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ADDENDUM TO "REFINEMENT INVARIANCE OF INTERSECTION (CO)HOMOLOGIES"

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Abstract

In a previous work we proved the refinement invariance of several intersection (co)homologies existing in the literature. Specifically, we worked with a refinement $f\colon (X,\mathcal{S}) \to (X,\mathcal{T})$ between two CS-sets where the strata of $\mathcal{S}$ were embedded in the strata of $\mathcal{T}$. However, in this paper, we establish that this embedding condition is not a requirement for the refinement invariance property.

1. Introduction

Let us see, in broad outline, the results obtained in [**14**]. We work with a CS-set X endowed with two stratifications $\mathcal{S}$ and $\mathcal{T}$, where the first one is finer than the second one. We study the relationship between the intersection homology[1] such that the identity $I\colon X \to X$ induces the isomorphism

$$H^{\overline{p}}_*(X,\mathcal{S}) \cong H^{\overline{q}}_*(X,\mathcal{T}), \tag{1}$$

The perversity $\overline{p}$ is induced by the perversity $\overline{q}$ or vice versa.

These results encompass some other already known results about topological invariance of the intersection (co)homology (cf. [**11, 13, 10, 6, 7, 2, 4, 17, 5**]). Recently, the topological/refinement invariance of the intersection homology has been extended to the more general setting of the torsion sensitive intersection homology (cf. [**8, 10**]).

The original proof of the classical topological invariance of the intersection homology given by King in [**13**] uses the intrinsic stratification $\mathcal{S}^*$. He proves that the identity map $I\colon (X,\mathcal{S}) \to (X,\mathcal{S}^*)$ induces the isomorphism (1). This gives the topological invariance since $\mathcal{S}^* = \mathcal{T}^*$. The main focus of the proof is on the local description. Near a point x of X the identity $I\colon (X,\mathcal{S}) \to (X,\mathcal{S}^*)$ essentially becomes a stratified map

$$h\colon \mathbb{R}^m \times \mathring{\mathtt{c}}W \to \mathbb{R}^k \times \mathring{\mathtt{c}}L \tag{2}$$

(see for example [**9**, Section 5.5]). Here, W is the link of x relatively to $\mathcal{S}$ and L denotes the link of x relatively to $\mathcal{S}^*$.

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[1] In this study, we analyze several intersection (co)homologies that are commonly found in literature and are instrumental in establishing Poincaré Duality in various contexts. Please refer to Paragraph 2.1 for a complete list of these homologies.

We do not have this nice local description for any refinement $I\colon (X,\mathcal{S}) \to (X,\mathcal{T})$. The main tool used by [**14**] is the construction of a finite sequence of CS-sets

$$(X,\mathcal{S}) = (X,\mathcal{R}_0) \xrightarrow{I_1} (X,\mathcal{R}_1) \xrightarrow{I_2} \cdots \xrightarrow{I_{\ell-1}} (X,\mathcal{R}_{\ell-1}) \xrightarrow{I_\ell} (X,\mathcal{R}_\ell) = (X,\mathcal{T}),$$

where each step $I\colon (X,\mathcal{R}_{j-1}) \to (X,R_j)$ is a *simple refinement*.

In this context, any point of X has the local description (2) with the following improvements:

a) W is a refinement of L.

b) W is a sphere and $L = \varnothing$.

c) $W = S^{k-m-1} * L$ where S^{k-m-1} is the $(k-m-1)$-dimensional sphere.

The refinement notion used in [**14**] requires that each stratum of $\mathcal{S}$ is embedded in a stratum of $\mathcal{T}$. However, in this work, we show that this condition is not necessary. Instead, we introduce the notion of w-refinement, where each stratum of $\mathcal{T}$ is a union of strata of $\mathcal{S}$. This involves considering homological spheres $\mathbb{S}^k$ instead of actual spheres S^k in the previous local description (as noted in Remark 2.1)). We prove the invariance of the intersection (co)homologies under w-refinement in this setting (see Theorem 3.5 and 3.7).

Nota bene. There is an error in [**14**, Paragraph 5.3] that renders the proof of Corollary 5.10 invalid. Specifically, the author is uncertain whether the intrinsic stratification satisfies the required strata embedding condition to apply Theorem A, op. cit.

Despite this, the intrinsic stratification is a w-refinement in the context of this paper and the proof of Corollary 3.8 is correct. In simpler terms, we can deduce topological invariance from w-refinement invariance.

For a topological space X, we denote by $\mathtt{c}X = X \times [0,1]/(X \times \{0\})$ the *cone* on X and $\mathring{\mathtt{c}}X = X \times [0,1[/(X \times \{0\})$ the *open cone* on X. A point of the cone is denoted by $[x,t]$. The apex of the cone is $\mathtt{v} = [-,0]$.

We shall write S^m the m-dimensional sphere and $\mathbb{S}^m$ an m-dimensional homological sphere, $m \in \mathbb{N}$.

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2. Weak refinements

We introduce a weaker notion of refinement, as defined in [**14**], and work with stratified spaces as defined in [**14**, Definition 2.1]. We define a stratified space $(X,\mathcal{S})$ as a *w-refinement* of the stratified space $(X,\mathcal{T})$, denoted by $(X,\mathcal{S}) \blacktriangleleft (X,\mathcal{T})$, if $\mathcal{S} \neq \mathcal{T}$ and satisfies the condition

(S6w) any stratum $T \in \mathcal{T}$ is a union of strata of $\mathcal{S}$.

Given a stratum S, we define $S^{^I}$ as the unique stratum in $\mathcal{T}$ that contains S. There are three important types of strata: source strata, virtual strata, and exceptional strata. A source stratum is defined as having the same dimension as its containing stratum $S^{^I}$, while a virtual stratum has a lower dimension than $S^{^I}$. An exceptional stratum is a virtual stratum where $S^{^I}$ is regular.

The key requirement of condition (S6w) is that each source stratum $S \in \mathcal{S}$ is an open subset of $S^{^I}$. In other words, each source stratum is an open subset of $S^{^I}$. Additionally, the union of all source strata of $S^{^I}$ is an open, dense subset of $S^{^I}$ due to the nature of the strata and their dimensions.

It is worth noting that the identity $I\colon (X,\mathcal{S}) \to (X,\mathcal{T})$ is a stratified map, and we also write $(X,\mathcal{S}) \blacktriangleleft_I (X,\mathcal{T})$ to emphasize this point.

Remark 2.1. The condition (S6w) introduced in this work is weaker than the condition (S6) used in [**14**]. While both conditions require that each stratum of $\mathcal{T}$ is a union of strata of $\mathcal{S}$, condition (S6) imposes a specific local structure.

When condition (S6) holds, we can describe the local structure of a point $x \in X$ belonging to the stratum $S \in \mathcal{S}$ as follows: $\begin{cases} x &= (0,\mathtt{v}) \\ S &= \mathbb{R}^a \times \{\mathtt{v}\} \\ S^{^I} &= \mathbb{R}^a \times \mathring{\mathtt{c}} S^{b-1}, \end{cases}$ where S^{b-1} is the $(b-1)$-dimensional sphere and $\mathtt{v}$ the apex of the cone.

See Remark 2.6 for a comparison between condition (S6) of [**14**] and the weaker condition (S6w) used in this work, which allows for a more flexible local structure of the stratification.

Every w-refinement can be decomposed into a sequence of simpler w-refinements. To define the depth of a family of strata used in the next definition, we recall the following notation and definitions. For a family of strata $\mathcal{S}' \subset \mathcal{S}$, the depth of $\mathcal{S}'$ is denoted by $\operatorname{depth}\mathcal{S}'$ and is defined as the supremum of all integers i such that there exist strata $S_0, S_1, \ldots, S_i$ in $\mathcal{S}'$ with $S_0 \prec S_1 \prec \cdots \prec S_i$. Here, the order relation $S \preceq S'$ is defined as $S \subset \overline{S'}$ on $\mathcal{S}$. For further details, refer to [**3**, Proposition A.22].

