

REMARKS ON THE EQUIVALENCE BETWEEN DIFFERENTIAL GRADED CATEGORIES AND A-INFINITY CATEGORIES

JAMES PASCALEFF

(communicated by Emily Riehl)

Abstract

We show that the homotopy theories of differential graded categories and A_∞ -categories over a field are equivalent at the $(\infty, 1)$ -categorical level. The results are corollaries of a theorem of Canonaco-Ornaghi-Stellari combined with general relationships between different versions of $(\infty, 1)$ -categories.

1. Introduction

The purpose of this note is to show that certain desirable propositions follow from a direct combination of published results. The propositions state in various forms a “homotopy equivalence” between the “homotopy theory of differential graded (DG) categories” and the “homotopy theory of A_∞ -categories.”

The equivalence between DG categories and A_∞ -categories has been studied for several decades, starting no later than the thesis of Lefèvre-Hasegawa [LH03]. Without recounting the whole history, we shall begin our discussion with one of the more recent versions of this equivalence, proven by Canonaco-Ornaghi-Stellari [COS19].

We fix a field k over which our categories are linear. Let DGCat denote the category whose objects are small DG categories and whose morphisms are DG functors. Let $A_\infty\text{Cat}$ denote the category whose objects are small strictly unital A_∞ -categories and whose morphisms are strictly unital A_∞ -functors. Both DGCat and $A_\infty\text{Cat}$ are ordinary (rather than homotopy coherent) categories. They are not small categories, but all of the categorical constructions we will perform on them may be construed as happening within the framework of Grothendieck universes.

A DG category is, by definition, the same as an A_∞ -category with vanishing higher operations. Likewise, a DG functor between DG categories is the same as an A_∞ -functor with vanishing higher components. Thus there is a functor

$$i : \text{DGCat} \rightarrow A_\infty\text{Cat}.$$

To a DG or A_∞ -category A we may associate a graded k -linear category $H(A)$ by replacing all hom-complexes by their cohomologies; if we retain only cohomology in degree zero, we obtain a k -linear category $H^0(A)$. A DG or A_∞ -functor $F : A \rightarrow B$ induces functors $H(F) : H(A) \rightarrow H(B)$ and $H^0(F) : H^0(A) \rightarrow H^0(B)$.

A DG or A_∞ -functor $F : A \rightarrow B$ is a *quasi-equivalence* if $H(F)$ and $H^0(F)$ are both equivalences of categories; this is equivalent to the conjunction of the conditions that F acts by quasi-isomorphisms on all hom-complexes and that $H^0(F)$ is essentially surjective. We denote by $W_{\text{qe}}^{\text{DG}}$ the subcategory of DGCat with the same objects but with only the DG quasi-equivalences as morphisms; similarly $W_{\text{qe}}^{A_\infty}$ denotes the subcategory of $A_\infty\text{Cat}$ with the same objects but with only the A_∞ -quasi-equivalences as morphisms.

We now state the results of Canonaco-Ornaghi-Stellari [COS19]. First is the construction of a functor

$$U : A_\infty\text{Cat} \rightarrow \text{DGCat}$$

which could be called a *rectification functor*. On objects, this functor takes an A_∞ -category A , forms the bar construction BA , which is a “DG cocategory”, and then forms the cobar construction $\Omega(BA)$ of that. The precise details are not necessary for our discussion, but only the following theorems.

Theorem 1.1 (Canonaco-Ornaghi-Stellari [COS19]). *The functors*

$$U : A_\infty\text{Cat} \rightleftarrows \text{DGCat} : i$$

form an adjunction such that

1. U is left adjoint to i ,
2. i takes $W_{\text{qe}}^{\text{DG}}$ into $W_{\text{qe}}^{A_\infty}$,
3. U takes $W_{\text{qe}}^{A_\infty}$ into $W_{\text{qe}}^{\text{DG}}$,
4. all components of the unit $\eta : \text{id}_{A_\infty\text{Cat}} \rightarrow iU$ belong to $W_{\text{qe}}^{A_\infty}$, and
5. all components of the counit $\epsilon : Ui \rightarrow \text{id}_{\text{DGCat}}$ belong to $W_{\text{qe}}^{\text{DG}}$.

Proof. All of these statements are contained in [COS19, Proposition 2.1], except for the statement that i takes $W_{\text{qe}}^{\text{DG}}$ into $W_{\text{qe}}^{A_\infty}$, which is obvious, and the statement that U takes $W_{\text{qe}}^{A_\infty}$ into $W_{\text{qe}}^{\text{DG}}$, which is demonstrated in the proof of [COS19, Theorem 2.2]. \square

Consider now the localizations $\text{DGCat}[(W_{\text{qe}}^{\text{DG}})^{-1}]$ and $A_\infty\text{Cat}[(W_{\text{qe}}^{A_\infty})^{-1}]$ formed by inverting the quasi-equivalences—these are the *homotopy categories* of DG categories and A_∞ -categories respectively.

