Homology, Homotopy and Applications, vol. 26(1), 2024, pp.141–149

COMPARING DIAGONALS ON THE ASSOCIAHEDRA

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(*communicated by Johannes Huebschmann*)

Abstract

We prove that the formula for the diagonal approximation Δ_K on J. Stasheff's n-dimensional associahedron K_{n+2} derived by the current authors in [7] agrees with the "magical formula" for the diagonal approximation Δ'_K derived by Markl and Shnider in [5], by J.-L. Loday in [4], and more recently by Masuda, Thomas, Tonks, and Vallette in [6].

Dedicated to the memory of Jean-Louis Loday

1. Introduction

Recently there has been renewed interest in explicit combinatorial diagonal approximations on J. Stasheff's n-dimensional associahedron K_{n+2} [8]. Markl and Shnider (M-S) in [5], J.-L. Loday in [4], and more recently Masuda, Thomas, Tonks, and Vallette (MTTV) in [6] constructed a diagonal Δ'_K on K_{n+2} whose components are "matching pairs" of faces, which in the words of Jean-Louis Loday, are "pairs of cells of matching dimensions and comparable under the Tamari order." By definition, every component of the combinatorial diagonal Δ_K on K_{n+2} constructed by the current authors (S-U) in [7] is a matching pair. In this paper we prove that every matching pair is a component of Δ_K. Thus the S-U formula for Δ_K and the "magical formula" for Δ'_K agree (see Definitions 4 and 5).

Historically, S-U were the first to derive a cellular combinatorial/differential graded formula for Δ_K, M-S were the first to prove the magical formula for Δ'_K, and MTTV were the first to construct a point-set topological diagonal map, which descends to the magical formula at the cellular level.

Using the geometric methods of MTTV, Laplante-Anfossi created a general framework for studying diagonals on any polytope in [3]. In this framework, a choice of diagonal on the n-dimensional permutahedron P_{n+1} is given by a choice of chambers in its fundamental hyperplane arrangement ([3], Def. 1.18). While the specific diagonal Δ'_P on P_{n+1} studied in [3] differs from the S-U diagonal Δ_P, the diagonal Δ'_K on K_{n+2} induced by Δ'_P agrees with Δ_K.

Received August 22, 2023, revised March 22, 2023; published on March 20, 2024.
2020 Mathematics Subject Classification: Primary 55P48, 55P99; Secondary 52B05, 52B11.
Key words and phrases: associahedron, permutahedron, diagonal approximation, magical formula.
Article available at http://dx.doi.org/10.4310/HHA.2024.v26.n1.a9

Acknowledgments

We wish to thank Bruno Vallette for sharing his perspective on the history of combinatorial diagonals on K_n, and Guillaume Laplante-Anfossi and our anonymous referee for their helpful editorial suggestions.

2. Diagonals induced by Δ_P

Let S_n be the symmetric group on the finite set $\underline{n} := \{1, 2, \dots, n\}$. The permutahedron P_n is the convex hull of $n!$ vertices $\{(\sigma(1), \dots, \sigma(n)) : \sigma \in S_n\} \subset \mathbb{R}^n$. As a cellular complex, P_n is an $(n-1)$-dimensional convex polytope whose $(n-p)$-faces are indexed by (ordered) partitions $A_1|\cdots|A_p$ of $\underline{n}$, $1 \leqslant p \leqslant n$. Denoting the set of ordered partitions of $\underline{n}$ by $P(n)$, the faces of P_n are identified with elements of $P(n)$ in the standard way.

Let X be an n-dimensional polytope that admits a (surjective) cellular projection map $p : P_{n+1} \to X$ and a realization as a subdivision of the n-cube I^n, i.e., for any $0 \leqslant k \leqslant n$, each k-cell (k-subcube) of I^n is a union of k-cells of X, any two of which intersect along their boundaries.