Definition 2.2. Let $\mathcal{V}$ the family of *virtual strata* We say that the w-refinement $(X,\mathcal{S}) \blacktriangleleft (X,\mathcal{T})$ is *simple* when $\operatorname{depth}\mathcal{V} = 0$.

A CS-set is a stratified space that has a local conical structure. For a detailed explanation of this concept, we refer the reader to [**14**, Section 3]. Using a similar argument as in [**14**, Proposition 3.10], we can establish the proof of Proposition 2.3.

Proposition 2.3. *A w-refinement $(X,\mathcal{S}) \blacktriangleleft (X,\mathcal{T})$ between two different CS-sets possesses a simple decomposition made up of CS-sets.*

To describe the construction of compatible conical charts associated with a simple w-refinement in Proposition 2.7 , we need two preliminary results. The first one is a useful finding about cones from Stallings [**16, 15**], which can be found in [**9**, Lemma 2.10.1] for a proof.

Lemma 2.4. *Let X and Y be two compact topological spaces, whose apexes are $\mathtt{x}$ and $\mathtt{y}$ respectively. If there is an open neighborhood U of $\mathtt{x}$ in $\mathring{\mathtt{c}}X$ so that $(U,\mathtt{x})$ and $(\mathring{\mathtt{c}}Y,\mathtt{y})$ are homeomorphic, then $(\mathring{\mathtt{c}}X,\mathtt{x})$ and $(\mathring{\mathtt{c}}Y,\mathtt{y})$ are homeomorphic.*

Lemma 2.5. *Let $(X,\mathcal{S}) \blacktriangleleft_I (X,\mathcal{T})$ be a w-refinement. Consider a stratum $S \in \mathcal{S}$ and a point $x \in S$. Let us suppose $b = \dim S^I - \dim S \geqslant 1$. Then, there exists*

i) an open neighborhood $U \subset S^I$ of x, as small as necessary, and

ii) two homeomorphisms $(U,x) \xleftarrow{g} (\mathbb{R}^a \times \mathring{\mathtt{c}}\mathbb{S}^{b-1},(0,\mathtt{v})) \xrightarrow{f} (\mathbb{R}^{a+b},0)$ with $g^{-1}(S \cap U) = \mathbb{R}^a \times \{\mathtt{v}\}$.

Here, $a = \dim S$ and $\mathtt{v}$ is the apex of the cone $\mathring{\mathtt{c}}\mathbb{S}^{b-1}$.

Proof. Let $g\colon (\mathbb{R}^a \times \mathring{\mathtt{c}}L, \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{L}) \to (V,\mathcal{S})$ be a $\mathcal{S}$-conical chart of $x \in S$ where V is small as necessary. Recall that $S \cap V = g(\mathbb{R}^a \times \{\mathtt{w}\})$ and $g(0,\mathtt{w}) = x$, where $\mathtt{w}$ is the apex of $\mathring{\mathtt{c}}L$. The subset $g^{-1}(V \cap S^I \backslash S)$ is of the form $\mathbb{R}^a \times A \times]0,1[$ for some subset $A \subset L$ (cf. (S6w)). We get the homeomorphism $g\colon \mathbb{R}^a \times \mathring{\mathtt{c}}A \to V \cap S^I$. We take $U = V \cap S^I$. Notice that $\dim S = a$ and $\dim S^I = a + b$.

Without loss of the generality we can suppose that U is included in an open subset of S^I homeomorphic to $\mathbb{R}^{a+b}$. So, the cone $\mathring{\mathtt{c}}(\mathbb{S}^{a-1} * A)$ is homeomorphic to an open subset of $\mathring{\mathtt{c}}S^{a+b-1}$ (cf. [**14**, (8)]). Then $\mathring{\mathtt{c}}(\mathbb{S}^{a-1} * A)$ is homeomorphic to $\mathring{\mathtt{c}}S^{a+b-1}$ by a homeomorphism preserving the apexes (cf. Lemma 2.4). This gives f. A standard calculation gives that A is a $(b-1)$-homological sphere. ☐

Remark 2.6. Condition (S6w) of a w-refinement gives the following local structure of a point $x \in X$ belonging to the stratum $S \in \mathcal{S}$: $\begin{cases} x &= (0,\mathtt{v}) \\ S &= \mathbb{R}^a \times \{\mathtt{v}\} \\ S^I &= \mathbb{R}^a \times \mathring{\mathtt{c}}\mathbb{S}^{b-1}, \end{cases}$ where $\mathbb{S}^{b-1}$ is a $(b-1)$-homological sphere and $\mathtt{v}$ is the apex of the cone. This is the difference with the refinement of Remark 2.1.

The subset $S^I \backslash S = \mathbb{R}^a \times \mathbb{S}^{b-1} \times]0,1[$ is a union of strata $Q \in \mathcal{S}$. Notice that $S \prec Q$.

Let us suppose that the w-refinement $(X,\mathcal{S}) \blacktriangleleft_I (X,\mathcal{T})$ is simple. Since depth $\mathcal{V} = 0$ then all the strata of $S^I \backslash S$ are source strata. If $b > 1$ then S^I/S is just a source stratum, when $b = 1$ the subset $S^I \backslash S$ is the union of two source strata.

Globally, S^I contains a discrete family of strata of $\mathcal{V} = \mathcal{M}$, the rest being source strata.

Proposition 2.7. *Let $(X,\mathcal{S}) \blacktriangleleft_I (X,\mathcal{T})$ be a simple w-refinement between two CS-sets. We consider a point $x \in X$ belonging to the stratum $S \in \mathcal{S}$. We distinguish three cases.*

a) S is a source stratum. *Then there exists*

- *a $\mathcal{S}$-conical chart (ψ, W) of $x \in S$, whose link is $(L,\mathcal{L})$,*
- *a $\mathcal{T}$-conical chart (ψ, W) of $x \in S^I$, whose link is $(L,\mathcal{L}')$ for some w-refinement $\mathcal{L}'$ of $\mathcal{L}$, and*
- *a commutative diagram*

$$\begin{array}{ccc} (\mathbb{R}^a \times \mathring{\mathtt{c}}L, \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{L}) & \xrightarrow{\psi} & (W,\mathcal{S}) \\ \downarrow I & & \downarrow I \\ (\mathbb{R}^a \times \mathring{\mathtt{c}}L, \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{L}') & \xrightarrow{\psi} & (W,\mathcal{T}) \end{array}$$

b) S is an exceptional stratum. *Let* $b = \dim S^{I} - \dim S \geqslant 1$. *Then there exists*

- *a* $\mathcal{S}$*-conical chart* (ϕ, W) *of* $x \in S$*, whose link is* $(\mathbb{S}^{b-1}, \mathcal{I})$*,*
- *a chart* (ψ, W) *of* $x \in S^{I}$*, and*
- *a commutative diagram*

$$\begin{array}{ccc} (\mathbb{R}^a \times \mathring{\mathtt{c}}\mathbb{S}^{b-1}, \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{I}) & \xrightarrow{\phi} & (W, \mathcal{S}) \\ \downarrow f & & \downarrow I \\ (\mathbb{R}^{a+b}, \mathcal{I}) & \xrightarrow{\psi} & (W, \mathcal{T}), \end{array}$$

where f *is a homeomorphism.*

c) S is a non-exceptional virtual stratum. *Let* $b = \dim S^{I} - \dim S \geqslant 1$. *Then there exists*

- *a* $\mathcal{S}$*-conical chart* (ϕ, W) *of* $x \in S$*, whose link*[2] *is* $(\mathbb{S}^{b-1} * E, \mathcal{E}_{\star b-1})$*,*
- *a* $\mathcal{T}$*-conical chart* (ψ, W) *of* $x \in S^{I}$*, whose link is* $(E; \mathcal{E})$*, and*
- *a commutative diagram*

$$\begin{array}{ccc} (\mathbb{R}^a \times \mathring{\mathtt{c}}(\mathbb{S}^{b-1} * E), \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{E}_{\star b-1}) & \xrightarrow{\phi} & (W, \mathcal{S}) \\ {\scriptstyle (f \times \mathrm{Ide})(\mathrm{Ide} \times h^{-1})} \downarrow & & \downarrow I \\ (\mathbb{R}^{a+b} \times \mathring{\mathtt{c}}E, \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{E}) & \xrightarrow{\psi} & (W, \mathcal{T}) \end{array}$$

where h, f *are defined in [**14**, (8)] and Lemma 2.5 respectively.*

Proof. a) We consider a $\mathcal{S}$-conical chart of $x \in S^{I}$:

$$\psi\colon (\mathbb{R}^a \times \mathring{\mathtt{c}}L, \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{L}) \to (W, \mathcal{S})$$