Theorem 1.2 ([COS19], Theorem 2.2). *The functors U and i induce mutually quasi-inverse equivalences of homotopy categories*

$$A_\infty\text{Cat}[(W_{\text{qe}}^{A_\infty})^{-1}] \rightleftarrows \text{DGCat}[(W_{\text{qe}}^{\text{DG}})^{-1}].$$

Theorem 1.2 follows directly from the formal properties stated in Theorem 1.1. Theorem 1.2 states that “the homotopy theory of DG categories” is equivalent to “the homotopy theory of A_∞ -categories” at the 1-categorical level: the homotopy categories are equivalent as ordinary categories.

The contribution of this note. One may then ask whether “the homotopy theory of DG categories” is equivalent to “the homotopy theory of A_∞ -categories” at the $(\infty, 1)$ -categorical level. We shall show both that the answer is affirmative, and that this already follows from Theorem 1.1 without any further work involving DG or A_∞ -categories.

That is to say, Theorem 1.1 implies not only Theorem 1.2, but also an equivalence of homotopy theories at the apparently stronger $(\infty, 1)$ -categorical level. We give several precise formulations of this result. They are Corollaries 2.5, 3.2, 4.3, 4.5, 5.1, and 5.2.

1.1. Context

The results of this note are presumably “folklore” or “well-known to experts.” On the other hand, I was not able to find any reference for these results, so I went ahead and put the pieces of the puzzle together myself. The main content of this note is the observation that a certain kind of adjunction between relative categories (called a “Dwyer-Kan adjunction of relative categories” below) implies an equivalence between their localizations at the $(\infty, 1)$ -categorical level.

I have not attempted to survey all of the literature that touches on the questions raised here, of which there is a great deal, both about DG and A_∞ -categories, and about relationships between different versions of $(\infty, 1)$ -categories. Nevertheless, I should note that the article of Vallette [Val20] was very helpful. To relate different versions of $(\infty, 1)$ -categories, I have attempted to cite the original sources for the various functors that we use, but the overall picture that emerges is essentially the same as the one presented by Hinich [Hin16].

One motivation for this note was a 2020 preprint of Oh-Tanaka.¹ I had wished to cite Theorem 1.1 from that preprint, but unfortunately it was withdrawn. Note that Oh and Tanaka work over a general commutative base ring. The results of this note show that—in the special case where the base ring is a field—Theorem 1.1 from that preprint is in fact true. However, the precise form of the equivalence appears to be different from the one proposed by Oh-Tanaka. Tanaka has also informed me of ongoing work to recover Theorem 1.1 over a general commutative base ring.

Note added in proof. Canonaco, Ornaghi, and Stellari have recently extended the results of [COS19] to the case of a general commutative base ring; see [COS24]. This allows the results of the present paper to be extended to that case as well.

Acknowledgments

I wish to thank Charles Rezk and Doron Grossman-Naples for very helpful conversations, and Hiro Lee Tanaka for his comments on this note. I also thank the referee for their comments.

2. Dwyer–Kan equivalence

In their works [DK80b, DK80a], Dwyer and Kan studied a simplicially enriched version of the localization of category at a subcategory. We shall isolate the concepts and results that we need.

By definition, a *relative category* is a pair (C, W) consisting of a category C and a subcategory W , subject only to the condition that W contain all objects (and hence all identity morphisms) of C . We refer to the morphisms in W as *weak equivalences*.

¹Yong-Geun Oh and Hiro Lee Tanaka. A-infinity-categories, their infinity-category, and their localizations. [arXiv:2003.05806](https://arxiv.org/abs/2003.05806), since withdrawn by the authors.

A *relative functor* $F : (C_1, W_1) \rightarrow (C_2, W_2)$ is an ordinary functor $F : C_1 \rightarrow C_2$ such that $F(W_1) \subseteq W_2$.

Recall that an adjunction $L \dashv R$ consists of two functors $L : C \rightarrow D$, $R : D \rightarrow C$ and natural transformations $\eta : \text{id}_C \rightarrow RL$ and $\epsilon : LR \rightarrow \text{id}_D$ called the unit and counit respectively, satisfying certain relations known as triangle identities.

An adjoint equivalence is an adjunction such that both the unit and counit are natural isomorphisms. This condition obviously implies that L and R are equivalences of categories.

We now isolate an important concept from [DK80a]; I am not aware of a standard term for this concept.

