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THE HOMOTOPY CLASS OF TWISTED L_∞-MORPHISMS

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Abstract

The global formality of Dolgushev depends on the choice of a torsion-free covariant derivative. We prove that the globalized formalities with respect to two different covariant derivatives are homotopic. More explicitly, we derive the statement by proving a more general homotopy equivalence between L_∞-morphisms that are twisted with gauge equivalent Maurer–Cartan elements.

1. Introduction

The celebrated formality theorem by Kontsevich [**18**] provides the existence of an L_∞-quasi-isomorphism from the differential graded Lie algebra (DGLA) of polyvector fields $T_{\mathrm{poly}}(\mathbb{R}^d)$ to the DGLA of polydifferential operators $D_{\mathrm{poly}}(\mathbb{R}^d)$. In [**7, 8**] Dolgushev globalized this result to general smooth manifolds M using a geometric approach. Being a quasi-isomorphism, this formality induces a bijective correspondence

$$\boldsymbol{U}\colon \mathsf{Def}(T_{\mathrm{poly}}(M)[[\hbar]]) \longrightarrow \mathsf{Def}(D_{\mathrm{poly}}(M)[[\hbar]]) \tag{1}$$

between equivalence classes $\mathsf{Def}(T_{\mathrm{poly}}(M)[[\hbar]])$ of formal Poisson structures on M, $\hbar\pi \in \Gamma^\infty(\Lambda^2 TM)[[\hbar]]$, and equivalence classes $\mathsf{Def}(D_{\mathrm{poly}}(M)[[\hbar]])$ of star products $\star$ on M, see also [**5, 21**] for more details on deformation theory. In particular, this associates to a classical Poisson structure π_{cl} a class of deformation quantizations $\boldsymbol{U}([\hbar\pi_{\mathrm{cl}}])$ in the sense of the seminal paper [**2**]. On the other hand, it also gives a way to assign to each star product a class of formal Poisson structures, the so-called *Kontsevich class* of the star product.

However, the above mentioned globalization procedure of the Kontsevich formality from $\mathbb{R}^d$ to a general manifold M discussed in [**7**] depends on the choice of a torsion-free covariant derivative. More explicitly, it uses the covariant derivative to obtain Fedosov resolutions of the polyvector fields and polydifferential operators between which one has a fiberwise Kontsevich formality. Recently, in [**3**, Theorem 2.6] it has been shown that the map $\boldsymbol{U}$ from (1) does not depend on the choice of the connection. In this paper we investigate the role of the covariant derivative at the level of the formality and not at the level of equivalence classes of Maurer–Cartan elements.

The key point is that changing the covariant derivative corresponds to twisting by a Maurer–Cartan element that is equivalent to zero, see [**3**, Appendix C] for this

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observation and [7, 8, 12, 13] for more details on the twisting procedure. This corresponds to a more general observation: Let $F\colon (\mathfrak{g}, \mathrm{d}, [\cdot,\cdot]) \to (\mathfrak{g}', \mathrm{d}', [\cdot,\cdot])$ be an L_∞-morphism between DGLAs with complete descending and exhaustive filtrations $\mathcal{F}^\bullet\mathfrak{g}$ resp. $\mathcal{F}^\bullet\mathfrak{g}'$. Moreover, let $\pi \in \mathcal{F}^1\mathfrak{g}^1$ be a Maurer–Cartan element equivalent to zero via $\pi = \exp([g,\cdot]) \triangleright 0$ with $g \in \mathcal{F}^1\mathfrak{g}^0$. The element $\pi' = \sum_{k=1}^\infty \frac{1}{k!} F_k^1(\pi \vee \dots \vee \pi) \in \mathcal{F}^1\mathfrak{g}'^1$ is a Maurer–Cartan element in $\mathfrak{g}'$ equivalent to zero. Let the equivalence be given by $g' \in \mathcal{F}^1\mathfrak{g}'^0$, then one obtains for the twisted L_∞-morphism F^π (see Proposition 2.17) Proposition 3.10:

Proposition 1.1. *The L_∞-morphisms F and $e^{[-g',\cdot]} \circ F^\pi \circ e^{[g,\cdot]}$ from $(\mathfrak{g}, \mathrm{d}, [\cdot,\cdot])$ to $(\mathfrak{g}', \mathrm{d}', [\cdot,\cdot])$ are homotopic, where F^π denotes the L_∞-morphism F twisted by π.*

By homotopic we mean here that the two L_∞-morphisms are equivalent Maurer–Cartan elements in the convolution DGLA, compare [9, Definition 3], see also [11] for a comparison of different notions of homotopies between L_∞-morphisms.

This general statement can be applied to the globalization of the Kontsevich formality. Our main result here is the following theorem, see Theorem 4.13:

Theorem 1.2. *Let ∇ and ∇' be two different torsion-free covariant derivatives. Then the two global formalities constructed via Dolgushev's globalization procedure are homotopic.*

This immediately implies that they induce the same map on the equivalence classes of formal Maurer–Cartan elements, i.e. [3, Theorem 2.6].

Note that there are many other similar globalization procedures of formalities based on Dolgushev's globalization of the Kontsevich formality [7, 8], e.g. [4] for Lie algebroids, [19] for differential graded manifolds and [6] for Hochschild chains. The above technique can be adapted to these cases and we plan to pursue them in further works.

Finally, we want to mention that the globalization proposed by Dolgushev is not the first one. In fact Kontsevich himself globalized his local Formality in the same paper he proved gave a construction for it, see [18, Section 7]. He is using the language of ∞-jet spaces of polyvector fields and polydifferential operators, respectively. However, these ∞-jet spaces are (non-canonically) isomorphic as vector bundles to the formally completed fiberwise polyvector fields and polydifferential operators, respectively. The corresponding isomorphisms are constructed by the choice of a connection. We strongly believe that the globalization procedure proposed by Kontsevich in [18] is homotopic to the globalization from Dolgushev [7, 8] we are using in this note.

The paper is organized as follows: In Section 2 we recall the basics concerning Maurer–Cartan elements in DGLAs and L_∞-algebras, the notions of gauge and homotopy equivalence as well as the twisting procedure. Then we recall in Section 3 the interpretation of L_∞-morphisms as Maurer–Cartan elements and the notion of homotopic L_∞-morphisms. We show that pre- and post-compositions of homotopic L_∞-morphisms with an L_∞-morphism are again homotopic, a statement that is probably well-known to the experts, but that we could not find in the literature. Moreover, we prove here Proposition 3.10, i.e. that the twisted L_∞-morphisms are homotopic for equivalent Maurer–Cartan elements. Finally, we apply these general results to the globalization of Kontsevich's formality theorem, proving Theorem 4.13 and also an equivariant version for Lie group actions with invariant covariant derivatives.

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2. Preliminaries: Maurer–Cartan elements and twisting

2.1. Maurer–Cartan elements in DGLAs

We want to recall the basics concerning differential graded Lie algebras (DGLAs), Maurer–Cartan elements and their equivalence classes. In order to make sense of the gauge equivalence we consider in this context DGLAs $(\mathfrak{g}^\bullet, \mathrm{d}, [\cdot,\cdot])$ with complete descending filtrations

$$\cdots \supseteq \mathcal{F}^{-2}\mathfrak{g} \supseteq \mathcal{F}^{-1}\mathfrak{g} \supseteq \mathcal{F}^{0}\mathfrak{g} \supseteq \mathcal{F}^{1}\mathfrak{g} \supseteq \cdots, \qquad \mathfrak{g} \cong \varprojlim \mathfrak{g}/\mathcal{F}^n\mathfrak{g} \tag{2}$$

and

$$\mathrm{d}(\mathcal{F}^k\mathfrak{g}) \subseteq \mathcal{F}^k\mathfrak{g} \qquad \text{and} \qquad [\mathcal{F}^k\mathfrak{g}, \mathcal{F}^\ell\mathfrak{g}] \subseteq \mathcal{F}^{k+\ell}\mathfrak{g}. \tag{3}$$

In particular, $\mathcal{F}^1\mathfrak{g}$ is a projective limit of nilpotent DGLAs. In most cases the filtration will be bounded below, i.e. bounded from the left with $\mathfrak{g} = \mathcal{F}^k\mathfrak{g}$ for some $k \in \mathbb{Z}$. If the filtration is unbounded, then we assume always that it is in addition exhaustive, i.e. that

$$\mathfrak{g} = \bigcup_n \mathcal{F}^n\mathfrak{g}, \tag{4}$$

even if we do not mention it explicitly. Moreover, we assume that the DGLA morphisms are compatible with the filtrations.

Example 2.1. One motivation to consider the case of filtered DGLAs is formal power series $\mathfrak{g}[[\hbar]]$ of a DGLA $\mathfrak{g}$ with filtration $\mathcal{F}^k(\mathfrak{g}[[\hbar]]) = \hbar^k(\mathfrak{g}[[\hbar]])$.

Definition 2.2 (Maurer–Cartan elements)**.** Let $(\mathfrak{g}, \mathrm{d}, [\,\cdot\,,\,\cdot\,])$ be a DGLA with complete descending filtration. Then $\pi \in \mathcal{F}^1\mathfrak{g}^1$ is called *Maurer–Cartan element* if it satisfies the Maurer–Cartan equation

$$\mathrm{d}\pi + \frac{1}{2}[\pi, \pi] = 0. \tag{5}$$

The set of Maurer–Cartan elements is denoted by $\mathsf{MC}(\mathfrak{g})$.

Maurer–Cartan elements π lead to *twisted* DGLA structures $(\mathfrak{g}, \mathrm{d} + [\pi,\cdot], [\cdot,\cdot])$ and one has a gauge action on the set of Maurer–Cartan elements.

Proposition 2.3 (Gauge action)**.** *Let* $(\mathfrak{g}, \mathrm{d}, [\,\cdot\,,\,\cdot\,])$ *be a DGLA with complete descending filtration. The* gauge group $\mathrm{G}^0(\mathfrak{g}) = \{\Phi = e^{[g,\,\cdot\,]} \colon \mathfrak{g} \longrightarrow \mathfrak{g} \mid g \in \mathcal{F}^1\mathfrak{g}^0\}$ *defines an* *action on* $\mathsf{MC}(\mathfrak{g})$ *via*

$$\begin{aligned}\exp([g,\,\cdot\,])\triangleright\pi &= \sum_{n=0}^{\infty}\frac{([g,\,\cdot\,])^n}{n!}(\pi)-\sum_{n=0}^{\infty}\frac{([g,\,\cdot\,])^n}{(n+1)!}(\mathrm{d}g)\\ &= \pi-\frac{\exp([g,\,\cdot\,])-\mathrm{id}}{[g,\cdot]}(\mathrm{d}g+[\pi,g]).\end{aligned}\tag{6}$$

The set of equivalence classes of Maurer–Cartan elements in $\mathfrak{g}$ *is denoted by*

$$\mathsf{Def}(\mathfrak{g})=\frac{\mathsf{MC}(\mathfrak{g})}{\mathrm{G}^0(\mathfrak{g})}.\tag{7}$$

Note that the gauge action is well-defined since $g\in\mathcal{F}^1\mathfrak{g}$ and as the filtration is complete. $\mathsf{Def}(\mathfrak{g})$ is the orbit space of the transformation groupoid $\mathsf{MC}(\mathfrak{g})$ of the gauge action and $\mathsf{MC}(\mathfrak{g})$ is also called *Goldman–Millson groupoid* or *Deligne groupoid* [**16**]. It plays an important role in deformation theory [**21**]. In particular, the definition of the gauge action implies that twisting with gauge equivalent Maurer–Cartan elements leads to isomorphic DGLAs.

Corollary 2.4. *Let* $(\mathfrak{g},\mathrm{d},[\,\cdot\,,\,\cdot\,])$ *be a DGLA with complete descending filtration and with gauge equivalent Maurer–Cartan elements* π',π *via* $g\in\mathrm{G}^0(\mathfrak{g})$*. Then one has*

$$\mathrm{d}+[\pi',\cdot]=\exp([g,\,\cdot\,])\circ(\mathrm{d}+[\pi,\cdot])\circ\exp([-g,\,\cdot\,]).\tag{8}$$

In other words, $\exp([g,\cdot])\colon(\mathfrak{g},\mathrm{d}+[\pi,\cdot],[\,\cdot\,,\,\cdot\,])\to(\mathfrak{g},\mathrm{d}+[\pi',\cdot],[\,\cdot\,,\,\cdot\,])$ *is an isomorphism of DGLAs.*

2.2. Maurer–Cartan elements in L_∞-algebras

Let us recall the basics of L_∞-algebras and L_∞-morphisms. Proofs and further details can be found in [**7, 8, 13**]. Note that in this work we only consider L_∞-morphisms between DGLAs.

