (2) follows from (1) and [6, Lem. 1.4(b)].

Now we turn to the proof of (3). Fix $\mu \in \Lambda(d, n)$. By the right adjointness of Rj_* to j^* and part (2) we have

$$\mathrm{Hom}_{\mathcal{DP}_d}(W_\lambda, Rj_*(S_\mu)[t]) \simeq \mathrm{Ext}_{\mathcal{P}_{d,n}}^t(j^*W_\lambda, S_\mu) \simeq \mathrm{Ext}_{\mathcal{P}_{d,n}}^t(W_\lambda, S_\mu) = 0$$

for any $t \neq 0$, $\lambda \in \Lambda(d, n)$. Therefore $Rj_*(S_\lambda) \simeq j_*S_\mu$ and it is a good object (i.e. it has a filtration with the quotients being Schur functors). Let us recall that in any highest weight category \mathcal{C} with poset I , for any $x, y \in I$ we have $\mathrm{Hom}_{\mathcal{C}}(W_x, S_y) = \delta_{xy}$. Therefore for a good $X \in \mathcal{P}_d$ we have $\ell(X) = \dim \mathrm{Hom}_{\mathcal{P}_d}(W, X)$, where $\ell(X)$ is the length of a corresponding good filtration of X and $W = \bigoplus_{\lambda \in \Lambda(d,n)} W_\lambda$. By the adjointness and part (2) again, we obtain

$$\begin{aligned} \ell(j_*(S_\mu)) &= \dim \mathrm{Hom}_{\mathcal{P}_d}(W, j_*(S_\mu)) = \dim \mathrm{Hom}_{\mathcal{P}_{d,n}}(j^*(W), S_\mu) = \\ &= \dim \mathrm{Hom}_{\mathcal{P}_{d,n}}(W, S_\mu) = \dim \mathrm{Hom}_{\mathcal{P}_{d,n}}(W_\mu, S_\mu) = 1. \end{aligned}$$

In consequence, $j_*(S_\mu) \simeq S_\mu$. The functor j_* preserves injectives, because it has an exact left adjoint. Let $G \in \mathcal{P}_{d,n}$ be the injective envelope of $F \in \mathcal{P}_{d,n}$. Then, since j_* preserves monomorphisms, we have an embedding $j_*(F) \subset j_*(G)$. Suppose that $j_*(G)$ is not the injective envelope of $j_*(F)$, i.e. there exists an injective $H \in \mathcal{P}_d$ such that $j_*(F) \subset H \subset j_*(G)$. Since H is injective, it is a direct summand of $j_*(G)$. By applying j^* we get $F \subset j^*(H) \subset G$, since $j^*j_* \simeq id$. This gives a contradiction with the minimal property of envelope, because $j^*(H)$ is a direct summand of G , hence it is injective. This shows that j_* preserves injective envelopes.

The proofs of the respective facts for $j_!(W_\lambda)$ are analogous. \square

Now we recall the block structure of the Schur algebra $S(n, d)$ (c.f. [8]). By Theorem 2.2 this is also the block structure of $\mathcal{P}_{d,n}$. For $\lambda \in \Lambda(d, n)$ with the conjugate Young diagram $\tilde{\lambda}$ of the form $\tilde{\lambda} = (\lambda'_1, \dots, \lambda'_n)$ we define

$$\alpha(\lambda) = \max \{ r \geq 0 : \forall 1 \leq i \leq n-1 \lambda'_i - \lambda'_{i+1} \equiv -1 \pmod{p^r} \}.$$

Then, identifying a block with a subset of $\Lambda(d, n)$, λ and μ are in the same block of $\mathcal{P}_{d,n}$ if and only if λ and μ have the same p -core and $\alpha(\lambda) = \alpha(\mu)$.

We end the section by recalling the standard definitions and facts about the algebraic group GL_n (see e.g. [13]). Let $T_n \subset \mathrm{GL}_n$ be the standard maximal torus of GL_n , i.e. the subgroup of diagonal matrices. Let B_n be a Borel subgroup of GL_n consisting of upper triangular matrices. The weight lattice of GL_n , i.e. the group $X(T_n)$ of rational characters of T_n , is identified with the group \mathbb{Z}^n . Elements of $X(T_n)$ are called weights. For $M \in \mathrm{GL}_n^{\mathrm{rat}}\text{-mod}$ and $\lambda \in X(T_n)$ the weight space M_λ is the linear space $M_\lambda = \{ m \in M : tm = \lambda(t)m \text{ for all } t \in T_n \}$. A weight $\lambda \in X(T_n)$ is a weight of M if and only if $M_\lambda \neq 0$. We have a decomposition $M = \bigoplus_{\lambda \in X(T_n)} M_\lambda$. The poset of dominant weights $X(T_n)^+ \subset X(T_n)$ is the set

$$X(T_n)^+ = \{ \lambda : \langle \lambda, \alpha \rangle \geq 0 \text{ for all } \alpha \in \Phi^+ \} = \{ \lambda = (\lambda_1, \dots, \lambda_n) : \lambda_1 \geq \dots \geq \lambda_n \}$$

with the partial order defined as follows: $\lambda \geq \mu$ if and only if

$$\lambda - \mu = \sum_{1 \leq i < j \leq n} c_{ij}(e_i - e_j) \quad \text{with } c_{ij} \geq 0.$$

Here, $\Phi^+ = \{e_i - e_j : 1 \leq i < j \leq n\} \subset X(T_n)$, where e_i is the i -th element of the standard basis of $X(T_n)$, is the system of positive roots corresponding to the Borel subgroup B_n . A rational GL_n -module M is polynomial if and only if for all the weights $\lambda = (\lambda_1, \dots, \lambda_n)$ of M each λ_i satisfies $\lambda_i \geq 0$ (c.f. [11, Prop. 3.5]). The category $\mathrm{GL}_n^{\mathrm{rat}}\text{-mod}$ with the poset $X(T_n)^+$ is a highest weight category (c.f. [19, Ex. 6.2]) and $\mathcal{P}_{d,n}$ is its highest weight subcategory, though we warn the reader that the labelings of simples in both categories differ by the conjugation. Now we observe that the image of $\Lambda(d, n)$ under the involution $\lambda \mapsto \bar{\lambda}$, where $\bar{\lambda}$ is the conjugate Young diagram of λ , is an order ideal of $X(T_n)^+$, hence by [7, Thm. 3.9(a)] the inclusion functor

$$ev: \mathcal{P}_{d,n} \rightarrow \mathrm{GL}_n^{\mathrm{rat}}\text{-mod}$$

given explicitly as $ev(F) := F(\mathbf{k}^n)$ is a full embedding and so is its derived functor. In particular, we have the isomorphisms of Ext-groups:

$$\mathrm{Ext}_{\mathcal{P}_{d,n}}^*(F, G) \simeq \mathrm{Ext}_{\mathrm{GL}_n^{\mathrm{rat}}\text{-mod}}^*(F(\mathbf{k}^n), G(\mathbf{k}^n))$$

for any $F, G \in \mathcal{P}_{d,n}$ (see also [11, Cor. 3.12.1], where it was re-proved without referring to [7]).