Definition 2.1. Let (C_1, W_1) and (C_2, W_2) be relative categories, and let (L, R, η, ϵ) be an adjunction, where $L : C_1 \rightarrow C_2$ and $R : C_2 \rightarrow C_1$. We say that (L, R, η, ϵ) is a *Dwyer-Kan adjunction of relative categories* if the following conditions hold:

1. $L(W_1) \subseteq W_2$ and $R(W_2) \subseteq W_1$, so that L and R are relative functors,
2. for every object X in C_1 , the component of the unit $\eta_X : X \rightarrow RLX$ is in W_1 , and
3. for every object Y in C_2 , the component of the counit $\epsilon_Y : LRY \rightarrow Y$ is in W_2 .

In the special case where $W_i = C_i$ for $i = 1, 2$, a Dwyer-Kan adjunction of relative categories is just an adjunction. In the special case where W_i consists precisely of the isomorphisms in C_i , a Dwyer-Kan adjunction of relative categories is the same as an adjoint equivalence. Alternately, a Dwyer-Kan adjunction of relative categories is “the same as” an adjoint equivalence where the phrase “natural isomorphism” is replaced by “natural transformation in W .”

Theorem 2.2 (Restatement of 1.1). *The functors*

$$U : (\mathcal{A}_\infty \text{Cat}, W_{\text{qe}}^{\mathcal{A}_\infty}) \rightleftarrows (\text{DGCat}, W_{\text{qe}}^{\text{DG}}) : i$$

form a Dwyer-Kan adjunction of relative categories.

Next, given a relative category (C, W) , we may form the localization $C[W^{-1}]$ as an ordinary category. In 1980, Dwyer and Kan [DK80b, DK80a] gave two constructions of simplicially enriched categories that enhance $C[W^{-1}]$ to what contemporary mathematicians consider to be a kind of $(\infty, 1)$ -category. In this sense, a relative category is perhaps the most elementary data structure that presents an $(\infty, 1)$ -category.

The version that we shall use is the so-called *hammock localization*. Given a relative category (C, W) , this construction produces a category enriched over simplicial sets $L^{\text{H}}(C, W)$. The objects of $L^{\text{H}}(C, W)$ are the same as those of C , and for each pair of objects X, Y there is a morphism simplicial set $L^{\text{H}}(C, W)(X, Y)$ whose simplicies correspond to certain diagrams called hammocks; the detailed definition will not concern us.

The simplicial sets $L^{\text{H}}(C, W)(X, Y)$ are to be regarded as spaces and as such they have homotopy groups. The category $\pi_0 L^{\text{H}}(C, W)$ is obtained by replacing each morphism space by its set of components. According to [DK80a, Proposition 3.1],

there is a canonical isomorphism of categories

$$\pi_0 L^H(C, W) \cong C[W^{-1}].$$

An important property of hammock localization is that it is manifestly functorial with respect to relative functors. That is, if $F : (C_1, W_1) \rightarrow (C_2, W_2)$ is a relative functor, then there is an induced simplicially enriched functor

$$L^H F : L^H(C_1, W_1) \rightarrow L^H(C_2, W_2).$$

We recall the appropriate notion of “weak homotopy equivalence” for simplicially enriched categories. Recall that a *weak homotopy equivalence of simplicial sets* is a map that, after passing to geometric realizations, induces a bijection on π_0 and an isomorphism on all homotopy groups with all possible basepoints.

Definition 2.3. Let D_1 and D_2 be simplicially enriched categories, with a simplicially enriched functor $F : D_1 \rightarrow D_2$. Then F is a *Dwyer-Kan equivalence* if the following conditions hold:

1. The induced functor $\pi_0 F : \pi_0 D_1 \rightarrow \pi_0 D_2$ is an equivalence of categories, and
2. for every ordered pair of objects (X, Y) in D_1 , the map of simplicial morphism sets $F : D_1(X, Y) \rightarrow D_2(FX, FY)$ is a weak homotopy equivalence of simplicial sets.

The following proposition asserts a relationship between Dwyer-Kan adjunctions of relative categories and Dwyer-Kan equivalences of localizations.

Proposition 2.4. Let (C_1, W_1) and (C_2, W_2) be relative categories, and let (L, R, η, ϵ) be a Dwyer-Kan adjunction of relative categories, where $L : C_1 \rightarrow C_2$ and $R : C_2 \rightarrow C_1$. Then

$$L^H(L) : L^H(C_1, W_1) \rightarrow L^H(C_2, W_2) \quad \text{and} \quad L^H(R) : L^H(C_2, W_2) \rightarrow L^H(C_1, W_1)$$

are Dwyer-Kan equivalences of simplicially enriched categories. Also,

$$\pi_0 L^H(L) : C_1[W_1^{-1}] \rightarrow C_2[W_2^{-1}] \quad \text{and} \quad \pi_0 L^H(R) : C_2[W_2^{-1}] \rightarrow C_1[W_1^{-1}]$$

are mutually quasi-inverse equivalences of categories.