An L_∞*-algebra* (L,Q) is a graded vector space L together with a degree $+1$ codifferential Q on the graded cocommutative cofree coalgebra $(\overline{\mathrm{S}}(L[1]),\overline{\Delta})$ without counit cogenerated by $L[1]$. We always consider a vector space over a field $\mathbb{K}$ of characteristic zero. The codifferential Q is uniquely determined by the Taylor components $Q_n\colon\mathrm{S}^n(L[1])\longrightarrow L[2]$ for $n\geqslant 1$. Sometimes we also write $Q_k=Q_k^1$ and following [**5**] we denote by Q_n^i the component $\mathrm{pr}_{\mathrm{S}^i(L[1])}\circ Q\big|_{\mathrm{S}^n(L[1])}\colon\mathrm{S}^n(L[1])\to\mathrm{S}^i(L[1])[1]$ of Q. The property $Q^2=0$ implies in particular that $Q_1^1\colon L\to L[1]$ is a cochain differential. Let us consider two L_∞-algebras (L,Q) and (L',Q'). A degree 0 coalgebra morphism $F\colon\overline{\mathrm{S}}(L[1])\longrightarrow\overline{\mathrm{S}}(L'[1])$ such that $FQ=Q'F$ is called L_∞*-morphism*. Just like the codifferential also the morphism F is also uniquely determined by its Taylor components $F_n\colon\mathrm{S}^n(L[1])\longrightarrow L'[1]$, where $n\geqslant 1$. We write again $F_k=F_k^1$ and we get coefficients $F_n^j\colon\mathrm{S}^n(L[1])\to\mathrm{S}^j(L'[1])$ of F. Note that F_n^j depends only on $F_k^1=F_k$ for $k\leqslant n-j+1$. In particular, the first structure map of F is a map of complexes $F_1^1\colon(L,Q_1^1)\to(L',(Q')_1^1)$ and one calls F L_∞*-quasi-isomorphism* if F_1^1 is a quasi-isomorphism of complexes. Note that in the following we use $\vee$ for the graded commutative on the symmetric algebra of a vector space.

Example 2.5 (DGLA). A DGLA $(\mathfrak{g},\mathrm{d},[\,\cdot\,,\,\cdot\,])$ is an L_∞-algebra with $Q_1=-\mathrm{d}$ and $Q_2(\gamma\vee\mu)=-(-1)^{|\gamma|}[\gamma,\mu]$, where $|\gamma|$ denotes the degree in $\mathfrak{g}[1]$.

In order to generalize the definition of Maurer–Cartan elements we consider again L_∞-algebras with complete descending and exhaustive filtrations on L. Moreover, we require for the codifferential Q of L

$$Q_k(\mathcal{F}^{i_1}L \vee \cdots \vee \mathcal{F}^{i_k}L) \subseteq \mathcal{F}^{i_1+\cdots+i_k}L.$$

We assume again that L_∞-morphisms are compatible with the filtrations.

Definition 2.6 (Maurer–Cartan elements II)**.** Let (L, Q) be an L_∞-algebra with compatible complete descending filtration. Then $\pi \in \mathcal{F}^1L[1]^0$ is called *Maurer–Cartan element* if it satisfies the Maurer–Cartan equation

$$\sum_{n>0} \frac{1}{n!} Q_n(\pi \vee \cdots \vee \pi) = 0. \tag{9}$$

The set of Maurer–Cartan elements is again denoted by $\mathsf{MC}(L)$.

Note that the sum in (9) is well-defined for $\pi \in \mathcal{F}^1L^1$ because of the completeness of L. We recall some useful properties from [**8**, Prop. 1]:

Lemma 2.7. *Let $F\colon (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ be an L_∞-morphism of DGLAs and $\pi \in \mathcal{F}^1\mathfrak{g}^1$.*

(i) $d\pi + \frac{1}{2}[\pi, \pi] = 0$ is equivalent to $Q(\overline{\exp}(\pi)) = 0$, where $\overline{\exp}(\pi) = \sum_{k=1}^\infty \frac{1}{k!}\pi^{\vee k}$.

(ii) $F(\overline{\exp}(\pi)) = \overline{\exp}(S)$ with $S = \sum_{n>0} \frac{1}{n!} F_n(\pi \vee \cdots \vee \pi)$.

(iii) If π is a Maurer–Cartan element, then so is S.

We recall the generalization of the gauge action to an equivalence relation on the set of Maurer–Cartan elements of L_∞-algebras. We follow [**5**, Section 4] but adapt the definitions to the case of L_∞-algebras with complete descending and exhaustive filtrations as in [**11**]. Let therefore (L, Q) be such an L_∞-algebra with complete descending and exhaustive filtration and consider $L[t] = L \otimes \mathbb{K}[t]$ which has again a descending and exhaustive filtration

$$\mathcal{F}^kL[t] = \mathcal{F}^kL \otimes \mathbb{K}[t].$$

We denote its completion by $\widehat{L[t]}$ and note that since Q is compatible with the filtration it extends to $\widehat{L[t]}$. Similarly, L_∞-morphisms extend to these completed spaces.

Remark 2.8. Note that one can define the completion as space of equivalence classes of Cauchy sequences with respect to the filtration topology. Alternatively, the completion can be identified with

$$\varprojlim L[t]/\mathcal{F}^nL[t] \subset \prod_n L[t]/\mathcal{F}^nL[t] \cong \prod_n L/\mathcal{F}^nL \otimes \mathbb{K}[t]$$

consisting of all coherent tuples $X = (x_n)_n \in \prod_n L[t]/\mathcal{F}^nL[t]$, where

$$L[t]/\mathcal{F}^{n+1}L[t] \ni x_{n+1} \longmapsto x_n \in L[t]/\mathcal{F}^nL[t]$$

under the obvious surjections. Moreover, $\mathcal{F}^n\widehat{L[t]}$ corresponds to the kernel of the projective limit $\varprojlim L[t]/\mathcal{F}^nL[t] \to L[t]/\mathcal{F}^nL[t]$ and thus

$$\widehat{L[t]}/\mathcal{F}^n\widehat{L[t]} \cong L[t]/\mathcal{F}^nL[t].$$

Since L is complete, we can also interpret $\widehat{L[t]}$ as the subspace of $L[[t]]$ such that

$X \bmod \mathcal{F}^n L[[t]]$ is polynomial in t. In particular, $\mathcal{F}^n \widehat{L[t]}$ is the subspace of elements in $\mathcal{F}^n L[[t]]$ that are polynomial in t modulo $\mathcal{F}^m L[[t]]$ for all m.

By the above construction of $\widehat{L[t]}$ it is clear that differentiation $\frac{\mathrm{d}}{\mathrm{d}t}$ and integration with respect to t extend to it since they do not change the filtration. Sometimes we write also $\dot{X}$ instead of $\frac{\mathrm{d}}{\mathrm{d}t}X$ and, moreover, the evaluation

$$\delta_s \colon \widehat{L[t]} \ni X \longmapsto X(s) = X\big|_{t=s} \in L$$

is well-defined for all $s \in \mathbb{K}$ since L is complete.

Example 2.9. In the case that the filtration of L comes from a grading $L^\bullet$, the completion is given by $\widehat{L[t]} \cong \prod_i L^i[t]$, i.e. by polynomials in each degree. A special case is here the case of formal power series $L = V[[\hbar]]$ with $\widehat{L[t]} \cong (V[t])[[\hbar]]$ as in [**3**, Appendix A].

Now we can introduce a general equivalence relation between Maurer–Cartan elements of L_∞-algebras.

Definition 2.10 (Homotopy equivalence)**.** Let (L, Q) be a L_∞-algebra with a complete descending filtration. The *homotopy equivalence relation* on the set $\mathsf{MC}(L)$ is the transitive closure of the relation $\sim$ defined by: $\pi_0 \sim \pi_1$ if and only if there exist $\pi(t) \in \mathcal{F}^1\widehat{L^1[t]}$ and $\lambda(t) \in \mathcal{F}^1\widehat{L^0[t]}$ such that

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}t}\pi(t) = Q^1(\lambda(t) \vee \exp(\pi(t))) = \sum_{n=0}^{\infty} \frac{1}{n!} Q^1_{n+1}(\lambda(t) \vee \pi(t) \vee \cdots \vee \pi(t)), \\ \pi(0) = \pi_0 \qquad \text{and} \qquad \pi(1) = \pi_1.\end{gathered} \tag{10}$$

The set of equivalence classes of Maurer–Cartan elements of L is denoted by $\mathsf{Def}(L) = \mathsf{MC}(L)/\sim$.

Note that in the case of nilpotent L_∞-algebras it suffices to consider polynomials in t as there is no need to complete $L[t]$, compare [**15**]. We check now that this is well-defined and even yields a curve $\pi(t)$ of Maurer–Cartan elements.

Proposition 2.11. *For every $\pi_0 \in \mathcal{F}^1 L^1$ and $\lambda(t) \in \mathcal{F}^1\widehat{L^0[t]}$ there exists a unique $\pi(t) \in \mathcal{F}^1\widehat{L^1[t]}$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\pi(t) = Q^1(\lambda(t) \vee \exp(\pi(t)))$ and $\pi(0) = \pi_0$. If $\pi_0 \in \mathsf{MC}(L)$, then $\pi(s) \in \mathsf{MC}(L)$ for all $s \in \mathbb{K}$.*

Proof. The proof for the nilpotent case can be found in [**5**, Prop. 4.8]. In our setting of complete filtrations we only have to show that the solution $\pi(t) = \sum_{k=0}^\infty \pi_k t^k$ in the formal power series $\mathcal{F}^1 L^1 \otimes \mathbb{K}[[t]]$ is an element of $\mathcal{F}^1\widehat{L^1[t]}$. By Remark 2.8 this is equivalent to $\pi(t) \bmod \mathcal{F}^n L^1[[t]] \in L^1[t]$ for all n. Indeed, we have inductively

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi(t) \mod \mathcal{F}^2 L^1[[t]] = Q^1(\lambda(t)) \mod \mathcal{F}^2 L^1[[t]] \in L^1[t].$$

For the higher orders we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi(t) \equiv \sum_{k=0}^{n-2} \frac{1}{k!} Q^1_{k+1}(\lambda(t) \vee \pi(t) \vee \cdots \vee \pi(t)) \mod \mathcal{F}^n L^1[[t]]$$

and thus $\pi(t) \bmod \mathcal{F}^n L^1[[t]] \in L^1[t]$. □

One can show that for DGLAs with complete filtrations the two notions of equivalences are equivalent, see e.g. [**21**, Thm. 5.5].

Theorem 2.12. *Two Maurer–Cartan elements in* $(\mathfrak{g}, \mathrm{d}, [\cdot,\cdot])$ *are homotopy equivalent if and only if they are gauge equivalent.*

One direction of this theorem can be made more explicit in the following proposition.