For example, $X = P_n$ can be realized as a subdivision of I^{n-1} inductively as follows: Identify P_1 with $1 \in P(1)$. If P_{n-1} has been constructed and $a = A_1|\cdots|A_p \in P(n-1)$ is a face, let $a_0 = 0$, $a_j = \#(A_{p-j+1} \cup \cdots \cup A_p)$ for $0 < j < p$, $a_p = \infty$, and define $\frac{1}{2^\infty} := 0$. Let $I(a) := I_1 \cup I_2 \cup \cdots \cup I_p$, where $I_j := [1 - \frac{1}{2^{a_{j-1}}}, 1 - \frac{1}{2^{a_j}}]$; then $P_n = \bigcup_{a \in P(n-1)} a \times I(a)$, where the identification of faces with partitions is given by

Face of $a \times I(a)$	**Partition in** $P(n)$	
$a \times 0$	$A_1|\cdots|A_p|n$	
$a \times (I_j \cap I_{j+1})$	$A_1|\cdots|A_{p-j}|n|A_{p-j+1}|\cdots|A_p,$	$1 \leqslant j \leqslant p-1$
$a \times 1$	$n|A_1|\cdots|A_p,$	
$a \times I_j$	$A_1|\cdots|A_{p-j+1} \cup n|\cdots|A_p,$	$1 \leqslant j \leqslant p$

(see Figures 1 and 2). We refer to a vertex common to P_n and I^{n-1} as a *cubical vertex*. Thus a is a cubical vertex of P_n if and only if $a|n$ and $n|a$ are cubical vertices of P_{n+1}. Indeed, a cubical vertex has the form $a = a_1|\cdots|a_{i-1}|1|a_{i+1}|\cdots|a_n$, where $a_1 > \cdots > a_{i-1}$ and $a_{i+1} < \cdots < a_n$.

We begin with a review of the diagonal Δ_P and the diagonal Δ_X induced by the projection p; then Δ_K is obtained by setting $X = K_{n+2}$. Whereas the vertices of P_{n+1} are identified with the permutations in S_{n+1}, the *weak order* on S_{n+1} given by $\cdots|x_i|x_{i+1}|\cdots < \cdots|x_{i+1}|x_i|\cdots$ if $x_i < x_{i+1}$ extends to a partial order (p-o) and the associated Hasse diagram orients the 1-skeleton of P_{n+1} [1]. Denote the minimal and maximal vertices of a face e of P_{n+1} by $\min e$ and $\max e$, respectively, and define $e \leqslant e'$ if there exists an oriented edge-path in P_{n+1} from $\max e$ to $\min e'$. Then p induces a p-o on the cells of X. For example, when the faces of P_{n+1} are indexed by planar leveled trees (PLTs) with $n+2$ leaves and the faces of K_{n+2} are indexed by planar rooted trees (PRTs) with $n+2$ leaves (without levels), Tonks' projection $p = \theta$ given by forgetting levels [9] induces the *Tamari order* on the faces $\{\theta(T_i)\}$ of K_{n+2} given by $\theta(T_i) \leqslant \theta(T_j)$ if $T_i \leqslant T_j$. In particular, the vertices of K_{n+1} form a

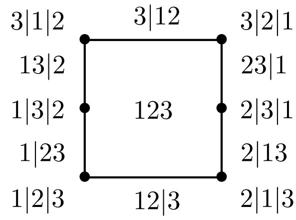

Figure 1: P_3 as a subdivision of $P_2 \times I$.