Each stratum of $\mathcal{T}$ is a union of strata of $\mathcal{S}$ (cf. (S6w)). Since $\dim S^{I} = \dim S$ then $S^{I} \cap W = \psi(\mathbb{R}^a \times \{\mathtt{w}\}) = S \cap W$ where $\mathtt{w}$ is the apex of the cone $\mathring{\mathtt{c}}L$. The other strata of $(W, \mathcal{T})$ are of the form $\psi(\mathbb{R}^a \times A \times]0,1[)$ where A is a union of some strata of $\mathcal{L}$. So, there exists a filtration $\mathcal{L}'$ on L such that

$$\psi\colon (\mathbb{R}^a \times]0,1[\times L, \mathcal{I} \times \mathcal{I} \times \mathcal{L}') \to (W \backslash S^{I}, \mathcal{T})$$

is a stratified homeomorphism. This is also the case for

$$\psi\colon (\mathbb{R}^a \times \mathring{\mathtt{c}}L, \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{L}') \to (W, \mathcal{T}).$$

We get that (ψ, W) is a $\mathcal{T}$-conical chart of $x \in S^{I}$ with link $(L, \mathcal{L}')$.

b) Take $(\phi, W) = (g, U)$ and $(\psi, W) = (g \circ f^{-1}, U)$ from Lemma 2.5. Since S^{I} is a regular stratum of $\mathcal{T}$, then W is an open subset of X. The pair (ψ, W) is a chart of $x \in S^{I}$.

[2]We find the definition of a join stratification in [**14**, Examples 2.2]).

Recall that $g^{-1}(U \cap S) = \mathbb{R}^a \times \{\mathtt{v}\}$, where $\mathtt{v}$ is the apex of $\mathring{c}\mathbb{S}^{b-1}$. Then,

$$\phi\colon (\mathbb{R}^a \times (\mathbb{S}^{b-1} \times]0,1[), \mathcal{I} \times \mathcal{I}) \xrightarrow{f} (f(\mathbb{R}^a \times \mathbb{S}^{b-1} \times]0,1[), \mathcal{I}) \xrightarrow{g\circ f^{-1}} (U\backslash S, \mathcal{T}) = (U\backslash S, \mathcal{S})$$

is a stratified homeomorphism (cf. Remark 2.6). We conclude that

$$\phi\colon (\mathbb{R}^a \times \mathring{c}\mathbb{S}^{b-1}, \mathcal{I} \times \mathring{c}\mathcal{I}) \to (W, \mathcal{S})$$

is a stratified homeomorphism and therefore (ϕ, W) is a $\mathcal{S}$-chart of $x \in S$ whose link is $(\mathbb{S}^{b-1}, \mathcal{I})$.
The diagram is commutative since, by construction, we have $\phi = \psi \circ f$.

c) We define $a = \dim S$, which gives $\dim S^{^I} = a + b$. Let

$$\varphi\colon (\mathbb{R}^{a+b} \times \mathring{c}E, \mathcal{I} \times \mathring{c}\mathcal{E}) \to (Q, \mathcal{T})$$

be a $\mathcal{T}$-conical chart of $x \in S^{^I}$. The set $Q \cap S^{^I} = \varphi(\mathbb{R}^{a+b} \times \{\mathtt{w}\})$, where $\mathtt{w}$ is the apex of the cone $\mathring{c}E$, is an open neighborhood of $x \in S^{^I}$. We consider (U, g) given by Lemma 2.5 with $U \subset Q \cap S^{^I}$. We define $W = \varphi(\mathrm{pr}\varphi^{-1}(U) \times \mathring{c}E) \subset Q$ which is an open subset containing x. Here $\mathrm{pr}\colon \mathbb{R}^{a+b} \times \mathring{c}E \to \mathbb{R}^{a+b}$ is the canonical projection. The stratified homeomorphism

$$\gamma = \varphi \circ ((\mathrm{pr} \circ \varphi^{-1} \circ g) \times \mathrm{Ide}_{\mathring{c}L})\colon (\mathbb{R}^a \times \mathring{c}\mathbb{S}^{b-1} \times \mathring{c}E, \mathcal{I} \times \mathcal{I} \times \mathring{c}\mathcal{E}) \to (W, \mathcal{T}),$$

verifies $\gamma^{-1}(S^{^I}) = \mathbb{R}^a \times \mathring{c}\mathbb{S}^{b-1} \times \{\mathtt{w}\}$ and $\gamma^{-1}(S) = \mathbb{R}^a \times \{(\mathtt{v}, \mathtt{w})\}$. We consider f given by Lemma 2.5. Notice that

$$\psi = \gamma \circ (f^{-1} \times \mathrm{Ide})\colon (\mathbb{R}^{a+b} \times \mathring{c}E, \mathcal{I} \times \mathring{c}\mathcal{E}) \to (W, \mathcal{T})$$

gives a $\mathcal{T}$-conical chart (ψ, W) of x, whose link is $(E, \mathcal{E})$.
From Remark 2.6 we get that the stratification $\mathcal{S}$ induces the stratification

$$\mathbb{R}^a \times \{(\mathtt{v}, \mathtt{w})\}, \mathbb{R}^a \times (\mathbb{S}^{b-1})_{cc} \times]0,1[\times \{\mathtt{v}\}$$

on $\gamma^{-1}(S^{^I})$. In other words, the map

$$\gamma\colon (\mathbb{R}^a \times \mathring{c}\mathbb{S}^{b-1} \times \{\mathtt{w}\}, \mathcal{I} \times \mathring{c}\mathcal{I} \times \mathcal{I}) \to (W \cap S^{^I}, \mathcal{S})$$

is a stratified homeomorphism. Since all the strata of $(W\backslash S, \mathcal{S})$ are source strata then $\mathcal{S} = \mathcal{T}$ on $W\backslash S^{^I}$. This gives that

$$\gamma\colon (\mathbb{R}^a \times \mathring{c}\mathbb{S}^{b-1} \times E \times]0,1[, \mathcal{I} \times \mathcal{I} \times \mathcal{E} \times I) \to (W\backslash S^{^I}, \mathcal{S})$$

is a stratified homeomorphism. Combining these two results, we get that

$$\gamma\colon (\mathbb{R}^a \times (\mathring{c}\mathbb{S}^{b-1} \times \mathring{c}E), \mathcal{I} \times (\mathcal{I} \times \mathring{c}\mathcal{E})_{(\mathtt{v},\mathtt{w})}) \to (W, \mathcal{S})$$

is a stratified homeomorphism. Finally, using the stratified homeomorphism h (cf. [**14**, (8)]) we get the conical chart

$$\phi = \gamma \circ (\mathrm{Id} \times h^{-1})\colon (\mathbb{R}^a \times \mathring{c}(\mathbb{S}^{b-1} * E), \mathcal{I} \times \mathring{c}\mathcal{E}_{\star b-1}) \to (W, \mathcal{S})$$

whose link is $(\mathbb{S}^{b-1} * E, \mathcal{E}_{\star b-1})$.

The diagram is commutative since, by construction, we have

$$\begin{aligned}\psi(f \times \mathrm{Ide}\,)(\mathrm{Ide}\, \times h^{-1}) &= \gamma(f^{-1} \times \mathrm{Ide}\,)(f \times \mathrm{Ide}\,)(\mathrm{Ide}\, \times h^{-1})\\ &= \gamma(\mathrm{Ide}\, \times h^{-1}) = \phi.\end{aligned}$$ □

We can consider b) as a special case of c) by taking $E = \varnothing$.