Proposition 2.13. *Let* $(\mathfrak{g}, \mathrm{d}, [\cdot,\cdot])$ *be a DGLA with complete descending filtration. Consider* $\pi_0 \sim \pi_1$ *with equivalence given by* $\pi(t) \in \mathcal{F}^1\widehat{\mathfrak{g}^1[t]}$ *and* $\lambda(t) \in \mathcal{F}^1\widehat{\mathfrak{g}^0[t]}$*. The formal solution of*

$$\lambda(t) = \frac{\exp([A(t),\cdot]) - \mathrm{id}}{[A(t),\cdot]}\frac{\mathrm{d}A(t)}{\mathrm{d}t}, \qquad A(0) = 0 \tag{11}$$

is an element $A(t) \in \mathcal{F}^1\widehat{\mathfrak{g}^0[t]}$ *and satisfies*

$$\pi(t) = e^{[A(t),\cdot]}\pi_0 - \frac{\exp([A(t),\cdot]) - \mathrm{id}}{[A(t),\cdot]}\mathrm{d}A(t). \tag{12}$$

In particular, for $g = A(1) \in \mathcal{F}^1\mathfrak{g}^0$ *one has*

$$\pi_1 = \exp([g,\cdot]) \triangleright \pi_0. \tag{13}$$

Proof. As formal power series in t Equation 11 has a unique solution

$$A(t) \in \mathcal{F}^1\mathfrak{g}^0 \otimes \mathbb{K}[[t]].$$

But one has even $A(t) \in \mathcal{F}^1\widehat{\mathfrak{g}^0[t]}$ since

$$\begin{aligned}
\frac{\mathrm{d}A(t)}{\mathrm{d}t} &\equiv \lambda(t) - \sum_{k=1}^{n-2} \frac{1}{(k+1)!}[A(t),\cdot]^k \frac{\mathrm{d}A(t)}{\mathrm{d}t} \mod \mathcal{F}^n\mathfrak{g}[[t]] \\
&\equiv \left(\lambda(t) - \sum_{k=1}^{n-2} \frac{1}{(k+1)!}[A(t) \mod \mathcal{F}^{n-1}\mathfrak{g}[[t]],\cdot]^k \left(\frac{\mathrm{d}A(t)}{\mathrm{d}t} \mod \mathcal{F}^{n-1}\mathfrak{g}[[t]]\right)\right) \\
&\qquad \mod \mathcal{F}^n\mathfrak{g}[[t]]
\end{aligned}$$

is by induction polynomial in t. Note that one has

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{[A(t),\cdot]} = \left[\frac{\exp([A(t),\cdot]) - \mathrm{id}}{[A(t),\cdot]}\frac{\mathrm{d}A(t)}{\mathrm{d}t}, \cdot\right] \circ \exp([A(t),\cdot]). \tag{$*$}$$

Our aim is now to show that $\pi'(t) = e^{[A(t),\cdot]}\pi_0 - \frac{\exp([A(t),\cdot])-\mathrm{id}}{[A(t),\cdot]}\mathrm{d}A(t)$, i.e. the right hand side of (12), satisfies

$$\begin{aligned}
\frac{\mathrm{d}\pi'(t)}{\mathrm{d}t} &= -\mathrm{d}\lambda(t) + \left[\lambda(t), e^{[A(t),\cdot]}\pi_0 - \frac{\exp([A(t),\cdot]) - \mathrm{id}}{[A(t),\cdot]}\mathrm{d}A(t)\right] \\
&= -\mathrm{d}\lambda(t) + [\lambda(t), \pi'(t)],
\end{aligned}$$

which is just (10) in the special case of DGLAs. Then we know $\pi'(t) = \pi(t) \in \mathcal{F}^1\widehat{\mathfrak{g}^1[t]}$ since the solution $\pi(t)$ is unique by Proposition 2.11, which immediately identifies

$\pi'(1) = \pi_1$. At first we compute

$$\begin{aligned} \mathrm{d}\lambda(t) &= \frac{\exp([A(t),\cdot]) - \mathrm{id}}{[A(t),\cdot]} \mathrm{d}\frac{\mathrm{d}A(t)}{\mathrm{d}t} \\ &\quad + \sum_{k=0}^{\infty}\sum_{j=0}^{k-1} \frac{1}{(k+1)!}\binom{k}{j+1}\left[\mathrm{ad}_A^j \mathrm{d}A(t), \mathrm{ad}_A^{k-1-j}\frac{\mathrm{d}A(t)}{\mathrm{d}t}\right] \end{aligned}$$

and using $(*)$ we get

$$\begin{aligned} \frac{\mathrm{d}\pi'(t)}{\mathrm{d}t} &= \left[\frac{\exp([A(t),\cdot]) - \mathrm{id}}{[A(t),\cdot]}\frac{\mathrm{d}A(t)}{\mathrm{d}t}, \exp([A(t),\cdot])\pi_0\right] - \frac{\exp([A(t),\cdot]) - \mathrm{id}}{[A(t),\cdot]}\mathrm{d}\frac{\mathrm{d}A(t)}{\mathrm{d}t} \\ &\quad - \sum_{k=0}^{\infty}\sum_{j=0}^{k-1} \frac{1}{(k+1)!}\binom{k}{j+1}\left[\mathrm{ad}_A^j \frac{\mathrm{d}A(t)}{\mathrm{d}t}, \mathrm{ad}_A^{k-1-j}\mathrm{d}A\right] \\ &= -\mathrm{d}\lambda(t) + \left[\lambda(t), e^{[A(t),\cdot]}\pi_0 - \frac{\exp([A(t),\cdot]) - \mathrm{id}}{[A(t),\cdot]}\mathrm{d}A(t)\right] \end{aligned}$$

and the proposition is proven. □

Remark 2.14. There are also different notions of homotopy resp. gauge equivalences for Maurer–Cartan elements in L_∞-algebras: 1 e.g. the above definition, sometimes also called *Quillen homotopy*, and the *gauge homotopy* where one requires $\lambda(t) = \lambda$ to be constant, compare [**9**]. In [**11**] it is shown that these notions are also equivalent for complete L_∞-algebras, extending the result for DGLAs.

One important property is that L_∞-morphisms map equivalence classes of Maurer–Cartan elements to equivalence classes, see [**5**, Prop. 4.9].

Proposition 2.15. *Let $F\colon (L,Q) \to (L',Q')$ be a morphism of L_∞-algebras with complete filtrations, and $\pi_0, \pi_1 \in \mathsf{MC}(L)$ with $\pi_0 \sim \pi_1$ via $\pi(t) \in \mathcal{F}^1\widehat{\mathfrak{g}^1[t]}$ along with $\lambda(t) \in \mathcal{F}^1\widehat{\mathfrak{g}^0[t]}$. Then F is compatible with the homotopy equivalence relation, i.e. one has $F^1(\overline{\exp}\pi_0) \sim F^1(\overline{\exp}\pi_1)$ via*

$$\pi'(t) = F^1(\overline{\exp}(\pi(t))) \quad \text{and} \quad \lambda'(t) = F^1(\lambda(t) \vee \exp(\pi(t))).$$

If F is an L_∞-quasi-isomorphism, then it is well-known that it induces a bijection on the equivalence classes of Maurer–Cartan elements. Finally, recall that also the twisting with Maurer–Cartan elements can be generalized to L_∞-algebras, see [**6**, Section 2.3].

Lemma 2.16. *Let (L,Q) be an L_∞-algebra and $\pi \in \mathcal{F}^1L[1]^0$ a Maurer–Cartan element. Then the map Q^π given by*

$$Q^\pi(X) = \exp((-\pi)\vee)Q(\exp(\pi\vee)X), \qquad X \in \overline{\mathrm{S}}(L[1]) \tag{14}$$

defines a codifferential on $\overline{\mathrm{S}}(L[1])$.

One can not only twist the DGLAs resp. L_∞-algebras, but also the L_∞-morphisms between them. Below we need the following result, see [**6**, Prop. 2] and [**8**, Prop. 1].

Proposition 2.17. *Let $F\colon (\mathfrak{g},Q) \to (\mathfrak{g}',Q')$ be an L_∞-morphism of DGLAs, with $\pi \in \mathcal{F}^1\mathfrak{g}^1$ a Maurer–Cartan element and $S = F^1(\overline{\exp}(\pi)) \in \mathcal{F}^1\mathfrak{g}'^1$.*

(i) The map

$$F^\pi = \exp(-S\vee) F \exp(\pi\vee) \colon \overline{\mathrm{S}}(\mathfrak{g}[1]) \longrightarrow \overline{\mathrm{S}}(\mathfrak{g}'[1])$$

defines an L_∞-morphism between the DGLAs $(\mathfrak{g}, \mathrm{d} + [\pi, \cdot\,])$ and $(\mathfrak{g}', \mathrm{d} + [S, \cdot\,])$.

(ii) The structure maps of F^π are given by

$$F_n^\pi(x_1, \dots, x_n) = \sum_{k=0}^{\infty} \frac{1}{k!} F_{n+k}(\pi, \dots, \pi, x_1, \dots, x_n). \tag{15}$$

(iii) Let F be an L_∞-quasi-isomorphism such that F_1^1 is not only a quasi-isomorphism of filtered complexes $L \to L'$ but even induces a quasi-isomorphism

$$F_1^1 \colon \mathcal{F}^k L \longrightarrow \mathcal{F}^k L'$$

for each k. Then F^π is an L_∞-quasi-isomorphism.

3. Relation between twisted morphisms

Here we prove the main results about the relation between twisted L_∞-morphisms. More explicitly, consider an L_∞-morphism $F \colon (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ between DGLAs and let $\pi_0, \pi_1 \in \mathcal{F}^1\mathfrak{g}^1$ be two equivalent Maurer–Cartan elements via $\pi_1 = \exp([g, \cdot]) \triangleright \pi_0$. We show that F^{π_0} and F^{π_1} can be interpreted as homotopic in the sense of [**9**, Definition 3].

3.1. L_∞-morphisms as Maurer–Cartan elements

At first, recall that we can interpret L_∞-morphisms as Maurer–Cartan elements in the convolution algebra. More explicitly, let $(L, Q), (L', Q')$ be two L_∞-algebras and denote the graded linear maps by $\operatorname{Hom}(\overline{\mathrm{S}}(L[1]), L')$. If L and L' are equipped with complete descending filtrations, then we require the maps to be compatible with the filtration. The L_∞-structures on L and L' lead to an L_∞-structure on this vector space of maps, see [**9**, Proposition 1 and Proposition 2] and also [**3**] for the case of DGLAs.

Proposition 3.1. *The coalgebra $\overline{\mathrm{S}}(\operatorname{Hom}(\overline{\mathrm{S}}(L[1]), L')[1])$ can be equipped with a codifferential $\widehat{Q}$ with structure maps*

$$\widehat{Q}_1^1 F = Q_1'^1 \circ F - (-1)^{|F|} F \circ Q \tag{16}$$

and

$$\widehat{Q}_n^1(F_1 \vee \cdots \vee F_n) = (Q')_n^1 \circ \vee^{n-1} \circ (F_1 \otimes F_2 \otimes \cdots \otimes F_n) \circ \overline{\Delta}^{n-1}. \tag{17}$$

It is called convolution L_∞-algebra *and its Maurer–Cartan elements are identified with L_∞-morphisms. Here $|F|$ denotes the degree in* $\operatorname{Hom}(\overline{\mathrm{S}}(L[1]), L')[1]$.

Example 3.2. Let $\mathfrak{g}, \mathfrak{g}'$ be two DGLAs. Then $\operatorname{Hom}(\overline{\mathrm{S}}(\mathfrak{g}[1]), \mathfrak{g}')$ is in fact a DGLA with differential

$$\partial F = \mathrm{d}' \circ F + (-1)^{|F|} F \circ Q \tag{18}$$

and Lie bracket

$$[F, G] = -(-1)^{|F|} (Q')_2^1 \circ (F \otimes G) \circ \overline{\Delta}. \tag{19}$$

Here $|F|$ denotes again the degree in $\operatorname{Hom}(\overline{\mathrm{S}}(\mathfrak{g}[1]), \mathfrak{g}')[1]$. This DGLA is also called *convolution DGLA.*

We note that the convolution L_∞-algebra $\mathcal{H} = \operatorname{Hom}(\overline{\mathrm{S}}(L[1]), L')$ is equipped with the following complete descending filtration:

$$\begin{aligned} \mathcal{H} =& \mathcal{F}^1\mathcal{H} \supset \mathcal{F}^2\mathcal{H} \supset \cdots \supset \mathcal{F}^k\mathcal{H} \supset \cdots \\ \mathcal{F}^k\mathcal{H} =& \left\{ f \in \operatorname{Hom}(\overline{\mathrm{S}}(L[1]), L') \mid f\big|_{\mathrm{S}^{<k}(L[1])} = 0 \right\}. \end{aligned} \tag{20}$$

Thus all twisting procedures are well-defined and one can define a notion of homotopic L_∞-morphisms.

Definition 3.3. Two L_∞-morphisms F, F' from (L, Q) to (L', Q') are called *homotopic* if they are homotopy equivalent Maurer–Cartan elements in the convolution L_∞-algebra $\mathcal{H}$. In this case one writes $F \sim F'$.

Remark 3.4. Note that there are several definitions of homotopies between L_∞-morphisms around which are in some situations equivalent, see e.g. [**11**], where they use a different notion of filtered L_∞-algebras which properly includes ours. More precisely their definition of an homotopy of two L_∞-morphism $F_0, F_1 \colon L \to L'$ is given by a map

$$F_t \colon L \to \widehat{L[t, \mathrm{d}t]}$$

such that the postcompostions with evaluations at 0 (resp. 1) is given by F_0 (resp. F_1). In [**11**] this is called concordance. This means in particular

$$F_t \in \operatorname{Hom}(\overline{\mathrm{S}}(L[1]), \widehat{L'[t, \mathrm{d}t]}) \supseteq \operatorname{Hom}(\overline{\mathrm{S}}(\widehat{L[1]}), L')[t, \mathrm{d}t],$$

where on the right hand side our homotopies are included. This means in particular, that the definition of homotopy we are using is more restrictive, and hence the results we are proving stay true for the more general definition.