Finally, let us recall that any finite dimensional rational GL_n -module M is of the form $M \simeq F(\mathbf{k}^n) \otimes \det^r$ for some $F \in \mathcal{P}_{d,n}$ and a non-positive integer r , where $\det^r = (\Lambda^n(\mathbf{k}^n))^{\otimes r}$ if $r > 0$, $\det^0 = id$ and $\det^r = ((\Lambda^n(\mathbf{k}^n))^{\otimes -r})^*$ otherwise (c.f. [22, Prop. 2.2.1]).

3. The homological algebra in $\mathcal{P}_{p,n}$

In the present section we study the homological algebra in the category $\mathcal{P}_{p,n}$, for a ground field \mathbf{k} of positive characteristic p . Since for $d < p$, $S(n, d)$ is a semisimple algebra (c.f. [17, Thm. 2.2.8]), $\mathcal{P}_{d,n}$ is a semisimple category for $d < p$. Thus, the first non-trivial case from the point of view of homological algebra is $d = p$. Moreover, if $n \geq p$ then $\mathcal{P}_{p,n} \simeq \mathcal{P}_p$ by Theorem 2.2. Therefore we assume in the sequel that $n < p$ and we call this situation *unstable*. Let us observe that $\alpha(\lambda) = 0$ for any $\lambda \in \Lambda(p, n)$. It is also easily seen that each diagram in $\Lambda(p, n)$ not being a p -hook is a p -core and all p -hooks in $\Lambda(p, n)$ have the same p -core, namely the empty set. As the consequence, the only non-trivial homological computations appear in the block of $\mathcal{P}_{p,n}$ corresponding to the subset of $\Lambda(p, n)$ consisting of p -hooks, which will be denoted by $\mathcal{P}_{p,n}^\emptyset$. Set $S_i = S_{(i+1, 1^{p-i-1})}$, $W_i = W_{(i+1, 1^{p-i-1})}$ and $F_i = F_{(i+1, 1^{p-i-1})}$ for $0 \leq i \leq n-1$.

We start with collecting the facts which are formal consequences of the recollement between \mathcal{P}_p and $\mathcal{P}_{p,n}$ described in Section 1 and the corresponding properties of \mathcal{P}_p established in [14].

Theorem 3.1.

1. The decomposition matrix $D = (d_{ij}) \in M_{n \times n}(\mathbb{Z})$ of $\mathcal{P}_{p,n}^\emptyset$ is given by

$$d_{ij} = [W_{i-1} : F_{j-1}] = \begin{cases} 1 & \text{if } j = i \text{ or } j = i + 1, \\ 0 & \text{otherwise.} \end{cases}$$

2. Let $0 \leq i, j \leq n - 1$. Then:

$$\mathrm{Ext}^q(F_i, S_j) = \begin{cases} \mathbf{k} & \text{if } j \geq i \text{ and } q = j - i, \\ 0 & \text{otherwise.} \end{cases}$$

3. The category $\mathcal{P}_{p,n}$ has a Kazhdan-Lusztig theory relative to the choice of function $l: \Lambda(p, n) \rightarrow \mathbb{Z}$ given by $l((i + 1, 1^{p-i-1})) = i$ for $0 \leq i \leq n - 1$ and $l(\lambda) = 0$ for λ not being a p -hook.

4.

$$\mathrm{Ext}^q(F_i, F_j) = \begin{cases} \mathbf{k} & \text{if } q = |i - j| + 2r, \text{ where } 0 \leq r \leq n - \max\{i, j\} - 1, \\ 0 & \text{otherwise.} \end{cases}$$

5.

$$\mathrm{Ext}^q(S_i, S_j) = \begin{cases} \mathbf{k} & \text{if } (i = j \wedge q = 0) \vee (j > i \wedge (q = j - i - 1 \vee q = j - i)), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (1) follows from the fact that j^* is exact while preserving both F_i and S_i for $0 \leq i \leq n - 1$. The computation in (2) follows from [14, Prop. 2.1], since the groups under consideration are the same in $\mathcal{P}_{p,n}$ and \mathcal{P}_p because of the equality $j_*(S_j) = S_j$. (3) follows from (2) (although it is also a formal consequence of the existence of the recollement setup). The computation of the Ext-groups in (4) is in turn a formal consequence of the existence of a Kazhdan-Lusztig theory in $\mathcal{P}_{p,n}$. The computation in (5) follows from [14, Prop. 2.6] and the fact that $j_*(S_i) = S_i$. \square

Now we investigate multiplicative structures and formality phenomena in $\mathcal{P}_{p,n}$. Analogous questions in \mathcal{P}_p were studied by the second author in [14]. We will again use the close relation between \mathcal{P}_p and $\mathcal{P}_{p,n}$. A general picture is quite similar in both cases, but the crucial constructions of [14] need to be aptly adopted to $\mathcal{P}_{p,n}$. Namely, the second author constructed in [14, Sec. 3.2] an explicit injective resolution \mathcal{R}_i of F_i in \mathcal{P}_p consisting of the parts of the de Rham and Koszul complexes (see e.g. [11, Sec. 4]) spliced together. Unfortunately, $j^*(\mathcal{R}_i)$ will not be suitable for us, since it is not injective anymore. Nevertheless, we shall obtain an injective resolution $\mathcal{R}_{i,n}$ of F_i in $\mathcal{P}_{p,n}$ by a similar construction. Namely, set

$$\mathcal{R}_{i,n}^{r,s} = \Omega^{i+s-r} \quad \text{if } 0 \leq r \leq n - 1, 0 \leq s \leq n - i - 1 \text{ and } r - s \leq i,$$

where $\Omega^i := S^{p-i} \otimes \Lambda^i$. The horizontal and vertical differentials are, respectively, the de Rham and Koszul differentials. Since $\kappa d + d\kappa = 0$, $\mathcal{R}_{i,n}^{**}$ is a double complex. We observe that $\mathcal{R}_{i,n}$ is the truncation of the double complex providing \mathcal{R}_i at the vertical degree $n - i - 1$. For instance, $\mathcal{R}_{3,5}$ for $p = 7$ is of the following form:

$$\begin{array}{ccccccccc} \Omega^4 & \xrightarrow{\kappa_4} & \Omega^3 & \xrightarrow{\kappa_3} & \Omega^2 & \xrightarrow{\kappa_2} & \Omega^1 & \xrightarrow{\kappa_1} & \Omega^0 \\ d_3 \uparrow & & d_2 \uparrow & & d_1 \uparrow & & d_0 \uparrow & & \\ \Omega^3 & \xrightarrow{\kappa_3} & \Omega^2 & \xrightarrow{\kappa_2} & \Omega^1 & \xrightarrow{\kappa_1} & \Omega^0 & & \end{array} .$$

