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SZCZARBA'S TWISTING COCHAIN IS COMULTIPLICATIVE

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Abstract

We prove that Szczarba's twisting cochain is comultiplicative. In particular, the induced map from the cobar construction $\mathbf{\Omega}\, C(X)$ of the chains on a 1-reduced simplicial set X to $C(GX)$, the chains on the Kan loop group of X, is a quasi-isomorphism of dg bialgebras. We also show that Szczarba's twisted shuffle map is a dgc map connecting a twisted Cartesian product with the associated twisted tensor product. This gives a natural dgc model for fibre bundles. We apply our results to finite covering spaces and to the Serre spectral sequence.

1. Introduction

Let X be a simplicial set and G a simplicial group. Given a twisting function

$$\tau\colon X_{>0} \to G, \tag{1.1}$$

Szczarba [26] has constructed an explicit twisting cochain

$$t\colon C(X) \to C(G). \tag{1.2}$$

In [10, Thm. 6.2] we showed that it agrees with the twisting cochain obtained by Shih [25, §II.1] using homological perturbation theory if one uses a slightly modified version of the Eilenberg–Mac Lane homotopy.

In this paper we consider the associated map of differential graded algebras (dgas)

$$\mathbf{\Omega}\, C(X) \to C(G) \tag{1.3}$$

where $\mathbf{\Omega}\, C(X)$ is the reduced cobar construction of the differential graded coalgebra (dgc) $C(X)$.

The diagonal of $C(G)$ is compatible with the multiplication, meaning that $C(G)$ is actually a dg bialgebra. By work of Baues [1] and Gerstenhaber–Voronov [11, Cor. 6], the same holds true for $\mathbf{\Omega}\, C(X)$. Here the diagonal can be expressed via the homotopy Gerstenhaber structure of $C(X)$, that is, in terms of certain cooperations

$$E^k\colon C(X) \to C(X)^{\otimes k} \otimes C(X) \tag{1.4}$$

with $k \geqslant 0$, see Section 3.

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The question arises as to whether the dga map (1.3) is comultiplicative, meaning compatible with the diagonals. Hess–Parent–Scott–Tonks [**14**, Thm. 4.4] showed that for 1-reduced X this is true up to homotopy in a strong sense, and they observed that it holds on the nose up to degree 3. In the case where X is a simplicial suspension the comultiplicativity was established by Hess–Parent–Scott [**13**, Thm. 4.11]. Our main result says that it is true in general.

Theorem 1.1. *Let $X \neq \varnothing$ be a simplicial set, and let G and τ be as above. The dga map $\mathbf{\Omega}\, C(X) \to C(G)$ induced by Szczarba's twisting cochain t is comultiplicative.*

This applies in particular to the canonical twisting cochain $\tau_X \colon X_{>0} \to GX$ of a 1-reduced simplicial set where GX denotes the Kan loop group of X. This gives the following.

Corollary 1.2. *For 1-reduced X, the map $\mathbf{\Omega}\, C(X) \to C(GX)$ induced by Szczarba's twisting cochain t is a quasi-isomorphism of dg bialgebras.*

Using Hess–Tonks' extended cobar construction $\tilde{\mathbf{\Omega}}\, C(X)$, we generalize this to reduced simplicial sets in Proposition 7.1. After the prepublication of this article, Medina-Mardones and Rivera showed that $\tilde{\mathbf{\Omega}}\, C(X)$ and $C(GX)$ are quasi-isomorphic as E_∞-coalgebras [**19**, Thm. 2]. This quasi-isomorphism involves a zigzag, however, not the extension $\tilde{\mathbf{\Omega}}\, C(X) \to C(GX)$ of the map above.

Given a left G-space F, one can consider the twisted Cartesian product $X \times_\tau F$ as well as the twisted tensor product $C(X) \otimes_t C(F)$. Dualizing a construction due to Kadeishvili–Saneblidze [**17**], we turn the latter into a dgc. Szczarba also defined a twisted shuffle map

$$\psi \colon C(X) \otimes_t C(F) \to C(X \times_\tau F) \tag{1.5}$$

and proved that it is a quasi-isomorphism of complexes [**26**]. In [**10**, Prop. 7.1] we showed that ψ is in fact a morphism of left $C(X)$-comodules, and also of right $C(G)$-modules in the case $F = G$. We strengthen the first aspect as follows.

Theorem 1.3. *Szczarba's twisted shuffle map ψ is a quasi-isomorphism of dgcs. In particular, the twisted tensor product $C(X) \otimes_t C(F)$ is a dgc model for $X \times_\tau F$.*

Using cubical chains, Kadeishvili–Saneblidze [**17**, Sec. 6] have previously obtained a dgc model for fibre bundles with simply connected base.[1]

Content and structure of this paper are as follows: We review background material in Section 2 and homotopy Gerstenhaber coalgebras in Section 3. After establishing a purely combinatorial result in Section 4 and discussing the Szczarba maps in Section 5 we prove Theorem 1.1 in Section 6. The generalization of Corollary 1.2 mentioned above appears in Section 7. In Section 8 we explain how homotopy Gerstenhaber coalgebra structures give rise to dgc structures on twisted tensor products, and in Section 9 we prove Theorem 1.3. In Section 10 we compare Szczarba's twisted tensor product with a similar one due to Shih [**25**]. In Section 11 we deduce from our results a dga model for finite covering spaces as well as certain spectral sequences

[1] The assumption of simple connectedness is omitted in the statement of [**17**, Thm. 6.1], but used in the proof. That proof is actually problematic because it refers to a map of monoidal cubical sets $\operatorname{Sing}^I \Omega Y \to \operatorname{Sing}^I G$ whose existence is doubtful.

studied by Papadima–Suciu [**21**] and Rüping–Stephan [**24**]. In a similar vein, we obtain the (co)multiplicative structure of the (co)homological Serre spectral sequence in Section 12. In the first appendix we relate our diagonal on $\mathbf{\Omega}\, C(X)$ to the one defined by Baues [**1**] for 1-reduced X. In the second we fill a gap in the literature by showing that Szczarba's twisting cochain (1.2) and twisted shuffle map (1.5) are in fact well-defined on normalized chain complexes.

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2. Preliminaries

2.1. Generalities

We write

$$[n] = \{0, \dots, n\} \quad \text{and} \quad \underline{n} = \{1, \dots, n\}$$

for $n \geqslant 0$. We work over a commutative ring $\Bbbk$ with unit; all tensor products and chain complexes are over $\Bbbk$. Unless specified otherwise, all chain complexes are homological. The degree of an element c of a graded module C is denoted by $|c|$. We write 1_C for the identity map of C and

$$T_{B,C} \colon B \otimes C \to C \otimes B, \qquad b \otimes c \mapsto (-1)^{|b||c|}\, c \otimes b \tag{2.1}$$

for the transposition of factors in a tensor product of graded modules. The suspension and desuspension operators are denoted by $\mathbf{s}$ and $\mathbf{s}^{-1}$, respectively. We systematically use the Koszul sign rule, compare [**9**, Secs. 2.2 & 2.3].

For clarity, we sometimes write 1_A for the unit of a dga A and 1_C for the unit of a coaugmented dgc C. A dg bialgebra is a chain complex A that is both a dga and a dgc in such a way that each pair of structure maps are morphisms with respect to the other structure.

We write $C(X)$ for the normalized chains on a simplicial set X. We also write $\tilde{\partial}$ for the last face map, that is, $\tilde{\partial} x = \partial_n x$ for $x \in X_n$ with $n \geqslant 1$.

2.2. The cobar construction

Let C be a dgc with coaugmentation $\iota \colon \Bbbk \hookrightarrow C$, so that $C = \Bbbk \oplus \bar{C}$ where $\bar{C} = \ker \varepsilon$. The (reduced) cobar construction of C is

$$\mathbf{\Omega}\, C = \bigoplus_{k \geqslant 0} \mathbf{\Omega}_k\, C \qquad \text{where} \qquad \mathbf{\Omega}_k\, C = (\mathbf{s}^{-1}\bar{C})^{\otimes k}, \tag{2.2}$$

compare [**16**, Sec. II.3] or [**1**, §0]. We write elements of $\mathbf{\Omega}\, C$ in the form

$$\langle c_1 | \dots | c_k \rangle = \mathbf{s}^{-1}\, c_1 \otimes \dots \otimes \mathbf{s}^{-1}\, c_k \tag{2.3}$$

with $c_1, \dots, c_k \in \bar{C}$. The cobar construction is an augmented dga with concatenation as product and unit $1 = \langle\rangle \in \mathbf{\Omega}_0\, C = \Bbbk$. The differential and augmentation are determined by

$$d\,\langle c\rangle = -\langle dc\rangle + (\mathbf{s}^{-1}\otimes\mathbf{s}^{-1})\,\bar\Delta\,c \qquad\text{and}\qquad \varepsilon(\langle c\rangle)=0 \tag{2.4}$$

for $\langle c\rangle\in\mathbf{\Omega}_1\,C$, where

$$\bar\Delta\colon C\xrightarrow{\Delta} C\otimes C\to\bar C\otimes\bar C \tag{2.5}$$

is the reduced diagonal.

2.3. Twisting cochains

Let C be a coaugmented dgc and A an augmented dga. Recall that the complex $\operatorname{Hom}(C,A)$ is an augmented dga via

$$d(f) = d_A\,f-(-1)^{|f|}\,f\,d_C, \qquad 1_{\operatorname{Hom}(C,A)} = \iota_A\,\varepsilon_C \tag{2.6}$$

$$f\cup g = \mu_A\,(f\otimes g)\,\Delta_C, \qquad \varepsilon(f)=\varepsilon_A\,f\,\iota_C(1) \tag{2.7}$$

for $f,\ g\in\operatorname{Hom}(C,A)$. Here $\iota_A\colon\Bbbk\to A$ is the unit map, ι_C is the coaugmentation of C, and ε_C and ε_A are the augmentations of C and A, respectively.

A twisting cochain is a map $t\in\operatorname{Hom}(C,A)$ of degree -1 (in the homological setting) such that

$$t\,\iota_C=0, \qquad \varepsilon_A\,t=0, \qquad d(t)=t\cup t. \tag{2.8}$$

It canonically induces the morphism of dgas

$$\mathbf{\Omega}\,C\to A, \quad \langle c_1|\dots|c_k\rangle\mapsto t(c_1)\cdots t(c_k). \tag{2.9}$$

For example, the canonical twisting cochain

$$t_C\colon C\to\mathbf{\Omega}\,C, \qquad c\mapsto\langle\bar c\rangle\in\mathbf{\Omega}_1 C \tag{2.10}$$

corresponds to the identity map on $\mathbf{\Omega}\,C$. Here we have written $\bar c = c-\iota\,\varepsilon(c)$ for the component of c in $\bar C$.

2.4. The shuffle map

We recall the definition of the shuffle map for an arbitrary number of factors. Given $k\geqslant 1$ non-negative integers $q_1,\ \dots,\ q_k$ with sum q, a $(q_1,\dots,q_k)$-shuffle is a partition $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k)$ of the set $[q-1]$. We write $(-1)^{(\boldsymbol{\alpha})}=(-1)^{(\alpha_1,\dots,\alpha_k)}$ for its signature and $\operatorname{Shuff}(q_1,\dots,q_k)$ for the set of all such shuffles. Observe that for $k=1$ there is only one (q)-shuffle.

For simplicial sets $X_1,\ \dots,\ X_k$ the shuffle map is given by

$$\begin{aligned}\nabla_{X_1,\dots,X_k}\colon C_{q_1}(X_1)\otimes\cdots\otimes C_{q_k}(X_k)&\to C_q(X_1\times\cdots\times X_k),\\ x_1\otimes\cdots\otimes x_k&\mapsto\sum_{\boldsymbol{\alpha}}(-1)^{(\boldsymbol{\alpha})}\,(s_{\bar\alpha_1}x_1,\dots,s_{\bar\alpha_k}x_k)\end{aligned} \tag{2.11}$$

where the sum is over all $\boldsymbol{\alpha}\in\operatorname{Shuff}(q_1,\dots,q_k)$, and $\bar\alpha_s=[q-1]\setminus\alpha_s$ for $1\leqslant s\leqslant k$.