We collect a few immediate consequences of Definition 3.3:

Proposition 3.5. *Let F, F' be two homotopic L_∞-morphisms from (L, Q) to (L', Q').*

(i) *F_1^1 and $(F')_1^1$ are chain homotopic.*

(ii) *If F is an L_∞-quasi-isomorphism, then so is F'.*

(iii) *If $L = \mathfrak{g}, L' = \mathfrak{g}'$ are two DGLAs equipped with complete descending filtrations, then F and F' induce the same maps from $\mathsf{Def}(\mathfrak{g})$ to $\mathsf{Def}(\mathfrak{g}')$.*

(iv) *In the case of DGLAs $\mathfrak{g}, \mathfrak{g}'$, compositions of homotopic L_∞-morphisms with a DGLA morphism of degree zero are again homotopic.*

Proof. The first three points are proven in [**3**] and the last one follows directly. □

We now aim to generalize the last point of the previous proposition to compositions with L_∞-morphisms. We start with the post-composition:

Proposition 3.6. *Let F_0, F_1 be two homotopic L_∞-morphisms between (L, Q) and (L', Q'). Let H be an L_∞-morphism from (L', Q') to (L'', Q'') and let G be an L_∞-morphism from (L''', Q''') to (L, Q), then $HF_0 \sim HF_1$ and $F_0G \sim F_1G$.*

Proof. The proof for pre-composing is a rather simple computation and does not involve any subtle points and this is why we omit it here. On the other hand there are some subtleties for the post-composition: For $F \in \operatorname{Hom}(\overline{\mathrm{S}}(L[1]), L')^1$ we define $\widehat{H}(F)\colon \overline{\mathrm{S}}(L[1]) \to L''[1]$ via

$$(\widehat{H}(F))_n = (HF)^1_n = \sum_{\ell=1}^{n} H^1_\ell F^\ell_n = \sum_{\ell=1}^{n} H^1_\ell \left(\frac{1}{\ell!} F^1 \vee \cdots \vee F^1\right) \circ \overline{\Delta}^{\ell-1}.$$

Here the $\vee$-product of maps is given by

$$F \vee G = \vee \circ (F \otimes G)\colon \overline{\mathrm{S}}(L[1]) \otimes \overline{\mathrm{S}}(L[1]) \to \overline{\mathrm{S}}(L'[1]).$$

Writing $\overline{\Delta}^{\bullet} = \sum_{k=0}^{\infty} \overline{\Delta}^{k}$ and defining all maps to be zero on the domains on which they were previously not defined, we can rewrite this as

$$\widehat{H}F = H^1 \circ \overline{\exp} F \circ \overline{\Delta}^{\bullet}.$$

Let

$$F(t) \in (\operatorname{Hom}(\widehat{\overline{\mathrm{S}}(L[1]), L'})[1])^0[t], \quad \text{and}$$
$$\lambda(t) \in (\operatorname{Hom}(\widehat{\overline{\mathrm{S}}(L[1]), L'})[1])^{-1}[t]$$

describe the homotopy equivalence between F_0 and F_1. Then

$$\widehat{H}F(t) \in (\operatorname{Hom}(\widehat{\overline{\mathrm{S}}(L[1]), L''})[1])^{-1}[t]$$

satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{H}F(t) &= \sum_{\ell=1}^{\infty} H^1_\ell \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\ell!} F(t) \vee \cdots \vee F(t)\right) \circ \overline{\Delta}^{l-1} \\ &= H^1 \circ \left(\widehat{Q}^1(\lambda(t) \vee \exp(F(t)) \vee \exp(F(t))\right) \circ \overline{\Delta}^{\bullet}.\end{aligned}$$

As in [**5**, Lemma 4.1] one can check

$$\begin{aligned}\widehat{Q}(\lambda(t) \vee \exp(F(t))) &= \exp(F(t)) \vee \widehat{Q}^1(\lambda(t) \vee \exp(F(t))) \\ &\quad - \lambda(t) \vee \exp(F(t)) \vee \widehat{Q}^1(\exp(F(t))) \\ &= \exp(F(t)) \vee \widehat{Q}^1(\lambda(t) \vee \exp(F(t)))\end{aligned}$$

since $F(t)$ is a Maurer–Cartan element. This gives the equality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{H}F(t) &= H^1 \circ \left(\widehat{Q}(\lambda(t) \vee \exp(F(t)))\right) \circ \overline{\Delta}^{\bullet} \\ &= H^1 \circ Q' \circ (\lambda(t) \vee \exp(F(t))) \circ \overline{\Delta}^{\bullet} + H^1 \circ (\lambda(t) \vee \exp(F(t))) \circ \overline{\Delta}^{\bullet} \circ Q \\ &= (Q'')^1 \circ H \circ (\lambda(t) \vee \exp(F(t))) \circ \overline{\Delta}^{\bullet} + H^1 \circ (\lambda(t) \vee \exp(F(t))) \circ \overline{\Delta}^{\bullet} \circ Q \\ &= (\widehat{Q}')^1_1 \left(H^1 \circ (\lambda(t) \vee \exp(F(t))) \circ \overline{\Delta}^{\bullet}\right) \\ &\quad + \sum_{\ell=2}^{\infty} (Q'')^1_\ell \circ H^\ell \circ (\lambda(t) \vee \exp(F(t))) \circ \overline{\Delta}^{\bullet}.\end{aligned}$$

Since F and H are of degree zero we get for the last term

$$
\begin{aligned}
&\frac{1}{k!} H^{\ell}_{k+1} \circ (\lambda(t) \vee F(t) \vee \cdots \vee F(t)) \circ \overline{\Delta}^{k}(X) \\
&= \frac{1}{k!\ell!} (H^1 \vee \cdots \vee H^1) \circ \overline{\Delta}^{\ell-1} \circ (\lambda(t) \vee F(t) \vee \cdots \vee F(t)) \circ \overline{\Delta}^{k}(X) \\
&= \frac{1}{k!\ell!} (H^1 \vee \cdots \vee H^1) \circ \sum_{\substack{i_1+\cdots+i_\ell=k+1 \\ i_j \geqslant 1}} \sum_{\sigma \in Sh(i_1,\ldots,i_\ell)} \\
&\quad \sigma \triangleleft \Big((\lambda(t) \vee F(t) \vee \cdots \vee F(t)) \circ \overline{\Delta}^{k}(X) \Big) \\
&= \frac{\ell}{k!\ell!} (H^1 \vee \cdots \vee H^1) \circ \sum_{\substack{i_1+\cdots+i_\ell=k+1 \\ i_j \geqslant 1}} \sum_{\substack{\sigma \in Sh(i_1,\ldots,i_\ell) \\ \sigma(1)=1}} \\
&\quad \sigma \triangleleft \Big((\lambda(t) \vee F(t) \vee \cdots \vee F(t)) \circ \overline{\Delta}^{k}(X) \Big) \\
&= \frac{1}{(\ell-1)!} \sum_{i_1+\cdots+i_\ell=k+1, i_j \geqslant 1} \Big(\frac{1}{(i_1-1)!} H^1_{i_1}(\lambda \vee F \cdots \vee F) \circ \overline{\Delta}^{i_1-1} \vee \frac{1}{i_2!} \\
&\quad H^1_{i_2}(F \vee \cdots \vee F) \circ \overline{\Delta}^{i_2-1} \vee \cdots \vee \frac{1}{i_\ell!} H^1_{i_\ell}(F \vee \cdots \vee F) \circ \overline{\Delta}^{i_\ell-1} \Big) \circ \overline{\Delta}^{\ell-1}(X).
\end{aligned}
$$

Here we wrote

$$
\sigma \triangleleft (x_1 \vee \cdots \vee x_{k+1}) = \epsilon(\sigma) x_{\sigma(1)} \vee \cdots \vee x_{\sigma(i_1)} \otimes \cdots \otimes x_{\sigma(k+1-i_\ell+1)} \vee \cdots \vee x_{\sigma(n)}
$$

with Koszul sign $\epsilon(\sigma)$. Therefore, it follows

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \widehat{H} F(t) = (\widehat{Q}')^1_1 \Big(H^1 \circ (\lambda(t) \vee \exp(F(t))) \circ \overline{\Delta}^{\bullet} \Big) \\
+ \sum_{\ell=2}^{\infty} (\widehat{Q}')^1_\ell \circ \Big((H^1 \circ (\lambda(t) \vee \exp F) \circ \overline{\Delta}^{\bullet}) \vee \exp(\widehat{H} F) \Big),
\end{aligned}
$$

which is just the desired (10) with $\lambda(t) = H^1 \circ (\lambda(t) \vee \exp F) \circ \overline{\Delta}^{\bullet}$ and the statement is shown. □

3.2. Homotopy classification of L_∞-algebras

The above considerations allow us to understand better the homotopy classification of L_∞-algebras from [**5**, **18**], which will help us in the application to the global formality. We define:

Definition 3.7. Two L_∞-algebras (L, Q) and (L', Q') are said to be *homotopy equivalent* if there are L_∞-morphisms $F\colon (L, Q) \to (L', Q')$ and $G\colon (L', Q') \to (L, Q)$ such that $F \circ G \sim \mathrm{id}_{L'}$ and $G \circ F \sim \mathrm{id}_L$. In such case F and G are said to be quasi-inverse to each other.

This definition coincides indeed with the usual definition of homotopy equivalence via L_∞-quasi-isomorphisms from [**5**].

Proposition 3.8. *Two L_∞-algebras (L, Q) and (L', Q') are homotopy equivalent if and only if there exists an L_∞-quasi-isomorphism between them.*

Proof. Due to [**5**, Prop. 2.8] every L_∞-algebra L is isomorphic to the product of a linear contractible one and a minimal one $(L,Q) \cong (V \oplus W, \widetilde{Q})$. This means one has $L \cong V \oplus W$ as vector spaces, such that V is an acyclic cochain complex with differential d_V and W is an L_∞-algebra with codifferential Q_W with $Q^1_{W,1} = 0$. The codifferential $\widetilde{Q}$ on $\overline{\mathrm{S}}((V \oplus W)[1])$ is given on $v_1 \vee \cdots \vee v_m$ with $v_1, \ldots, v_k \in V$ and $v_{k+1}, \ldots, v_m \in W$ by

$$\widetilde{Q}^1(v_1 \vee \cdots \vee v_m) = \begin{cases} -\mathrm{d}_V(v_1), & \text{for } k = m = 1 \\ Q^1_W(v_1 \vee \cdots \vee v_m), & \text{for } k = 0 \\ 0, & \text{else.} \end{cases}$$

This implies in particular that the canonical maps

$$I_W \colon W \longrightarrow V \oplus W \quad \text{and} \quad P_W \colon V \oplus W \longrightarrow W$$

are L_∞-morphisms. We want to show now that $I_W \circ P_W \sim \mathrm{id}$. Choose a contracting homotopy $h_V \colon V \to V[-1]$ with $h_V \mathrm{d}_V + \mathrm{d}_V h_V = \mathrm{id}_V$ and define the maps

$$P(t) \colon V \oplus W \ni (v,w) \longmapsto (tv, w) \in V \oplus W$$

and

$$H(t) = H \colon V \oplus W \ni (v,w) \longmapsto (-h_V(v), 0) \in V \oplus W.$$

Note that $P(t)$ is a path of L_∞-morphisms by the explicit form of the codifferential. We clearly have

$$\frac{\mathrm{d}}{\mathrm{d}t} P^1_1(t) = \mathrm{pr}_V = \widetilde{Q}^1_1 \circ H(t) + H(t) \circ \widetilde{Q}^1_1 = \widehat{Q}^1_1(H(t))$$

since h_V is a contracting homotopy. This implies

$$\frac{\mathrm{d}}{\mathrm{d}t} P(t) = \widehat{Q}^1 \left(H(t) \vee \exp(P(t)) \right)$$

since $\mathrm{im}(H(t)) \subseteq V$ and as the higher brackets of $\widetilde{Q}$ vanish on V. Since $P(0) = I_W \circ P_W$ and $P(1) = \mathrm{id}$ we conclude that $I_W \circ P_W \sim \mathrm{id}$. We choose an analogue splitting for L', i.e. $L' = V' \oplus W'$ such that V' is an acyclic cochain complex with differential $\mathrm{d}_{V'}$ and such that W' is an L_∞-algebra with codifferential $Q_{W'}$ with $Q^1_{W',1} = 0$. Let us now consider an L_∞-quasi-isomorphism $F \colon L \to L'$. Since $I_W, I_{W'}, P_W$ and $P_{W'}$ are L_∞-quasi-isomorphisms, we know that

$$F_W = P_{W'} \circ F \circ I_W \colon W \longrightarrow W'$$

is an L_∞-quasi-isomorphism. But since W and W' are minimal its first Taylor coefficient $F^1_{W,1}$ is even an isomorphism. By [**5**, Corollary 2.3] it follows that F_W is an L_∞-isomorphism and we denote the inverse by $G_{W'}$. We define now

$$G = I_W \circ G_{W'} \circ P_{W'} \colon L' \longrightarrow L.$$

Since by Proposition 3.6 compositions of homotopic L_∞-morphisms with an L_∞-morphism are again homotopic, we get

$$\begin{aligned} F \circ G = F \circ I_W \circ G_{W'} \circ P_{W'} &\sim I_{W'} \circ P_{W'} \circ F \circ I_W \circ G_{W'} \circ P_{W'} \\ &= I_{W'} \circ F_W \circ G_{W'} \circ P_{W'} = I_{W'} \circ P_{W'} \sim \mathrm{id} \end{aligned}$$

and similarly $G \circ F \sim \mathrm{id}$.