Proof. This is a restatement in our terminology of [DK80a, Corollary 3.6]. □

We remark that $\pi_0 L^H(L)$ and $\pi_0 L^H(R)$ actually fit into an adjoint equivalence of the ordinary localizations, with unit and counit induced from η and ϵ .

According to Bergner [Ber07], there is a model category structure on the category of simplicially enriched categories such that the weak equivalences are precisely the Dwyer-Kan equivalences. So Proposition 2.4 states that $L^H(C_1, W_1)$ and $L^H(C_2, W_2)$ are weakly equivalent in the Bergner model structure.

Corollary 2.5. *The functors*

$$L^H(U) : L^H(\mathbf{A}_\infty \text{Cat}, W_{\text{qe}}^{\mathbf{A}_\infty}) \rightleftarrows L^H(\mathbf{DGCat}, W_{\text{qe}}^{\mathbf{DG}}) : L^H(i)$$

are both Dwyer-Kan equivalences of simplicially enriched categories. That is to say, $L^H(U)$ and $L^H(i)$ are both weak equivalences for the Bergner model structure on the category of simplicially enriched categories.

3. The Barwick–Kan model structure on RelCat

Let RelCat denote the category whose objects are small relative categories and whose morphisms are relative functors. Instead of taking simplicial localization of relative categories as above, we may regard relative categories themselves as a kind of $(\infty, 1)$ -categories. In this vein Barwick–Kan [BK12a, BK12b] defined a model category structure directly on RelCat. This model category structure is designed so as to have a specific relationship to Rezk’s theory of complete Segal spaces [Rez01]; we denote by sSpc the category of simplicial spaces (that is, bisimplicial sets) on which Rezk’s complete Segal space model structure is defined.

Theorem 3.1 ([BK12a, BK12b]). *There is a model structure on RelCat, called the Barwick–Kan model structure, with the following properties:*

1. *there is an adjunction $K_\xi : \text{sSpc} \rightleftarrows \text{RelCat} : N_\xi$ where K_ξ is left adjoint to N_ξ , and this adjunction is a Quillen equivalence between the Barwick–Kan model structure on RelCat and Rezk’s complete Segal space model structure on sSpc,*
2. *a morphism in RelCat is a weak equivalence if and only if its image under N_ξ is a weak equivalence in sSpc, and*
3. *a morphism $F : (C_1, W_1) \rightarrow (C_2, W_2)$ is a weak equivalence if and only if the induced morphism $L^H F : L^H(C_1, W_1) \rightarrow L^H(C_2, W_2)$ is a Dwyer–Kan equivalence of simplicially enriched categories.*

Proof. This is a simplified paraphrase of [BK12b, Theorem 6.1] and [BK12a, Theorem 1.8]. \square

Corollary 3.2. *The relative functors $U : (\text{A}_\infty \text{Cat}, W_{\text{qe}}^{\text{A}_\infty}) \rightleftarrows (\text{DGCat}, W_{\text{qe}}^{\text{DG}}) : i$ are both weak equivalences for the Barwick–Kan model structure on RelCat. Also, the functors $N_\xi(U)$ and $N_\xi(i)$ are weak equivalences for Rezk’s complete Segal space model structure on sSpc.*

4. Fibrant replacement, complete Segal spaces, and quasicategories

So far we have considered relative categories and simplicially enriched categories as models for $(\infty, 1)$ -categories. It is now understood that many different models of $(\infty, 1)$ -categories are equivalent: see [Ber10, Ber18] for surveys of these relationships.

Two further models for $(\infty, 1)$ -categories are complete Segal spaces and quasicategories. The former has already made a passing appearance. The phrase “Rezk’s complete Segal space model structure” refers to a model structure on sSpc, but a *complete Segal space* is a fibrant object for this model structure. Similarly, there is a model structure on the category sSet of simplicial sets called the *Joyal model structure*, and a *quasicategory* is a fibrant object for the Joyal model structure.