Since Ω^k is injective for $0 \leq k \leq n - 1$ (as being a summand in the injective cogenerator $S^{p,n}$) and $H^*(\mathrm{Tot}(\mathcal{R}_{i,n})) = H^0(\mathrm{Tot}(\mathcal{R}_{i,n})) = F_i$ (c.f [14, Sec. 3.2]), $\mathrm{Tot}(\mathcal{R}_{i,n})$ is an injective resolution of F_i . We use for the simplicity the same symbol $\mathcal{R}_{i,n}$ for that. Set

$\mathcal{R} = \bigoplus_{0 \leq i \leq n-1} \mathcal{R}_{i,n}$ and $F = \bigoplus_{0 \leq i \leq n-1} F_i$. We now observe that by truncation we get a quasi-isomorphism $j^*(\mathcal{R}_i) \simeq \mathcal{R}_{i,n}$, which, in particular, shows that j^* induces an epimorphism $\text{Ext}_{\mathcal{P}_p}^*(F, F) \rightarrow \text{Ext}_{\mathcal{P}_{p,n}}^*(F, F)$. Also, since the construction of $\mathcal{R}_{i,n}$ is analogous to that of \mathcal{R}_i , the arguments of [14, Sec. 3.2] apply to the current situation and we obtain:

Theorem 3.2.

1. Let $B = \bigoplus_{t \in \mathbb{N}} B_t$ be the graded algebra with grading

$$B_t = \text{span}\{b_{ji}^t : 0 \leq i, j \leq n-1 \text{ and } t = |i-j| + 2r, \text{ where} \\ 0 \leq r \leq n - \max\{i, j\} - 1\}$$

for $0 \leq t \leq 2n-2$ and $B_t = 0$ for $t \geq 2n-1$, where b_{ji}^t are the formal symbols for i, j, t satisfying the above conditions. The multiplication on B is given by

$$b_{lm}^t \cdot b_{ji}^u = \begin{cases} b_{mi}^{t+u} & \text{if } j = l \text{ and } u + t \leq 2n - i - m - 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then there is a graded algebra isomorphism $\text{Ext}_{\mathcal{P}_{p,n}}^*(F, F) \simeq B$.

2. There is a graded algebra isomorphism

$$\text{Ext}_{\mathcal{P}_{p,n}}^*(F_i, F_i) \simeq \mathbf{k}[x]/(x^{n-i})$$

for $0 \leq i \leq n-1$ and x of degree 2. In particular, $\text{Ext}_{\mathcal{P}_{p,n}}^*(F_i, F_i)$ is a commutative algebra.

3. The algebra $\text{End}^*(\mathcal{R})$ is a formal DG algebra, i.e. there exists a quasi-isomorphism of DG algebras $\eta: \text{Ext}_{\mathcal{P}_{p,n}}^*(F, F) \rightarrow \text{End}^*(\mathcal{R})$, where $\text{Ext}_{\mathcal{P}_{p,n}}^*(F, F)$ is regarded as the DG algebra with zero differential.
4. There is an equivalence of triangulated categories

$$\mathcal{D}^b \mathcal{P}_{p,n}^{\mathcal{O}} \simeq \mathcal{D}^b(\text{Ext}_{\mathcal{P}_{p,n}}^*(F, F) - \text{mod}^{gr}).$$

We remark that similar results concerning the Yoneda algebra of Schur functors, analogous to [14, Thm. 3.2, Cor. 3.3], also hold, though we leave their formulation to the interested reader.

The computations in $\mathcal{P}_{p,n}$ obtained so far could be hardly called surprising. However, when we try to compute the groups $\text{Ext}_{\mathcal{P}_{p,n}}^*(F_i, W_j)$ and $\text{Ext}_{\mathcal{P}_{p,n}}^*(S_i, W_j)$, we encounter some interesting phenomena. Again, analogously to [14] we will use the minimal injective resolution of W_i , let us call it $L_{i,n}$ (in fact [14, (10)] gives the minimal projective resolution of S_i , hence we consider the Kuhn dual case). Again we cannot just apply j^* to the dual of [14, (10)], since the resulting complex is not injective. Instead, we need to adapt the idea of its construction to our context. Namely, this time we concatenate the truncated dual Koszul and Koszul complexes, but we cut them off earlier than it was done in [14, (10)] in order to get an injective complex. Thus, we put as $L_{i,n}$ the following:

$$(\Omega^{i+1})^\# \longrightarrow (\Omega^{i+2})^\# \longrightarrow \dots \longrightarrow (\Omega^{n-1})^\# \xrightarrow{\alpha} \Omega^{n-1} \longrightarrow \Omega^{n-2} \longrightarrow \dots \longrightarrow \Omega^0,$$

where $\alpha = (\Omega^{n-1})^\# \xrightarrow{k^\#} (\Omega^n)^\# \simeq \Omega^n \xrightarrow{k} \Omega^{n-1}$. We remark that $L_{i,n}$ is the minimal injective resolution of W_i in $\mathcal{P}_{p,n}$, since Ω^q is the injective envelope of F_q for every $0 \leq q \leq n-1$ by [14, Prop. 1.1] and [6, Lem. 1.2].