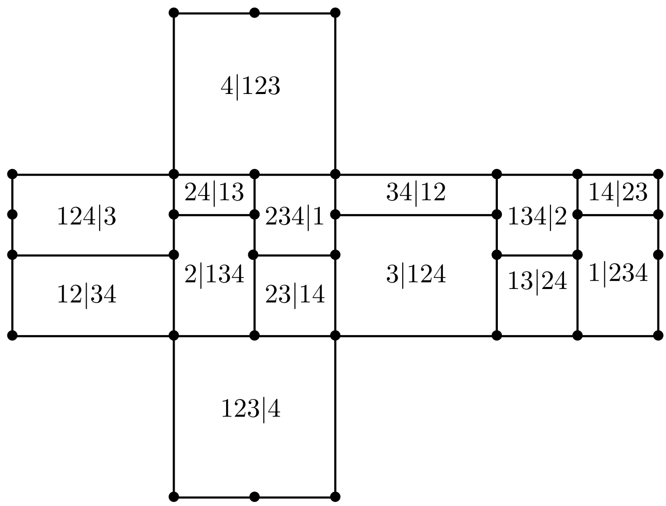

Figure 2: The facets of P_4 as a subdivision of I^3.

subset of the vertices of P_n and the Tamari order restricted to this subset agrees with the weak order.

Let e be a cell of X and let $|e|$ denote its dimension. A *k-subdivision cube of* e is a set of faces of e whose union is a k-subcube of I^n for some $k \leqslant n$. For example, when e is the top dimensional cell of P_4, the facets in $\{2|134, 24|13\}$ and $\{2|134, 24|13, 23|14, 234|1\}$ form 2-subdivision cubes of e, but any three in the latter do not (see Figure 2). Denote the set of vertices of e by $\mathcal{V}_e$ (when $e = X$ we suppress the subscript e). Given a vertex $v \in \mathcal{V}_e$, let $I_{v,1}^{k_1}$ and $I_{v,2}^{k_2}$ be k_i-subdivision cubes of e such that $\max I_{v,1}^{k_1} = \min I_{v,2}^{k_2} = v$ and $k_1 + k_2 = |e|$; then $\left(I_{v,1}^{k_1}, I_{v,2}^{k_2}\right)$ is a *pair of* (k_1, k_2)*-subdivision cubes of* e. Denote the set of all such pairs by e_v and let $(\mathbf{I}_{v,1}^{k_1}, \mathbf{I}_{v,2}^{k_2})_e$ denote its unique maximal element; then $\left(I_{v,3}^{k_3}, I_{v,4}^{k_4}\right) \subseteq (\mathbf{I}_{v,1}^{k_1}, \mathbf{I}_{v,2}^{k_2})_e$ for all $(I_{v,3}^{k_3}, I_{v,4}^{k_4}) \in e_v$. For example, when e is the top dimensional cell of P_4 and $v = 4|2|3|1$, we have $(\mathbf{I}_{v,1}^{2}, \mathbf{I}_{v,2}^{1})_e = (\{2|134, 24|13\}, \{4|23|1\})$. For an explicit description of $\left(\mathbf{I}_{v,1}^{k_1}, \mathbf{I}_{v,2}^{k_2}\right)_e$ when $e \subseteq P_n$ see (3) below.

If in addition, the cellular projection $p: P_{n+1} \to X$ preserves maximal pairs of (k_1, k_2)-subdivision cubes, i.e., for every cell e of P_{n+1} we have

$$p\left(\mathbf{I}_{v,1}^{k_1}, \mathbf{I}_{v,2}^{k_2}\right)_e = \left(\mathbf{I}_{p(v),1}^{k_1}, \mathbf{I}_{p(v),2}^{k_2}\right)_{p(e)},$$

the components of the induced diagonal Δ_X on a cell $f \subseteq X$ form the set of product cells

$$\Delta_X(f) := \bigcup_{\substack{\left(e^{k_1},\, e^{k_2}\right) \in \left(\mathbf{I}_{v,1}^{k_1}, \mathbf{I}_{v,2}^{k_2}\right)_f \\ v \in \mathcal{V}_f}} \{e^{k_1} \times e^{k_2}\}. \tag{1}$$