This means that in order to convert $(\text{DGCat}, W_{\text{qe}}^{\text{DG}})$ and $(\text{A}_\infty \text{Cat}, W_{\text{qe}}^{\text{A}_\infty})$ into complete Segal spaces or quasicategories, we must consider fibrant replacements. In general, if \mathcal{M} is a model category and X is an object of \mathcal{M} , we shall denote by X^{fib} a fibrant replacement of X , which is a fibrant object X^{fib} of \mathcal{M} equipped with an acyclic

cofibration $j_X : X \rightarrow X^{\text{fib}}$. Given a morphism $\phi : X \rightarrow Y$ in \mathcal{M} , and fibrant replacements $j_X : X \rightarrow X^{\text{fib}}$ and $j_Y : Y \rightarrow Y^{\text{fib}}$, the composite $j_Y \phi$ is a morphism whose target object is fibrant. Therefore it extends along the acyclic cofibration j_X to a morphism $\phi^{\text{fib}} : X^{\text{fib}} \rightarrow Y^{\text{fib}}$ such that $\phi^{\text{fib}} j_X = j_Y \phi$. The morphism ϕ^{fib} is uniquely determined up to homotopy. If ϕ is a weak equivalence, then ϕ^{fib} is as well, by the two-out-of-three property.

A natural question is whether the objects of RelCat we are interested in, namely $(\text{DGCat}, W_{\text{qe}}^{\text{DG}})$ and $(A_\infty \text{Cat}, W_{\text{qe}}^{A_\infty})$, are already fibrant for the Barwick–Kan model structure on RelCat. We have the following proposition.

Proposition 4.1. *The object $(\text{DGCat}, W_{\text{qe}}^{\text{DG}})$ is a fibrant object in RelCat.*

Proof. According to Tabuada [Tab05], DGCat itself admits the structure of a model category such that $W_{\text{qe}}^{\text{DG}}$ is the class of weak equivalences. The desired conclusion then follows from a result of Meier [Mei16, Main Theorem]. \square

Remark 4.2. I was unable to find any result implying that $(A_\infty \text{Cat}, W_{\text{qe}}^{A_\infty})$ is fibrant in RelCat, but this may very well be true. See Section 6 for further discussion.

In order to proceed, let us choose a fibrant replacement

$$j_{A_\infty} : (A_\infty \text{Cat}, W_{\text{qe}}^{A_\infty}) \rightarrow (A_\infty \text{Cat}, W_{\text{qe}}^{A_\infty})^{\text{fib}}$$

in RelCat. As explained above, we obtain morphisms

$$U^{\text{fib}} : (A_\infty \text{Cat}, W_{\text{qe}}^{A_\infty})^{\text{fib}} \rightleftarrows (\text{DGCat}, W_{\text{qe}}^{\text{DG}}) : i^{\text{fib}}$$

such that $i^{\text{fib}} = j_{A_\infty} i$ and $U^{\text{fib}} j_{A_\infty} = U$. Both i^{fib} and U^{fib} are weak equivalences between fibrant objects in RelCat.

Recall that a right Quillen functor (= right adjoint in a Quillen adjunction) always preserves fibrant objects and weak equivalences between fibrant objects.

Corollary 4.3. *Let $N_\xi : \text{RelCat} \rightarrow \text{sSpc}$ be the right Quillen functor appearing in Theorem 3.1. Then $N_\xi(\text{DGCat}, W_{\text{qe}}^{\text{DG}})$ and $N_\xi(A_\infty \text{Cat}, W_{\text{qe}}^{A_\infty})^{\text{fib}}$ are complete Segal spaces, and $N_\xi(i^{\text{fib}})$ and $N_\xi(U^{\text{fib}})$ are equivalences between them.*

To pass to quasicategories we use the following result of Joyal–Tierney [JT07].

Theorem 4.4 ([JT07]). *There is a pair of adjoint functors $p_1^* : \text{sSet} \rightleftarrows \text{sSpc} : i_1^*$, with p_1^* left adjoint to i_1^* , which is a Quillen equivalence between the Joyal model structure on sSet and Rezk’s complete Segal space model structure on sSpc.*

By composing the Quillen equivalence in Theorem 4.4 with the one in Theorem 3.1, we obtain a Quillen equivalence between RelCat with the Barwick–Kan model structure and sSet with the Joyal model structure (such that RelCat is the domain of the right adjoint). This equivalence was also explicitly described by Barwick–Kan [BK11].

Corollary 4.5. *The simplicial sets $i_1^* N_\xi(\text{DGCat}, W_{\text{qe}}^{\text{DG}})$ and $i_1^* N_\xi(A_\infty \text{Cat}, W_{\text{qe}}^{A_\infty})^{\text{fib}}$ are quasicategories, and $i_1^* N_\xi(i^{\text{fib}})$ and $i_1^* N_\xi(U^{\text{fib}})$ are equivalences between them.*

Let us now summarize what we have achieved. Corollary 4.5 is a lift of Theorem 1.2 to the level of quasicategories. Indeed, all of the functors that we are applying preserve the homotopy category $C[W^{-1}]$ of a relative category (C, W) . The homotopy category of the quasicategory $i_1^*N_\xi(\mathrm{DGCat}, W_{\mathrm{qe}}^{\mathrm{DG}})$ is $\mathrm{DGCat}[(W_{\mathrm{qe}}^{\mathrm{DG}})^{-1}]$, and likewise the homotopy category of $i_1^*N_\xi(\mathrm{A}_\infty\mathrm{Cat}, W_{\mathrm{qe}}^{\mathrm{A}_\infty})^{\mathrm{fib}}$ is $\mathrm{A}_\infty\mathrm{Cat}[(W_{\mathrm{qe}}^{\mathrm{A}_\infty})^{-1}]$. So the equivalence of categories in Theorem 1.2 follows from Corollary 4.5 by taking the homotopy category.