Suppose we have a simple w-refinement with only one virtual stratum $V \in \mathcal{S}$. We describe the local situation around a point x belonging to a stratum $S \in \mathcal{S}$. We have several possibilities

- $S \prec V$. The local description is the same as in case a) with $\mathcal{L}' \neq \mathcal{L}$.
- $S = V$. The local description is that of item b) (if $\dim S^{I} = \dim X$) or that of c) (if $\dim S^{I} < \dim X$).
- $S \npreceq V$. The local description is the same as in case a) with $\mathcal{L}' = \mathcal{L}$. In fact, on the open subset $X \backslash \overline{V}$, the stratifications $\mathcal{S}$ and $\mathcal{T}$ are equal.

2.1. (Co)homologies

The main aim of this paper is to prove the invariance of intersection (co)homologies and related (co)homologies under w-refinement. These (co)homologies are associated with a perverse CS-set $(X, \mathcal{S}, \overline{p})$ (refer to [**14**, Section 3.1]) and are as follows:

$H_*^{\overline{p}}(X;\mathcal{S})$	Intersection homology
$H^*_{\overline{p}}(X;\mathcal{S})$	Intersection cohomology
$H_*^{BM,\overline{p}}(X;\mathcal{S})$	Borel-Moore intersection homology
$H^*_{\overline{p},c}(X;\mathcal{S})$	Intersection cohomology with compact supports
$\mathfrak{H}_*^{\overline{p}}(X;\mathcal{S})$	Tame intersection homology
$\mathfrak{H}^*_{\overline{p}}(X;\mathcal{S})$	Tame intersection cohomology
$\mathfrak{H}_*^{BM,\overline{p}}(X;\mathcal{S})$	Borel-Moore tame intersection homology
$\mathfrak{H}^*_{\overline{p},c}(X;\mathcal{S})$	Tame intersection cohomology with compact supports
$\mathscr{H}^*_{\overline{p}}(X;\mathcal{S})$	Blown-up intersection cohomology
$\mathscr{H}^*_{\overline{p},c}(X;\mathcal{S})$	Blown-up intersection cohomology with compact supports

The (co)homologies under discussion arise when investigating the Poincaré duality concerning pseudo-manifolds. For a comprehensive presentation of these notions, we refer to [**14**, Section 1]. Because of the new local description of the conical chart for a point of a virtual stratum (cf. Lemma 2.7) we require the subsequent computation.

Let $(X, \mathcal{S})$ be a CS-set. We consider a perversity $\overline{p}$ on the join $\mathbb{S}^m * X$ (cf. [**14**, Examples 2.2]) taking the same value $\overline{p}(\mathbb{S}^m)$ on each connected component $(\mathbb{S}^m)_{cc}$ of $\mathbb{S}^m$. A such perversity is determined by a perversity $\overline{p}$ on the cone $\mathring{c}X$. It also defines a perversity $\overline{p}$ on X and $\mathring{\mathtt{c}}X$ as follows

$$\overline{p}(\mathbb{S}^m) = \overline{p}(\mathtt{v}) \quad \text{and} \quad \overline{p}(\mathring{\mathtt{c}}\mathbb{S}^m \times S) = \overline{p}(S\times]0,1[) = \overline{p}(S)$$

where $\mathtt{v}$ is the apex of the cone $\mathring{\mathtt{c}}X$ and $S \in \mathcal{S}$. Since $\operatorname{codim}_{\mathbb{S}^m * X} \mathbb{S}^m = \operatorname{codim}_{\mathring{\mathtt{c}}X}\{\mathtt{v}\}$ and $\operatorname{codim}_{\mathbb{S}^m * X}(\mathring{\mathtt{c}}\mathbb{S}^m \times S) = \operatorname{codim}_{\mathring{\mathtt{c}}X}(S\times]0,1[) = \operatorname{codim}_X S$ then

$$D\overline{p}(\mathbb{S}^m) = D\overline{p}(\mathtt{v}) \quad \text{and} \quad D\overline{p}(\mathring{\mathtt{c}}\mathbb{S}^m \times S) = D\overline{p}(S\times]0,1[) = D\overline{p}(S).$$

Lemma 2.8. *Let $(X, \mathcal{S})$ be a CS-set. Consider $\overline{p}$ a perversity on the join $\mathbb{S}^m * X$ for some integer $m \in \mathbb{N}$. We suppose that $\overline{p}$ takes the same value $\overline{p}(\mathbb{S}^m)$ on each connected component $(\mathbb{S}^m)_{cc}$ of $\mathbb{S}^m$. We have*

$$H_k^{\overline{p}}(\mathbb{S}^m * X, \mathcal{S}_{\star m}) = \begin{cases} H_k^{\overline{p}}(X,\mathcal{S}) & \text{if } k \leqslant D\overline{p}(\mathbb{S}^m), \\ G & \text{if } 0 = k > D\overline{p}(\mathbb{S}^m), \\ 0 & \text{if } D\overline{p}(\mathbb{S}^m) < k \leqslant D\overline{p}(\mathbb{S}^m) + m + 1, k \neq 0 \\ \widetilde{H}_{k-m-1}^{\overline{p}}(X,\mathcal{S}) & \text{if } k \geqslant D\overline{p}(\mathbb{S}^m) + m + 2, k \neq 0 \end{cases}$$

$$\mathfrak{H}_k^{\overline{p}}(\mathbb{S}^m * X, \mathcal{S}_{\star m}) = \begin{cases} \mathfrak{H}_k^{\overline{p}}(X;\mathcal{S}) & \text{if } k \leqslant D\overline{p}(\mathbb{S}^m), , \\ 0 & \text{if } D\overline{p}(\mathbb{S}^m) < k \leqslant D\overline{p}(\mathbb{S}^m) + m + 1, \\ \mathfrak{H}_{k-m-1}^{\overline{p}}(X,\mathcal{S}) & \text{if } k \geqslant D\overline{p}(\mathbb{S}^m) + m + 2, \end{cases}$$

$$\mathscr{H}_{\overline{p}}^{k}(S^m * X, \mathcal{S}_{\star m+1}) = \begin{cases} \mathscr{H}_{\overline{p}}^{k}(X,\mathcal{S}) & \text{if } k \leqslant \overline{p}(\mathbb{S}^m), \\ 0 & \text{if } \overline{p}(\mathbb{S}^m) < k \leqslant \overline{p}(\mathbb{S}^m) + m + 1, \\ \mathscr{H}_{\overline{p}}^{k-m-1}(X,\mathcal{S}) & \text{if } k \geqslant \overline{p}(\mathbb{S}^m) + m + 2, \end{cases}$$

where the first line isomorphisms come from the natural inclusion $X \hookrightarrow \mathbb{S}^m * X$.

Proof. The three cases use the same technics: Mayer–Vietoris and local calculations. We prove just the first one. The join stratification is $\mathcal{S}_{\star m} = \{(\mathbb{S}^m)_{cc}, \mathring{\mathtt{c}}\mathbb{S}^m \times S \mid S \in \mathcal{S}\}$.

We consider the open covering $\mathbb{S}^m * X = U \cup V$ where

$$U = (\mathbb{S}^m * X)\backslash\mathbb{S}^m = \mathring{\mathtt{c}}\mathbb{S}^m \times X \quad \text{and} \quad V = (\mathbb{S}^m * X)\backslash X = \mathbb{S}^m \times \mathring{\mathtt{c}}X.$$

This last equality comes from the homeomorphism $[[z,t],y] \mapsto (z,[y,1-t])$. We have $U \cap V = \mathbb{S}^m \times]0,1[\times X$. The inclusion $U \cap V \hookrightarrow U$ induces the morphism

$$\mathrm{pr}_* \colon H_*^{\overline{p}}(\mathbb{S}^m \times X, \mathcal{I} \times \mathcal{S}) \to H_*^{\overline{p}}(X,\mathcal{S}),$$

where $\mathrm{pr}\colon \mathbb{S}^m \to X$ is la projection canonic, and the inclusion $U \cap V \hookrightarrow V$ induces the morphism $I_* \colon H_*^{\overline{p}}(\mathbb{S}^m \times X, \mathcal{I} \times \mathcal{S}) \to H_*^{\overline{p}}(\mathbb{S}^m \times \mathring{\mathtt{c}}X, \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{S})$, where

$$I \colon \mathbb{S}^m \times X \hookrightarrow \mathbb{S}^m \times \mathring{\mathtt{c}}X$$

is the inclusion $(\theta, x) \mapsto (\theta, [x, 1/2])$ (cf. [**5**, Corollary 3.14]).