In particular, $p = \theta$ preserves maximal pairs of (k_1, k_2)-subdivision cubes and $\Delta_K(e)$ is given by setting $X = K_{n+2}$ (see (4) below). Note that $\left(e^{k_1}, e^{k_2}\right) \in \left(\mathbf{I}_{v,1}^{k_1}, \mathbf{I}_{v,2}^{k_2}\right)_X$ implies $e^{k_1} \leqslant e^{k_2}$. Thus $e^{k_1} \times e^{k_2}$ is a "matching pair" in the sense of MTTV (see Definition 3.1). Furthermore, since $f = p\,(e)$ for some

$$e = P_{n_1} \times \cdots \times P_{n_s} \quad \text{and} \quad p\,(e) = p(P_{n_1}) \times \cdots \times p(P_{n_s}),$$

the diagonal $\Delta_X(f)$ is automatically the comultiplicative extension of its values on the factors of f, i.e.,

$$\Delta_X(f) = \Delta_X(p(P_{n_1})) \times \cdots \times \Delta_X(p(P_{n_s})).$$

The subset $\mathcal{V}_e \subseteq S_n$ determines the components of $\Delta_P(e)$ in the following way: Let $\sigma = x_1|\cdots|x_n \in \mathcal{V}_e$. Reading σ from left-to-right and from right-to-left, construct the partitions $\overleftarrow{\sigma}_1|\cdots|\overleftarrow{\sigma}_p$ and $\overrightarrow{\sigma}_q|\cdots|\overrightarrow{\sigma}_1$ of maximal decreasing subsets and form the *Strong Complementary Pair* (SCP)

$$a_\sigma \times b_\sigma := \overleftarrow{\sigma}_1|\cdots|\overleftarrow{\sigma}_p \times \overrightarrow{\sigma}_q|\cdots|\overrightarrow{\sigma}_1 \in P(n) \times P(n).$$

Then

$$\begin{gathered}\sigma = \max a_\sigma = \min b_\sigma, \quad \min \overleftarrow{\sigma}_j < \max \overleftarrow{\sigma}_{j+1} \text{ for all } j < p, \text{ and} \\ \min \overrightarrow{\sigma}_i < \max \overrightarrow{\sigma}_{i+1} \text{ for all } i < q.\end{gathered}$$

Thus, for $\sigma = 2|1|3|5|4$ we have $\overleftarrow{\sigma}_1|\overleftarrow{\sigma}_2|\overleftarrow{\sigma}_3 = 21|3|54$ and $\overrightarrow{\sigma}_3|\overrightarrow{\sigma}_2|\overrightarrow{\sigma}_1 = 2|135|4$ so that $a_\sigma \times b_\sigma = 21|3|54 \times 2|135|4$.

Let $a = A_1|\cdots|A_p \in P(n)$. For $1 \leqslant j < p$, let $M_j \subseteq (A_j \setminus \{\min A_j\})$ such that $\min M_j > \max A_{j+1}$ when $M_j \neq \varnothing$. Define the *right-shift* M_j *action*

$$R_{M_j}(a) := \begin{cases} A_1|\cdots|A_j \setminus M_j|A_{j+1} \cup M_j|\cdots|A_k, & M_j \neq \varnothing \\ a, & M_j = \varnothing. \end{cases}$$

Let $\mathbf{M}{:=}\,(M_1, M_2, \ldots, M_{p-1})$ and denote the composition $R_{M_{p-1}} \cdots R_{M_2} R_{M_1}(a)$ by $R_{\mathbf{M}}\,(a)$.

Dually, let $b = B_q|\cdots|B_1 \in P(n)$. For $1 \leqslant i < q$, let $N_i \subseteq (B_i \setminus \{\min B_i\})$ such that $\min N_i > \max B_{i+1}$ when $N_i \neq \varnothing$. Define the *left-shift* N_i *action*

$$L_{N_i}(b) := \begin{cases} B_q|\cdots|B_{i+1} \cup N_i|B_i \setminus N_i|\cdots|B_1, & N_i \neq \varnothing \\ b, & N_i = \varnothing. \end{cases}$$

Let $\mathbf{N}{:=}\,(N_1, N_2, \ldots, N_{q-1})$ and denote the composition $L_{N_{q-1}} \cdots L_{N_2} L_{N_1}(b)$ by $L_{\mathbf{N}}\,(b)$.