5. Other quasicategory models

Since all models of $(\infty, 1)$ -categories are related, sometimes in several ways, there are other “paths” from RelCat to the category of quasicategories.

One path goes through the simplicial localization. Given a relative category (C, W) , we form $L^H(C, W)$. This may not be fibrant (the morphism simplicial sets may not be Kan complexes), so we should take a fibrant replacement $(L^H(C, W))^{\mathrm{fib}}$ for the Bergner model structure.² Then we apply the homotopy coherent nerve N_c . Again this is a right Quillen functor in a Quillen equivalence between the Bergner model structure on simplicially enriched categories and the Joyal model structure on sSet , so $N_c(L^H(C, W))^{\mathrm{fib}}$ is a quasicategory.

We remark that the quasicategory $N_c(L^H(C, W))^{\mathrm{fib}}$ is equivalent to $i_1^*N_\xi(C, W)^{\mathrm{fib}}$, the version considered previously [Mei16, p. 3277], [SP12].

From Corollary 2.5, we have a pair of Dwyer-Kan equivalences $L^H(i)$ and $L^H(U)$ relating the simplicial localizations $L^H(\mathrm{DGCat}, W_{\mathrm{qe}}^{\mathrm{DG}})$ and $L^H(\mathrm{A}_\infty\mathrm{Cat}, W_{\mathrm{qe}}^{\mathrm{A}_\infty})$. After fibrant replacement, we obtain

$$L^H(U)^{\mathrm{fib}} : L^H(\mathrm{A}_\infty\mathrm{Cat}, W_{\mathrm{qe}}^{\mathrm{A}_\infty})^{\mathrm{fib}} \rightleftarrows L^H(\mathrm{DGCat}, W_{\mathrm{qe}}^{\mathrm{DG}})^{\mathrm{fib}} : L^H(i)^{\mathrm{fib}},$$

where both morphisms are again weak equivalences for the Bergner model structure.

Corollary 5.1. *The maps $N_cL^H(i)^{\mathrm{fib}}$ and $N_cL^H(U)^{\mathrm{fib}}$ are equivalences between the quasicategories $N_c(L^H(\mathrm{DGCat}, W_{\mathrm{qe}}^{\mathrm{DG}}))^{\mathrm{fib}}$ and $N_c(L^H(\mathrm{A}_\infty\mathrm{Cat}, W_{\mathrm{qe}}^{\mathrm{A}_\infty}))^{\mathrm{fib}}$.*

It is also possible to take a relative category (C, W) and directly produce a quasicategorical localization by localizing the nerve $N(C)$ with respect to the collection of edges W . Let us denote such a localization by $N(C)_{(W)}$. An explicit construction shown to me by Rezk consists of first taking the pushout of the diagram of simplicial sets

$$\coprod_{w \in W} J \leftarrow \coprod_{w \in W} \Delta^1 \rightarrow N(C),$$

where J is the walking isomorphism, and then taking fibrant replacement for the Joyal model structure. However, the object $N(C)_{(W)}$ is really intended to be characterized by a quasicategorical universal property, namely: for any quasicategory D , the map $\mathrm{Fun}(N(C)_{(W)}, D) \rightarrow \mathrm{Fun}(N(C), D)$ induced by $N(C) \rightarrow N(C)_{(W)}$ is an equivalence of quasicategories from $\mathrm{Fun}(N(C)_{(W)}, D)$ to the full sub-quasicategory of $\mathrm{Fun}(N(C), D)$ spanned by functors that take edges in W to equivalences in D .

²Two explicit constructions are given by applying to each morphism space either $\mathrm{Sing}|\cdot|$ or Kan’s Ex^∞ functor.

Given a relative functor $F : (C_1, W_1) \rightarrow (C_2, W_2)$, we obtain a map of nerves $N(F) : N(C_1) \rightarrow N(C_2)$. Since $F(W_1) \subseteq W_2$, this extends to a map of localizations $N(F)_{\text{loc}} : N(C_1)_{(W_1)} \rightarrow N(C_2)_{(W_2)}$.