Then by repeating the arguments from the proofs of [14, Prop. 2.8, Cor. 2.9, Prop. 2.10] we obtain:

Proposition 3.3. *We have the following computations in $\mathcal{P}_{p,n}$:*

1. For $0 \leq i, j \leq n-1$

$$\text{Ext}^q(S_i, F_j) = \begin{cases} \mathbf{k} & \text{if } (i < j \wedge q = j - i - 1) \vee q = 2n - i - j - 2, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\text{Ext}^q(F_i, W_j) = \begin{cases} \mathbf{k} & \text{if } (i > j \wedge q = i - j - 1) \vee q = 2n - i - j - 2, \\ 0 & \text{otherwise.} \end{cases}$$

2. For $0 \leq i, j \leq n-1$

$$\text{Ext}^q(S_i, W_j) = \begin{cases} \mathbf{k} & \text{if } (i = j \wedge q = 0) \vee q = 2n - i - j - 3 \vee \\ & q = 2n - i - j - 2, \\ 0 & \text{otherwise.} \end{cases}$$

We point out here that although our construction is similar to that in \mathcal{P}_p , we cannot say that the result is just the truncation of the corresponding computation in \mathcal{P}_p , since the Ext-groups are shifted. In particular, in this case j^* is not epimorphic on the Ext-groups. Some even more striking examples of non-obvious behavior of j^* will be provided in the next paragraph.

It was observed in [1, p. 183–184] that $I^{\otimes d}$ is not projective if $d > n$. The conceptual reason for this is that it is not a direct summand in a projective generator of $\mathcal{P}_{d,n}$ anymore. Hence, let us compute some Ext-groups in $\mathcal{P}_{p,n}$ involving $I^{\otimes p}$. To this end, we shall use the adjunction $\{j^*, Rj_*\}$ between the categories \mathcal{DP}_d and $\mathcal{DP}_{d,n}$. Let us start with an easy general observation:

Proposition 3.4. *For any $F \in \mathcal{P}_{d,n}$ and projective $P \in \mathcal{P}_d$ we have:*

$$\text{Ext}_{\mathcal{P}_{d,n}}^q(j^*(P), F) \simeq \text{Hom}_{\mathcal{P}_d}(P, H^q(Rj_*(F))).$$

Proof. It follows from the afore-mentioned adjunction and the fact that the functor $\text{Hom}_{\mathcal{P}_d}(P, -)$ is exact, hence it commutes with cohomology. \square

Thus, we see that the task of computing $\text{Ext}_{\mathcal{P}_{p,n}}^*(I^{\otimes p}, F)$ is essentially reduced to that of computing $H^*(Rj_*(F))$. For this we shall use for various $F \in \mathcal{P}_{p,n}$ their j_* -acyclic resolutions, which are often simpler than the injective ones. One of these is $K_{i,n}$, the truncated complex of “Koszul kernels” in $\mathcal{P}_{p,n}$ given by:

$$S_i \xrightarrow{d} S_{i+1} \xrightarrow{d} \cdots \xrightarrow{d} S_{n-1},$$

which is a resolution of F_i (cf. [14, (8)]). The second is the truncated de Rham complex $M_{i,n}$:

$$\Omega^{i+1} \xrightarrow{d} \Omega^{i+2} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n,$$

which is a resolution of W_i . In fact $M_{i,n}$ is the first part of the complex $L_{i,n}$, because $(\Omega^j)^\# \simeq \Omega^j$ for $j > 0$. It is not injective, since Ω^n is not injective, but it is, as we will

see, j_* -acyclic. We shall also need the complex $M'_{i,n}$ in \mathcal{P}_p given by:

$$\Omega^{i+1} \xrightarrow{d} \Omega^{i+2} \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-1} \xrightarrow{\beta} S_{n-1},$$

where the map β is the composition of the canonical projection and the de Rham differential:

$$\Omega^{n-1} \longrightarrow S_{n-2} \xrightarrow{d} S_{n-1}.$$

In the next theorem we still use the convention introduced in Theorem 2.5 that for “generally known functors” we denote their restriction j^* by the same symbol. Thus, formula like $j_*(F) \simeq F$ actually means that for some $F \in \mathcal{P}_p$ we have $j^*(j_*(F)) \simeq F$.

Now we have:

Theorem 3.5. *For any $0 \leq i \leq n-1$ we have:*

1. $Rj_*(F_i) = j_*(K_{i,n}) \simeq K_{i,n}$. Hence $j_*(F_{n-1}) \simeq Rj_*(F_{n-1}) \simeq S_{n-1}$ and for all $0 \leq i < n-1$, we have:

$$H^q(Rj_*(F_i)) \simeq \begin{cases} F_i & \text{for } q = 0, \\ F_n & \text{for } q = n - i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

2. $Rj_*(W_i) = j_*(M_{i,n}) \simeq M'_{i,n}$. Hence $j_*(W_{n-1}) \simeq Rj_*(W_{n-1}) \simeq S_{n-1}$ and for all $0 \leq i < n-1$, we have:

$$H^q(Rj_*(W_i)) = \begin{cases} W_i & \text{for } q = 0, \\ F_n & \text{for } q = n - i - 2, \ q = n - i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We first observe that, by Theorem 2.5(3), the complex $K_{i,n}$ is j_* -acyclic, hence it may be used for computing $Rj_*(F_i)$. Since $j_*(S_q) \simeq S_q$ for $q \leq n-1$ (Theorem 2.5(3) again), we see that the complexes $j_*(K_{i,n})$ and $K_{i,n}$ consist of isomorphic objects. Then by the faithfulness of j_* , $j_*(d) \neq 0$, but since $\dim(\text{Hom}_{\mathcal{P}_p}(S_q, S_{q+1})) = 1$ by [14, Prop. 2.6], we conclude that $j_*(d) = d$ up to a non-zero scalar. This shows that $Rj_*(K_{i,n}) \simeq K_{i,n}$ in \mathcal{DP}_p . The formula for cohomology of $Rj_*(F_i)$ follows from the fact that the kernel (resp. cokernel) of the map $d: S_q \rightarrow S_{q+1}$ in \mathcal{P}_p for $q < p$ is isomorphic to F_q (resp. F_{q+1}).