Applying Mayer–Vietoris (cf. [**5**, Proposition 4.1]) we get the short exact sequence

$$0 \to \operatorname{Coker} F_k \to H_k^{\overline{p}}(\mathbb{S}^m * X, \mathcal{S}_{\star m}) \to \operatorname{Ker} F_{k-1} \to 0,$$

for each $k \in \mathbb{Z}$, where

$$F_k = \mathrm{pr}_k \oplus I_k \colon H_k^{\overline{p}}(\mathbb{S}^m \times X, \mathcal{I} \times \mathcal{S}) \to H_k^{\overline{p}}(X,\mathcal{S}) \oplus H_k^{\overline{p}}(\mathbb{S}^m \times \mathring{\mathtt{c}}X, \mathcal{I} \times \mathring{\mathtt{c}}\mathcal{S}).$$

Intersection homology verifies Kunneth (cf. [**5**, Proposition 4.7]), so this maps becomes

$$F_k = H_k^{\overline{p}}(X,\mathcal{S}) \oplus H_{k-m}^{\overline{p}}(X,\mathcal{S}) \to H_k^{\overline{p}}(X,\mathcal{S}) \oplus H_k^{\overline{p}}(\mathring{\mathtt{c}}X, \mathring{\mathtt{c}}\mathcal{S}) \oplus H_{k-m}^{\overline{p}}(\mathring{\mathtt{c}}X, \mathring{\mathtt{c}}\mathcal{S})$$

where

$$F_k(a,b) = \begin{cases} (a, J_k(a), J_{k-m}(b)) & \text{if } m \neq 0, \\ (a-b, J_k(a), J_k(b)) & \text{if } m = 0. \end{cases}$$

Here, $J_\ell \colon H^{\overline{p}}_{\ell}(X,\mathcal{S}) \to H^{\overline{p}}_{\ell}(\mathring{c}X,\mathcal{S})$ is induced by the inclusion $x \mapsto [x,1/2]$. Notice that $\operatorname{Ker} F_k = \operatorname{Ker} J_{k-m}$ and $\operatorname{Coker} F_k = H^{\overline{p}}_{k}(\mathring{c}X,\mathring{c}\mathcal{S}) \oplus \operatorname{Coker} J_{k-m}$.

Let us consider the long exact sequence associated to the pair $(\mathring{c}X, X)$

$$\cdots H^{\overline{p}}_{k-1}(\mathring{c}X,X,\mathring{c}\mathcal{S}) \to H^{\overline{p}}_{k}(X,\mathcal{S}) \xrightarrow{J_*} H^{\overline{p}}_{k}(\mathring{c}X,\mathring{c}\mathcal{S}) \to H^{\overline{p}}_{k}(\mathring{c}X,X,\mathring{c}\mathcal{S}) \to \cdots$$

(cf. [**5**, Definition 4.4]). Notice that for each $k \in \mathbb{Z}$

– the map F_{k-1} is a monomorphism or the map F_k is an epimorphism and
– the map J_{k-1} is a monomorphism or the map J_k is an epimorphism

We conclude that

$$\begin{aligned} &H^{\overline{p}}_{k}(\mathbb{S}^m * X, \mathcal{S}_{\star m}) = \operatorname{Coker} F_k \oplus \operatorname{Ker} F_{k-1} = H^{\overline{p}}_{k}(\mathring{c}X,\mathring{c}\mathcal{S}) \oplus \operatorname{Coker} J_{k-m} \oplus \operatorname{Ker} J_{k-m-1} \\ &= H^{\overline{p}}_{k}(\mathring{c}X,\mathring{c}\mathcal{S}) \oplus H^{\overline{p}}_{k-m}(\mathring{c}X,X,\mathring{c}\mathcal{S}) \\ &=_{(1)} \begin{cases} H^{\overline{p}}_{k}(X,\mathcal{S}) & \text{if } k \leqslant D\overline{p}(\mathbb{S}^m) \\ 0 & \text{if } 0 \neq k > D\overline{p}(\mathbb{S}^m) \\ G & \text{if } 0 = k > D\overline{p}(\mathbb{S}^m) \end{cases} \oplus \begin{cases} 0 & \text{if } k \leqslant D\overline{p}(\mathbb{S}^m) + m + 1 \\ \widetilde{H}^{\overline{p}}_{k-m-1}(X,\mathcal{S}) & \text{if } k \geqslant D\overline{p}(\mathbb{S}^m) + m + 2 \end{cases} \end{aligned}$$

which gives the claim. Here, (1) is given by [**5**, Proposition 5.2, Corollary 5.3].

When $k \leqslant D\overline{p}(\mathbb{S}^m)$ the isomorphism $H^{\overline{p}}_{k}(X,\mathcal{S}) = H^{\overline{p}}_{k}(\mathbb{S}^m * X, \mathcal{S}_{\star m})$ comes from the map $X \xrightarrow{f} U \xrightarrow{g} \mathbb{S}^m * X$ with $f(x) = ([z,0],x)$ and $g([z,t],x) = [[z,t],x]$. In other words, the natural inclusion $X \hookrightarrow \mathbb{S}^m * X$, $x \mapsto [[z,0],x]$. □

3. Refinement invariance for CS-sets

Our primary focus in this work is to demonstrate the w-refinement invariance of all the homologies and cohomologies listed in Paragraph 2.1. We have accomplished this via Theorem 3.5 for w-coarsenings and Theorem 3.7 for w-refinements. To apply the former, we rely on a specific variant of perversities called K-perversities.

3.1. K-perversities

The specific types of perversities that satisfy w-refinement invariance are referred to as K-perversities. In essence, they are perversities formulated on the left-hand side of a w-refinement $(X,\mathcal{S}) \blacktriangleleft (X,\mathcal{T})$, wherein their restriction to the strata of the right-hand side conforms to a classical perversity and meets the growing condition of a Goresky–MacPherson perversity.

Definition 3.1. Let $(X,\mathcal{S}) \blacktriangleleft (X,\mathcal{T})$ be a w-refinement. A perversity $\overline{p}$ on $(X,\mathcal{S})$ is a *K-perversity* if it verifies conditions (K1) and (K2).

(K1) We have, for any strata $S, Q \in \mathcal{S}$ with $S \preceq Q$ and $S^{I} = Q^{I}$,

$$\overline{p}(Q) \leqslant \overline{p}(S) \leqslant \overline{p}(Q) + \overline{t}(S) - \overline{t}(Q), \tag{3}$$

(K2) We have, for any strata $S, Q \in \mathcal{S}$ with $\dim S = \dim Q$ and $S^{^I} = Q^{^I}$,

$$\overline{p}(Q) = \overline{p}(S), \tag{4}$$

Remark 3.2.

i) Notice these two conditions are equivalent to conditions

$$D\overline{p}(Q) \leqslant D\overline{p}(S) \leqslant D\overline{p}(Q) + \overline{t}(S) - \overline{t}(Q) \quad \text{and} \quad D\overline{p}(Q) = D\overline{p}(S). \tag{5}$$

Also, the condition (3) holds for all regular strata S and Q. If the stratum Q is regular and the stratum S is singular (i.e., S is an exceptional stratum), then the condition (3) is replaced by

$$0 \leqslant \overline{p}(S) \leqslant \overline{t}(S). \tag{6}$$

In particular, the existence of a K-perversity implies the absence of 1-exceptional strata, since it is not possible to have $0 \leqslant \overline{t}(S) = -1$.

ii) Let W be an open subset as in Proposition 2.7 (c). The $\mathcal{S}$-stratification on W is given by the family

$$\{S \cap W, (S^{^I} \backslash S)_{cc} \cap W\}.$$

Let $\overline{p}$ be a K-perversity on $(X, \mathcal{S})$. Condition (4) implies that $\overline{p}$ takes the same value on each $(S^{^I} \backslash S)_{cc}$. In other words, the value of $\overline{p}((\mathbb{S}^{b-1})_{cc})$ does not depend on the choice of connected component.