The other direction follows from Proposition 3.5. If $F \circ G \sim \mathrm{id}$ and $G \circ F \sim \mathrm{id}$, then we know that $F_1^1 \circ G_1^1$ and $G_1^1 \circ F_1^1$ are both chain homotopic to the identity. Therefore, F and G are L_∞-quasi-isomorphisms. $\square$

Proposition 3.6 directly imply the uniqueness of quasi-inverses up to homotopy.

Corollary 3.9. *Let $F\colon (L, Q) \to (L', Q')$ be an L_∞-quasi-isomorphism with two given quasi-inverses $G, G'\colon (L', Q') \to (L, Q)$ in the sense of Definition 3.7. Then one has $G \sim G'$.*

Proof. One has $G \sim G \circ (F \circ G') = (G \circ F) \circ G' \sim G'$. $\square$

3.3. Homotopy equivalence between twisted morphisms

Let now $F\colon (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ be an L_∞-morphism between DGLAs with complete descending and exhaustive filtrations. Instead of comparing the twisted morphisms F^π and $F^{\pi'}$ with respect to two equivalent Maurer–Cartan elements π and π', we consider for simplicity just a Maurer–Cartan element $\pi \in \mathcal{F}^1\mathfrak{g}^1$ equivalent to zero via $\pi = \exp([g, \cdot]) \triangleright 0$, i.e. $\lambda(t) = g = \dot{A}(t) \in \mathcal{F}^1\widehat{\mathfrak{g}^0[t]}$, where we use the notation from Section 2. Then we know that 0 and $S = \sum_{n>0} \frac{1}{n!} F_n(\pi^{\vee n})$ are equivalent Maurer–Cartan elements in $(\mathfrak{g}', \mathrm{d}')$. Let the equivalence be implemented by an $A'(t) \in \mathcal{F}^1\widehat{(\mathfrak{g}')^0[t]}$ as in Proposition 2.13. Then we have the diagram

$$\begin{array}{ccccc} & & (\mathfrak{g}', \mathrm{d}') & & \\ & \overset{F}{\nearrow} & & \overset{e^{[A'(1),\cdot]}}{\searrow} & \\ (\mathfrak{g}, \mathrm{d}) & & \Downarrow & & (\mathfrak{g}', \mathrm{d}' + [S, \cdot]) \\ & \underset{e^{[A(1),\cdot]}}{\searrow} & & \underset{F^\pi}{\nearrow} & \\ & & (\mathfrak{g}, \mathrm{d} + [\pi, \cdot]) & & \end{array} \tag{21}$$

where $e^{[A(1),\cdot]}$ and $e^{[A'(1),\cdot]}$ are well-defined by the completeness of the filtrations, and where we omit the Lie brackets. In the following we prove one of our main results, stating that this diagram commutes up to homotopy, which is indicated by the vertical arrow:

Proposition 3.10. *The L_∞-morphisms F and $e^{[-A'(1),\cdot]} \circ F^\pi \circ e^{[A(1),\cdot]}$ are homotopic, i.e. gauge equivalent Maurer–Cartan elements in $\mathrm{Hom}(\overline{\mathrm{S}}(\mathfrak{g}[1]), \mathfrak{g}')$.*

The candidate for the path between F and $e^{[-A'(1),\cdot]} \circ F^\pi \circ e^{[A(1),\cdot]}$ is

$$F(t) = e^{[-A'(t),\cdot]} \circ F^{\pi(t)} \circ e^{[A(t),\cdot]}.$$

However, $F(t)$ is not necessarily in the completion $\mathrm{Hom}(\widehat{\overline{\mathrm{S}}(\mathfrak{g}[1]), \mathfrak{g}')^1[t]}$ with respect to the filtration from (20) since for example

$$F(t) \mod \mathcal{F}^2\mathrm{Hom}(\overline{\mathrm{S}}(\mathfrak{g}[1]), \mathfrak{g}')[[t]] = e^{[-A'(t),\cdot]} \circ F_1^{\pi(t)} \circ e^{[A(t),\cdot]}$$

is in general not polynomial in t. To solve this problem we introduce a new filtration on the convolution DGLA $\mathfrak{h} = \mathrm{Hom}(\overline{\mathrm{S}}(\mathfrak{g}[1]), \mathfrak{g}')$ that takes into account the filtrations on $\overline{\mathrm{S}}(\mathfrak{g}[1])$ and $\mathfrak{g}'$:

$$\begin{aligned}\mathfrak{h} =& \mathfrak{F}^1\mathfrak{h} \supset \mathfrak{F}^2\mathfrak{h} \supset \cdots \supset \mathfrak{F}^k\mathfrak{h} \supset \cdots \\ \mathfrak{F}^k\mathfrak{h} =& \sum_{n+m=k} \left\{ f \in \operatorname{Hom}(\overline{\mathrm{S}}(\mathfrak{g}[1]),\mathfrak{g}') \mid f\big|_{\mathrm{S}^{<n}(\mathfrak{g}[1])} = 0 \quad \text{and} \quad f\colon \mathcal{F}^\bullet \to \mathcal{F}^{\bullet+m} \right\}.\end{aligned} \tag{22}$$

Here the filtration on $\overline{\mathrm{S}}(\mathfrak{g}[1])$ is the product filtration induced by

$$\mathcal{F}^k(\mathfrak{g}[1]\otimes\mathfrak{g}[1]) = \sum_{n+m=k} \operatorname{im}\left(\mathcal{F}^n\mathfrak{g}[1]\otimes\mathcal{F}^m\mathfrak{g}[1] \to \mathfrak{g}[1]\otimes\mathfrak{g}[1]\right),$$

see e.g. [**12**, Section 2].

Proposition 3.11. *The above filtration* (22) *is a complete descending filtration on the convolution DGLA* $\operatorname{Hom}(\overline{\mathrm{S}}(\mathfrak{g}[1]),\mathfrak{g}')$.

Proof. The filtration is obviously descending and $\mathfrak{h} = \mathfrak{F}^1\mathfrak{h}$ since we consider in the convolution DGLA only maps that are compatible with respect to the filtration. It is compatible with the convolution DGLA structure and complete since $\mathfrak{g}'$ is complete. ∎

Thus we can finally prove Proposition 3.10.

of Prop. 3.10. The path $F(t) = e^{[-A'(t),\cdot]} \circ F^{\pi(t)} \circ e^{[A(t),\cdot]}$ is an element in the completion $(\widehat{\operatorname{Hom}(\overline{\mathrm{S}}(\mathfrak{g}[1]),\mathfrak{g}')}[1])^0[t]$ with respect to the filtration from (22). This is clear since $A(t) \in \mathcal{F}^1\widehat{\mathfrak{g}^0[t]}$, $A'(t) \in \mathcal{F}^1\widehat{(\mathfrak{g}')^0[t]}$ and $\pi(t) \in \mathcal{F}^1\widehat{\mathfrak{g}^1[t]}$ imply that

$$\sum_{i=1}^{n-1} e^{[-A'(t),\cdot]} \circ F_i^{\pi(t)} \circ e^{[A(t),\cdot]} \mod \mathfrak{F}^n(\operatorname{Hom}(\overline{\mathrm{S}}(\mathfrak{g}[1]),\mathfrak{g}')[1])[[t]]$$

is polynomial in t. Moreover, $F(t)$ satisfies by (11)

$$\begin{aligned}\frac{\mathrm{d}F(t)}{\mathrm{d}t} =& -\exp([-A'(t),\cdot]) \circ [\lambda'(t),\cdot] \circ F^{\pi(t)} \circ e^{[A(t),\cdot]} \\ &+ e^{[-A'(t),\cdot]} \circ F^{\pi(t)} \circ [\lambda(t),\cdot] \circ e^{[A(t),\cdot]} + e^{[-A'(t),\cdot]} \circ \frac{\mathrm{d}F^{\pi(t)}}{\mathrm{d}t} \circ e^{[A(t),\cdot]}.\end{aligned}$$

But we have

$$\begin{aligned}&\frac{\mathrm{d}F_k^{\pi(t)}}{\mathrm{d}t}(X_1 \vee \cdots \vee X_k) = F_{k+1}^{\pi(t)}(Q_1^{\pi(t),1}(\lambda(t)) \vee X_1 \vee \cdots \vee X_k) \\ &= F_{k+1}^{\pi(t)}(Q_{k+1}^{\pi(t),k+1}(\lambda(t) \vee X_1 \vee \cdots \vee X_k)) + F_{k+1}^{\pi(t)}(\lambda(t) \vee Q_k^{\pi(t),k}(X_1 \vee \cdots \vee X_k)) \\ &= Q_1^{S(t),1}F_{k+1}^{\pi(t),1}(\lambda(t) \vee X_1 \vee \cdots \vee X_k) + Q_2^{S(t),1}F_{k+1}^{\pi(t),2}(\lambda(t) \vee X_1 \vee \cdots \vee X_k) \\ &\quad - F_k^{\pi(t),1} \circ Q_{k+1}^{\pi(t),k}(\lambda(t) \vee X_1 \vee \cdots \vee X_k) + F_{k+1}^{\pi(t)}(\lambda(t) \vee Q_k^{\pi(t),k}(X_1 \vee \cdots \vee X_k)).\end{aligned}$$

Setting now $\lambda_k^F(t)(\cdots) = F_{k+1}^{\pi(t)}(\lambda(t) \vee \cdots)$ we get

$$\frac{\mathrm{d}F_k^{\pi(t)}}{\mathrm{d}t} = \widehat{Q}_1^{t,1}(\lambda_k^F(t)) + \widehat{Q}_2^{t,1}(\lambda^F(t) \vee F^{\pi(t)}) - F_k^{\pi(t)} \circ [\lambda(t),\cdot] + [\lambda'(t),\cdot] \circ F_k^{\pi(t)}.$$

Thus we get

$$\begin{aligned}\frac{\mathrm{d}F(t)}{\mathrm{d}t} &= e^{[-A'(t),\cdot]} \circ \Big(\widehat{Q}_1^{t,1}(\lambda^F(t)) + \widehat{Q}_2^{t,1}(\lambda^F(t) \vee F^{\pi(t)})\Big) \circ e^{[A(t),\cdot]} \\ &= \widehat{Q}_1^{1}(e^{[-A'(t),\cdot]}\lambda^F(t)e^{[A(t),\cdot]}) + \widehat{Q}_2^{1}(e^{[-A'(t),\cdot]}\lambda^F(t)e^{[A(t),\cdot]} \vee F(t))\end{aligned}$$

since $\exp([A(t),\cdot])$ and $\exp([A'(t),\cdot])$ commute with the brackets and intertwine the differentials. This is the desired homotopy equivalence between $F(0) = F$ and $F(1)$ as in (10) with $\lambda(t) = e^{[-A'(t),\cdot]}\lambda^F(t)e^{[A(t),\cdot]}$. By Theorem 2.12 F and $F(1)$ are also gauge equivalent. ☐

4. Application: homotopic globalizations of the Kontsevich formality

Now we want to apply the above general results to the globalization of the Kontsevich formality to smooth real manifolds M. More precisely, the globalization procedure proved by Dolgushev in [**7, 6**] depends on the choice of a torsion-free covariant derivative on M and we show that the globalizations with respect to two different covariant derivatives are homotopic. Note that we are not aiming to give a complete overview on Dolgushev's construction, since this would go beyond the scope of this note. The actual aim is to quickly arrive to the point of the construction where we can use the above results.