For the second part, we start by invoking Theorem 2.5(3) again to show that $M_{i,n}$ is j_* -acyclic, since all its terms except Ω^n are injective while $\Omega^n \simeq S_{n-1}$ in $\mathcal{P}_{p,n}$. More precisely, by the fact mentioned before Proposition 3.3 that for $q < n$, Ω^q is the injective envelope of S_q and Theorem 2.5(3), we obtain that $j_*(\Omega^q) \simeq \Omega^q$ for $0 \leq q < n$ and $j_*(\Omega^n) \simeq S_{n-1}$. Then we conclude that $j_*(M_{i,n}) \simeq M'_{i,n}$ by the similar argument to that used in the proof of the first part, since the Hom-spaces between the functors under inspection are one-dimensional by e.g. an elementary computation using the sum-diagonal adjunction. Then the fact that $H^0(M'_{i,n}) \simeq W_i$ is the main point of [14, (10)]. The rest of the formula for $H^*(Rj_*(W_i))$ could be derived from the description of the kernel/cokernel of β , but it also follows immediately from the first part of the Theorem and the long exact sequence of cohomology applied to the distinguished

triangle

$$Rj_*(F_{i+1}) \rightarrow Rj_*(W_i) \rightarrow Rj_*(F_i)$$

for $i < n - 1$ coming from the short exact sequence in $\mathcal{P}_{p,n}$:

$$0 \rightarrow F_{i+1} \rightarrow W_i \rightarrow F_i \rightarrow 0,$$

which is a consequence of Theorem 3.1(1). \square

Now we can apply our computation and Proposition 3.4 to $P = I^{\otimes p}$ and compute the corresponding Ext-groups. In order to describe an extra structure carried by these Exts, which comes from the action of the symmetric group Σ_p on $I^{\otimes p}$, we shall formulate our results in terms of “the derived Schur functor” Rs . Namely, classically (see e.g. [17, Sec. 4]) one considers “the Schur functor” $s: \mathcal{P}_d \rightarrow \mathbf{k}[\Sigma_d]\text{-mod}$ given by the formula:

$$s(F) := \text{Hom}_{\mathcal{P}_d}(I^{\otimes d}, F).$$

In fact, over a field of characteristic zero s establishes an equivalence between \mathcal{P}_d and the category $\mathbf{k}[\Sigma_d]\text{-mod}$ of finite dimensional \mathbf{k} -representations of Σ_d . Over a field of positive characteristic s is not an equivalence anymore, but it still preserves an important information. For example, for a p -regular λ , s takes the simple functor F_λ to the simple $\mathbf{k}[\Sigma_d]$ -module associated to λ , which we shall denote by G_λ (see eg. [12, Chap. 8.4] where an explicit construction of simple Σ_d -modules is given, or [16] where the relevant recollement diagram is studied). In particular, we denote by G_i the simple $\mathbf{k}[\Sigma_p]$ -module associated to the corresponding hook diagram. Also, $s(S_\lambda)$ and $s(W_\lambda)$ can be explicitly described. They are called, respectively, the Specht and dual Specht modules and denoted by Sp_λ and Sp'_λ [12, Chap. 7] (again, we will also use the notations: Sp_i and Sp'_i).

Now, since in our situation $I^{\otimes p}$ is not projective, it is natural to consider its derived functor operating on the bounded derived categories:

$$Rs: \mathcal{DP}_{p,n} \rightarrow \mathcal{Dk}[\Sigma_p]\text{-mod},$$

given by the formula:

$$Rs(F) := \text{RHom}_{\mathcal{P}_{p,n}}(I^{\otimes d}, F),$$

which translates to Ext-computations via the formula:

$$H^t(Rs(F)) \simeq \text{Ext}_{\mathcal{P}_{p,n}}^t(I^{\otimes p}, F).$$

Thus, within the formalism of total derived functors, we can formulate our Ext-computations as follows (for completeness, we also add the obvious computation of $Rs(S_\lambda)$):

Corollary 3.6. *There are the following isomorphisms in $\mathcal{Dk}[\Sigma]_p\text{-mod}$:*

1. For any $0 \leq i \leq n - 1$, $Rs(S_i) = Sp_i$.
2. $Rs(F_0) = G_n[-(n - 1)]$, $Rs(F_{n-1}) = Sp_{n-1}$, and for $0 < i < n - 1$:

$$\text{Ext}_{\mathcal{P}_{p,n}}^q(I^{\otimes p}, F_i) \simeq H^q(Rs(F_i)) = \begin{cases} G_i & \text{for } q = 0, \\ G_n & \text{for } q = n - 1 - i, \\ 0 & \text{otherwise.} \end{cases}$$

3. $Rs(W_{n-1}) = Sp_{n-1}$, and for $0 \leq i < n - 1$:

$$\text{Ext}_{\mathcal{P}_{p,n}}^q(I^{\otimes p}, W_i) \simeq H^q(Rs(W_i)) = \begin{cases} Sp'_i & \text{for } q = 0, \\ G_n & \text{for } q = n - i - 2, \ q = n - i - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Straightforward. \square

Remark 3.7. Let us point out for some interesting phenomena here. Firstly, our formulas are overlapping at the end of the poset, since we have $S_{n-1} \simeq F_{n-1} \simeq W_{n-1}$ in $\mathcal{P}_{p,n}$ (because $(n, 1^{p-n})$ is minimal in its block). This, for example, shows that j^* is not full, since

$$\text{Hom}_{\mathcal{P}_p}(I^{\otimes p}, F_{n-1}) = G_{n-1} \quad \text{while} \quad \text{Hom}_{\mathcal{P}_{p,n}}(I^{\otimes p}, F_{n-1}) = Sp_{n-1}.$$

At the other end of the scale something interesting also happens. Namely, since (1^p) is p -singular we have $s(F_0) = 0$, but, as we see, $Rs(F_0)$ is not only nonzero but it is the shifted G_n , which could not be hit classically, since we do not have F_n in $\mathcal{P}_{p,n}$.

These phenomena have quite a surprising consequence for $n = p - 1$. We recall that for an abelian category \mathcal{A} , $K_0(\mathcal{D}\mathcal{A})$ denotes the abelian group generated by objects of the bounded derived category of \mathcal{A} with the relations coming from distinguished triangles (see e.g. [5, Sec. 2]). Since we have the relation $X[1] = -X$, $K_0(\mathcal{D}\mathcal{A})$ is isomorphic to the usual $K_0(\mathcal{A})$, hence, for an artinian abelian category \mathcal{A} , it is a free abelian group of rank equal to the number of non-isomorphic simples in \mathcal{A} . Now we have:

Corollary 3.8. *Rs induces an isomorphism:*

$$K_0(Rs) : K_0(\mathcal{DP}_{p,p-1}) \simeq K_0(\mathcal{D}\mathbf{k}[\Sigma_p]\text{-mod}).$$

Proof. It suffices to show that $K_0(Rs)$ is onto. To this end, we first observe that $\pm[G_{p-1}]$ lies in the image, since $Rs(F_0) = G_{p-1}[-(p-2)]$. Then, by the rest of computations in Corollary 3.6, we see that all $[G_i]$ for $0 \leq i < p - 1$ lie in the image of Rs too. \square

This can be contrasted with the fact that Rs is extremely far from being an isomorphism. It is not only because $\mathbf{k}[\Sigma_p]\text{-mod}$ has infinite homological dimension, but often s acts trivially on Ext-groups just for a dimension reason. For example, we have $\text{Ext}_{\mathcal{P}_{p,p-1}}^*(F_0, F_0) = \mathbf{k}[x]/x^{p-1}$ with $|x| = 2$ while

$$\text{Ext}_{\mathbf{k}[\Sigma_p]\text{-mod}}^*(G_{p-1}, G_{p-1}) \simeq H^*(\Sigma_p, \mathbf{k}),$$

which is a subring of $\mathbf{k}[x] \otimes \Lambda^*(y)$ with $|x| = 2, |y| = 1$ consisting of elements of degree congruent to 0 or -1 modulo $2p$.