Using the shuffle map, one turns the chain complex of a simplicial group G into a dga. For $m\geqslant 0$, the m-fold iterated multiplication is given by

$$C(G)^{\otimes m}\xrightarrow{\nabla_{G,\dots,G}} C(G\times\cdots\times G)\xrightarrow{\mu^{[m]}_*} C(G) \tag{2.12}$$

where $\mu^{[m]}\colon G\times\cdots\times G\to G$ is the m-fold product map. This gives the identity map of $C(G)$ for $m=1$ and the unit map $\Bbbk\hookrightarrow C(G)$ for $m=0$.

2.5. Twisted Cartesian products

Twisted Cartesian products are simplicial versions of fibre bundles, compare [**18**, Sec. 18] or [**26**, Sec. 1]. More precisely, let X and F be simplicial sets, and assume that the simplicial group G acts on F from the left. The twisted Cartesian product $X \times_\tau F$ differs from the usual Cartesian product $X \times F$ only by the zeroeth face map, which is

$$\partial_0\,(x, y) = \big(\partial_0\, x, \tau(x)\,\partial_0\, y\big). \tag{2.13}$$

The twisting function

$$\tau \colon X_{>0} \to G \tag{2.14}$$

is of degree -1 and for any $x \in X$ of dimension $n > 0$ satisfies

$$\partial_0\,\tau(x) = \tau(\partial_0\, x)^{-1}\,\tau(\partial_1\, x) \qquad \text{if } n > 1, \tag{2.15}$$

$$\partial_k\,\tau(x) = \tau(\partial_{k+1}\, x) \qquad \text{for } 0 < k < n, \tag{2.16}$$

$$s_k\,\tau(x) = \tau(s_{k+1}\, x) \qquad \text{for } 0 \leqslant k < n, \tag{2.17}$$

and for any $x \in X$ of dimension $n \geqslant 0$ also

$$\tau(s_0\, x) = 1 \in G_n, \tag{2.18}$$

see [**26**, eq. (1.1)], [**18**, Def. 18.3] or [**15**, Sec. 1.3].

2.6. Interval cut operations

Let k, $l \geqslant 0$, and let $u \colon \underline{k+l} \to \underline{k}$ be a surjection such that $u(i) \neq u(i+1)$ for all $0 \leqslant i < k + l$. Berger–Fresse [**3**, Sec. 2] have associated to u an *interval cut operation*

$$AW_u \colon C(X) \to C(X)^{\otimes k}, \tag{2.19}$$

natural in the simplicial set X. On an n-simplex $x \in X$, it is given by

$$AW_u\, x = \sum_{\boldsymbol{p}} (-1)^{\mathrm{pos}(\boldsymbol{p}) + \mathrm{perm}(\boldsymbol{p})}\, x_1^{\boldsymbol{p}} \otimes \cdots \otimes x_k^{\boldsymbol{p}}. \tag{2.20}$$

Here the sum runs over all decompositions $\boldsymbol{p} = (0 = p_0, p_1, \ldots, p_{k+l} = n)$ of $[n]$ into $k + l$ intervals. If we think of these intervals as being labelled via u, then

$$x_s^{\boldsymbol{p}} = x(p_{i_1-1}, \ldots, p_{i_1}, p_{i_2-1}, \ldots, p_{i_2}, \ldots, p_{i_m-1}, \ldots, p_{i_m}) \tag{2.21}$$

where $i_1, \ldots, i_m$ enumerate the intervals with label s. We refer to [**3**, §2.2.4] for the definitions of the position sign exponent $\mathrm{pos}(\boldsymbol{p})$ and the permutation sign exponent $\mathrm{perm}(\boldsymbol{p})$.

Whenever we talk about the *length* of an interval $[p_{i-1}, \ldots, p_i]$ in this paper, we always mean its naive length $p_i - p_{i-1}$, not the possibly different length defined in [**3**, §2.2.3] to compute the position and permutation sign exponents.

3. Homotopy Gerstenhaber coalgebras

Homotopy Gerstenhaber coalgebras (hgcs) are defined such that their duals are homotopy Gerstenhaber algebras (hgas), see Remark 3.2 below and also [**17**, p. 223].

More precisely, an *hgc* is a coaugmented dgc C together with a family of cooperations

$$E^k\colon C \to C^{\otimes k} \otimes C \tag{3.1}$$

for $k \geqslant 0$ such that

$$E^0 = 1_C, \tag{3.2}$$

$$\operatorname{im} E^k \subset \bar{C}^{\otimes k} \otimes \bar{C} \qquad \text{for } k > 0, \tag{3.3}$$

$$E^k(c) = 0 \qquad \text{for } |c| < k. \tag{3.4}$$

Recall that $\bar{C} = \ker \varepsilon$ is the augmentation ideal and $\bar{c} = c - \iota\,\varepsilon(c)$ the component of c in $\bar{C}$. There are further conditions on the maps E^k. Defining

$$\mathbf{E}^k\colon C \to (\mathbf{s}^{-1}\,\bar{C})^{\otimes k} \otimes \mathbf{s}^{-1}\,\bar{C} = \mathbf{\Omega}_k C \otimes \mathbf{\Omega}_1 C \subset \mathbf{\Omega}\, C \otimes \mathbf{\Omega}\, C \tag{3.5}$$

for $k \geqslant 0$ via

$$\mathbf{s}^{\otimes(k+1)}\,\mathbf{E}^k(c) = E^k(\bar{c}), \tag{3.6}$$

the assignment

$$\begin{aligned} \mathbf{E}\colon C &\to \mathbf{\Omega}\, C \otimes \mathbf{\Omega}\, C, \\ c &\mapsto \langle \bar{c} \rangle \otimes 1 + \sum_{k=0}^{\infty} \mathbf{E}^k(c) = \langle \bar{c} \rangle \otimes 1 + 1 \otimes \langle \bar{c} \rangle + \sum_{k=1}^{\infty} \mathbf{E}^k(c) \end{aligned} \tag{3.7}$$

is well-defined by (3.4). We require $\mathbf{E}$ to be a twisting cochain and the associated dga map

$$\Delta\colon \mathbf{\Omega}\, C \to \mathbf{\Omega}\, C \otimes \mathbf{\Omega}\, C, \qquad \big\langle\, c_1 \,\big|\, \dots \,\big|\, c_k \,\big\rangle \mapsto \mathbf{E}(c_1) \cdots \mathbf{E}(c_k) \tag{3.8}$$

to be coassociative, so that $\mathbf{\Omega}\, C$ becomes a dg bialgebra.

It will be convenient to rephrase these conditions in terms of the function

$$\begin{aligned} \mathfrak{E}\colon C &\to \mathbf{\Omega}\, C \otimes C, \\ c &\mapsto (1 \otimes p_C)\,\mathbf{E}(c) + 1 \otimes \iota\,\varepsilon(c) = 1 \otimes c + (1 \otimes p_C) \sum_{k=1}^{\infty} \mathbf{E}^k(c) \end{aligned} \tag{3.9}$$

of degree 0 where

$$p_C\colon \mathbf{\Omega}\, C \longrightarrow \mathbf{\Omega}_1 C = \mathbf{s}^{-1} \bar{C} \xrightarrow{\ \mathbf{s}\ } \bar{C} \hookrightarrow C \tag{3.10}$$

is the composition of the canonical projection, the suspension map and the canonical inclusion. Like the suspension map, p_C is of degree 1.

Lemma 3.1. *Let* $\mathbf{E}$ *and* $\mathfrak{E}$ *be as in* (3.7) *and* (3.9)*.*

(i) *That* $\mathbf{E}$ *is a twisting cochain is equivalent to the two identities*

$$d(\mathfrak{E}) = (\mu_{\mathbf{\Omega}C} \otimes 1_C)(t_C \otimes \mathfrak{E})\,\Delta_C - (\mu_{\mathbf{\Omega}C} \otimes 1_C)\,(1_{\mathbf{\Omega}C} \otimes T_{C,\mathbf{\Omega}C})\,(\mathfrak{E} \otimes t_C)\,\Delta_C,$$

$$(1_{\mathbf{\Omega}C} \otimes \Delta_C)\,\mathfrak{E} = (\mu_{\mathbf{\Omega}C} \otimes 1_C \otimes 1_C)\,(1_{\mathbf{\Omega}C} \otimes T_{C,\mathbf{\Omega}C} \otimes 1_C)\,(\mathfrak{E} \otimes \mathfrak{E})\,\Delta_C.$$

(ii) *Assume that* $\mathbf{E}$ *is a twisting cochain. The coassociativity of the diagonal* (3.8) *then is equivalent to the formula*

$$(\Delta_{\mathbf{\Omega}C} \otimes 1_C)\,\mathfrak{E} = (1_{\mathbf{\Omega}C} \otimes \mathfrak{E})\,\mathfrak{E}.$$

Proof. For the first part, we note that both sides of the twisting cochain condition $d(\mathbf{E}) = \mathbf{E} \cup \mathbf{E}$ only have components in $\mathbf{\Omega}\, C \otimes \mathbf{\Omega}_l\, C$ with $l \leqslant 2$. We project onto these components separately. The projections for $l = 0$ are always equal. A direct calculation shows that the projections for $l = 1$ and $l = 2$ correspond to the two identities for $\mathfrak{E}$ given above. It is helpful to distinguish the two cases $c = 1$ and $c \in \bar{C}$, and in the second one to split up the diagonal as $\Delta\, c = c \otimes 1 + 1 \otimes c + \bar{\Delta}\, c$ where $\bar{\Delta}$ is the reduced diagonal (2.5). For the first identity one also uses $d(p_C) = 0$.

The second claim follows similarly by projecting the coassociativity condition

$$(\Delta \otimes 1)\, \Delta = (1 \otimes \Delta)\, \Delta \quad \text{to} \quad \mathbf{\Omega}\, C \otimes \mathbf{\Omega}\, C \otimes \mathbf{\Omega}_1\, C. \qquad \square$$

Remark 3.2. Let $A = \operatorname{Hom}(C, \Bbbk)$ be the augmented dga dual to the coaugmented dgc C. For $k \geqslant 0$ define the transpose

$$E_k \colon A^{\otimes k} \otimes A \to A \tag{3.11}$$

of the cooperation E^k by

$$\left\langle E_k(a_1, \ldots, a_k; b), c \right\rangle = (-1)^{k(|a_1| + \cdots + |a_k| + |b|)} \left\langle a_1 \otimes \cdots \otimes a_k \otimes b, E^k(c) \right\rangle \tag{3.12}$$

for $c \in C$, compare [**9**, eq. (4)]. The operations E_k then form an hga structure on A that satisfies the analogues of the identities stated in [**8**, Sec. 6.1]. Note that in [**8**] operations of the form $E_k(a; b_1, \ldots, b_k)$ are used; see [**8**, Rem. 6.1] for their relation to the braces used by Gerstenhaber–Voronov [**11**]. The explicit signs given there remain unchanged, except for an additional overall minus sign in the formula for $d(E_k)$.

Let $t \colon C \to A$ be a twisting cochain, where C is an hgc and A a dg bialgebra. We say that t is *comultiplicative* if the induced dga map $\mathbf{\Omega}\, C \to A$ is a morphism of dgcs and therefore of dg bialgebras. This definition is dual to Kadeishvili–Saneblidze's notion of a multiplicative twisting cochain [**17**, Def. 7.2]. For example, the canonical twisting cochain $t_C \colon C \to \mathbf{\Omega}\, C$ is comultiplicative.

The normalized chains $C(X)$ on a simplicial set $X \neq \varnothing$ form an hgc in a natural way, for any coaugmentation $\Bbbk \hookrightarrow C(X)$ sending $1 \in \Bbbk$ to some basepoint $x_0 \in X$. In terms of interval cut operations, the structure maps are given by

$$E^k = (-1)^k\, AW_{e_k}, \tag{3.13}$$

that is,

$$\mathbf{E}^k = (-1)^{k(k-1)/2}\, (\mathbf{s}^{-1})^{\otimes(k+1)}\, AW_{e_k} \tag{3.14}$$

where

$$e_k = (k+1, 1, k+1, 2, k+1, \ldots, k+1, k, k+1). \tag{3.15}$$

The sign difference in (3.14) compared to (3.13) stems from the fact that the (de)suspension operators have degree ± 1, so that

$$(\mathbf{s}^{-1})^{\otimes(k+1)}\, \mathbf{s}^{\otimes(k+1)} = (-1)^{k(k+1)/2} \tag{3.16}$$

because the sign changes each time an $\mathbf{s}^{-1}$ is moved past an $\mathbf{s}$ for a different tensor factor. Note that AW_{e_0} is the identity map as required by condition (3.2). A look at the formula (2.21) moreover shows that the intervals labelled $1, \ldots, k$ in the surjection must have length at least 1 in order for the last factor $x^{\boldsymbol{p}}_{k+1}$ of each term in the sum (2.20) for AW_{e_k} to be non-degenerate, which confirms (3.4) and also (3.3).