Before presenting the main results of this work, it is necessary to establish some technical lemmas regarding pull-back and push-forward perversities. These lemmas can be found in [**14**, Lemma 5.3, 5.4, and 5.5].

Lemma 3.3. *Let $(X, \mathcal{S}) \blacktriangleleft_I (X, \mathcal{T})$ be a w-refinement. Let $\overline{p}$ be a K-perversity. Then*

i) $I_\star\overline{p}(T) = \overline{p}(S)$ for each $T \in \mathcal{T}$ where $S \in \mathcal{S}$ is a source stratum of T.

ii) $I^\star D I_\star \overline{p} \leqslant D\overline{p}$.

iii) $I^\star I_\star \overline{p} \leqslant \overline{p}$.

Lemma 3.4. *Let $(X, \mathcal{S}) \blacktriangleleft_I (X, \mathcal{R}) \blacktriangleleft_J (X, \mathcal{T})$ be two w-refinements. For any choice of K-perversity $\overline{p}$ on $(X, \mathcal{S})$, relatively to the w-refinement $E = J \circ I$, we have*

a) $\overline{p}$ is a K-perversity, relatively to the w-refinement I, and

b) $I_\star\overline{p}$ is a K-perversity, relatively to the w-refinement J.

3.2. Main results

We give the two invariance results of the various intersection (co)homologies: by w-coarsening and by w-refinement.

Theorem 3.5 (Invariance by w-coarsening)**.** *Let $(X, \mathcal{S}) \blacktriangleleft (X, \mathcal{T})$ be a w-refinement between two CS-sets. For any K-perversity $\overline{p}$ on $(X, \mathcal{S})$ the identity $I \colon X \to X$ induces* *the isomorphisms*

(R1) $H_*^{\overline{p}}(X;\mathcal{S}) \cong H_*^{I_\star\overline{p}}(X;\mathcal{T})$, (R2) $H^*_{\overline{p}}(X;\mathcal{S}) \cong H^*_{I_\star\overline{p}}(X;\mathcal{T})$,

(R3) $H^*_{\overline{p},c}(X;\mathcal{S}) \cong H^*_{I_\star\overline{p},c}(X;\mathcal{T})$, (R4) $\mathfrak{H}_*^{\overline{p}}(X;\mathcal{S}) \cong \mathfrak{H}_*^{I_\star\overline{p}}(X;\mathcal{T})$,

(R5) $\mathfrak{H}^*_{\overline{p}}(X;\mathcal{S}) \cong \mathfrak{H}^*_{I_\star\overline{p}}(X;\mathcal{T})$, (R6) $\mathfrak{H}^*_{\overline{p},c}(X;\mathcal{S}) \cong \mathfrak{H}^*_{I_\star\overline{p},c}(X;\mathcal{T})$.

If in addition, X is second countable then

(R7) $H_*^{BM,\overline{p}}(X;\mathcal{S}) \cong H_*^{BM,I_\star\overline{p}}(X;\mathcal{T})$, (R8) $\mathfrak{H}_*^{BM,\overline{p}}(X;\mathcal{S}) \cong \mathfrak{H}_*^{BM,I_\star\overline{p}}(X;\mathcal{T})$,

(R9) $\mathscr{H}^*_{\overline{p}}(X;\mathcal{S}) \cong \mathscr{H}^*_{I_\star\overline{p}}(X;\mathcal{T})$, (R10) $\mathscr{H}^*_{\overline{p},c}(X;\mathcal{S}) \cong \mathscr{H}^*_{I_\star\overline{p},c}(X;\mathcal{T})$.

Proof. Proceeding as in [**14**, Theorem 5.7] it suffices to consider the cases *(R1), (R4) and (R9)* supposing that the w-refinement is simple. We verify the conditions of [**14**, Proposition 4.5], for (R1) and (R4), and [**14**, Proposition 4.6], for (R9). The functor Φ comes from $I\colon X \to X$.

a) Consider the Mayer–Vietoris sequences coming from [**5**, Proposition 4.1] and [**2**, Corollary 10.1].[3]

b) Since chains have compact support we get (R1), (R4). The case (R9) is straightforward.

d) Since $\mathcal{S}_U = \mathcal{I}$ implies $\mathcal{T}_U = \mathcal{I}$ then property (d) becomes a tautology.

c) Consider a singular point $x \in X$. We assume that we have a chart containing x that is small enough, as discussed in [**14**, Remark 4.7]. We distinguish three cases based on this assumption.

(C-a) $\boldsymbol{x \in S}$, **source stratum of $\boldsymbol{\mathcal{S}}$**. We can use Proposition 2.7 (a) and the local calculations from [**5**, Corollary 3.14, Proposition 5.1] and [**2**, Theorems D, E] to restate question (c) in terms of three implications: (R1), (R4), and (R9). These implications are as follows:
Considering Proposition 2.7 (a) and using the *local calculations* [**5**, Corollary 3.14, Proposition 5.1]) and [**2**, Theorems D, E]) we can restate question (c) in terms of three implications, denoted as (R1), (R4), and (R9). The implications are as follows:

$$(R1)\quad H_*^{\overline{p}}(L,\mathcal{L})\cong H_*^{I_\star\overline{p}}(L,\mathcal{L}') \implies H_*^{\overline{p}}(\mathring{c}L,\mathring{c}\mathcal{L})\cong H_*^{I_\star\overline{p}}(\mathring{c}L,\mathring{c}\mathcal{L}')$$

$$(R4)\quad \mathfrak{H}_*^{\overline{p}}(L,\mathcal{L})\cong \mathfrak{H}_*^{I_\star\overline{p}}(L,\mathcal{L}') \implies \mathfrak{H}_*^{\overline{p}}(\mathring{c}L,\mathring{c}\mathcal{L})\cong \mathfrak{H}_*^{I_\star\overline{p}}(\mathring{c}L,\mathring{c}\mathcal{L}')$$

$$(R9)\quad \mathscr{H}_*^{\overline{p}}(L,\mathcal{L})\cong \mathscr{H}_*^{I_\star\overline{p}}(L,\mathcal{L}') \implies \mathscr{H}_*^{\overline{p}}(\mathring{c}L,\mathring{c}\mathcal{L})\cong \mathscr{H}_*^{I_\star\overline{p}}(\mathring{c}L,\mathring{c}\mathcal{L}').$$

Since the perversity $\overline{p}$ verifies $\overline{p}(S) = I_\star\overline{p}(S^I)$ (cf. Lemma 3.3 i), we have $\overline{p}(\mathtt{v}) = I_\star\overline{p}(\mathtt{v})$ (cf. [**14**, (17)]). The result follows directly from the above local calculations.

(C-b) $\boldsymbol{x \in S}$, **exceptional stratum of $\boldsymbol{\mathcal{S}}$**. We can restate question (c) in terms of three implications, denoted as (R1), (R4), and (R9), by using Proposition 2.7

[3]Notice that X is second countable, Hausdorff and locally compact ([**14**, Remark 4.7]). Then, the pseudomanifold X is paracompact (cf. [**1**, II.12.12]).

(b) and the local calculations mentioned above. The implications are as follows:

$(R1)\ H_*^{\overline{p}}\big(\mathring{\mathtt{c}}\mathbb{S}^{b-1}, \mathring{\mathtt{c}}\mathcal{I}\big)\cong G,\ (R4)\ \mathfrak{H}_*^{\overline{p}}\big(\mathring{\mathtt{c}}\mathbb{S}^{b-1}, \mathring{\mathtt{c}}\mathcal{I}\big)\cong G,\ (R9)\ \mathscr{H}_{\overline{p}}^{*}\big(\mathring{\mathtt{c}}\mathbb{S}^{b-1}, \mathring{\mathtt{c}}\mathcal{I}\big)\cong R.$

Here, $b = \operatorname{codim} S \geqslant 1$. Since

$$0 \leqslant \overline{p}(S) \leqslant \overline{t}(S) = b - 2$$

(as given by equation (6)), we have $0 \leqslant \overline{p}(\mathtt{u}) \leqslant b - 2$ (cf. [**14**, (18)]). The result follows directly from the local calculations mentioned above.