Now given $\sigma \in \mathcal{V}_e$ and the SCP $a_\sigma \times b_\sigma$, the pair $R_{\mathbf{M}}(a_\sigma) \times L_{\mathbf{N}}(b_\sigma)$ is a *Complementary Pair* (CP) *on* $a_\sigma \times b_\sigma$. Define

$$A_\sigma \times B_\sigma := \bigcup_{\mathbf{M},\mathbf{N}} \{R_{\mathbf{M}}(a_\sigma) \times L_{\mathbf{N}}(b_\sigma)\}$$

and

$$\Delta_P(e) := \bigcup_{\sigma \in \mathcal{V}_e} A_\sigma \times B_\sigma. \tag{2}$$

Example 2.1. On the top dimensional cell $e^2 \subseteq P_3$, $\Delta_P(e^2)$ is the union of

$$\begin{array}{ll} A_{1|2|3} \times B_{1|2|3} = \{1|2|3 \times 123\}, & A_{1|3|2} \times B_{1|3|2} = \{1|32 \times 13|2\}, \\ A_{2|1|3} \times B_{2|1|3} = \{21|3 \times 2|13,\ 21|3 \times 23|1\}, & A_{2|3|1} \times B_{2|3|1} = \{2|31 \times 23|1\}, \\ A_{3|1|2} \times B_{3|1|2} = \{31|2 \times 3|12,\ 1|32 \times 3|12\}, & A_{3|2|1} \times B_{3|2|1} = \{321 \times 3|2|1\}. \end{array}$$

Remark 2.2. Note that the matrix representation of a CP introduced in [7] conveniently organizes and systematizes the combinatorial calculation of Δ_P. An SCP is represented by a *step matrix* and a general CP is represented by a *derived matrix*, given by left-shift and down-shift actions on a step matrix.

When $X = P_{n+1}$, Formulas (1) and (2) are equivalent. The maximal (k_1, k_2)-subdivision pair with respect to a vertex σ of P_{n+1} is

$$\left(\mathbf{I}_{\sigma,1}^{k_1}, \mathbf{I}_{\sigma,2}^{k_2}\right) = \left(\bigcup_{e_1 \in A_\sigma} e_1,\ \bigcup_{e_2 \in B_\sigma} e_2\right). \tag{3}$$

Definition 2.3. A positive dimensional face e of P_n is **non-degenerate** if $|\theta(e)| = |e|$. A positive dimensional partition $a = A_1|\cdots|A_p \in P(n)$ is **degenerate** if for some j and some $k > 0$, there exist $x, z \in A_j$ and $y \in A_{j+k}$ such that $x < y < z$; otherwise a is **non-degenerate**. A CP $\alpha \times \beta$ is **non-degenerate** if α and β are non-degenerate.

Define $\Delta_K(K_{n+1}) = \Delta_K(\theta(P_n)) := (\theta \times \theta)\Delta_P(P_n)$; then

$$\Delta_K(e^{n-1}) = \bigcup_{\substack{\text{non-degenerate CPs} \\ \alpha \times \beta \in A_\sigma \times B_\sigma \\ \sigma \in S_n}} \{\theta(\alpha) \times \theta(\beta)\}. \tag{4}$$

3. Agreement of Δ_K and Δ'_K

Definition 3.1. A pair of faces $a \times b \subseteq K_{n+1} \times K_{n+1}$ is a **Matching Pair** (MP) if $a \leqslant b$ and $|a| + |b| = n - 1$.

The "magical formula" derived in [5] and [6] is

$$\Delta'_K\left(e^{n-1}\right) = \bigcup_{\substack{\text{MPs of faces} \\ a \times b \subseteq K_{n+1} \times K_{n+1}}} \{a \times b\}. \tag{5}$$

Tonks' projection θ sends every non-degenerate CP to an MP. The converse is our main result: *Every MP is the image of a unique non-degenerate CP under θ; thus Δ'_K and Δ_K agree.* Our proof of this fact views P_n as a subdivision of K_{n+1}.