Explicitly, the induced diagonal on $\mathbf{\Omega}\, C(X)$ can be written as

$$\Delta \langle x \rangle = \mathbf{E}(x) = \langle x \rangle \otimes 1 + \sum_{k=0}^{n} \sum_{\boldsymbol{p}} (-1)^{\varepsilon(\boldsymbol{p})} \left\langle x_1^{\boldsymbol{p}} \,\middle|\, \dots \,\middle|\, x_k^{\boldsymbol{p}} \right\rangle \otimes \left\langle x_{k+1}^{\boldsymbol{p}} \right\rangle \tag{3.17}$$

for $x \in X_n$, where $\boldsymbol{p}$ runs through the cuts of $[n]$ prescribed by e_k. The sign exponent is given by

$$\varepsilon(\boldsymbol{p}) = \frac{k(k-1)}{2} + \operatorname{des}(\boldsymbol{p}) + \operatorname{pos}(\boldsymbol{p}) + \operatorname{perm}(\boldsymbol{p}) \tag{3.18}$$

where

$$\operatorname{des}(\boldsymbol{p}) = \sum_{s=1}^{k} (k+1-s)\, |x_s^{\boldsymbol{p}}| = \sum_{s=1}^{k} (k+1-s)(p_{2s} - p_{2s-1}) \tag{3.19}$$

is the sign exponent incurred by the desuspension operators in (3.7). Note that for $n=0$ the formula (3.17) boils down to $\Delta \langle x \rangle = \langle x \rangle \otimes 1 + 1 \otimes \langle x \rangle$, so that $\langle x \rangle$ is primitive for any 0-simplex $x \in X_0$.[2]

In Appendix A we show that for 1-reduced X the diagonal (3.17) on $\mathbf{\Omega}\, C(X)$ agrees with those defined by Baues [**1**] and Hess–Parent–Scott–Tonks [**14**].

Lemma 3.3. *Let k, $n \geqslant 1$, and let $\boldsymbol{p} = (p_0, \dots, p_{2k+1})$ be an interval cut of $[n]$ for the surjection e_k such that all intervals with label $k+1$ have length 0. Then*

$$\varepsilon(\boldsymbol{p}) \equiv \sum_{s=1}^{k} (s-1)(p_{2s} - p_{2s-1} - 1) \pmod 2.$$

Proof. Modulo 2, we have

$$\operatorname{pos}(\boldsymbol{p}) = p_1 + p_3 + \dots + p_{2k-1}, \tag{3.20}$$

$$\begin{aligned} \operatorname{perm}(\boldsymbol{p}) &= (p_3 - p_1) + 2 \cdot (p_5 - p_3) + \dots + k \cdot (p_{2k+1} - p_{2k-1}) \\ &\equiv p_1 + p_3 + \dots + p_{2k-1} + n\,k, \end{aligned} \tag{3.21}$$

$$\begin{aligned} \operatorname{des}(\boldsymbol{p}) &= \sum_{s=1}^{k} (k+1-s)(p_{2s} - p_{2s-1}) \\ &\equiv n\,k + \sum_{s=1}^{k} (s-1)(p_{2s} - p_{2s-1}), \end{aligned} \tag{3.22}$$

$$\frac{k(k-1)}{2} = \sum_{s=1}^{k} (s-1), \tag{3.23}$$

which gives the desired result. □

Lemma 3.4. *Let $0 \leqslant m \leqslant k$ and $n \geqslant 1$, and let*

$$\boldsymbol{p} : p_0 \overset{k+1}{\text{---}} p_1 \overset{1}{\text{---}} \cdots \overset{m}{\text{---}} p_{2m} \overset{\boldsymbol{k+1}}{\text{---}} p_{2m+1} \overset{m+1}{\text{---}} \cdots \overset{k}{\text{---}} p_{2k} \overset{k+1}{\text{---}} p_{2k+1}$$

be an interval cut of $[n]$ corresponding to the surjection e_k. Assume that the interval corresponding to the $(m+1)$-st occurrence of $k+1$ (highlighted above) has length at

[2] Strictly speaking, we should write $\langle \bar{x} \rangle$, so that $\langle \bar{x} \rangle = 0$ for the basepoint $x = x_0$.

least 1. *Let* $\boldsymbol{p}'$ *be the interval cut for* e_{k+1} *that is obtained from* $\boldsymbol{p}$ *by replacing this interval by*

$$\cdots \xrightarrow{\;m\;} p_{2m} \xrightarrow{\;\boldsymbol{k+2}\;} q \xrightarrow{\;\boldsymbol{m+1}\;} q+1 \xrightarrow{\;\boldsymbol{k+2}\;} p_{2m+1} \xrightarrow{\;m+2\;} \cdots$$

for some $p_{2m} \leqslant q < p_{2m+1}$. *Then*

$$\varepsilon(\boldsymbol{p}') = \varepsilon(\boldsymbol{p}).$$

Proof. One verifies directly that modulo 2 the exponent for the position sign changes by q, the one for the permutation sign by

$$p_0 + \cdots + p_{2m} + q + m + 1, \tag{3.24}$$

the one coming from desuspensions by

$$p_0 + \cdots + p_{2m} + k + m + 1 \tag{3.25}$$

and the one for the explicit sign by k. Hence there is no sign change in total. □

4. A bijection

For $0 \leqslant l \leqslant n$ we define

$$\begin{aligned} S_{n,l} &= \left\{ \boldsymbol{i} = (i_1, \ldots, i_l) \in \mathbb{N}^l \;\middle|\; 0 \leqslant i_s \leqslant n-s \text{ for any } 1 \leqslant s \leqslant l \right\} \\ &= [n-1] \times [n-2] \times \cdots \times [n-l] \end{aligned} \tag{4.1}$$

as well as $S_n = S_{n,n}$. The degree of an element $\boldsymbol{i} \in S_{n,l}$ is

$$|\boldsymbol{i}| = i_1 + \cdots + i_l. \tag{4.2}$$

Note that $S_{n,n}$ has $n!$ elements, and $S_{n,0}$ has the empty sequence $\varnothing$ as unique element.

Let $1 \leqslant k \leqslant n$ and $\boldsymbol{p} = (p_0, \ldots, p_k)$ where $0 = p_0 < p_1 < \cdots < p_k = n$. We set $l = n - k$ and define

$$S_{n-1}(\boldsymbol{p}) = \left\{ \boldsymbol{i} \in S_{n-1,l} \;\middle|\; \partial_{i_l+1} \cdots \partial_{i_1+1}\,[n] = \boldsymbol{p} \right\}. \tag{4.3}$$

Here $[n]$ denotes the standard n-simplex, to which the given face operators are applied in the specified order. We also set $q_s = p_s - p_{s-1}$ for $1 \leqslant s \leqslant k$.

We define a function

$$\begin{aligned} \Psi_{\boldsymbol{p}} \colon S_{n-1}(\boldsymbol{p}) &\to \operatorname{Shuff}(q_1 - 1, \ldots, q_k - 1) \times S_{q_1-1} \times \cdots \times S_{q_k-1} \\ \boldsymbol{i} &\mapsto \big(\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k), \boldsymbol{j}_1, \ldots, \boldsymbol{j}_k\big) \end{aligned} \tag{4.4}$$

as follows: Considering the condition (4.3), we think of an element $\boldsymbol{i} \in S_{n-1}(\boldsymbol{p})$ as describing a way of removing the $l = n - k$ elements not appearing in the sequence $\boldsymbol{p}$ from the n-simplex $[n]$. For $1 \leqslant s \leqslant k$ the element $\boldsymbol{j}_s \in S_{q_s-1}$ similarly records the order in which the elements between p_{s-1} and p_s are removed by $\boldsymbol{i}$, ignoring all other removed elements. The shuffle $\boldsymbol{\alpha}$ keeps track of how the element removals of the intervals $(p_{s-1}, \ldots, p_s)$ are interleaved. More precisely, we declare $q - 1 \in \alpha_s$ if and only if the face operator ∂_{i_q+1} in (4.3) removes an element between p_{s-1} and p_s.

Example 4.1. Take $n = 7$, $k = 3$, $\boldsymbol{p} = (0, 3, 4, 7)$ and $\boldsymbol{i} = (5, 0, 0, 2)$. The missing elements in $(0, 3, 4, 7)$ are removed in the order 6, 1, 2, 5. Those missing in $(0, 3)$

are removed in the order 1, 2, and those missing in $(4,7)$ in the order 6, 5. We therefore have $\boldsymbol{j}_1=(0,0)$, $\boldsymbol{j}_2=\varnothing$ and $\boldsymbol{j}_3=(1,0)$ as well as $\alpha_1=\{1,2\}$, $\alpha_2=\varnothing$ and $\alpha_3=\{0,3\}$.

Note that for $k=1$ the map $\Psi_{\boldsymbol{p}}$ boils down to the identity map on S_{n-1} because $\operatorname{Shuff}(n-1)$ is a singleton. Moreover, for $k=2$ we have $S_{n-1,l}=S_{n-1,n-2}\cong S_n$ (since any $\boldsymbol{i}\in S_{n-1}$ ends in $i_{n-1}=0$), and the maps $\Psi_{\boldsymbol{p}}$ with $0<p_1<n$ combine to the bijection

$$S_{n-1}\cong\bigcup_{q_1+q_2=n}\operatorname{Shuff}(q_1-1,q_2-1)\times S_{q_1-1}\times S_{q_2-1} \tag{4.5}$$

described by Szczarba [**26**, Lemma 3.3].

Proposition 4.2. *The map $\Psi_{\boldsymbol{p}}$ is bijective, and in the notation of* (4.4) *we have*

$$|\boldsymbol{i}|\equiv(\boldsymbol{\alpha})+\sum_{s=1}^{k}|\boldsymbol{j}_s|+\sum_{s=1}^{k}(s-1)(q_s-1)\pmod 2.$$

Remember from Section 2.4 that given a shuffle $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k)$ we write $(\boldsymbol{\alpha})$ for the exponent of its signature. For $k=2$ the above identity appears already in [**26**, Lemma 3.3] and [**15**, Lemma 6].[3]

Proof. It is clear how to reverse the construction to obtain the inverse of $\Psi_{\boldsymbol{p}}$.

Regarding the claimed formula, we assume first that $\boldsymbol{i}$ is of the form

$$\boldsymbol{i}=\big(\underbrace{0,\dots,0}_{q_1-1},\underbrace{1,\dots,1}_{q_2-1},\dots,\underbrace{k-1,\dots,k-1}_{q_k-1}\big). \tag{4.6}$$

Then the shuffle $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k)$ is the identity map on $[l-1]$ and $\boldsymbol{j}_s=(0,\dots,0)$ for all s, from which we conclude that the formula holds.

Consider two elements from $[l-1]$ that are removed one right after the other. Changing the order of the removals changes the degree of $\boldsymbol{i}$ by ±1. If the two removed values belong to the same, say the s-th, interval of $\boldsymbol{p}$, then the degree of $\boldsymbol{j}_s$ also changes by ±1, and $\boldsymbol{\alpha}$ remains fixed. If the values belong to different intervals, then all $\boldsymbol{j}_s$ remain the same, but the sign of the shuffle changes. Hence in any case the claimed identity is preserved.