4. Duality

In this section we introduce a duality relating certain subcategories of $\mathcal{P}_{d,n}$'s. Let $\Lambda(d, n, k)$ be the set of Young diagrams of weight d consisting of at most n columns and at most k rows. We consider the category $\mathcal{P}_{d,n,k}$, which is the full subcategory of

$\mathcal{P}_{d,n}$ generated by the simples F_λ for λ in $\Lambda(d, n, k)$. Since $\Lambda(d, n, k)$ is an order ideal in $\Lambda(d, n)$, we are again in the framework of [7]. In particular, $\mathcal{P}_{d,n,k}$ inherits from $\mathcal{P}_{d,n}$ the structure of a highest weight category and the embedding $i'_* : \mathcal{P}_{d,n,k} \rightarrow \mathcal{P}_{d,n}$ extends to a recollement of abelian categories. Let $ev' := ev \circ i'_*$. Then the essential image of ev' can be characterized as consisting of representations with all the weights having all entries non-negative but not exceeding k .

Let us consider the bijection between $\Lambda(d, n, k)$ and $\Lambda(nk - d, n, k)$, which sends a Young diagram λ to its complement in the rectangle $n \times k$ turned upside down, i.e. $\lambda = (\lambda_1, \dots, \lambda_k) \mapsto \hat{\lambda} := (n - \lambda_k, \dots, n - \lambda_1)$. Our goal is to extend this bijection to an equivalence of highest weight categories $\Phi_{d,n,k} : \mathcal{P}_{d,n,k} \simeq \mathcal{P}_{nk-d,n,k}$. At the end of this section we will explain that our duality resembles the Spanier-Whitehead Duality from stable homotopy theory (although it takes place in the context of abelian categories).

Theorem 4.1. *There exists a functor $\Phi_{d,n,k} : \mathcal{P}_{d,n,k} \rightarrow \mathcal{P}_{nk-d,n,k}$ satisfying the following properties:*

1. Let $t_{n,k} : \mathrm{GL}_n^{\mathrm{rat}}\text{-mod} \rightarrow \mathrm{GL}_n^{\mathrm{rat}}\text{-mod}$ be the functor given by $M \mapsto M \otimes \det^k$. Then there is an isomorphism of contravariant functors:

$$ev' \circ \Phi_{d,n,k} \circ (-)^\# \simeq t_{n,k} \circ (-)^* \circ ev'.$$

2. $\Phi_{d,n,k}$ is an equivalence of \mathbf{k} -linear abelian categories.
3. We have: $\Phi_{d,n,k}(S_\lambda) \simeq S_{\hat{\lambda}}$, $\Phi_{d,n,k}(W_\lambda) \simeq W_{\hat{\lambda}}$, $\Phi_{d,n,k}(F_\lambda) \simeq F_{\hat{\lambda}}$, i.e. $\Phi_{d,n,k}$ is an equivalence of highest weight categories.

Proof. We define $\Phi_{d,n,k}$ by the formula:

$$\Phi_{d,n,k}(F)(V) := \mathrm{Hom}_{\mathcal{P}_{nk,n}}(W_P, i'_*(F) \otimes S_V^{nk-d})$$

for the Young diagram $P = (n^k)$. It is clear from this definition that $\Phi_{d,n,k}$ targets $\mathcal{P}_{nk-d,n,k}$, however, it is a priori not obvious that its image is in fact contained in $\mathcal{P}_{nk-d,n,k}$. This will be showed in the course of the proof. We start with establishing the first assertion of Theorem. Since the weights of M^* are the negatives of those of M , we see that for $F \in \mathcal{P}_{d,n,k}$ the weights of $F(\mathbf{k}^n)^* \otimes \det^k$ have all entries non-negative, hence it belongs to the image of ev . Then we consider the functor ev' right adjoint to ev . Its existence follows from the Special Adjoint Functor Theorem, but, by the Yoneda lemma, it can also be explicitly defined as:

$$ev'(M)(V) := \mathrm{Hom}_{\mathrm{GL}_n^{\mathrm{rat}}\text{-mod}}(ev(\Gamma_{V^*}^d), M).$$

Since ev is a full embedding, we have: $ev' \circ ev \simeq id$, hence

$$F(\mathbf{k}^n)^* \otimes \det^k \simeq ev \circ ev'(F(\mathbf{k}^n)^* \otimes \det^k).$$

This shows that in order to obtain the first part of the Theorem it suffices to show that

$$ev'(F(\mathbf{k}^n)^* \otimes \det^k) \simeq \Phi_{d,n,k}(F^\#).$$

Applying the formula for ev' and using the tensor-hom adjunction in $\mathrm{GL}_n^{\mathrm{rat}}\text{-mod}$ we

obtain:

$$\begin{aligned} ev^!(F(\mathbf{k}^n)^* \otimes \det^k)(V) &= \mathrm{Hom}_{\mathrm{GL}_n^{\mathrm{rat}}\text{-mod}}(\Gamma_{V^*}^{nk-d}(\mathbf{k}^n), F(\mathbf{k}^n)^* \otimes \det^k) \simeq \\ &\mathrm{Hom}_{\mathrm{GL}_n^{\mathrm{rat}}\text{-mod}}(\Gamma_{V^*}^{nk-d}(\mathbf{k}^n) \otimes F(\mathbf{k}^n), \det^k). \end{aligned}$$

Now, since $\det^k = ev(S_P)$ and ev is faithfully full, we get:

$$\begin{aligned} ev^!(F(\mathbf{k}^n)^* \otimes \det^k)(V) &= \mathrm{Hom}_{\mathcal{P}_{nk,n}}(\Gamma_{V^*}^{nk-d} \otimes i'_*(F), S_P) \simeq \\ &\mathrm{Hom}_{\mathcal{P}_{nk,n}}(W_P, S_V^{nk-d} \otimes i'_*(F^\#)), \end{aligned}$$

which gives the first part of Theorem.