4.1. Preliminaries: globalization procedure

Starting points are the DGLA of polyvector fields $(T_{\mathrm{poly}}(M), 0, [\cdot,\cdot]_S)$ on a smooth manifold M with zero differential and Schouten bracket, and the DGLA of polydifferential operators $(D_{\mathrm{poly}}(M), \partial, [\cdot,\cdot]_G)$ with Hochschild differential and Gerstenhaber bracket.

The idea of Dolgushev in [**7, 6**] was to replace the algebra of functions $\mathcal{C}^\infty(M)$ by the completed symmetric algebra of the cotangent bundle

$$SM := \prod_i \Gamma^\infty(\mathrm{S}^i T^*M).$$

In a coordinate chart SM looks like formal power series of $\dim M$ variables with coefficients valued in the smooth function on M. This algebra behaves now well enough to apply the local Kontsevich formality. Let us briefly recall the construction in [**7, 6**] in order to set up notation.

- $\mathcal{T}^k_{poly}$ denotes the bundle of *formal fiberwise polyvector fields* of degree k over M. Its sections are $\mathcal{C}^\infty(M)$-linear operators $v\colon \Lambda^{k+1}\Gamma^\infty(SM) \to \Gamma^\infty(SM)$ of the local form

$$v = \sum_{p=0}^{\infty} v^{j_0\dots j_k}_{i_1\dots i_p}(x) y^{i_1}\cdots y^{i_p} \frac{\partial}{\partial y^{j_0}} \wedge \dots \wedge \frac{\partial}{\partial y^{j_k}}$$

  and with the fiberwise Schouten bracket one obtains a DGLA structure.
- Analogously, the sections of *formal fiberwise differential operators* $\mathcal{D}^k_{poly}$ are

$\mathcal{C}^\infty(M)$-linear operators $X\colon \bigotimes^{k+1}\Gamma^\infty(SM) \to \Gamma^\infty(SM)$ of the local form

$$X = \sum_{\alpha_0,\dots,\alpha_k}\sum_{p=0}^{\infty} X^{\alpha_0\dots\alpha_k}_{i_1\dots i_p}(x) y^{i_1}\cdots y^{i_p} \frac{\partial}{\partial y^{\alpha_0}} \otimes \cdots \otimes \frac{\partial}{\partial y^{\alpha_k}}.$$

Here $X^{\alpha_0\dots\alpha_k}_{i_1\dots i_p}$ are symmetric in the indices $i_1,\dots,i_p$ and α are multi-indices $\alpha = (j_0,\dots,j_k)$. Moreover, the sum in the orders of the derivatives is finite. Together with the fiberwise Hochschild differential ∂_M and the fiberwise Gerstenhaber bracket one also obtains a DGLA structure.

- $D = -\delta + \nabla + [A,\cdot] = \mathrm{d} + [B,\cdot]$ is the Fedosov differential, where $\delta = [\mathrm{d}x^i \frac{\partial}{\partial y^i},\cdot]$, $\nabla = \mathrm{d}x^i\frac{\partial}{\partial x^i} - [\mathrm{d}x^i \Gamma^k_{ij}(x) y^j \frac{\partial}{y^k},\cdot]$ with Christoffel symbols Γ^k_{ij} of a torsion-free connection on M with curvature $R = -\frac{1}{2}\mathrm{d}x^i\mathrm{d}x^j (R_{ij})^k_l(x) y^l \frac{\partial}{\partial y^k}$, and $A \in \Omega^1(M,\mathcal{T}^0_{poly}) \subseteq \Omega^1(M,\mathcal{D}^0_{poly})$ is the unique solution of

$$\begin{cases} \delta(A) &= R + \nabla A + \frac{1}{2}[A,A], \\ \delta^{-1}(A) &= r, \\ \sigma(A) &= 0. \end{cases}$$

  Here $r \in \Omega^0(M,\mathcal{T}^0_{\mathrm{poly}})$ is arbitrary but fixed and has vanishing constant and linear term with respect to the y-variables. We refer to (∇, r) as globalization data.

- $\tau\colon \Gamma^\infty_\delta(\mathcal{T}_{\mathrm{poly}}) \to Z^0(\Omega(M,\mathcal{T}_{\mathrm{poly}}), D) \subset \Omega(M,\mathcal{T}_{\mathrm{poly}})$ denotes the Fedosov Taylor series, given by

$$\tau(a) = a + \delta^{-1}(\nabla\tau(a) + [A,\tau(a)]).$$

  Here one has $\Gamma^\infty_\delta(\mathcal{T}_{\mathrm{poly}}) = \{v = \sum_k v^{j_0\cdots j_k}(x)\frac{\partial}{\partial y^{j_0}} \wedge\cdots\wedge \frac{\partial}{\partial y^{j_k}}\}$, analogously for the polydifferential operators.

- The isomorphism $\nu\colon \Gamma^\infty_\delta(\mathcal{T}_{\mathrm{poly}}) \to T_{\mathrm{poly}}(M)$ is given by

$$\nu(w)(f_0,\dots,f_k) = \sigma w(\tau(f_0),\dots,\tau(f_k)) \quad \text{for} \quad f_1,\dots,f_k \in \mathcal{C}^\infty(M),$$

  where $\sigma\colon \Omega(M,\mathcal{T}_{\mathrm{poly}}) \to \Gamma^\infty_\delta(\mathcal{T}_{\mathrm{poly}})$ sets the $\mathrm{d}x^i$ and y^j coordinates to zero, analogously for the polydifferential operators.

- $\mathcal{U}^B$ is the fiberwise formality of Kontsevich $\mathcal{U}$ twisted by

$$B = D - \mathrm{d} = -\mathrm{d}x^i\frac{\partial}{\partial y^i} - \mathrm{d}x^i\Gamma^k_{ij}(x)y^j\frac{\partial}{\partial y^k} + \sum_{p\geqslant 1}\mathrm{d}x^i A^k_{ij_1\dots j_p}(x) y^{j_1}\cdots y^{j_p}\frac{\partial}{\partial y^k}.$$

  By the properties of the Kontsevich formality the first two summands do not contribute, i.e. $\mathcal{U}^B = \mathcal{U}^A$.

One obtains the diagram

$$T_{\mathrm{poly}}(M) \xrightarrow{\tau\circ\nu^{-1}} (\Omega(M,\mathcal{T}_{\mathrm{poly}}), D) \xrightarrow{\mathcal{U}^B} (\Omega(M,\mathcal{D}_{\mathrm{poly}}), D+\partial_M) \xleftarrow{\tau\circ\nu^{-1}} (D_{\mathrm{poly}}(M),\partial), \tag{23}$$

where $\tau\circ\nu^{-1}$ are quasi-isomorphisms of DGLAs and where $\mathcal{U}^B$ is an L_∞-quasi-isomorphism, where the Lie brackets on each term in (23) were defined above. In a next step, the morphism $\mathcal{U}^B \circ \tau \circ \nu^{-1}$ is modified to a quasi-isomorphism

$$U\colon (T_{\mathrm{poly}}(M), 0, [\cdot,\cdot]_S) \longrightarrow (Z^0_D(\Omega(M, \mathcal{D}_{\mathrm{poly}})), \partial_M, [\,\cdot\,,\,\cdot\,]_G),$$

see [7, Prop. 5]. By [9, Lemma 1] we know that $\mathcal{U}^B \circ \tau \circ \nu^{-1}$ and U are homotopic. The desired quasi-isomorphism

$$\boldsymbol{U}^{(\nabla,r)} = \nu \circ \sigma \circ U\colon (T_{\mathrm{poly}}(M), 0, [\cdot,\cdot]_S) \longrightarrow (D_{\mathrm{poly}}(M), \partial, [\cdot,\cdot]_G) \tag{24}$$

is then the composition of U with the DGLA isomorphism

$$\nu \circ \sigma\colon (Z^0_D(\Omega(M, \mathcal{D}_{\mathrm{poly}})), \partial_M, [\,\cdot\,,\,\cdot\,]_G) \longrightarrow (D_{\mathrm{poly}}(M), \partial, [\cdot,\cdot]_G).$$

Corollary 4.1. *The formality $\boldsymbol{U}^{(\nabla,r)}$ induces a one-to-one correspondence between equivalent formal Poisson structures on M and equivalent differential star products on $\mathcal{C}^\infty(M)[[\hbar]]$, i.e. a bijection*

$$\boldsymbol{U}^{(\nabla,r)}\colon \mathsf{Def}(T_{\mathrm{poly}}(M)[[\hbar]]) \longrightarrow \mathsf{Def}(D_{\mathrm{poly}}(M)[[\hbar]]). \tag{25}$$

4.2. Explicit construction of the projection L_∞-morphism

As an alternative to the modification of the formality in [7, Prop. 5] we want to construct the L_∞-quasi-inverse of $\tau \circ \nu^{-1}$ on the polydifferential operator side of (23). We want to use the construction from [14, Prop. 3.2] that gives a formula for the L_∞-quasi-inverse of an inclusion of DGLAs, see also [20] for the existence in more general cases. In our setting we have the contraction

$$(D_{\mathrm{poly}}(M), \partial) \underset{\nu\circ\sigma}{\overset{\tau\circ\nu^{-1}}{\rightleftarrows}} (\Omega(M, \mathcal{D}_{\mathrm{poly}}), \partial_M + D)\,, \quad \circlearrowleft h \tag{26}$$

where the homotopy h with respect to $\partial_M + D$ is constructed as follows: As in the Fedosov construction in the symplectic setting one has a homotopy D^{-1} for the differential D, see also [7, Thm. 3]:

Proposition 4.2. *The map*

$$D^{-1} = -\delta^{-1} \frac{1}{\mathrm{id} - [\delta^{-1}, \nabla + [A,\cdot]]} = -\frac{1}{\mathrm{id} - [\delta^{-1}, \nabla + [A,\cdot]]}\delta^{-1} \tag{27}$$

is a homotopy between the identity and $\tau\sigma$ on $\Omega(M, \mathcal{D}_{\mathrm{poly}})$, i.e. one has

$$X = DD^{-1}X + D^{-1}DX + \tau\sigma(X). \tag{28}$$

Proof. The proof is the same as in the symplectic setting, see e.g. [24, Prop. 6.4.17]. □

If this homotopy is also compatible with the Hochschild differential ∂_M, then we can indeed apply [14, Prop. 3.2] to describe the L_∞-morphism extending $\nu \circ \sigma$. Let us denote by $(D^{-1})_{k+1}$ the extended homotopy on $\mathrm{S}^{k+1}(\Omega(M, \mathcal{D}_{\mathrm{poly}})[1])$ and let us write $Q_{\mathcal{D}_{\mathrm{poly}}}, Q_{D_{\mathrm{poly}}}$ for the induced codifferentials on the symmetric algebras. Then we get:

Proposition 4.3. *The homotopy D^{-1} anticommutes with ∂_M, whence it is also a homotopy for $\partial_M + D$. Therefore, one obtains an L_∞-quasi-isomorphism*

$$P\colon \mathrm{S}(\Omega(M, \mathcal{D}_{\mathrm{poly}})[1]) \to \mathrm{S}(D_{\mathrm{poly}}(M)[1])$$

with recursively defined structure maps

$$P_1^1 = \nu \circ \sigma \qquad \text{and} \qquad P_{k+1}^1 = (Q^1_{D_{\mathrm{poly}},2} \circ P^2_{k+1} - P^1_k \circ Q^k_{\mathcal{D}_{\mathrm{poly}},k+1}) \circ (D^{-1})_{k+1}. \tag{29}$$

Proof. The fact that D^{-1} anticommutes with ∂_M is clear as $\nabla + [A, \cdot]$ and δ^{-1} anticommute with ∂, and the rest follows directly from [**14**, Prop. 3.2]. □

Summarizing, we obtain another global formality:

Corollary 4.4. *Given globalization data (∇, r) there exists an L_∞-quasi-isomorphism*

$$F^{(\nabla,r)} = P \circ \mathcal{U}^B \circ \tau \circ \nu^{-1}\colon (T_{\mathrm{poly}}(M), 0, [\cdot,\cdot]_S) \longrightarrow (D_{\mathrm{poly}}(M), \partial, [\cdot,\cdot]_G) \tag{30}$$

with F_1^1 being the Hochschild–Kostant–Rosenberg map.