Definition 3.2. Let $0 \leqslant k < n$. An **associahedral** k**-cell of** P_n is a k-cell of K_{n+1}. A **subdivision** k**-cell of** P_n is a k-cell of some associahedral k-cell of P_n. The **maximal (**respt. **minimal) subdivision** k**-cell** of an associahedral k-cell a, denoted by $a_{\max}$ (respt. $a_{\min}$), satisfies $\max a_{\max} = \max a$ (respt. $\min a_{\min} = \min a$). A **non-degenerate vertex of** P_n is an associahedral vertex.

Thus a subdivision k-cell of P_n has the form $A_1|\cdots|A_{n-k}$. In fact, a vertex v of P_n is associahedral if and only if the $(n-q)$-cell $\overrightarrow{v}_q|\cdots|\overrightarrow{v}_1$ is non-degenerate, in which case $\min \overrightarrow{v}_q > \cdots > \min \overrightarrow{v}_1$. If $k > 0$, an associahedral k-cell a is a subdivision k-cell if and only if $a = a_{\min}$.

Proposition 3.3. *If a is an associahedral k-cell and u is a subdivision k-cell of a, then*

(i) $a_{\min}$ *is non-degenerate.*

(ii) If $u \neq a_{\min}$, *then u is degenerate and* $u = L_{\mathbf{N}}(a_{\min})$ *for some* $\mathbf{N}$.

(iii) $a_{\min} = R_{\mathbf{M}}(a_{\max})$ *for some* $\mathbf{M}$.

Proof. Set $p = n - k$ and consider an associahedral k-cell a of P_n. If a is also a subdivision k-cell, then $a = a_{\min} = \theta(a)$ is non-degenerate and $\mathbf{M} = \varnothing$. Otherwise, conclusions (i) and (ii) follow from the construction of P_n as a subdivision of K_{n+1}. For part (iii), given a subdivision k-cell $u = A_1|\cdots|A_p$ of a, let

$$N_p := \left\{x \in A_p \setminus \{\min A_p\} : x > \max A_{p-1}\right\}.$$

Inductively, if $1 < i < p$ and N_{i+1} has been constructed, let $A_i' := A_i \cup N_{i+1}$ and let

$$N_i := A_i' \setminus \left\{x \in A_i' \setminus \{\min A_i'\} : x > \max A_{i-1}\right\}.$$

Then $a_{\max} = L_{(N_p,\ldots,N_2)}(a_{\min})$. Set $\mathbf{M} = (M_1,\ldots,M_{p-1}) := (N_2,\ldots,N_p)$; then $a_{\min} = R_{\mathbf{M}}(a_{\max})$. □

Example 3.4. Consider the associahedral facet

$$a = 1|234 \cup 13|24 \cup 14|23 \cup 134|2;$$

then $a_{\min} = 1|234$ is non-degenerate,

$$13|24 = L_{\{3\}}(a_{\min}), \quad 14|23 = L_{\{4\}}(a_{\min}), \quad \text{and} \quad a_{\max} = 134|2 = L_{\{3,4\}}(a_{\min}).$$

Furthermore, $a_{\min} = 1|234 = R_{\{3,4\}}(134|2)$.

Proposition 3.5. *Let v be an associahedral vertex of P_n and let $a = \overrightarrow{v}_q|\cdots|\overrightarrow{v}_1$. If b is a non-degenerate cell of P_n such that $|b| = |a|$ and* $\min a \leqslant \min b$, *then* $b = L_{\mathbf{N}}(a)$ *for some* $\mathbf{N}$.