Starting from (4.6), we can reach any $\boldsymbol{i}\in S_{n-1}(\boldsymbol{p})$ by repeating this swapping procedure. This completes the proof. □

5. The Szczarba operators

5.1. The twisting cochain

We review the definition of Szczarba's twisting cochain [**26**, pp. 200–201] in the formulation given by Hess–Tonks [**15**, Sec. 1.4]. Let X be a simplicial set and G a

[3] Recall that Szczarba writes the signature of the shuffle (ν,μ) as $\operatorname{sgn}(\mu,\nu)$ and also from [**15**, p. 1866] that his sign exponent $\varepsilon(i,n+1)$ equals $n+|\boldsymbol{i}|$. Also note that the subscripts of the degeneracy operators s_μ and s_ν in [**15**] should be swapped.

simplicial group, and let

$$\tau\colon X_{>0} \to G \tag{5.1}$$

be a twisting function. It will be convenient in what follows to write $\sigma(x) = \tau(x)^{-1}$ for $x \in X_{>0}$.

Szczarba [**26**, Thm. 2.1] has introduced the operators

$$\begin{aligned} \mathrm{Sz}_{\boldsymbol{i}}\colon X_n &\to G_{n-1} \\ x &\mapsto D_{\boldsymbol{i},0}\,\sigma(x)\,D_{\boldsymbol{i},1}\,\sigma(\partial_0 x)\cdots D_{\boldsymbol{i},n-1}\,\sigma((\partial_0)^{n-1}x) \end{aligned} \tag{5.2}$$

for $n \geqslant 1$ and $\boldsymbol{i} \in S_{n-1}$. In particular, one has $\mathrm{Sz}_\varnothing\, x = \sigma(x)$. We follow Hess–Tonks [**15**, Def. 5] in using the symbol $\mathrm{Sz}_{\boldsymbol{i}}$ and the name *Szczarba operator*. In terms of these operators, Szczarba's twisting cochain $t\colon C(X) \to C(G)$ is given for $x \in X_n$ by

$$t(x) = \begin{cases} 0 & \text{if } n = 0, \\ \mathrm{Sz}_\varnothing\, x - 1 = \sigma(x) - 1 & \text{if } n = 1, \\ \sum_{\boldsymbol{i}\in S_{n-1}} (-1)^{|\boldsymbol{i}|}\,\mathrm{Sz}_{\boldsymbol{i}}\, x & \text{if } n \geqslant 2. \end{cases} \tag{5.3}$$

In Appendix B we recall the definition of the simplicial operators $D_{\boldsymbol{i},k}$, and we show that t is well-defined on normalized chains.

Example 5.1. In low degrees, Szczarba's twisting cochain looks as follows. Simplices are indicated by vertex numbers. For example, a 2-simplex $x \in X_2$ is written as 012 and $s_1\,\partial_0\,x$ as 122. Note that the products are taken in the simplicial group G, not in the dga $C(G)$.

$$\begin{aligned}
t(01) &= +\sigma(01) - 1, && \boldsymbol{i} = \varnothing && (5.4)\\
t(012) &= +\sigma(012)\,\sigma(122), && \boldsymbol{i} = (0) && (5.5)\\
t(0123) &= +\sigma(0123)\,\sigma(1223)\,\sigma(2333) && \boldsymbol{i} = (0,0) && (5.6)\\
&\quad -\sigma(0113)\,\sigma(1233)\,\sigma(2333), && \boldsymbol{i} = (1,0) &&\\
t(01234) &= +\sigma(01234)\,\sigma(12234)\,\sigma(23334)\,\sigma(34444) && \boldsymbol{i} = (0,0,0) && (5.7)\\
&\quad -\sigma(01224)\,\sigma(12224)\,\sigma(23344)\,\sigma(34444) && \boldsymbol{i} = (0,1,0) &&\\
&\quad -\sigma(01134)\,\sigma(12334)\,\sigma(23334)\,\sigma(34444) && \boldsymbol{i} = (1,0,0) &&\\
&\quad +\sigma(01114)\,\sigma(12344)\,\sigma(23344)\,\sigma(34444) && \boldsymbol{i} = (1,1,0) &&\\
&\quad +\sigma(01124)\,\sigma(12224)\,\sigma(23444)\,\sigma(34444) && \boldsymbol{i} = (2,0,0) &&\\
&\quad -\sigma(01114)\,\sigma(12244)\,\sigma(23444)\,\sigma(34444) && \boldsymbol{i} = (2,1,0) &&
\end{aligned}$$

We need to understand how the Szczarba operators relate to the bijection $\Psi_{\boldsymbol{p}}$ introduced in Section 4. Let $n = k + l$ with $1 \leqslant k \leqslant n$. Any $\boldsymbol{i} = (i_1, \dots, i_{n-1}) \in S_{n-1}$ can be written in the form in the form

$$\boldsymbol{i} = (\boldsymbol{i}_1, \boldsymbol{i}_2) = (i_{1,1}, \dots, i_{1,l}, i_{2,1}, \dots, i_{2,k-1}) \tag{5.8}$$

with $\boldsymbol{i}_1 \in S_{n-1}(\boldsymbol{p})$ and $\boldsymbol{i}_2 \in S_{k-1}$, where

$$\boldsymbol{p} = (p_0, \dots, p_k) = \partial_{i_l+1} \cdots \partial_{i_1+1}\,[n]. \tag{5.9}$$

Lemma 5.2. *Using this notation, we have*

$$(\partial_0)^l\,\mathrm{Sz}_{\boldsymbol{i}}\, x = \mathrm{Sz}_{\boldsymbol{i}_2}\, x(p_0, p_1, \dots, p_k),$$

$$\tilde\partial^{k-1}\operatorname{Sz}_{\boldsymbol{i}} x = s_{\bar\alpha_1}\operatorname{Sz}_{\boldsymbol{j}_1} x(p_0,\dots,p_1)\cdots s_{\bar\alpha_k}\operatorname{Sz}_{\boldsymbol{j}_k} x(p_{k-1},\dots,p_k)$$

where $\Psi_{\boldsymbol{p}}(\boldsymbol{i}_1)=(\boldsymbol{\alpha},\boldsymbol{j}_1,\dots,\boldsymbol{j}_k)$ *and* $\bar\alpha_s=[l-1]\smallsetminus\alpha_s$ *for* $1\leqslant s\leqslant k$.

Proof. The case $l=0$ of the first identity is void. Given the definition (4.3), it reduces for $l=1$ to the formula

$$\partial_0\operatorname{Sz}_{\boldsymbol{i}} x = \operatorname{Sz}_{(i_2,\dots,i_{n-1})}\partial_{i_1+1}x, \tag{5.10}$$

which is stated in [**15**, Lemma 6]. The case $l\geqslant 2$ follows by iteration.

The second identity is trivial for $k=1$, compare the discussion of $\Psi_{\boldsymbol{p}}$ following Example 4.1. For $k=2$ it is again given in [**15**, Lemma 6]. For larger k it follows by induction:

Assume the identity proven for k and l and consider $k'=k+1$ and $l'=l-1$. The other values for the new situation are also written with a prime, that is, $\boldsymbol{p}'$, $\boldsymbol{i}'=(\boldsymbol{i}'_1,\boldsymbol{i}'_2)$ and $\Psi_{\boldsymbol{p}'}(\boldsymbol{i}'_1)=(\boldsymbol{\alpha}',\boldsymbol{j}'_1,\dots,\boldsymbol{j}'_2)$.

Let $\boldsymbol{p}=\partial_{i'_{2,1}+1}\boldsymbol{p}'$, and let $\hat p$ be the removed value. We split $\boldsymbol{i}'$ as $\boldsymbol{i}'=(\boldsymbol{i}_1,\boldsymbol{i}_2)$ with $\boldsymbol{i}_1=(i'_{1,1},\dots,i'_{1,l-1},i'_{2,1})$ and $\boldsymbol{i}_2=(i'_{2,2},\dots,i'_{2,k-1})$ and corresponding values $\boldsymbol{\alpha}$ and $\boldsymbol{j}_1$, …, $\boldsymbol{j}_k$. Then

$$\begin{aligned}\tilde\partial^{k}\operatorname{Sz}_{\boldsymbol{i}'} x &= \tilde\partial\,\tilde\partial^{k-1}\operatorname{Sz}_{\boldsymbol{i}'} x \\ &= \tilde\partial\Big(s_{\bar\alpha_1}\operatorname{Sz}_{\boldsymbol{j}_1} x(p_0,\dots,p_1)\cdots s_{\bar\alpha_k}\operatorname{Sz}_{\boldsymbol{j}_k} x(p_{k-1},\dots,p_k)\Big)\end{aligned} \tag{5.11}$$

By the definition of the shuffle $\boldsymbol{\alpha}$, we have $l-1\in\alpha_r$ if ∂_{i_l+1} removes an element between p_{r-1} and p_r. Hence $l-1\notin\alpha_s$ for $s\neq r$ and therefore

$$\begin{aligned}&= s_{\bar\alpha_1\smallsetminus\{l-1\}}\operatorname{Sz}_{\boldsymbol{j}_1} x(p_0,\dots,p_1)\\ &\qquad\cdots s_{\bar\alpha_r}\tilde\partial\operatorname{Sz}_{\boldsymbol{j}_r} x(p_{r-1},\dots,p_r)\cdots s_{\bar\alpha_k\smallsetminus\{l-1\}}\operatorname{Sz}_{\boldsymbol{j}_k} x(p_{k-1},\dots,p_k).\end{aligned}$$

Set $\hat q_1=\hat p-p_{r-1}$ and $\hat q_2=p_r-\hat p$. Again by the case $l=2$ we have

$$\tilde\partial\operatorname{Sz}_{\boldsymbol{j}_r} x(p_{r-1},\dots,p_r) = s_{\bar\beta_2}\operatorname{Sz}_{\boldsymbol{k}_1} x(p_{r-1},\dots,\hat p)\cdot s_{\bar\beta_1}\operatorname{Sz}_{\boldsymbol{k}_2} x(\hat p,\dots,p_r) \tag{5.12}$$

for a $(\hat q_1-1,\hat q_2-1)$-shuffle (β_1,β_2) and sequences $\boldsymbol{k}_1\in S_{\hat q_1-1}$, $\boldsymbol{k}_2\in S_{\hat q_2-1}$. We thus obtain the desired formula since

$$\boldsymbol{j}'_1=\boldsymbol{j}_1,\ \ \dots,\ \ \boldsymbol{j}'_r=\boldsymbol{k}_1,\ \ \boldsymbol{j}'_{r+1}=\boldsymbol{k}_2,\ \ \dots,\ \ \boldsymbol{j}'_{k+1}=\boldsymbol{j}_k, \tag{5.13}$$

$$\boldsymbol{\alpha}'=(\alpha_1,\dots,\alpha_{r-1},\gamma_1,\gamma_2,\alpha_{r+1},\dots,\alpha_k) \tag{5.14}$$

where the subsets γ_1, $\gamma_2\subset[l-2]$ are defined by

$$s_{\bar\gamma_1}=s_{\bar\alpha_r}s_{\bar\beta_1},\qquad s_{\bar\gamma_2}=s_{\bar\alpha_r}s_{\bar\beta_2}. \tag{5.15}$$

□

5.2. The twisted shuffle map

Let F be a left G-space. We recall the definition of Szczarba's twisted shuffle map [**26**, Thm. 2.3]

$$\psi=\psi_F\colon C(X)\otimes_t C(F)\to C(X\times_\tau F) \tag{5.16}$$

in a notation inspired by Hess–Tonks. For any $n\geqslant 0$ and $\boldsymbol{i}\in S_n$ we define the operator

$$\begin{aligned}\widehat{\operatorname{Sz}}_{\boldsymbol{i}}\colon X_n&\to (X\times_\tau G)_n = X_n\times G_n,\\ x&\mapsto\big(D_{\boldsymbol{i},0}\,x, D_{\boldsymbol{i},1}\,\sigma(x)\,D_{\boldsymbol{i},2}\,\sigma(\partial_0 x)\cdots D_{\boldsymbol{i},n}\,\sigma((\partial_0)^{n-1}x)\big),\end{aligned} \tag{5.17}$$

which is interpreted as $\widehat{\mathrm{S}}\mathrm{z}_{\varnothing}\, x = (x, 1) \in X_0 \times G_0$ for $n = 0$ and $\boldsymbol{i} = \varnothing$. Based on this we define the map

$$\psi(x \otimes y) = \sum_{\boldsymbol{i} \in S_n} (-1)^{|\boldsymbol{i}|} \, (\mathrm{id}_X, \mu_F)_* \, \nabla \left(\widehat{\mathrm{S}}\mathrm{z}_{\boldsymbol{i}}\, x \otimes y \right), \tag{5.18}$$

where $n = |x|$ as before, $\nabla\colon C(X \times_\tau G) \otimes C(F) \to C(X \times_\tau G \times F)$ is the shuffle map and $\mu_F\colon G \times F \to F$ the group action.[4] For a proof that ψ descends to normalized chains see again Appendix B.