Proof. We immediately get

$$F_1^{(\nabla,r),1} = P_1^1 \circ (\mathcal{U}^B)_1^1 \circ \tau \circ \nu^{-1} = \nu \circ \sigma \circ \mathcal{U}_1^1 \circ \tau \circ \nu^{-1}$$

and the statement follows since $\mathcal{U}_1^1$ is the fiberwise Hochschild–Kostant–Rosenberg map. □

The higher structure maps of P^1_{k+1} of the L_∞-projection contain copies of the homotopy D^{-1} that decrease the antisymmetric form degree. Therefore, they vanish on $\Omega^0(M, \mathcal{D}_{\mathrm{poly}})$ and are needed to get rid of the form degrees arising from the twisting with B, analogously to the modifying of the formality from $\mathcal{U}^B \circ \tau \circ \nu^{-1}$ to U.

As a last point, we want to remark that we can use the L_∞-projection P to obtain a splitting of $\Omega(M, \mathcal{D}_{\mathrm{poly}})$ similar to the one used in the proof of Proposition 3.8: Instead of splitting into the product of the cohomology as minimal L_∞-algebra and a linear contractible one, we can prove in our setting:

Lemma 4.5. *One has an L_∞-isomorphism*

$$L\colon (\Omega(M, \mathcal{D}_{\mathrm{poly}}), D + \partial_M, [\cdot,\cdot]_G) \longrightarrow D_{\mathrm{poly}}(M) \oplus \mathrm{im}[D, D^{-1}]. \tag{31}$$

Here the L_∞-structure on $D_{\mathrm{poly}}(M)$ is the usual one consisting of Gerstenhaber bracket and Hochschild differential ∂ and on $\mathrm{im}[D, D^{-1}]$ the L_∞-structure is just given by the differential $\partial_M + D$. The L_∞-structure on $D_{\mathrm{poly}}(M) \oplus \mathrm{im}[D, D^{-1}]$ is the product L_∞-structure.

Proof. Using Proposition 4.3, we already have an L_∞-morphism

$$P\colon \Omega(M, \mathcal{D}_{\mathrm{poly}}) \to D_{\mathrm{poly}}(M)$$

with first structure map $\nu \circ \sigma$. Now we construct an L_∞-morphism

$$F\colon \Omega(M, \mathcal{D}_{\mathrm{poly}}) \to \mathrm{im}[D, D^{-1}].$$

We set $F_1^1 = DD^{-1} + D^{-1}D$ and $F_n^1 = -D^{-1} \circ F^1_{n-1} \circ (Q_{\mathcal{D}_{\mathrm{poly}}})^{n-1}_n$ for $n \geqslant 2$ and note that in particular $F_k^1 = 0$ for $k \geqslant 3$ by $D^{-1}D^{-1} = 0$. In the following, we denote by $Q_{\mathcal{D}_{\mathrm{poly}}}$ the L_∞-structure on $\Omega(M, \mathcal{D}_{\mathrm{poly}})$ and by $\widetilde{Q}$ the L_∞-structure on $\operatorname{im}[D, D^{-1}]$ with $\widetilde{Q}_1^1 = -(\partial_M + D)$ as only vanishing component. We have $F_n^1 = D^{-1} \circ L_{\infty,n}$ with $L_{\infty,n} = -F_{n-1}^1 \circ (Q_{\mathcal{D}_{\mathrm{poly}}})_n^{n-1}$. By [**14**, Lemma 3.1] we know that if F is an L_∞-morphism up to order n, i.e. if $(\widetilde{Q}F)_k^1 = (FQ_{\mathcal{D}_{\mathrm{poly}}})_k^1$ for all $k \leqslant n$, then one has $\widetilde{Q}_1^1 \circ L_{\infty,n+1} = -L_{\infty,n+1} \circ (Q_{\mathcal{D}_{\mathrm{poly}}})_{n+1}^{n+1}$. By Proposition 4.3 we know that F is an L_∞-morphism up to order one. Assuming it defines an L_∞-morphism up to order n, then we get with (28)

$$
\begin{aligned}
\widetilde{Q}_1^1 \circ F_{n+1}^1 &= -(\partial_M + D) \circ D^{-1} \circ L_{\infty,n+1} \\
&= D^{-1} \circ \partial_M \circ L_{\infty,n+1} - L_{\infty,n+1} + D^{-1} \circ D \circ L_{\infty,n+1} + \tau \circ \sigma \circ L_{\infty,n+1} \\
&= -L_{\infty,n+1} - D^{-1} \circ \widetilde{Q}_1^1 \circ L_{\infty,n+1} \\
&= -L_{\infty,n+1} + F_{n+1}^1 \circ (Q_{\mathcal{D}_{\mathrm{poly}}})_{n+1}^{n+1}.
\end{aligned}
$$

Thus F is an L_∞-morphism up to order $n+1$ and therefore an L_∞-morphism.

The universal property of the product gives the desired L_∞-morphism $L = P \oplus F$ which is even an isomorphism since its first structure map $(\nu \circ \sigma) \oplus (DD^{-1} + D^{-1}D)$ is an isomorphism with inverse $(\tau \circ \nu^{-1}) \oplus \mathrm{id}$, see e.g. [**5**, Prop. 2.2]. □

4.3. Homotopic global formalities

The above globalization of the Kontsevich Formality depends on the choice of a covariant derivative. We want to show that globalizations with respect to different covariant derivatives are homotopic in the sense of Definition 3.3. The ideas are similar to those in the proof of [**3**, Theorem 2.6]: observe that changing the covariant derivative corresponds to twisting with a Maurer–Cartan element which is equivalent to zero, and apply Proposition 3.10.

Remark 4.6 (Filtrations on Fedosov resolutions). In order to apply Proposition 3.10 we need complete descending and exhaustive filtrations on the Fedosov resolutions. As in [**3**, Appendix C] we assign to $\mathrm{d}x^i$ and y^i the degree 1 and to $\frac{\partial}{\partial y^i}$ the degree -1 and consider the induced descending filtration. The filtration on $\Omega(M, \mathcal{T}_{\mathrm{poly}})$ is complete and bounded below since

$$
\Omega(M, \mathcal{T}_{\mathrm{poly}}) \cong \varprojlim \Omega(M, \mathcal{T}_{\mathrm{poly}})/\mathcal{F}^k\Omega(M, \mathcal{T}_{\mathrm{poly}}) \quad \text{and} \quad \Omega(M, \mathcal{T}_{\mathrm{poly}}) = \mathcal{F}^{-d}\Omega(M, \mathcal{T}_{\mathrm{poly}}),
$$

where d is the dimension of M. In the case of the differential operators the filtration is unbounded in both directions. Instead of $\mathcal{D}_{\mathrm{poly}}$ we consider from now on its y-adic completion without changing the notation. This is the completion with respect to the filtration induced by assigning y^i the degree 1 and $\frac{\partial}{\partial y^i}$ the degree -1. The globalization of the formality works just the same and one obtains the desired properties

$$
\Omega(M, \mathcal{D}_{\mathrm{poly}}) \cong \varprojlim \Omega(M, \mathcal{D}_{\mathrm{poly}})/\mathcal{F}^k\Omega(M, \mathcal{D}_{\mathrm{poly}}) \quad \text{and}
$$

$$
\Omega(M, \mathcal{D}_{\mathrm{poly}}) = \bigcup_k \mathcal{F}^k\Omega(M, \mathcal{D}_{\mathrm{poly}}).
$$

Let (∇', r') be a second pair of globalization data, then, analogously to (23), there is a second diagram

$$T_{\mathrm{poly}}(M) \overset{\tau'\circ\nu^{-1}}{\longrightarrow} (\Omega(M,\mathcal{T}_{\mathrm{poly}}),D') \overset{\mathcal{U}^{B'}}{\longrightarrow} (\Omega(M,\mathcal{D}_{\mathrm{poly}}),D'+\partial_M) \overset{\tau'\circ(\nu')^{-1}}{\longleftarrow} (D_{\mathrm{poly}}(M),\partial),$$

where

$$D' = -\delta + \nabla' + [A',\cdot] = \mathrm{d} + [B',\cdot],$$

and where A' is the unique solution of

$$\begin{cases} \delta(A') &= R' + \nabla' A' + \frac{1}{2}[A',A'], \\ \delta^{-1}(A') &= r', \\ \sigma(A') &= 0. \end{cases}$$

In the case of polyvector fields one easily sees that $\nu = \nu'$. Note

$$\nabla' - \nabla = \left[-\mathrm{d}x^i y^j (\Gamma'^k_{ij} - \Gamma^k_{ij}) \frac{\partial}{\partial y^k}, \cdot \right] = [\delta s, \cdot]$$

for $s = -\frac{1}{2} y^i y^j (\Gamma'^k_{ij} - \Gamma^k_{ij}) \frac{\partial}{\partial y^k}$. Thus we get

$$D' = -\delta + \nabla + [A' + \delta s, \cdot] = -\delta + \nabla + [\widetilde{A}, \cdot] \tag{32}$$

and since $R' = R + \nabla \delta s + \frac{1}{2}[\delta s, \delta s]$ we know that $\widetilde{A} = A' + \delta s$ is the unique solution of

$$\begin{cases} \delta(\widetilde{A}) &= R + \nabla \widetilde{A} + \frac{1}{2}[\widetilde{A},\widetilde{A}], \\ \delta^{-1}(\widetilde{A}) &= r' + s, \\ \sigma(\widetilde{A}) &= 0. \end{cases}$$

As in [**3**, Appendix C] one can now show that B and B' can be interpreted as equivalent Maurer–Cartan elements:

Proposition 4.7. *There exists an element*

$$h \in \mathcal{F}^2\Omega^0(M,\mathcal{T}^0_{\mathrm{poly}}) \subset \mathcal{F}^2\Omega^0(M,\mathcal{D}^0_{\mathrm{poly}}) \tag{33}$$

that is at least quadratic in the fiber coordinates y such that one has

$$B' - B = \widetilde{A} - A = -\frac{\exp([h,\cdot]) - \mathrm{id}}{[h,\cdot]} Dh \in \mathcal{F}^1\Omega^1(M,\mathcal{T}^0_{\mathrm{poly}}) \subset \mathcal{F}^1\Omega^1(M,\mathcal{D}^0_{\mathrm{poly}}) \tag{34}$$

and

$$\exp([h,\cdot]) \circ D \circ \exp([-h,\cdot]) = D'. \tag{35}$$

Thus the difference $B' - B$ is gauge equivalent to zero in $(\Omega(M,\mathcal{T}_{\mathrm{poly}}),D)$ as well as in $(\Omega(M,\mathcal{D}_{\mathrm{poly}}),D+\partial_M)$, where h implements in both cases the gauge equivalence.

Proof. For the existence of the element $h \in \mathcal{F}^1\Omega^0(M,\mathcal{T}^0_{\mathrm{poly}})$ encoding the gauge equivalence in the polyvector fields see [**3**, Appendix C]. Thus we have a path

$$B(t) = -\frac{\exp([th,\cdot]) - \mathrm{id}}{[th,\cdot]} D(th) \in \mathcal{F}^1\Omega^1(\widehat{M,\mathcal{T}^0_{\mathrm{poly}}})[t]$$

that satisfies $B(0) = 0$, $B(1) = B' - B$ and

$$\frac{\mathrm{d}B(t)}{\mathrm{d}t} = Q^1(\lambda(t) \vee \exp(B(t))) \qquad \text{with} \qquad \lambda(t) = h.$$

The formality $\mathcal{U}^B$ satisfies in the notation of Proposition 2.15

$$\widetilde{B}(t) = \mathcal{U}^{B,1}(\overline{\exp}(B(t))) = B(t) \qquad \text{and} \qquad \widetilde{\lambda}(t) = \mathcal{U}^{B,1}(h \vee \exp(B(t))) = h$$

since the higher orders of the Kontsevich formality vanish if one only inserts vector fields. Thus h implements indeed the gauge equivalence between 0 and $B' - B$ in both DGLAs, in the fiberwise polyvector fields and in the fiberwise polydifferential operators. □

Remark 4.8. Note that one can show that the constructed h from Proposition 4.7 is unique, since it is given by a recursion formula (in symmetric degrees). Using the fact that $\mathcal{F}^2\Omega^0(M, \mathcal{T}^0_{\mathrm{poly}})$ acts as a group via the Baker–Campbell–Hausdorff formula on Maurer–Cartan elements, one can show that the assignment

$$((\nabla, r), (\nabla', r')) \mapsto (B, B', h)$$

(with the notation as in Proposition 4.7) is canonical in the following way: given three globalization data (∇^i, r^i) for $i = 1, 2, 3$, with

$$((\nabla^i, r^i), (\nabla^j, r^j)) \mapsto (B^i, B^j, h^{ij})$$

we get

$$h^{13} = \mathrm{BCH}(h^{12}, h^{23}),$$

where BCH is the Baker–Campbell–Hausdorff series.