The results of Hinich [Hin16, Mar21] show that the previously considered versions $i_1^* N_\xi(C, W)^{\text{fib}}$ and $N_c(L^H(C, W))^{\text{fib}}$ are equivalent to $N(C)_{(W)}$. More precisely, given a relative category (C, W) , there is a map of simplicial sets

$$N(C) \rightarrow N_c(L^H(C, W))^{\text{fib}},$$

which extends to a map $N(C)_{(W)} \rightarrow N_c(L^H(C, W))^{\text{fib}}$, and this latter map is an equivalence of quasicategories. These equivalences are natural with respect to relative functors.

Corollary 5.2. *The maps $N(i)_{\text{loc}}$ and $N(U)_{\text{loc}}$ are equivalences between the quasi-categorical localizations $N(\text{DGCat})_{(W_{\text{qe}}^{\text{DG}})}$ and $N(\text{A}_\infty \text{Cat})_{(W_{\text{qe}}^{\text{A}_\infty})}$.*

6. The problem of fibrancy of $(\text{A}_\infty \text{Cat}, W_{\text{qe}}^{\text{A}_\infty})$

We conclude with a few remarks about the problem raised in Remark 4.2, namely the question of whether $(\text{A}_\infty \text{Cat}, W_{\text{qe}}^{\text{A}_\infty})$ is a fibrant object in RelCat . If this were true, it would slightly simplify some of the preceding constructions.

The main result of Meier [Mei16] is that, in order for (C, W) to be fibrant in RelCat , it is sufficient that (C, W) admits the structure of a *fibration category*. This means that in addition to the class of weak equivalences W , one must also specify a subcategory $\text{Fib} \subset C$ of *fibrations*, such that a certain list of axioms is satisfied. See [Mei16, Definition 3.1] for the precise definition.

So in order to show that $(\text{A}_\infty \text{Cat}, W_{\text{qe}}^{\text{A}_\infty})$ is fibrant, we would need a notion of fibration of A_∞ -categories. A somewhat naive guess is the following: An A_∞ -functor $F : A \rightarrow B$ between A_∞ -categories is said to be a *fibration* if the following conditions are satisfied:

1. for every pair of objects X, Y in A and each integer p , the first component $F^1 : \text{hom}_A^p(X, Y) \rightarrow \text{hom}_B^p(FX, FY)$ is a surjective map of vector spaces, and
2. $H^0(F) : H^0(A) \rightarrow H^0(B)$ is an isofibration.

We recall that an *isofibration* is a functor $F : C \rightarrow D$ between ordinary categories with the following property:

- If $\phi : F(X) \rightarrow Z$ is an isomorphism in D whose source object is in the image of F , then there is an isomorphism $\tilde{\phi} : X \rightarrow \tilde{Z}$ in C such that $F\tilde{\phi} = \phi$.

This definition of fibration of A_∞ -categories is directly analogous both to the definition of fibration of simplicially enriched categories for the Bergner model structure, and to the definition of fibration of DG categories for the Tabuada model structure.³

The problem is now to verify the axioms (F1)–(F5) of [Mei16, Definition 3.1] using this notion of fibration. Axioms (F1), (F2), and (F3) are straightforward (note that W and Fib both contain all isomorphisms and that all objects are fibrant). The

³Even so, other definitions of “fibration of A_∞ -categories” are possible. A less naive proposal is to take from [LH03] the fibrations of conilpotent DG coalgebras, generalize that concept to DG cocategories somehow, and then take the preimage of that class under the bar construction.

factorization axiom (F5) follows from axiom (F4) and the existence of path objects in $A_\infty \text{Cat}$ [Dri04, Appendix D].

The trickiest axiom turns out to (F4); in the present context, it is equivalent to the assertion that the pullback of a fibration of A_∞ -categories along any A_∞ -functor exists and is a fibration, and also that the pullback of an acyclic fibration (fibration which is also a weak equivalence) is an acyclic fibration. I do not know any proof of this assertion.

In fact the mere existence of such pullbacks is a nontrivial problem, because the category $A_\infty \text{Cat}$ is **not complete**. Indeed, the category $A_\infty \text{Alg}$ of A_∞ -algebras (A_∞ -categories with a single object) is not complete either, as was already well-appreciated by Lefèvre-Hasegawa [LH03]. See [COS19, §1.5] for a simple counterexample. The reason is elementary: an A_∞ -functor or A_∞ -morphism of A_∞ -algebras is precisely not “a map of underlying vector spaces that preserves the operations”—it only preserves the operations up to specified homotopy. This means that the naive attempt to form the equalizer of a parallel pair of A_∞ -morphisms $F, G : A \rightarrow B$ —take the equalizer of F^1 and G^1 in vector spaces and then define A_∞ -operations on the result—fails catastrophically. Instead, one must take the equalizer in the category of DG cocategories or DG coalgebras, and then prove that the resulting object lies in the essential image of the bar construction.