Given a decomposition $n = k + l$ with k, $l \geqslant 0$, we can write any $\boldsymbol{i} \in S_n$ in the form $\boldsymbol{i} = (\boldsymbol{i}_1, \boldsymbol{i}_2)$ with $\boldsymbol{i}_1 \in S_n(\boldsymbol{p})$ and $\boldsymbol{i}_2 \in S_k$, where

$$\boldsymbol{p} = (0 = p_0, p_1, \dots, p_{k+1} = n+1) = \partial_{i_l+1} \cdots \partial_{i_1+1}\, [n+1]. \tag{5.19}$$

We also write $q_s = p_s - p_{s-1}$ for $1 \leqslant s \leqslant k+1$.

Lemma 5.3. *In the notation above, we have*

$$\begin{aligned}(\partial_0)^l\, \widehat{\mathrm{S}}\mathrm{z}_{\boldsymbol{i}}\, x &= \widehat{\mathrm{S}}\mathrm{z}_{\boldsymbol{i}_2}\, x(p_1 - 1, \dots, p_{k+1} - 1), \\ \tilde{\partial}^k\, \widehat{\mathrm{S}}\mathrm{z}_{\boldsymbol{i}}\, x &= s_{\bar{\alpha}_1}\, \widehat{\mathrm{S}}\mathrm{z}_{\boldsymbol{j}_1}\, x(0, \dots, p_1 - 1) \cdot s_{\bar{\alpha}_2}\, \mathrm{Sz}_{\boldsymbol{j}_2}\, x(p_1 - 1, \dots, p_2 - 1) \\ &\qquad \cdots s_{\bar{\alpha}_{k+1}}\, \mathrm{Sz}_{\boldsymbol{j}_{k+1}}\, x(p_k - 1, \dots, p_{k+1} - 1),\end{aligned}$$

where $\Psi_{\boldsymbol{p}}(\boldsymbol{i}_1) = (\boldsymbol{\alpha}, \boldsymbol{j}_1, \dots, \boldsymbol{j}_{k+1})$ *and* $\bar{\alpha}_s = [l-1] \smallsetminus \alpha_s$ *for* $1 \leqslant s \leqslant k+1$.

Proof. Apart from the trivial case $l = 0$, the first formula follows by induction from the case $l = 1$, that is,

$$\partial_0\, \widehat{\mathrm{S}}\mathrm{z}_{\boldsymbol{i}}\, x = \widehat{\mathrm{S}}\mathrm{z}_{(i_2, \dots, i_n)}\, \partial_{i_1} x, \tag{5.20}$$

which can be found in [**26**, pp. 205–206] as the discussion of the "first term of (4.1)" there.

The second formula is also trivial for $k = 0$, and for $k = 1$ it is contained in [**26**, eq. (4.5)]. The extension to larger k follows again by induction, based on the case $k = 2$ of the present claim as well as the case $k = 2$ of Lemma 5.2, using the same kind of reasoning as given there. □

6. Proof of Theorem 1.1

Let X be a simplicial set, G a simplicial group and $\tau\colon X_{>0} \to G$ a twisting function. Explicitly, the Szczarba map (1.3) is given by

$$\mathrm{Sz}\colon \boldsymbol{\Omega}\, C(X) \to C(G), \qquad \left\langle\, x_1 \mid \cdots \mid x_m \,\right\rangle \mapsto t(x_1) \cdots t(x_m) \tag{6.1}$$

where $t\colon C(X) \to C(G)$ is Szczarba's twisting cochain as defined in (5.3). Since we are looking at a multiplicative map between bialgebras, we only have to show

$$\Delta_{C(G)}\, \mathrm{Sz}\, \langle x \rangle = (\mathrm{Sz} \otimes \mathrm{Sz})\, \Delta_{\boldsymbol{\Omega} C(X)}\, \langle x \rangle \tag{6.2}$$

for any $x \in X$, say of degree n. If $n = 0$, then $\langle x \rangle$ is primitive and annihilated by Sz, so that (6.2) holds. We therefore assume $n \geqslant 1$ for the rest of the proof.

[4] In the definition of ψ in [**26**, p. 201] the upper summation index should read "$p!$".

The left-hand side of (6.2) equals

$$\Delta\ \mathrm{Sz}\,\langle x\rangle = \Delta\, t(x) = \sum_{k=1}^{n} \tilde\partial^{k-1}\, t(x)\otimes(\partial_0)^l\, t(x) \tag{6.3}$$
$$= \sum_{k=1}^{n}\sum_{\boldsymbol{i}\in S_{n-1}} (-1)^{|\boldsymbol{i}|}\,\tilde\partial^{k-1}\ \mathrm{Sz}_{\boldsymbol{i}}\, x\otimes(\partial_0)^l\ \mathrm{Sz}_{\boldsymbol{i}}\, x$$

where we have again used the abbreviation $l = n-k$. Using the explicit formula (3.17) for the diagonal, we can write the right-hand side of (6.2) in the form

$$(\mathrm{Sz}\otimes\mathrm{Sz})\,\Delta\,\langle x\rangle = t(x)\otimes 1 + \sum_{k=0}^{n}\sum_{\boldsymbol{p}} (-1)^{\varepsilon(\boldsymbol{p})}\, t(x_1^{\boldsymbol{p}})\cdots t(x_k^{\boldsymbol{p}})\otimes t(x_{k+1}^{\boldsymbol{p}}) \tag{6.4}$$

where $\boldsymbol{p} = (p_0, p_1, \dots, p_{2k+1})$ ranges over the cuts of $[n]$ into $2k+1$ intervals corresponding to the surjection e_k. We are going to pair off the summands of the expressions (6.3) and (6.4). We write $q_s = p_{2s} - p_{2s-1}$ for $1\leqslant s\leqslant k$ and $\ell(\boldsymbol{p})$ for the sum of the lengths of the intervals in $\boldsymbol{p}$ corresponding to the final value $k+1$.

Assume $\ell(\boldsymbol{p}) = 0$, so that the k intervals labelled $1, \dots, k$ cover the whole interval $[n]$. From the definition of t we get

$$t(x_{k+1}^{\boldsymbol{p}}) = \sum_{\boldsymbol{i}_2\in S_{k-1}} (-1)^{|\boldsymbol{i}_2|}\ \mathrm{Sz}_{\boldsymbol{i}_2}\, x_{k+1}^{\boldsymbol{p}}, \tag{6.5}$$

and together with that of the shuffle map (2.11) also

$$\begin{aligned}(-1)^{\varepsilon(\boldsymbol{p})}\, t(x_1^{\boldsymbol{p}})\cdots t(x_k^{\boldsymbol{p}}) &= \\ &\sum (-1)^{\varepsilon(\boldsymbol{p})+(\boldsymbol{\alpha})+\sum_s |\boldsymbol{j}_s|}\, s_{\bar\alpha_1}\ \mathrm{Sz}_{\boldsymbol{j}_1}\, x_1^{\boldsymbol{p}}\cdots s_{\bar\alpha_k}\ \mathrm{Sz}_{\boldsymbol{j}_k}\, x_k^{\boldsymbol{p}} \\ &\qquad + \text{additional terms with fewer than } k \text{ factors.}\end{aligned} \tag{6.6}$$

Here the sum is over all $(q_1-1,\dots,q_k-1)$-shuffles $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_k)$ as well as over all $\boldsymbol{j}_1\in S_{q_1-1}, \dots, \boldsymbol{j}_k\in S_{q_k-1}$. The additional terms indicated above arise whenever we have $q_s = 1$ for some s because of the extra term $-1\in C(G)$ produced by t in the case of a 1-simplex.

Consider the case $k>1$. As a consequence of Lemma 5.2, the expressions $(\partial_0)^l\, t(x)$ in (6.3) that give terms of the form (6.5) are indexed by the $\boldsymbol{i} = (\boldsymbol{i}_1, \boldsymbol{i}_2)\in S_{n-1}$ with $\boldsymbol{i}_1\in S_{n-1}(\boldsymbol{p})$ and $\boldsymbol{i}_2\in S_{k-1}$. By the same lemma, the terms $\tilde\partial^{k-1}\, t(x)$ for all such $\boldsymbol{i}_1$ give exactly the terms in the sum formula of (6.6). Lemma 3.3 and Proposition 4.2 show that also the signs work out correctly since $|\boldsymbol{i}| = |\boldsymbol{i}_1| + |\boldsymbol{i}_2|$ and

$$|\boldsymbol{i}_1| = (\boldsymbol{\alpha}) + \sum_{s=1}^{k}|\boldsymbol{j}_s| + \sum_{s=1}^{k}(s-1)(q_s-1) = \varepsilon(\boldsymbol{p}) + (\boldsymbol{\alpha}) + \sum_{s=1}^{k}|\boldsymbol{j}_s|. \tag{6.7}$$

If $k=1$, then $x_1^{\boldsymbol{p}} = x$, $x_2^{\boldsymbol{p}} = x(0,n)$ is of degree 1, and $\varepsilon(\boldsymbol{p}) = 0$. In addition to the terms discussed in the preceding paragraph, we get a -1 on the right-hand side of (6.5) and therefore $-t(x)\otimes 1$ in (6.4), which cancels with the very first term in the same formula.

We now argue that the decompositions $\boldsymbol{p}$ with $\ell(\boldsymbol{p})>0$ (including the only possible decomposition for $k=0$) lead to terms in the sum (6.4) that cancel out with the additional terms in (6.6) for $\ell(\boldsymbol{p})=0$.

Given two decompositions $\boldsymbol{p}$ and $\boldsymbol{p}'$ for the surjections e_k and $e_{k'}$, respectively, we write $\boldsymbol{p}'\geqslant\boldsymbol{p}$ if $\boldsymbol{p}'$ can be obtained from $\boldsymbol{p}$ by zero or more applications of the "refinement procedure" described in Lemma 3.4. This gives a partial order on the set of all such decompositions.

For any decomposition $\boldsymbol{p}$ there are exactly $2^{\ell(\boldsymbol{p})}$ decompositions $\boldsymbol{p}'\geqslant\boldsymbol{p}$. In the maximal such $\boldsymbol{p}'$, all intervals with the final label $k+1$ in $\boldsymbol{p}$ have been subdivided into intervals of length 1 and relabelled with non-final values, separated by intervals of length 0 labelled $k+1$. In particular, $\ell(\boldsymbol{p}')=0$. Conversely, there are exactly $2^{\ell_1(\boldsymbol{p})}$ decompositions $\boldsymbol{p}'\leqslant\boldsymbol{p}$, where $\ell_1(\boldsymbol{p})$ is the number of intervals of length 1 having non-final labels. The minimal such $\boldsymbol{p}'$ has no intervals of this kind.