At this point we can at last see that the image of $\Phi_{d,n,k}$ lies in $\mathcal{P}_{nk-d,n,k}$, since by the first part of Theorem 4.1 it consists of functors having (after taking ev) all the weights with the entries not exceeding k .

We turn to the second part. Obviously, it suffices to show that $\Phi_{d,n,k} \circ (-)^\#$ is an equivalence. By using the first part again and the fact that $\det^k \otimes (\det^k)^* \simeq \mathbf{k}$, we obtain

$$ev' \circ \Phi_{nk-d,n,k} \circ (-)^\# \circ \Phi_{d,n,k} \circ (-)^\# \simeq t_{n,k} \circ (-)^* \circ t_{n,k} \circ (-)^* \circ ev' \simeq ev',$$

which shows that $\Phi_{-,n,k} \circ (-)^\#$ is an involution.

Now we turn to the third part. The crucial is

Lemma 4.2. *For any $\lambda \in \Lambda(d, n, k)$ there is an isomorphism in \mathcal{P}_{nk-d} (hence in $\mathcal{P}_{nk-d,n}$): $S_{\hat{\lambda}} \simeq S_{P/\lambda}$, where $S_{P/\lambda}$ stands for the Schur functor associated to the skew Young diagram P/λ .*

Proof. By the Littlewood-Richardson Formula [4], there is a filtration of $S_{P/\lambda}$ with the graded object consisting of Schur functors S_μ . Our task is to show that among S_μ 's the only one with $\mu_1 \leq n$ is $\hat{\lambda}$ and it has multiplicity 1. Combinatorially, it means that there is the exactly one Yamanouchi word of shape P/λ and content μ , and its content is $\hat{\lambda}$. This is an elementary exercise. We recall that a Yamanouchi word of shape P/λ and content μ is a semistandard tableau of shape P/λ and content μ satisfying an extra condition (the Yamanouchi condition). This condition says that when we order its rows in the decreasing order (i.e. we start from the last one) and list their contents (from left to right), the obtained sequence of numbers obeys the following condition. When we cut the sequence from the right at any place, for any $i < j$ the number of occurrences of i is greater or equal to the number of occurrences of j . For example, the sequence 1212 satisfies the condition while 1221 does not, since in the subsequence 122 we have one 1 but two 2's. Coming back to the skew diagram P/λ , we see that such a word must have 1's in the places (i, j) for $\lambda_i < j \leq \lambda_{i-1}$ by the Yamanouchi condition. Thus, we see that the number of 1's is equal to $n - \lambda_k$. Then by the same argument the positions of 2's are uniquely determined. By continuing in this manner we construct the single word, which is of content $\hat{\lambda}$. \square

Now we recall that by [2, Thm II.4.11] we have the Decomposition Formula, i.e. the following decomposition, up to filtration, of the Weyl functor applied to the sum

of spaces:

$$W_P(V \oplus W) = \sum_{\mu \subseteq P} W_\mu \otimes W_{P/\mu}.$$

Hence by using the relationship between $\mathcal{P}_{nk,n}$ and \mathcal{P}_{nk} and the sum-diagonal adjunction [9, Thm. 1.7] we obtain:

$$\begin{aligned} \Phi_{d,n,k}(S_\lambda)(V) &= \text{Hom}_{\mathcal{P}_{nk,n}}(W_P, S_\lambda \otimes S_V^{nk-d}) \simeq \text{Hom}_{\mathcal{P}_{nk}}(W_P, S_\lambda \otimes S_V^{nk-d}) \simeq \\ &\text{Hom}_{\mathcal{P}_d \times \mathcal{P}_{nk-d}}(W_P(- \oplus -), S_\lambda \hat{\otimes} S_V^{nk-d}) \simeq \\ &\text{Hom}_{\mathcal{P}_d \times \mathcal{P}_{nk-d}} \left(\sum_{\mu \in \Lambda(d,n,k)} W_\mu \hat{\otimes} W_{P/\mu}, S_\lambda \hat{\otimes} S_V^{nk-d} \right) \simeq \text{Hom}_{\mathcal{P}_{nk-d}}(W_{P/\lambda}, S_V^{nk-d}) \simeq \\ &S_{P/\lambda}(V) \simeq S_{\hat{\lambda}}(V), \end{aligned}$$

which gives the first isomorphism of the third part of the theorem. In order to obtain the second isomorphism we recall that co-good objects in a highest weight category are characterized by the condition that they have trivial higher Ext's with costandard objects. This shows that $\Phi_{d,n,k}(W_\lambda)$ is co-good, because $\Phi_{d,n,k}$ induces isomorphisms on Ext's. Now we conclude that $\Phi_{d,n,k}(W_\lambda) = W_{\hat{\lambda}}$, because $\text{Hom}_{\mathcal{P}_{nk-d,n,k}}(\Phi_{d,n,k}(W_\lambda), S_\mu) = \delta_{\lambda\hat{\mu}}$. The third isomorphism follows from the fact that a simple object in a highest weight category is the socle of the respective costandard object. \square

Let us discuss some corollaries to Theorem 4.1 and its relation to the \mathbf{k} -linear duality in GL_n^{rat} -mod.

Corollary 4.3. *Let $\lambda, \mu \in \Lambda(d, n, k)$, $s \geq 0$ and G_λ, G_μ be the Schur, Weyl or simple functors associated to λ, μ (the choice is the same in a given formula). Then:*

1. *There is an isomorphism in GL_n^{rat} -mod:*

$$G_\lambda((\mathbf{k}^n)^*) \otimes \det^k \simeq G_{\hat{\lambda}}(\mathbf{k}^n).$$

2. *There is an isomorphism of graded vector spaces:*

$$\text{Ext}_{\mathcal{P}_{d,n,k}}^*(G_\lambda, G_\mu) \simeq \text{Ext}_{\mathcal{P}_{nk-d,n,k}}^*(G_{\hat{\lambda}}, G_{\hat{\mu}}).$$

Analogous isomorphisms of Ext-groups hold in the categories: $\mathcal{P}_{d,n}$ vs. $\mathcal{P}_{nk-d,n}$, GL_n^{rat} -mod and also, with the exception of the case of simple functors, in \mathcal{P}_d vs. \mathcal{P}_{nk-d} .