(C-c) $\boldsymbol{x} \in \boldsymbol{S}$, **non-exceptional virtual stratum of** $\boldsymbol{\mathcal{S}}$. To address question (c) using Proposition 2.7 (c), we can use the local calculations above to rephrase it in the following forms:

$$\begin{array}{lll}(R1) & H_*^{\overline{p}}\big(\mathring{\mathtt{c}}(\mathbb{S}^{b-1} * E), \mathring{\mathtt{c}}\mathcal{E}_{\star b-1}\big) & \cong & H_*^{I_\star\overline{p}}(\mathring{\mathtt{c}}E, \mathring{\mathtt{c}}\mathcal{E}) \\ (R4) & \mathfrak{H}_*^{\overline{p}}\big(\mathring{\mathtt{c}}(\mathbb{S}^{b-1} * E), \mathring{\mathtt{c}}\mathcal{E}_{\star b-1}\big) & \cong & \mathfrak{H}_*^{I_\star\overline{p}}(\mathring{\mathtt{c}}E, \mathring{\mathtt{c}}\mathcal{E}) \\ (R9) & \mathscr{H}_{\overline{p}}^{*}\big(\mathring{\mathtt{c}}(\mathbb{S}^{b-1} * E), \mathring{\mathtt{c}}\mathcal{E}_{\star b-1}\big) & \cong & \mathscr{H}_{I_\star\overline{p}}^{*}(\mathring{\mathtt{c}}E, \mathring{\mathtt{c}}\mathcal{E}),\end{array}$$

where $b = \dim S^{^I} - \dim S \geqslant 1$.

Since $S \preceq (S^{^I}\backslash S)_{cc}$ (cf. Remark 2.6) and

$$\overline{p}((S^{^I}\backslash S)_{cc}) \leqslant \overline{p}(S) \leqslant \overline{p}((S^{^I}\backslash S)_{cc}) + b$$

(cf. (3)) then we have $\overline{p}((\mathbb{S}^{b-1})_{cc}) \leqslant \overline{p}(\mathtt{u}) \leqslant \overline{p}((\mathbb{S}^{b-1})_{cc}) + b$ and

$$D\overline{p}((\mathbb{S}^{b-1})_{cc}) \leqslant D\overline{p}(\mathtt{u}) \leqslant D\overline{p}((\mathbb{S}^{b-1})_{cc}) + b$$

(cf. (5)).

On the other hand, we have $I_\star\overline{p}(\mathtt{w}) = I_\star\overline{p}(S^{^I}) = \overline{p}((S^{^I}\backslash S)_{cc}) = \overline{p}(\big(\mathbb{S}^{b-1}\big)_{cc})$ (cf. Lemma 3.3 i) and (cf. [**14**, (19)]). Since

$$\dim(\mathbb{R}^a \times \mathbb{S}^{b-1} \times]0, 1[) = \dim(\mathbb{R}^{a+b} \times \{\mathtt{w}\})$$

then $DI_\star\overline{p}(\mathtt{w}) = D\overline{p}(\big(\mathbb{S}^{b-1}\big)_{cc})$. We conclude that

$$DI_\star\overline{p}(\mathtt{w}) \leqslant D\overline{p}(\mathtt{u}) \leqslant DI_\star\overline{p}(\mathtt{w}) + b.$$

By utilizing the previous local computations and Lemma 2.8 (see Remark 3.2 ii) for more details), we can reformulate question (c) in the following manner:

$$(R1)\ H_*^{\overline{p}}(E, \mathcal{E})\cong H_*^{I_\star\overline{p}}(E, \mathcal{E}), \quad (R4)\ \mathfrak{H}_*^{\overline{p}}(E, \mathcal{E})\cong\mathfrak{H}_*^{I_\star\overline{p}}(E, \mathcal{E})$$
$$(R9)\ \mathscr{H}_{\overline{p}}^{*}(E, \mathcal{E})\cong\mathscr{H}_{I_\star\overline{p}}^{*}(E, \mathcal{E}),$$

Any other stratum $R \in \mathcal{S}$ verifying $S \prec R$ is a source stratum (cf. Definition 2.2). So, $\overline{p}(R) = I_\star\overline{p}(R)$ (cf. Lemma 3.3 i) and we get $\overline{p} = I_\star\overline{p}$ on E (see [**14**, (20)]). The claim is proved. □

Remark 3.6. Note that the above isomorphisms may not hold if there exist 1-exceptional strata (see also Remark 3.2). This is the case for (R4), ... (R10). For example,

$$\mathfrak{H}_*^{\overline{0}}\big(\mathring{\mathtt{c}}S^0, \mathring{\mathtt{c}}\mathcal{I}\big) = 0 \neq G = \mathfrak{H}_*^{\overline{0}}(]-1, 1[, \mathcal{I}).$$

But we have $H_*^{\overline{0}}\big(\mathring{\mathtt{c}}S^0\big) = G = H_*^{\overline{0}}(]-1, 1[, \mathcal{I}, \mathring{\mathtt{c}}\mathcal{I})$. In fact, the local calculations

$H_0^{\overline{p}}(\mathring{\mathtt{c}}S^0, \mathcal{S})$ and $\mathfrak{H}_0^{\overline{p}}(\mathring{\mathtt{c}}S^0; \mathcal{S})$ are different:

$$H_0^{\overline{p}}(\mathring{\mathtt{c}}S^0, \mathring{\mathtt{c}}\mathcal{I}) = \begin{cases} H_0(S^0) & \text{if } D\overline{p}(\mathtt{v}) \geqslant 0 \\ G & \text{if } D\overline{p}(\mathtt{v}) < 0 \end{cases}$$

$$\mathfrak{H}_0^{\overline{p}}(\mathring{\mathtt{c}}S^0, \mathring{\mathtt{c}}\mathcal{I}) = \begin{cases} H_0(S^0) & \text{if } D\overline{p}(\mathtt{v}) \geqslant 0 \\ 0 & \text{if } D\overline{p}(\mathtt{v}) < 0. \end{cases}$$

We note that the condition $\mathfrak{H}_*^{\overline{p}}(\mathring{\mathtt{c}}S^0, \mathring{\mathtt{c}}\mathcal{I}) \cong G$ in (C-c) is never satisfied, while only $D\overline{p}(\mathtt{v}) < 0$ is required for $H_*^{\overline{p}}(\mathring{\mathtt{c}}S^0, \mathring{\mathtt{c}}\mathcal{I}) \cong G$ to hold in (C-c).

We can relax condition (3) for cases (R1), (R2), and (R3) in the presence of an 1-exceptional stratum S. In these cases, it suffices to require that $D\overline{p}(S) < 0$, which means that $\overline{p}(S) \geqslant 0$, rather than $0 \leqslant \overline{p}(S) \leqslant \overline{t}(S)$. Thus, 1-exceptional strata are allowed in (R1), (R2), and (R3).

Theorem 3.7 (Invariance by w-refinement)**.** *Let $(X, \mathcal{S}) \blacktriangleleft (X, \mathcal{T})$ be a w-refinement between two CS-sets. We assume that there are no 1-exceptional strata. Then, for any perversity $\overline{q}$ on $(X, \mathcal{T})$, the identity map $I \colon X \to X$ induces the following isomorphisms:*

(R1) $H_*^{I^\star\overline{q}}(X; \mathcal{S}) \cong H_*^{\overline{q}}(X; \mathcal{T})$, (R2) $H^*_{I^\star\overline{q}}(X; \mathcal{S}) \cong H^*_{\overline{q}}(X; \mathcal{T})$,

(R3) $H^*_{I^\star\overline{q},c}(X; \mathcal{S}) \cong H^*_{\overline{q},c}(X; \mathcal{T})$, (R4) $\mathfrak{H}_*^{I^\star\overline{q}}(X; \mathcal{S}) \cong \mathfrak{H}_*^{\overline{q}}(X; \mathcal{T})$,

(R5) $\mathfrak{H}^*_{I^\star\overline{q}}(X; \mathcal{S}) \cong \mathfrak{H}^*_{\overline{q}}(X; \mathcal{T})$, (R6) $\mathfrak{H}^*_{I^\star\overline{q},c}(X; \mathcal{S}) \cong \mathfrak{H}^*_{\overline{q},c}(X; \mathcal{T})$.