Example 6.1. Take $k=1$ and the decomposition

$$\boldsymbol{p}\colon 0\xrightarrow{\;2\;}0\xrightarrow{\;1\;}1\xrightarrow{\;2\;}3. \tag{6.8}$$

The maximal $\boldsymbol{p}'\geqslant\boldsymbol{p}$ and the minimal $\boldsymbol{p}''\leqslant\boldsymbol{p}$ are as follows. Subdivided or combined intervals are indicated in boldface.

$$\boldsymbol{p}'\colon 0\xrightarrow{\;4\;}0\xrightarrow{\;1\;}1\xrightarrow{\;\mathbf{4}\;}1\xrightarrow{\;\mathbf{2}\;}2\xrightarrow{\;\mathbf{4}\;}2\xrightarrow{\;\mathbf{3}\;}3\xrightarrow{\;\mathbf{4}\;}3 \qquad (k'=3), \tag{6.9}$$

$$\boldsymbol{p}''\colon 0\xrightarrow{\;\mathbf{1}\;}3 \qquad (k''=0). \tag{6.10}$$

Note that we have

$$x^{\boldsymbol{p}}_{k+1}=x^{\boldsymbol{p}'}_{k'+1} \tag{6.11}$$

whenever $\boldsymbol{p}$ and $\boldsymbol{p}'$ are comparable. We therefore look at a minimal $\boldsymbol{p}$ in our ordering and the term $x^{\boldsymbol{p}}_{k+1}$ it produces. As the added intervals of any $\boldsymbol{p}'\geqslant\boldsymbol{p}$ are all of length 1, the corresponding terms $t(x^{\boldsymbol{p}'}_s)$ in

$$(-1)^{\varepsilon(\boldsymbol{p}')}\,t(x^{\boldsymbol{p}'}_1)\cdots t(x^{\boldsymbol{p}'}_{k'}) \tag{6.12}$$

all contain $-1\in C(G)$. The summand

$$(-1)^{\varepsilon(\boldsymbol{p}')+(\boldsymbol{\alpha}')+\sum_{s'}|\boldsymbol{j}'_{s'}|+\ell_1(\boldsymbol{p}')}\prod_{\substack{1\leqslant s'\leqslant k'\\ q'_{s'}\neq 1}} s_{\bar\alpha_{s'}}\,\mathrm{Sz}_{\boldsymbol{j}'_{s'}}\,x^{\boldsymbol{p}'}_{s'}=:(-1)^{\ell_1(\boldsymbol{p}')}\,a \tag{6.13}$$

therefore appears in the product (6.12). We claim that the expression a only depends on $\boldsymbol{p}$. More precisely, we have

$$a=(-1)^{\varepsilon(\boldsymbol{p})+(\boldsymbol{\alpha})+\sum_s|\boldsymbol{j}_s|}\,s_{\bar\alpha_1}\,\mathrm{Sz}_{\boldsymbol{j}_1}\,x^{\boldsymbol{p}}_1\cdots s_{\bar\alpha_k}\,\mathrm{Sz}_{\boldsymbol{j}_k}\,x^{\boldsymbol{p}}_k. \tag{6.14}$$

This is because an interval of length $q'_{s'}=1$ leads to $\alpha'_{s'}=\varnothing$ and $\boldsymbol{j}'_{s'}=\varnothing$, while the remaining $\alpha'_{s'}$ and $\boldsymbol{j}'_{s'}$ are not affected and appear as α_s and $\boldsymbol{j}_s$ for some index $s\leqslant s'$. Moreover, we have $\varepsilon(\boldsymbol{p}')=\varepsilon(\boldsymbol{p})$ by a repeated application of Lemma 3.4.

If $\ell(\boldsymbol{p})>0$, then we get $2^{\ell(\boldsymbol{p})}$ terms with alternating signs, so that

$$\sum_{\boldsymbol{p}'\geqslant\boldsymbol{p}}(-1)^{\ell_1(\boldsymbol{p}')}\,a\otimes t(x^{\boldsymbol{p}'}_{k'+1})=\sum_{\boldsymbol{p}'\geqslant\boldsymbol{p}}(-1)^{\ell_1(\boldsymbol{p}')}\,a\otimes t(x^{\boldsymbol{p}}_{k+1})=0. \tag{6.15}$$

The only terms in (6.4) not appearing in such a sum are $t(x)\otimes 1$ plus those written out in (6.6) for $\boldsymbol{p}$ with $\ell(\boldsymbol{p})=0$, and we have seen already that they add up to (6.3). This completes the proof.

7. The extended cobar construction and the loop group

Let X be a reduced simplicial set (that is, having a unique 0-simplex), and let GX be its Kan loop group, compare [**18**, Def. 26.3]. (Its topological realization $|GX|$ is a model for the based loop space $\Omega|X|$ as a topological monoid, see [**2**, §1.8, Prop. 3.3].) Let $\tau\colon X_{>0} \to GX$ be the canonical twisting function, and let t be Szczarba's twisting cochain associated to it.

Hess–Tonks have defined an extended cobar construction $\tilde{\mathbf{\Omega}}\, C(X)$ such that the canonical dga map $\mathbf{\Omega}\, C(X) \to C(GX)$ extends to a dga map

$$\phi\colon \tilde{\mathbf{\Omega}}\, C(X) \to C(GX), \tag{7.1}$$

see [**15**, Thm. 7]. They moreover showed that ϕ is a strong deformation retract of chain complexes such that all maps involved are natural in X [**15**, Thm. 15].

Let us recall the definition of $\tilde{\mathbf{\Omega}}\, C(X)$ in the form given by Rivera–Saneblidze [**23**, Sec. 4.2]. Write $C = C(X)$, and let G be the free group on generators g_x where x runs through the non-degenerate 1-simplices of X. We define a new dgc $\tilde{C}$ by $\tilde{C}_n = C_n$ for $n \ne 1$ and $\tilde{C}_1 = \Bbbk[G]$, the group algebra of G. We set $d\, g = 0$, $\varepsilon(g) = 0$ and $\Delta\, g = g \otimes 1_C + 1_C \otimes g$ for any $g \in G$. We embed C into $\tilde{C}$ by sending x as before to $g_x - 1_G$. The dga $\tilde{\mathbf{\Omega}}\, C(X)$ is the quotient of the usual cobar construction $\mathbf{\Omega}\, \tilde{C}$ by the two-sided dg ideal generated by the cycles $\langle a|b\rangle - \langle ab\rangle$ for a, $b \in \tilde{C}_1$ as well as $\langle 1_G\rangle - 1_{\mathbf{\Omega}\tilde{C}}$. By abuse of notation, we write elements of $\tilde{\mathbf{\Omega}}\, C(X)$ like those of $\mathbf{\Omega}\, \tilde{C}$.

We extend Szczarba's twisting cochain t to a linear map $\tilde{t}\colon \tilde{C} \to C(GX)$ by defining $\tilde{t}(g_x) = \sigma(x)$ for any non-degenerate 1-simplex $x \in X$ and taking its multiplicative extension to $G \subset \tilde{C}_1$. The result is again a twisting cochain. The induced dga morphism $\mathbf{\Omega}\, \tilde{C} \to C(GX)$ descends to $\tilde{\mathbf{\Omega}}\, C(X)$, where it defines the map ϕ from (7.1).

We extend the augmentation and the diagonal from $\mathbf{\Omega}\, C(X)$ to $\mathbf{\Omega}\, \tilde{C}$ by setting

$$\varepsilon(\langle g\rangle) = 1 \qquad \text{and} \qquad \Delta\, \langle g\rangle = \langle g\rangle \otimes \langle g\rangle \tag{7.2}$$

for any $g \in G$. This induces well-defined maps on $\tilde{\mathbf{\Omega}}\, C(X)$.

Proposition 7.1. *Let X be a reduced simplicial set. With the structure maps given above, $\tilde{\mathbf{\Omega}}\, C(X)$ becomes a dg bialgebra and ϕ a quasi-isomorphism of dg bialgebras.*

Proof. The maps (7.2) are compatible with ϕ because analogous formulas hold for the 0-simplices $\phi(\langle g\rangle) \in GX$. Since ϕ is a deformation retract, it is an injective quasi-isomorphism and its image a direct summand of $C(GX)$. Because the latter is a dg bialgebra, so is $\tilde{\mathbf{\Omega}}\, C(X)$, and ϕ is a morphism of dg bialgebras. □

Remark 7.2. The extended cobar construction $\tilde{\mathbf{\Omega}}\, C(X)$ is in fact the normalized chain complex of a certain cubical monoid $Y = \tilde{\mathbf{\Omega}} X$, see [**23**, Sec. 3.5]. This cubical monoid can be (formally) triangulated to a simplicial monoid $\mathcal{T}\, Y$. Sending each n-cube to the $n!$ simplices in its triangulation gives a well-defined quasi-isomorphism of dg bialgebras $\mathbb{T}\colon C(Y) \to C(\mathcal{T}\, Y)$. After the prepublication of this article, Minichiello–Rivera–Zeinalian [**20**, Cor. 5.20] have shown that there is a morphism of simplicial monoids $f\colon \mathcal{T}\, Y \to GX$ such that $\phi = f_* \circ \mathbb{T}$. This gives a different proof that ϕ is morphism of dg bialgebras.

8. Twisted tensor products

Let C be an hgc and A a dg bialgebra, and let M be an A-dgc. By the latter we mean a dgc M that is also a left A-module such that the diagonal $\Delta_M\colon M \to M \otimes M$ and the augmentation $\varepsilon_M\colon M \to \Bbbk$ are A-equivariant. (Recall that A acts on $M \otimes M$ via its diagonal $\Delta_A\colon A \to A \otimes A$ and on $\Bbbk$ via its augmentation $\varepsilon_A\colon A \to \Bbbk$.)

Let $t\colon C \to A$ be a twisting cochain. The differential of the twisted tensor product $C \otimes_t M$ is given by

$$d_t = d_C \otimes 1 + 1 \otimes d_M - \delta_t \tag{8.1}$$

where

$$\delta_t = (1 \otimes \mu_M)\,(1 \otimes t \otimes 1)\,(\Delta_C \otimes 1) \tag{8.2}$$

and $\mu_M\colon A \otimes M \to M$ is the structure map of the A-module M. In the Sweedler notation this is expressed as

$$d_t(c \otimes m) = d\,c \otimes m + (-1)^{|c|}\,c \otimes d\,m - \sum_{(c)} (-1)^{|c_{(1)}|}\,c_{(1)} \otimes t(c_{(2)})\,m \tag{8.3}$$

for $c \otimes m \in C \otimes_t M$.

The purpose of this section is to observe that $C \otimes_t M$ can again be turned into a dgc if t is comultiplicative. The dual situation of a multiplication on the twisted tensor product of an hga and a dg bialgebra has already been considered by Kadeishvili–Saneblidze [**17**, Thm. 7.1].

Let $f\colon \mathbf{\Omega}\,C \to A$ be the map of dg bialgebras induced by the comultiplicative twisting cochain t. Based on f and on the map $\mathfrak{E}$ from (3.9), we introduce the map of degree 0

$$\mathfrak{F}\colon C \xrightarrow{\mathfrak{E}} \mathbf{\Omega}\,C \otimes C \xrightarrow{f \otimes 1} A \otimes C. \tag{8.4}$$

The diagonal of $C \otimes_t M$ then is defined as

$$\begin{aligned}\Delta = (1_C \otimes \mu_M \otimes 1_C \otimes 1_M)&(1_C \otimes 1_A \otimes T_{C,M} \otimes 1_M)\\ &\big(1_C \otimes \mathfrak{F} \otimes 1_M \otimes 1_M\big)\,(\Delta_C \otimes \Delta_M)\end{aligned} \tag{8.5}$$

where $\mu_M\colon A \otimes M \to M$ is the action. In terms of the Sweedler notation this means

$$\Delta(c \otimes m) = \sum_{(c),(m)} \sum_i (-1)^{|c_i||m_{(1)}|}\,\big(c_{(1)} \otimes a_i \cdot m_{(1)}\big) \otimes \big(c_i \otimes m_{(2)}\big) \tag{8.6}$$

for $c \otimes m \in C \otimes_t M$ and $\mathfrak{F}(c_{(2)}) = \sum_i a_i \otimes c_i \in A \otimes C$.

Proposition 8.1. *Let $t\colon C \to A$ be a comultiplicative twisting cochain, and let M be an A-dgc. Then the twisted tensor product $C \otimes_t M$ is a dgc with the diagonal given above and the augmentation $\varepsilon_C \otimes \varepsilon_M$.*

Proof. This is a lengthy computation based on the analogues

$$d(\mathfrak{F}) = (\mu_A \otimes 1_C)(t \otimes \mathfrak{F})\,\Delta_C - (\mu_A \otimes 1_C)\,(1_A \otimes T_{C,A})\,(\mathfrak{F} \otimes t)\,\Delta_C, \tag{8.7}$$

$$(1_A \otimes \Delta_C)\,\mathfrak{F} = (\mu_A \otimes 1_C \otimes 1_C)\,(1_A \otimes T_{C,A} \otimes 1_C)\,(\mathfrak{F} \otimes \mathfrak{F})\,\Delta_C, \tag{8.8}$$

$$(\Delta_A \otimes 1_C)\,\mathfrak{F} = (1_A \otimes \mathfrak{F})\,\mathfrak{F}. \tag{8.9}$$

of the identities for $\mathfrak{E}$ stated in Lemma 3.1. One additionally uses the formula

$$\Delta_A\, t = (1 \otimes t)\, \mathfrak{F} + t \otimes \iota_A, \tag{8.10}$$

which can be seen as follows: Since f is a morphism of coalgebras, one has

$$\Delta_A\, t = \Delta_A\, f\, t_C = (f \otimes f)\, \Delta_{\mathbf{\Omega} C}\, t_C = (f \otimes f)\, \mathbf{E}. \tag{8.11}$$

The image of $\mathbf{E}$ lies in $\mathbf{\Omega}\, C \otimes \mathbf{\Omega}_l\, C$ with $l \leqslant 1$. Considering the terms for $l = 0$ and $l = 1$ separately as in the proof of Lemma 3.1 gives (8.10).