In the case of $A_\infty \text{Alg}$, one of the main results of [LH03] is that the category of conilpotent DG coalgebras admits a model structure such that the subcategory of fibrant objects is equivalent to $A_\infty \text{Alg}$. As explained in [Val20], it follows that $A_\infty \text{Alg}$ is a “category of fibrant objects,” implying that it is indeed a fibration category in the sense of [Mei16], where the weak equivalences are the A_∞ -quasi-isomorphisms and the fibrations are degreewise surjections.

There is one more case that deserves mention. Let O be set, and let $A_\infty \text{Cat}_O$ be the category whose objects are small A_∞ -categories with object set O , and whose morphisms are A_∞ -functors that are identity on O . As stated in [COS19], the results of [LH03] apply *mutatis mutandis* to this case. Thus the problem of extending the results of [LH03] to $A_\infty \text{Cat}$ is the simple fact that a functor, even if it is an equivalence, need not be a bijection on objects.

References

- [Ber07] J. E. Bergner. A model category structure on the category of simplicial categories. *Trans. Amer. Math. Soc.*, 359(5):2043–2058, 2007.
- [Ber10] Julia E. Bergner. A survey of $(\infty, 1)$ -categories. In *Towards higher categories*, volume 152 of *IMA Vol. Math. Appl.*, pages 69–83. Springer, New York, 2010.
- [Ber18] J. E. Bergner. *The homotopy theory of $(\infty, 1)$ -categories*, volume 90 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2018.
- [BK11] C. Barwick and D. M. Kan. A Thomason-like Quillen equivalence between quasi-categories and relative categories, 2011.

- [BK12a] C. Barwick and D. M. Kan. A characterization of simplicial localization functors and a discussion of DK equivalences. *Indag. Math. (N.S.)*, 23(1-2):69–79, 2012.
- [BK12b] C. Barwick and D. M. Kan. Relative categories: another model for the homotopy theory of homotopy theories. *Indag. Math. (N.S.)*, 23(1-2):42–68, 2012.
- [COS19] A. Canonaco, M. Ornaghi, and P. Stellari. Localizations of the category of A_∞ categories and internal Homs. *Doc. Math.*, 24:2463–2492, 2019.
- [COS24] A. Canonaco, M. Ornaghi, and P. Stellari. Localizations of the category of A_∞ categories and internal Homs over a ring. [arXiv:2404.06610](https://arxiv.org/abs/2404.06610).
- [DK80a] W. G. Dwyer and D. M. Kan. Calculating simplicial localizations. *J. Pure Appl. Algebra*, 18(1):17–35, 1980.
- [DK80b] W. G. Dwyer and D. M. Kan. Simplicial localizations of categories. *J. Pure Appl. Algebra*, 17(3):267–284, 1980.
- [Dri04] V. Drinfeld. DG quotients of DG categories. *J. Algebra*, 272(2):643–691, 2004.
- [Hin16] V. Hinich. Dwyer-Kan localization revisited. *Homology Homotopy Appl.*, 18(1):27–48, 2016.
- [JT07] A. Joyal and M. Tierney. Quasi-categories vs Segal spaces. In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pages 277–326. Amer. Math. Soc., Providence, RI, 2007.
- [LH03] K. Lefèvre-Hasegawa. *Sur les A_∞ -catégories*. Thèse de doctorat, Université Denis Diderot–Paris 7, November 2003.
- [Mar21] A. Marino. Is hammock localization a localization in the sense of Lurie? MathOverflow, 2021. Available at <https://mathoverflow.net/questions/395468/is-hammock-localization-a-localization-in-the-sense-of-lurie>.
- [Mei16] L. Meier. Fibration categories are fibrant relative categories. *Algebr. Geom. Topol.*, 16(6):3271–3300, 2016.
- [Rez01] C. Rezk. A model for the homotopy theory of homotopy theory. *Trans. Amer. Math. Soc.*, 353(3):973–1007, 2001.
- [SP12] C. Schommer-Pries. Does the classification diagram localize a category with weak equivalences? MathOverflow, 2012. Available at <http://tinyurl.com/q-93139>.
- [Tab05] G. Tabuada. Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories. *C. R. Math. Acad. Sci. Paris*, 340(1):15–19, 2005.
- [Val20] B. Vallette. Homotopy theory of homotopy algebras. *Ann. Inst. Fourier (Grenoble)*, 70(2):683–738, 2020.

James Pascaleff jpascale@illinois.edu

Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL, 61801, United States