Proof. The first statement follows immediately from the parts 1 and 3 of Theorem 4.1 applied to $G^\#$. The second statement for $\mathcal{P}_{d,n,k}$ follows from the parts 2 and 3 of Theorem 4.1. The case of $\mathcal{P}_{d,n}$ and the case of GL_n^{rat} -mod are consequences of the full embeddings of respective derived categories, which in turn follow from the existence of the recollement setups. The result is that the Ext-groups under inspection are just the same in all these categories. The case of the category \mathcal{P}_d is a bit more specific. The Ext-groups between Schur functors and between Weyl functors are still the same in \mathcal{P}_d and $\mathcal{P}_{d,n}$ because the left adjoint to the projection $\mathcal{P}_d \rightarrow \mathcal{P}_{d,n}$ preserves the Weyl functors while the right adjoint preserves the Schur functors. \square

The first part of Theorem 4.1 indicates to a very close connection between $\Phi_{d,n,k}$ and the linear duality in $\mathrm{GL}_n^{\mathrm{rat}}$ -mod. Also, our proof of the fact that $\Phi_{d,n,k}$ is an equivalence heavily uses the fact that $(-)^*$ is an equivalence. Thus, one should not expect that our duality sheds a new light on representations of GL_n . For example, the parts of Corollary 4.3 concerning GL_n seem to be well-known to experts (see, e.g., [22, Exercise 2.18]). That being said, one should not think that $\Phi_{d,n,k}$ is “just twisted $(-)^*$ restricted to $\mathcal{P}_{d,n,k}$ ”. We would rather say that it points out for a certain categorical phenomenon behind the isomorphisms of Corollary 4.3(1). To explain this point, let us observe that although both sides of the isomorphism are expressed in terms of natural constructions, a priori there is no evident way of interpreting it functorially. The most obvious reason is that the left-hand side depends on \mathbf{k}^n contravariantly while the right-hand side does covariantly. But even disregarding this observation, we see that just dimensions of $G_\lambda(V^*) \otimes \Lambda^n(V)^{\otimes k}$ and $G_{\hat{\lambda}}(V)$ differ drastically (unless $\dim(V) = n$). Thus, we found it quite surprising that the equivalence $(-)^*$, which is “just taking the linear dual”, leads to the equivalence $\Phi_{d,n,k}$ on the level of functor categories, which changes dimensions a lot.

In order to see how non-trivially $\Phi_{d,n,k}$ works in practice, let us look at the following example. Let $\lambda = (1^p) \in \Lambda(p, n, p)$ for $n > 1$. Then $\hat{\lambda} = ((n-1)^p)$. Now, at first sight, the functors: $S_\lambda = S^p$, $F_\lambda = I^{(1)}$ and $S_{\hat{\lambda}} = S_{((n-1)^p)}$, $F_{\hat{\lambda}} = (\Lambda^{n-1})^{(1)}$ apparently have nothing in common. However, by Theorem 4.1, e.g. S_λ and $S_{\hat{\lambda}}$ have the composition series of the same length, with the factors related by the operation $\mu \mapsto \hat{\mu}$. Now, it is well known that S^p has the composition series of length 2 with $S^p/I^{(1)} = F_{(2,1^{p-1})}$ (see eg. [14]). Therefore, by Theorem 4.1 we obtain that

$$S_{((n-1)^p)}/(\Lambda^{n-1})^{(1)} = F_{(n,(n-1)^{p-2},n-2)}$$

which is a priori a highly non-obvious fact.

The same example shows that the excluding the case of simple functors in \mathcal{P}_d from Corollary 4.3(2) is necessary. Namely, by Corollary 4.3(2) we have:

$$\mathrm{Ext}_{\mathcal{P}_{p,n}}^*(I^{(1)}, I^{(1)}) \simeq \mathrm{Ext}_{\mathcal{P}_{pn-p,n}}^*((\Lambda^{n-1})^{(1)}, (\Lambda^{n-1})^{(1)}).$$

On the other hand, the analogous Ext-groups, when computed in \mathcal{P}_p and \mathcal{P}_{np-p} , are clearly very different by [9, Thm. 5.6].

Yet another interesting phenomenon is that since $\mathcal{P}_{d,n,k}$ is semisimple for $d < p$, by Theorem 4.1(2) the same holds for $\mathcal{P}_{nk-d,n,k}$. Thus, we obtain a certain semisimple subcategory in the category of functors of large degree.

In a similar spirit, if $n \geq d$ then we have $\mathcal{P}_{d,n} \simeq \mathcal{P}_d$, hence in such a case $\mathcal{P}_{d,n,k}$ is just a subcategory of \mathcal{P}_d . However, for the corresponding category $\mathcal{P}_{nk-d,n,k}$ “the stability condition” $n \geq nk - d$ is never satisfied unless $k = 1$, or $k = 2$ and $d = n$. Let us look at the last case. Since we have $\lambda = \hat{\lambda}$ for any $\lambda \in \Lambda(n, n, 2)$, $\Phi_{n,n,2}: \mathcal{P}_{n,n,2} \rightarrow \mathcal{P}_{n,n,2}$ is just an auto-equivalence isomorphic to the identity. Therefore we see that our duality is a truly unstable phenomenon.

We finish the article by pointing out a formal similarity between our duality and the Spanier-Whitehead Duality. We first recall how the classical S-W duality works. We start with, say, the category of CW-complexes with the monoidal structure given by the smash product. Then we obtain the stable category by formally inverting the operation of smashing with S^1 . In this framework, the Spanier-Whitehead duality

says that the monoidal dual in the stable category of a small enough space X (i.e. a good subspace of S^k) has the explicit description: it is $S^k \setminus X$ smashed with S^{-k} .

Now one can see that our duality $\Phi_{d,n,k}$ formally resembles that topological construction. This time, we start with $\mathcal{P}_{-,n} := \bigoplus_{d \geq 0} \mathcal{P}_{d,n}$ with the monoidal structure given by tensoring over \mathbf{k} . Then $\mathrm{GL}_n^{\mathrm{rat}}\text{-mod}$ is obtained from $\mathcal{P}_{-,n}$ by formally inverting the operation of tensoring with Λ^n . Thus, Theorem 4.1(1) may be thought of as follows: for an object F small enough (i.e. coming from $\mathcal{P}_{-,n,k}$) we express the monoidal dual of F as applying a certain explicit construction in $\mathcal{P}_{-,n}$ (i.e. $\Phi_{-,n,k} \circ (-)^\#$) to F and then tensoring with $(\Lambda^n)^{-k}$. To make this analogy even more striking, let us recall that our basic combinatorial construction $\lambda \mapsto \widehat{\lambda}$ is just taking the complement of λ in the rectangle P which labels $S_P \simeq \det^k$, which is the analogue of k -dimensional sphere.

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