Now it follows directly from Proposition 3.10 and $(\mathcal{U}^B)^{B'-B} = \mathcal{U}^{B'}$ that the twisted formalities are homotopic.

Corollary 4.9. *The L_∞-morphisms $\mathcal{U}^B$ and $e^{-[h,\cdot]} \circ \mathcal{U}^{B'} \circ e^{[h,\cdot]}$ are homotopic.*

Moreover, the Fedosov Taylor series is compatible in the following sense:

Corollary 4.10. *For all $X \in T_{\mathrm{poly}}(M)$ one has*

$$e^{[h,\cdot]} \circ \tau \circ \nu^{-1}(X) = \tau' \circ (\nu')^{-1}(X). \tag{36}$$

Proof. By the above proposition $\exp([h, \cdot])$ maps the kernel of D into the kernel of D'. Therefore,

$$e^{[h,\cdot]} \circ \tau \circ \nu^{-1}(X) = \tau' \circ \sigma \circ e^{[h,\cdot]} \circ \tau \circ \nu^{-1}(X) = \tau' \circ \nu^{-1}(X)$$

since h is at least quadratic in the y coordinates. □

Similarly, one has on the differential operator side the following identity:

Lemma 4.11. *For all $X \in Z^0_{D'}(\Omega(M, \mathcal{D}_{\mathrm{poly}}))$ one has*

$$\nu \circ \sigma \circ e^{-[h,\cdot]}(X) = \nu' \circ \sigma(X). \tag{37}$$

Proof. Using the definition of ν, we compute for $f_1, \ldots, f_n \in \mathcal{C}^\infty(M)$ and for any $X \in Z^0_{D'}(\Omega(M, \mathcal{D}^{n-1}_{\mathrm{poly}}))$

$$
\begin{aligned}
(\nu \circ \sigma \circ e^{-[h,\cdot]} X)(f_1, \ldots, f_n) &= \sigma((\sigma \circ e^{-[h,\cdot]} X)(\tau(f_1), \ldots, \tau(f_n))) \\
&= \sigma(e^{-h}(X(e^h \tau(f_1), \ldots, e^h \tau(f_n)))) \\
&= \sigma((\sigma \circ X)(\tau'(f_1), \ldots, \tau'(f_n))) \\
&= (\nu' \circ \sigma X)(f_1, \ldots, f_n)
\end{aligned}
$$

and the statement is shown. □

As a last preparation we want to compare the two different L_∞-projections P' and $P \circ e^{-[h,\cdot]}$ from $(\Omega(M, \mathcal{D}_{\mathrm{poly}}), \partial_M + D')$ to $(D_{\mathrm{poly}}(M), \partial)$.

Lemma 4.12. *The L_∞-projections P' and $P \circ e^{-[h,\cdot]}$ are homotopic.*

Proof. Since the higher structure maps of P and P' vanish on the zero forms, we have by Lemma 4.11

$$
P \circ e^{-[h,\cdot]} \circ \tau' \circ (\nu')^{-1} = \nu \circ \sigma \circ e^{-[h,\cdot]} \circ \tau' \circ (\nu')^{-1} = \nu' \circ \sigma \circ \tau' \circ (\nu')^{-1} = \mathrm{id}_{D_{\mathrm{poly}}(M)}.
$$

Instead of directly using Proposition 3.8, we recall the splitting from Lemma 4.5 and adapt the proof of Proposition 3.8. Define

$$
M(t)\colon D_{\mathrm{poly}}(M) \oplus \mathrm{im}[D, D^{-1}] \ni (D_1, D_2) \longmapsto (D_1, tD_2) \in D_{\mathrm{poly}}(M) \oplus \mathrm{im}[D, D^{-1}]
$$

which is an L_∞-morphism with respect to the product L_∞-structure. Setting

$$
H(t)\colon D_{\mathrm{poly}}(M) \oplus \mathrm{im}[D, D^{-1}] \ni (D_1, D_2) \longmapsto (0, -D^{-1}D_2) \in D_{\mathrm{poly}}(M) \oplus \mathrm{im}[D, D^{-1}]
$$

we obtain again

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} M(t) &= \mathrm{pr}_{\mathrm{im}[D,D^{-1}]} = 0 \oplus (DD^{-1} + D^{-1}D) \\
&= -\partial \oplus (\partial_M + D) \circ H(t) - H(t) \circ (\partial \oplus (\partial_M + D)) \\
&= \widehat{Q}^1(H(t) \vee \exp(M(t))).
\end{aligned}
$$

Therefore, it follows that

$$
L(t) = L^{-1} \circ M(t) \circ L\colon \Omega(M, \mathcal{D}_{\mathrm{poly}}) \longrightarrow \Omega(M, \mathcal{D}_{\mathrm{poly}})
$$

encodes the homotopy between

$$
L(0) = \tau \circ \nu^{-1} \circ P \quad \text{and} \quad L(1) = \mathrm{id}.
$$

But this implies with Proposition 3.6

$$
P \circ e^{-[h,\cdot]} \sim P \circ e^{-[h,\cdot]} \circ \tau' \circ (\nu')^{-1} \circ P' = P'
$$

and the statement is shown. □

Combining all the above statements we can show that the globalizations with respect to different covariant derivatives are homotopy equivalent.

Theorem 4.13. *Let (∇, r) and (∇', r') be two pairs of globalization data. Then the formalities constructed via Dolgushev's globalization as in (24) and the globalized formalities via the L_∞-projection as in (30) are all homotopic, i.e. one has*

$$
\boldsymbol{U}^{(\nabla,r)} \sim F^{(\nabla,r)} \sim F^{(\nabla',r')} \sim \boldsymbol{U}^{(\nabla',r')}. \tag{38}
$$

Proof. By Proposition 3.5 we already know that compositions of homotopic L_∞-morphisms with DGLA morphisms are homotopic, which yields

$$\begin{aligned} U &\sim \mathcal{U}^B \circ \tau \circ \nu^{-1} \sim e^{-[h,\cdot]} \circ \mathcal{U}^{B'} \circ e^{[h,\cdot]} \circ \tau \circ \nu^{-1} \\ &= e^{-[h,\cdot]} \circ \mathcal{U}^{B'} \circ \tau' \circ (\nu')^{-1} \sim e^{-[h,\cdot]} \circ U', \end{aligned}$$

where we used Corollary 4.9. It follows from Lemma 4.12 and Proposition 3.6

$$\begin{aligned} \boldsymbol{U}^{(\nabla,r)} &= \nu \circ \sigma \circ U = P \circ U \sim P \circ \mathcal{U}^B \circ \tau \circ \nu^{-1} \\ &= F^{(\nabla,r)} \sim P \circ e^{-[h,\cdot]} \circ \mathcal{U}^{B'} \circ \tau' \circ (\nu')^{-1} \sim P' \circ \mathcal{U}^{B'} \circ \tau' \circ (\nu')^{-1} \\ &= F^{(\nabla',r')} \sim P' \circ U' = \nu' \circ \sigma \circ U' = \boldsymbol{U}^{(\nabla',r')} \end{aligned}$$

and the theorem is shown. □

Remark 4.14. It is not clear to us how canonical the homotopies are: given three globalization data (∇^i, r^i) we can construct $h^{ij} \in \mathrm{Hom}(\mathrm{S}(T_{\mathrm{poly}}(M)[1]), D_{\mathrm{poly}}(M))$ with $|h^{ij}| = 0$ such that

$$F^{(\nabla^i,r^i)} = h^{ij} \triangleright F^{(\nabla^j,r^j)}.$$

It would be desirable to have $h^{ij} = \mathrm{BCH}(h^{ik}, h^{kj})$ in order to show that the chosen homotopies are in some sense natural. Remark 4.8 gives already a hint that it might be possible, but in some of the parts of the construction afterwards there were choices involved and abstract arguments were used to show that there are homotopies at all (i.e. the homotopy between P' and $P \circ e^{-[h,\cdot]}$ in Lemma 4.12): Nevertheless, we strongly believe that one can construct the h^{ij}s in a coherent manner.

Corollary 4.15. *Let M be a smooth manifold and let (∇, r) be a globalization data. For every coordinate patch (U, x)*

$$F^{(\nabla,r)}\big|_U \sim K\big|_U,$$

holds, where K denotes the Kontsevich formality on $\mathbb{R}^d$, and where d is the dimension of M.

Proof. The formalities themselves are differential operators and can therefore be restricted to open neighbourhoods. Moreover, the Kontsevich formality coincides with the Dolgushev formality on $\mathbb{R}^d$ for the choice of the canonical flat covariant derivative and $r = 0$. □

This allows us to recover [**3**, Theorem 2.6], i.e. that the induced maps on equivalence classes of Maurer–Cartan elements are independent of the choice of the covariant derivative. It implies in particular that globalizations with respect to different covariant derivatives lead to equivalent star products.

Corollary 4.16. *The induced map*

$$\mathsf{Def}(T_{\mathrm{poly}}(M)[[\hbar]]) \longrightarrow \mathsf{Def}(D_{\mathrm{poly}}(M)[[\hbar]])$$

does not depend on the choice of a covariant derivative.

Proof. The statement follows directly from Theorem 4.13 and Proposition 3.5. □

Remark 4.17. Note that 4.16 does not use the full strength of our results, in fact since one can easily adapt our proofs to the case of Q-manifolds, we get even an equivalence of derived deformation functors à la Pridham, see [**23**].

Finally, note that Theorem 4.13 also holds in the equivariant setting of an action of a Lie group G on M with G-invariant torsion-free covariant derivatives ∇ and ∇'.

Proposition 4.18. *Let* G *act on* M *and consider two pairs of globalization data* (∇, r) *and* (∇', r')*, where* ∇ *and* ∇' *are two* G*-invariant torsion-free covariant derivatives and where* r *and* r' *are* G*-invariant. Then the formalities are equivariant and equivariantly homotopic*

$$\boldsymbol{U}^{(\nabla,r)} \sim_{\mathrm{G}} F^{(\nabla,r)} \sim_{\mathrm{G}} F^{(\nabla',r')} \sim_{\mathrm{G}} \boldsymbol{U}^{(\nabla',r')}, \tag{39}$$

i.e. all paths encoding the equivalence relation from (10) *are* G*-equivariant.*

Proof. The formalities are equivariant since all involved maps are [**7**, Theorem 5]. Moreover, $\mathcal{U}^B$ and $e^{-[h,\cdot]} \circ \mathcal{U}^{B'} \circ e^{[h,\cdot]}$ are equivariantly homotopic by the explicit form of the homotopy from Proposition 3.10. Moreover, again by [**7**, Theorem 5] we know that U and $\mathcal{U}^B \circ \tau \circ \nu^{-1}$ are equivariantly homotopic, the same holds for the (∇', r') case. Thus by Theorem 4.13 it only remains to show that $P \circ e^{-[h,\cdot]}$ and P' are equivariantly homotopic. But this follows directly from Lemma 4.12 since all involved maps are equivariant. □

In the case of proper Lie group actions one knows that invariant covariant derivatives always exist and one has even an invariant Hochschild–Kostant–Rosenberg theorem, compare [**22**, Theorem 5.11]. Thus the L_∞-morphisms from (39) restrict to the invariant DGLAs and one obtains homotopic formalities from $(T_{\mathrm{poly}}(M))^{\mathrm{G}}$ to $(D_{\mathrm{poly}}(M))^{\mathrm{G}}$.

5. Final remarks

A lot of work has been done on the classification of formality morphisms from $T_{\mathrm{poly}}(\mathbb{R}^d)$ to $D_{\mathrm{poly}}(\mathbb{R}^d)$ or their formal counterparts up to homotopy and remarkable results have been achieved, see e.g. [**1**], [**10**], [**25**] and references therein.

Our above discussion shows now that globalizing does not produce any new homotopy classes, note that in fact we did not use the exact shape of the local Kontsevich formality, but rather one specific property of it, which makes possible to globalize à la Dolgushev (see [**7**, Theorem 1, property 4]). In fact, in [**17**] the author shows that the L_∞-part of Tamarkin's Gerstenhaber up to homotopy quasi-isomorphism can be globalized in this way.

Note that however, one can show that there are more formality maps, at least if one requires that their Taylor coefficients are local, on a manifold with non-trivial second de Rham cohomology than globalized ones.

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