In order to prove that $\Delta = \Delta_{C \otimes M}$ as given in (8.5) is a chain map, it is convenient to use the tensor product differential $d_\otimes = d_C \otimes 1 + 1 \otimes d_M$ on $C \otimes M$ and analogously on $(C \otimes M) \otimes (C \otimes M)$ and to show that

$$d_\otimes(\Delta_{C\otimes M}) - (\delta_t \otimes 1_{C\otimes M})\, \Delta_{C\otimes M} - (1_{C\otimes M} \otimes \delta_t)\, \Delta_{C\otimes M} + \Delta_{C\otimes M}\, \delta_t = 0. \tag{8.12}$$

With respect to these differentials, $\mathfrak{F}$ is the only map appearing in (8.5) that is not a chain map. The boundary $d_\otimes(\Delta)$ therefore has two summands coming from the right-hand side of (8.7). The first of them cancels with $(\delta_t \otimes 1)\, \Delta$. Using (8.10), the term $\Delta\, \delta_t$ splits up into two. Taking (8.8) into account, the first one cancels with $(1 \otimes \delta_t)\, \Delta$ and the second one with the second summand in $d_\otimes(\Delta)$.

The coassociativity of $\Delta_{C\otimes M}$ is a consequence of (8.8) and (8.9). The properties involving the augmentation follow directly from the definitions. □

Corollary 8.2. *Let $t\colon C(X) \to C(G)$ be Szczarba's twisting cochain determined by a twisting function $\tau\colon X_{>0} \to G$, and let F be a left G-space. Then $C(X) \otimes_t C(F)$ is a dgc.*

The diagonal is independent of the chosen coaugmentation of $C(X)$ and looks explicitly as follows: For $x \in X_n$ and $y \in F_m$ we have

$$\begin{aligned} \Delta\,(x \otimes y) = \sum_{i=0}^{n} \sum_{j=0}^{m} \sum_{k=0}^{n-i} \sum_{\boldsymbol{p}} (-1)^{\varepsilon(\boldsymbol{p}) + i + (m-j-1)|z^{\boldsymbol{p}}_{k+1}|} \\ \cdot \Big(\tilde\partial^i\, x \otimes t(z^{\boldsymbol{p}}_1) \cdots t(z^{\boldsymbol{p}}_k)\, \tilde\partial^j\, y\Big) \otimes \Big(z^{\boldsymbol{p}}_{k+1} \otimes (\partial_0)^{m-j}\, y\Big) \end{aligned} \tag{8.13}$$

where $z = (\partial_0)^{n-i}\, x$, and the last sum is over all interval cuts $\boldsymbol{p}$ of $[i]$ corresponding to e_k. (Recall that the unit $1 \in \mathbf{\Omega}\, C$ is annihilated by the map p_C implicit in $\mathfrak{F}$ and defined in (3.10), hence so is the term $\langle z\rangle \otimes 1$ appearing in $\Delta\,\langle z\rangle$ by $1 \otimes p_C$.)

9. Proof of Theorem 1.3

This proof is similar to the one for Theorem 1.1 given in Section 6. Since Szczarba proved that ψ_F is a chain map [**26**, Thm. 2.4], we only need to show that ψ_F is a morphism of coalgebras. We start by observing that it is enough to consider the case $F = G$ because we can write the twisted shuffle map ψ_F in the form

$$\begin{aligned} C(X) \otimes_t C(F) = C(X) \otimes_t C(G) \underset{C(G)}{\otimes} C(F) \\ \xrightarrow{\psi_G \otimes 1} C(X \times_\tau G) \underset{C(G)}{\otimes} C(F) \xrightarrow{\nabla} C\big(X \times_\tau G \underset{G}{\times} F\big) = C(X \times_\tau F). \end{aligned} \tag{9.1}$$

Hence if ψ_G is a dgc map, then so is ψ_F. (Recall from [**6**, (17.6)] that the shuffle map ∇ is a morphism of dgcs. This also implies that the tensor product of a left and a right A-dgc over a dg bialgebra A is again a dgc, compare [**7**, p. 848].)

The diagonal on the right $C(G)$-module $C(X \times_\tau G)$ is $C(G)$-equivariant, and inspection of the formula (8.13) shows that so is the diagonal on $C(X) \otimes_t C(G)$. Because $\psi = \psi_G$ is also $C(G)$-equivariant, we may assume $y = 1 \in C(G)$. In other words, it suffices to consider elements of the form $x \otimes 1 \in C(X) \otimes_t C(G)$ when checking the claimed identity

$$\Delta\,\psi = (\psi \otimes \psi)\,\Delta. \tag{9.2}$$

We therefore need to look at $\Delta\,\psi(x) = (-1)^{|\boldsymbol{i}|}\,\Delta\,\widehat{\mathrm{S}}\mathrm{z}_{\boldsymbol{i}}\,x$. Combining Lemma 5.2 with Proposition 4.2, we have

$$(\partial_0)^l\,\widehat{\mathrm{S}}\mathrm{z}_{\boldsymbol{i}}\,x = \widehat{\mathrm{S}}\mathrm{z}_{\boldsymbol{i}_2}\,x(p_1 - 1, \dots, p_{k+1} - 1) \tag{9.3}$$

and

$$\sum_{\boldsymbol{i}_1 \in S_n(\boldsymbol{p})} (-1)^{|\boldsymbol{i}|}\,\tilde{\partial}^k\,\widehat{\mathrm{S}}\mathrm{z}_{\boldsymbol{i}}\,x = \sum (-1)^{\varepsilon}\,\widehat{\mathrm{S}}\mathrm{z}_{\boldsymbol{j}_1}\,x(0, \dots, p_1 - 1)$$
$$\cdot\,\mathrm{Sz}_{\boldsymbol{j}_2}\,x(p_1 - 1, \dots, p_2 - 1) \cdots \mathrm{Sz}_{\boldsymbol{j}_{k+1}}\,x(p_k - 1, \dots, p_{k+1} - 1), \tag{9.4}$$

where the sum on the right-hand side is over all $\boldsymbol{j}_1 \in S_{q_1 - 1}$, …, $\boldsymbol{j}_{k+1} \in S_{q_{k+1} - 1}$, and

$$\varepsilon = |\boldsymbol{j}_1| + \dots + |\boldsymbol{j}_k| + \sum_{s=1}^{k} (s-1)(q_s - 1). \tag{9.5}$$

Also, formula (8.13) for the diagonal on $C(X) \otimes_t C(G)$ boils for $x \otimes 1$ down to

$$\Delta\,(x \otimes 1) = \sum_{i=0}^{n} \sum_{k=0}^{n-i} \sum_{\boldsymbol{p}} (-1)^{\varepsilon(\boldsymbol{p}) + i - |z^{\boldsymbol{p}}_{k+1}|}$$
$$\cdot \Bigl(\tilde{\partial}^i\,x \otimes t(z^{\boldsymbol{p}}_1) \cdots t(z^{\boldsymbol{p}}_k)\Bigr) \otimes \Bigl(z^{\boldsymbol{p}}_{k+1} \otimes 1\Bigr) \tag{9.6}$$

where $x \in X_n$, $z = (\partial_0)^{n-i}\,x \in X_i$, and the last sum is over all interval cuts $\boldsymbol{p}$ of $[i]$ corresponding to e_k. To this expression we have to apply the map $\psi \otimes \psi$. Note that the first tensor factor above is of the form

$$\tilde{\partial}^i\,x \otimes t(z^{\boldsymbol{p}}_1) \cdots t(z^{\boldsymbol{p}}_k) = \tilde{\partial}^i\,x \otimes \mathrm{Sz}\,z^{\boldsymbol{p}}_1 \cdots \mathrm{Sz}\,z^{\boldsymbol{p}}_k$$
$$+ \text{additional terms with fewer than } k \text{ factors in the second component.} \tag{9.7}$$

As in Section 6, these additional terms arise whenever a $z^{\boldsymbol{p}}_s$ with $1 \leqslant s \leqslant k$ is of degree 1 because of the extra term $-1 \in C(G)$ in the definition of t in this case.

We first consider the cuts $\boldsymbol{p}$ in (9.6) with $\ell(\boldsymbol{p}) = 0$, that is, where the intervals with labels 1 to k cover all of $[i]$. In this case we conclude the following from (9.3) and (9.4): If we apply $\psi \otimes \psi$ to the terms in (9.6) that correspond to the first line of (9.7), then we exactly get the terms appearing in

$$\sum_{\boldsymbol{i}_1 \in S_n(\boldsymbol{p})} (-1)^{|\boldsymbol{i}|}\,\tilde{\partial}^k\,\widehat{\mathrm{S}}\mathrm{z}_{\boldsymbol{i}}\,x \otimes (\partial_0)^l\,\widehat{\mathrm{S}}\mathrm{z}_{\boldsymbol{i}}\,x \tag{9.8}$$

if we set $i = p_1 - 1$ and $z = x(n - i, \dots, n)$. Moreover, the formula (9.5) tells us that the sign above corresponds with the one in (9.6).

We now proceed to showing that the decompositions $\boldsymbol{p}$ with $\ell(\boldsymbol{p})>0$ lead to summands in (9.6) that cancel out with the additional terms in (9.7) for the $\boldsymbol{p}$ with $\ell(\boldsymbol{p})=0$. The variable $i\in[n]$ in (9.6) is fixed during the following discussion.

We look at a minimal decomposition $\boldsymbol{p}$ of $[i]$ according to the partial ordering introduced in Section 6 and at the $2^{\ell_1(\boldsymbol{p})}$ decompositions $\boldsymbol{p}'\geqslant\boldsymbol{p}$. They all lead to the same $z^{\boldsymbol{p}'}_{k'+1}=z^{\boldsymbol{p}}_{k+1}$, hence to the same second tensor factor $\widehat{\mathrm{S}}\mathrm{z}\,z^{\boldsymbol{p}}_{k+1}$ in

$$(\psi\otimes\psi)\,\Delta\,(x\otimes 1). \tag{9.9}$$

For each such $\boldsymbol{p}'$, the first tensor factor in (9.6),

$$(-1)^{\varepsilon(\boldsymbol{p}')+i+|z^{\boldsymbol{p}'}_{k'+1}|}\,\tilde\partial^i\,x\otimes t(z^{\boldsymbol{p}'}_1)\cdots t(z^{\boldsymbol{p}'}_{k'}), \tag{9.10}$$

contains the term

$$(-1)^{+\ell_1(\boldsymbol{p}')}\Big((-1)^{\varepsilon(\boldsymbol{p})+i+|z^{\boldsymbol{p}}_{k+1}|}\,\tilde\partial^i\,x\otimes\mathrm{Sz}\,z^{\boldsymbol{p}}_1\cdots\mathrm{Sz}\,z^{\boldsymbol{p}}_k\Big), \tag{9.11}$$

because of the contributions $-1\in C(G)$ of each interval of length 1, and also because we have $\varepsilon(\boldsymbol{p}')=\varepsilon(\boldsymbol{p})$ by Lemma 3.4. As before, these terms add up to 0 for $\ell_1(\boldsymbol{p})>0$, which completes the proof.

10. Comparison with Shih's twisted tensor product

We have mentioned in the introduction already that Szczarba's twisting cochain agrees with the one constructed by Shih [**25**, §II.1] using homological perturbation theory. In [**10**, Sec. 7] we pointed out that despite this agreement their approaches lead to different twisted tensor products and different twisted shuffle maps.

Recall that given any cochain $t\colon C\to A$, one can define the twisted tensor products

$$C\otimes_t M \qquad\text{and}\qquad M\otimes_t C \tag{10.1}$$

for a left or, respectively, right A-module, see [**16**, Def. II.1.4] for instance. The twisted tensor products considered so far have been of the first kind.