Proof. Let $a = A_{n-k}|\cdots|A_1$ and let $r_i = \min A_i$. Since v is associahedral, it follows that $r_{n-k} > \cdots > r_1$. Since $\min a \leqslant \min b$, there is a product of p-o increasing transpositions $\tau := \tau_t \cdots \tau_2\tau_1$ such that $\tau(\min a) = \min b$ and τ_i preserves the inequality $r_j > r_{j-1}$ for $1 \leqslant i \leqslant t$ and $1 \leqslant j \leqslant n-k$. Define $\tau_0 := \mathbf{Id}$ and consider the vertex $v_i := \tau_{t_i} \cdots \tau_1\tau_0(\min a)$ for each $1 \leqslant t_i \leqslant t$. For each i, there is the (possibly degenerate) cell $u_i := \overrightarrow{v_i}_q|\cdots|\overrightarrow{v_i}_1$, where $q \in \{n-k, n-k+1\}$. Thus there is the sequence

$\{a = u_0, u_1, \ldots, u_t = b\}$ and its subsequence of k-cells $\{a = u_{i_0}, u_{i_1}, \ldots, u_{i_{s-1}}, u_{i_s} = b\}$. By construction, for $1 \leqslant j \leqslant s$, there exists $n_j \in \underline{n}$ such that $u_{i_j} = L_{\{n_j\}}(u_{i_{j-1}})$. For $1 \leqslant i < s$, let

$$N_i = \left\{n_j \in A_i \cup N_1 \cup \cdots \cup N_{i-1} : u_{i_j} = L_{\{n_j\}}(u_{i_{j-1}}) \text{ for some } j\right\}$$

and form the sequence of sets $\mathbf{N} := (N_{s-1}, \ldots, N_1)$. Since b is non-degenerate, the action $L_{\mathbf{N}}(a)$ is defined and $L_{\mathbf{N}}(a) = b$. □

Identify a k-face $F \subset K_{n+1}$ with its corresponding associahedral k-cell of P_n and label F with its minimal subdivision k-cell $F_{\min}$; then $\theta(F_{\min}) = F$ (compare Figures 2 and 3).

Example 3.6. Consider the associahedral vertex $v = 5|3|1|2|4|6$, the associated 3-cell $a = \overrightarrow{v}_3|\overrightarrow{v}_2|\overrightarrow{v}_1 = 5|3|1246$ and the non-degenerate 3-cell $b = 56|34|12$. Then

$$\min a = 5|3|1|2|4|6 < 5|6|3|4|1|2 = \min b,$$

and there is the product of p-o increasing transpositions

$$\tau = \tau_6 \cdots \tau_1 := (3,6)\,(4,6)\,(1,6)\,(2,6)\,(1,4)\,(2,4)$$

such that

$$\Big\{v_1 = \tau_1(\min a) = 5|3|1|4|2|6,\ v_2 = \tau_2(v_1) = 5|3|4|1|2|6,\ v_3 = \tau_3(v_2) = 5|3|4|1|6|2,$$
$$v_4 = \tau_4(v_3) = 5|3|4|6|1|2,\ v_5 = \tau_5(v_4) = 5|3|6|4|1|2,\ v_6 = \tau_6(v_5) = 5|6|3|4|1|2\Big\}.$$

There is the sequence of cells

$$\Big\{u_0 = 5|3|1246,\ u_1 = 5|3|14|26,\ u_2 = 5|34|126,\ u_3 = 5|34|16|2,$$
$$u_4 = 5|346|12,\ u_5 = 5|36|4|12,\ u_6 = 56|34|12\Big\}$$

and its subsequence of 3-cells

$$\Big\{u_0 = 5|3|1246,\ u_2 = 5|34|126,\ u_4 = 5|346|12,\ u_6 = 56|34|12\Big\}.$$

Thus

$$N_1 = \left\{n_j \in A_1 : u_{i_j} = L_{\{n_j\}}\left(u_{i_{j-1}}\right) \text{ for some } j\right\} = \{4,6\},$$

and

$$N_2 = \left\{n_j \in A_2 \cup N_1 : u_{i_j} = L_{\{n_j\}}\left(u_{i_{j-1}}\right) \text{ for some } j\right\} = \{6\}.$$

Conclude that $56|34|12 = L_{(\{4,6\},\{6\})}(5|3|1246)$.