If in addition, X is second countable then

(R7) $H_*^{BM, I^\star\overline{q}}(X; \mathcal{S}) \cong H_*^{BM,\overline{q}}(X; \mathcal{T})$, (R8) $\mathfrak{H}_*^{BM, I^\star\overline{q}}(X; \mathcal{S}) \cong \mathfrak{H}_*^{BM,\overline{q}}(X; \mathcal{T})$,

(R9) $\mathscr{H}^*_{I^\star\overline{q}}(X; \mathcal{S}) \cong \mathscr{H}^*_{\overline{q}}(X; \mathcal{T})$, (R10) $\mathscr{H}^*_{I^\star\overline{q},c}(X; \mathcal{S}) \cong \mathscr{H}^*_{\overline{q},c}(X; \mathcal{T})$

Proof. Apply Theorem 3.5 and proceed as in the proof of [**14**, Theorem 5.9]. □

In cases (R1), (R2), and (R3), it is possible for 1-exceptional strata S to appear if $\overline{p}(S) \geqslant 0$ (see Remark 3.6).

3.3. Topological invariance

One of the most important properties of intersection homology is its topological invariance, which has been well established in [**11**].

The following Corollary demonstrates that in certain cases, w-refinement invariance implies topological invariance. Specifically, we establish the well-known topological invariance of intersection homology [**11**] (also see [**13, 10**]) and of tame intersection homology [**6**] (for closed supports) and [**7**] (for compact supports). Additionally, we obtain the topological invariance of blown-up intersection cohomology [**2**, Theorem G] (for closed supports) and [**4**, Theorem A] (for compact supports).

Before presenting the result, it is important to highlight two key tools.

- *Intrinsic stratification* (cf. [**12, 13**]). Any stratified space $(X, \mathcal{S})$ has a smallest w-refinement: the *intrinsic stratified space* $(X, \mathcal{S}^*)$. It is a canonical object: we have $\mathcal{S}^* = \mathcal{T}^*$ for any stratification $\mathcal{T}$ defined on X. If $(X, \mathcal{S})$ is a CS-set then $(X, \mathcal{S}^*)$ is also a CS-set.

- *Classical perversities versus perversities.* The former depend on the codimension of the strata while the latter are defined on the strata themselves.
  A *King perversity* is a map $\overline{p}\colon \mathbb{N} \to \mathbb{Z}$ verifying $\overline{p}(0) = 0$ and
  $$\overline{p}(k) \leqslant \overline{p}(k+1) \leqslant \overline{p}(k) + 1$$
  for each $k \in \mathbb{N}^*$ (cf. [13]). It verifies
  $$\overline{p}(k) \leqslant \overline{p}(\ell) \leqslant \overline{p}(k) + \ell - k, \tag{7}$$
  if $1 \leqslant k \leqslant \ell$. A King perversity $\overline{p}$ induces a perversity, still denoted by $\overline{p}$: $\overline{p}(S) = \overline{p}(\operatorname{codim} S)$.
  A *Goresky–MacPherson perversity* is a King perversity $\overline{p}$ with $\overline{p}(0) = \overline{p}(1) = \overline{p}(2) = 0$ (cf. [11]). It verifies, for each $k \geqslant 2$,
  $$\overline{0} \leqslant \overline{p}(k) \leqslant k - 2 = \overline{t}(k) \tag{8}$$

Corollary 3.8. *Let $(X, \mathcal{S})$ be a CS-set endowed with a positive King perversity $\overline{p}$. Consider the intrinsic w-refinement $(X, \mathcal{S}) \blacktriangleleft (X, \mathcal{S}^*)$. The identity map $I\colon X \to X$ induces the isomorphisms*
$$H_*^{\overline{p}}(X;\mathcal{S}) \cong H_*^{\overline{p}}(X;\mathcal{S}^*) \quad H^*_{\overline{p}}(X;\mathcal{S}) \cong H^*_{\overline{p}}(X;\mathcal{S}^*) \quad H^*_{\overline{p},c}(X;\mathcal{S}) \cong H^*_{\overline{p},c}(X;\mathcal{S}^*),$$
if $\overline{p}(\ell) \geqslant 0$ when ℓ is the codimension of an exceptional stratum. We also have
$$\mathfrak{H}_*^{\overline{p}}(X;\mathcal{S}) \cong \mathfrak{H}_*^{\overline{p}}(X;\mathcal{S}^*) \quad \mathfrak{H}^*_{\overline{p}}(X;\mathcal{S}) \cong \mathfrak{H}^*_{\overline{p}}(X;\mathcal{S}^*) \quad \mathfrak{H}^*_{\overline{p},c}(X;\mathcal{S}) \cong \mathfrak{H}^*_{\overline{p},c}(X;\mathcal{S}^*),$$
if $0 \leqslant \overline{p}(\ell) \leqslant \overline{t}(\ell)$. If in addition, X is second countable then we have
$$\mathscr{H}^*_{\overline{p}}(X;\mathcal{S}) \cong \mathscr{H}^*_{\overline{p}}(X;\mathcal{S}^*),\ \mathscr{H}^*_{\overline{p},c}(X;\mathcal{S}) \cong \mathscr{H}^*_{\overline{p},c}(X;\mathcal{S}^*),$$
$$H_*^{BM,\overline{p}}(X;\mathcal{S}) \cong H_*^{BM,\overline{p}}(X;\mathcal{S}^*).$$

Proof. Same proof as in reference [14, Corollary 5.10]. □

Remark 3.9.

i) Consider a CS-set $(X, \mathcal{S})$ equipped with a Goresky–MacPherson perversity $\overline{p}$, where $\overline{p} \geqslant \overline{0}$ (cf. (8)). As a consequence of the previous Corollary, it follows that the cohomologies $H_*^{\overline{p}}(X;\mathcal{S})$, $H^*_{\overline{p}}(X;\mathcal{S})$ and $H^*_{\overline{p},c}(X;\mathcal{S})$ are independent of the stratification $\mathcal{S}$. However, we cannot extend this result to tame intersection homologies, since the condition $\overline{0} \leqslant \overline{p} \leqslant \overline{t}$ (cf. (8)) implies that the tame intersection homology coincides with the usual intersection homology.
   Assuming X is second countable and there are no 1-exceptional strata, we can apply the aforementioned Corollary to conclude that each of the cohomologies $\mathscr{H}^*_{\overline{p}}(X;\mathcal{S})$, $\mathscr{H}^*_{\overline{p},c}(X;\mathcal{S})$ and $H_*^{BM,\overline{p}}(X;\mathcal{S})$ are stratification-independent (cf. (8)).
ii) Consider a K-perversity, denoted as $\overline{p}$. Condition (K2) implies that the restriction of $\overline{p}$ to the $\mathcal{S}$-stratification lying on each stratum $T \in \mathcal{T}$ is actually a classical perversity, except for the condition $\overline{p}(0) = 0$. Meanwhile, property (K1) can be thought of as a growing condition of type (7), but even weaker. Although not entirely precise, we can understand a K-perversity as a perversity whose restriction to any stratum $T \in \mathcal{T}$ is a King perversity.

iii) There is a mistake in [**14**, Paragraph 5.3] that renders the proof of Corollary 5.10, op. cit., incorrect. In fact, it is unclear whether the intrinsic stratification satisfies the strata embedding condition necessary to apply Theorem A, op. cit., according to the author. However, Corollary 3.8 demonstrates that changing the refinement to a w-refinement is sufficient to obtain topological invariance.

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