Theorem 3.7. *Let $F \times G \subset K_{n+1} \times K_{n+1}$ be an MP. Then $F_{\min} \times G_{\min} \subset P_n \times P_n$ is a CP and $F \times G = \theta(F_{\min}) \times \theta(G_{\min})$. Consequently, the diagonals Δ'_K and Δ_K agree.*

Proof. Let $\sigma = \max F$; then $F_{\max} = \overleftarrow{\sigma}_1|\cdots|\overleftarrow{\sigma}_p$ for some p and $F_{\min} = R_{\mathbf{M}}(F_{\max})$ for some $\mathbf{M}$ by Proposition 3.3. Let $\beta = \overrightarrow{\sigma}_q|\cdots|\overrightarrow{\sigma}_1$ and consider the SCP $F_{\max} \times \beta$.

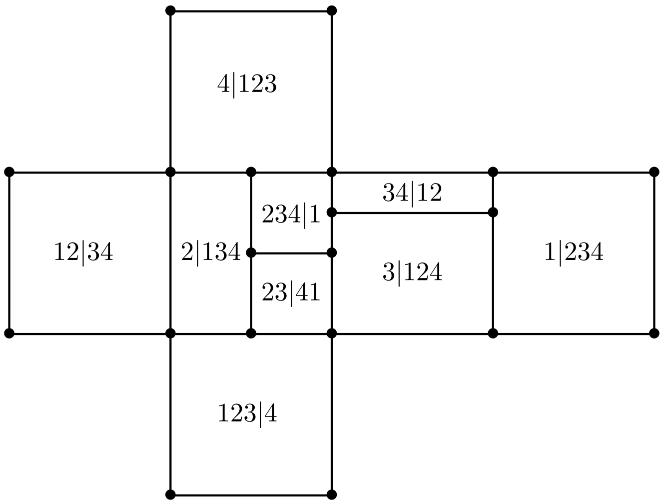

Figure 3: The facets of K_5 labeled with their minimal subdivision 2-cells in P_4.

Since σ is an associahedral vertex and $\min\beta \leqslant \min G_{\min}$ the hypotheses of Proposition 3.5 is satisfied; hence $G_{\min} = L_{\mathbf{N}}(\beta)$ for some $\mathbf{N}$. Therefore $F_{\min} \times G_{\min} = R_{\mathbf{M}}(F_{\max}) \times L_{\mathbf{N}}(\beta)$ is a CP and $F \times G = \theta(F_{\min}) \times \theta(G_{\min})$. $\square$

Example 3.8. Consider the diagonal component

$$F \times G = (\bullet\bullet\bullet)\bullet\bullet \times \bullet(\bullet\bullet(\bullet\bullet))$$

of $\Delta'_K(K_5)$. Then $F = 21|43 \cup 421|3$ is an associahedral 2-cell, $\sigma = \max F = 4|2|1|3$ is an associahedral vertex,

$$F_{\max} = \overleftarrow{\sigma}_1|\overleftarrow{\sigma}_2 = 421|3, \quad \text{and} \quad F_{\min} = 21|43 = R_{\{4\}}(421|3).$$

Furthermore,

$$\beta = \overrightarrow{\sigma}_3|\overrightarrow{\sigma}_2|\overrightarrow{\sigma}_1 = 4|2|13, \quad \min\beta_1 = 4|2|1|3 = \max F, \text{ and}$$
$$G_{\min} = L_{\{3\}}(4|2|13) = 4|23|1.$$

Thus $F \times G = \theta(21|43) \times \theta(4|23|1)$.

Addendum. After this paper was written, B. Delcroix-Oger, G. Laplante-Anfossi, V. Pilaud, and K. Stoeckl proved that Δ_P can be recovered from Δ'_P by an appropriate choice of chambers in the fundamental hyperplane arrangements of the permutahedra (see [**2**]). The fact that all known diagonals on the associahedra agree (up to mirror symmetry) follows immediately.

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