In Section 9 we have proven that Szczarba's twisted shuffle map

$$\psi\colon C(X)\otimes_t C(F)\to C(X\times_\tau F) \tag{10.2}$$

is a morphism of dgcs, and it is not difficult to see that for $F=G$ it is also a morphism of right $C(G)$-modules [**10**, Prop. 7.1].

Shih on the other hand uses the twisted tensor product $C(F)\otimes_t C(X)$ (where the fibre F is considered as a right G-space). His twisted shuffle map

$$\nabla^\tau\colon C(F)\otimes_t C(X)\to C(F\times_\tau X) \tag{10.3}$$

is part of a contraction that is a homotopy equivalence of right $C(X)$-comodules and, in the case $F=G$, of left $C(G)$-modules, see [**25**, Props. II.4.2 & II.4.3] and [**12**, Lemma 4.5*]. In this sense his result is stronger because it is not known whether Szczarba's map ψ is part of such a homotopy equivalence.[5]

[5] Since the underlying complexes are free and defined for $\Bbbk=\mathbb{Z}$, the map ψ is at least a homotopy equivalence of complexes, *cf.* [**5**, Prop. II.4.3].

On the other hand, there does not seem to be a dgc structure on $C(F) \otimes_t C(X)$. The "mirror image" of (8.5) gives a chain map

$$C(F) \otimes_t C(X) \to \Big(C(F) \otimes_t C(X)\Big) \otimes \Big(C(F) \otimes_t C(X)\Big), \tag{10.4}$$

but it is not coassociative in general because of the asymmetry inherent in the definition of the cooperations E^k. We expect, however, that (10.4) extends to an A_∞-coalgebra structure.

There is a different definition of an hgc, based on cooperations

$$\tilde{E}^k \colon C \to C \otimes C^{\otimes k}, \tag{10.5}$$

which for simplicial sets is realized by the interval cut operations $\tilde{E}^k = AW_{\tilde{e}_k}$ based on the surjections $\tilde{e}_k = (1, 2, 1, \ldots, 1, k, 1)$, *cf.* [**9**, Sec. 4]. In this setting $C(F) \otimes_t C(X)$ would become a dgc with the diagonal (10.4) if Szczarba's twisting cochain t were comultiplicative with respect to this new hgc structure. This is not the case, however, as can be seen for $\langle x \rangle \in \mathbf{\Omega}\, C(X)$ with $x \in X_2$ already.

11. Discrete fibres

In this section we dualize the dgc model from Theorem 1.3 to a dga model for bundles with finite fibres. We also derive a certain spectral sequence converging to the homology of a bundle with discrete fibre that in the context of CW complexes was constructed by Papadima–Suciu [**21**]. For finite fibres we again consider the dual spectral sequence converging to the cohomology of the bundle, which turns out to be a spectral sequence of algebras. In the special case of a p-group it has recently been studied by Rüping–Stephan [**24**].

11.1. The homological spectral sequence

Let G be (the simplicial group associated to) a discrete group, so that $C(G) = C_0(G) = \Bbbk[G]$ is the group ring with coefficients in $\Bbbk$. We write $\mathfrak{a} \lhd \Bbbk[G]$ for the augmentation ideal. For a discrete space F it gives rise to an increasing filtration of $C(F) = C_0(F)$ by the $\Bbbk[G]$-submodules

$$\mathcal{F}_{-p}(F) = \mathfrak{a}^p\, C(F) \tag{11.1}$$

with $p \in \mathbb{N}$ (and the convention $\mathfrak{a}^0 = \Bbbk[G]$). We write $\operatorname{gr}_*(F)$ for the associated graded module over the graded algebra $\operatorname{gr}_*(G)$ with structure map $\operatorname{gr}_* \mu$ induced by the action $\mu \colon G \times F \to F$.

Given a bundle $X \times_\tau F$, we consider the increasing filtration

$$\mathcal{F}_{-p}(X, F) = C(X) \otimes_t \mathcal{F}_{-p}(F) \tag{11.2}$$

of the twisted tensor product $C(X) \otimes_t C(F)$ by subcomplexes. The zeroeth page of the associated spectral sequence is of the form

$$\mathcal{E}^0_{p,q} = C_p(X) \otimes \operatorname{gr}_q(F) \tag{11.3}$$

and lives in the lower half-plane as $q \leqslant 0$.

Since G is discrete, any twisting cochain mapping to $C(G)$ vanishes in all degrees different from 1. It furthermore takes values in the augmentation ideal $\mathfrak{a}$ by the second defining identity in (2.8). Hence the twisting term in the differential of $C(X) \otimes_t C(F)$ lowers the filtration degree. As a result, the induced differential on $\mathcal{E}^0$ is $d^0 = d \otimes 1$, and the first page of the spectral sequence is of the form

$$\mathcal{E}^1_{p,q} = H_{p+q}(X; \operatorname{gr}_q(F)). \tag{11.4}$$

The convergence of this spectral sequence is delicate in general, see [**21**, Sec. 5.3]. However, if the augmentation ideal $\mathfrak{a}$ is nilpotent, meaning that $\mathfrak{a}^L = 0$ for some L, then the filtration is finite and convergence is not an issue.

Let us assume that that $\Bbbk$ is a field or, more generally, that $H(X)$ is torsion-free over the principal ideal domain $\Bbbk$. We then have

$$\mathcal{E}^1_{p,q} = H_{p+q}(X) \otimes \operatorname{gr}_q(F). \tag{11.5}$$

Moreover, $H(X)$ is a graded coalgebra in this case via the composition

$$H(X) \longrightarrow H(X \times X) \xrightarrow{\cong} H(X) \otimes H(X) \tag{11.6}$$

where the second map is the inverse of the Künneth isomorphism.

We need the following observation.

Lemma 11.1. *Let C be a dgc and G a discrete group, and let $t\colon C \to \Bbbk[G]$ be a twisting cochain. Then t induces a well-defined twisting cochain*

$$t_*\colon H(C) \to \operatorname{gr}_*(G), \qquad [c] \mapsto \begin{cases} [t(c)] \in \operatorname{gr}_{-1}(G) & \text{if } |c| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For well-definedness we have to show $t(d\,c) \in \mathcal{F}_{-2}(G)$ for $c \in C_1$. Since there is no differential on $\Bbbk[G]$, we get from the twisting cochain condition (2.8) that

$$t(d\,c) = d\,t(c) + t(d\,c) = (t \cup t)(c) \in \mathcal{F}_{-2}(G), \tag{11.7}$$

again because t takes values in the augmentation ideal $\mathfrak{a} = \mathcal{F}_{-1}(G)$.

For degree reasons this also shows that t_* is a twisting cochain. □

The differential on the first page of the spectral sequence is given by the twisting term (8.2). Using the lemma above and the fact that $H(X)$ is a coalgebra, we can see that this differential is the composition

$$\begin{aligned} \delta_{t_*}\colon H(X) \otimes \operatorname{gr}_*(F) &\xrightarrow{\Delta\otimes 1} H(X) \otimes H(X) \otimes \operatorname{gr}_*(F) \\ &\xrightarrow{1\otimes t_*\otimes 1} H(X) \otimes \operatorname{gr}_*(G) \otimes \operatorname{gr}_*(F) \xrightarrow{1\otimes \operatorname{gr}_* \mu} H(X) \otimes \operatorname{gr}_*(F). \end{aligned} \tag{11.8}$$

In other words, we have an isomorphism of complexes

$$\mathcal{E}^1 = H(X) \otimes_{t_*} \operatorname{gr}_*(F). \tag{11.9}$$

We thus recover the description of the spectral sequence of an equivariant chain complex as given by Papadima–Suciu [**21**, Thm. A], up to the order of the tensor factors.

We now look at coalgebra structures. The filtration $\mathcal{F}(G)$ is comultiplicative in the sense that we have

$$\Delta\,\mathcal{F}_{-p}(F) \subset \sum_{q+r=p} \mathcal{F}_{-q}(F) \otimes \mathcal{F}_{-r}(F) \tag{11.10}$$

for all p. In fact, the claim holds for the bialgebra $F = G$ by induction, starting with the case $p = 1$, which says that

$$\begin{aligned}\Delta(g-1) &= g \otimes g - 1 \otimes 1 = g \otimes (g-1) + (g-1) \otimes 1 \\ &\in \mathcal{F}_0(G) \otimes \mathcal{F}_{-1}(G) + \mathcal{F}_{-1}(G) \otimes \mathcal{F}_0(G)\end{aligned} \tag{11.11}$$

for any $g \in G$. It carries over to F as $C(F)$ is a $\Bbbk[G]$-dgc.

Moreover, inspection of formula (8.5) or (8.13) for the diagonal of $C(X) \otimes_t C(F)$ shows that the filtration $\mathcal{F}(X, G)$ is comultiplicative, too. Taking again into account that the twisting cochain t takes values in the augmentation ideal $\mathfrak{a}$, we see that the diagonal on the page $\mathcal{E}^0$ of the spectral sequence is componentwise,

$$\Delta(c \otimes m) = \sum_{(c),(m)} \bigl(c_{(1)} \otimes m_{(1)}\bigr) \otimes \bigl(c_{(2)} \otimes m_{(2)}\bigr) \tag{11.12}$$

for $c \in C_p(X)$ and $m \in \mathcal{F}_{-q}(F)/\mathcal{F}_{-q-1}(F)$. This implies the following.

Proposition 11.2. *Assume that $\Bbbk$ is a field. The filtration (11.2) gives rise to a spectral sequence of coalgebras. As a dgc, its first page is given by*

$$\mathcal{E}^1_{p,q} = H_{p+q}(X) \otimes_{t_*} \operatorname{gr}_q(F)$$

with the componentwise coproduct. If the augmentation ideal $\mathfrak{a}$ is nilpotent, then the spectral sequence converges to $H(X \times_\tau F)$ as a graded coalgebra.

11.2. Dga models and the cohomological spectral sequence

We now turn to cohomology. For the following purely algebraic reason we restrict to finite structure groups G and finite fibres F.

The dual C^* of a dgc C with coproduct Δ is a dga with the transpose Δ^* as multiplication, or more precisely, with the composition

$$C^* \otimes C^* \to (C \otimes C)^* \xrightarrow{\Delta^*} C^*. \tag{11.13}$$

However, the dual of a dga A is not a dgc in general, but it is so if C is finitely generated free $\Bbbk$-module in each degree. The coproduct is the transpose μ^* of the multiplication or rather its composition with the isomorphism $(A \otimes A)^* \cong A^* \otimes A^*$.

So let us assume that G is finite.[6] Then $C^*(G)$ is a dgc, and of course $C^*(X)$ is a dga for any X. Because of the definition

$$d_{C^*} = -d_C^* \tag{11.14}$$

of the differential on a dual complex as the *negative* of the transpose of the original

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[6]This restriction is missing for the multiplicative model stated in [**17**, p. 219]. Together with the assumption of simple connectedness made there for the base space X (see Footnote 1), that model boils down to the tensor product $C^*(X) \otimes C^*(F)$ for Cartesian products satisfying an appropriate finiteness condition.

one (compare [**9**, Sec. 2.3]), the transpose

$$t^*\colon C^*(G) \to C^*(X) \tag{11.15}$$

of Szczarba's twisting cochain satisfies

$$\begin{aligned} d(t^*) &= d_{C^*(X)}\, t^* + t^*\, d_{C^*(G)} = -\bigl(d^*_{C(X)}\, t^* + t^*\, d^*_{C(G)}\bigr) \\ &= -\bigl(t\, d_{C(X)} + d_{C(G)}\, t\bigr)^* = -d(t)^* = -(t \cup t)^* = -t^* \cup t^*. \end{aligned} \tag{11.16}$$

In other words, $u = -t^*$ is again a twisting cochain in our sense.

The quasi-isomorphism $C(X) \otimes_t C(F) \to C(X \times_\tau F)$ from Theorem 1.3 dualizes to a quasi-isomorphism of dgas between $C^*(X \times_\tau F)$ and the dual of $C(X) \otimes_t C(F)$. If the fibre F is finite, then we have an isomorphism of complexes

$$\bigl(C(X) \otimes_t C(F)\bigr)^* = C^*(X) \otimes_u C^*(F), \tag{11.17}$$

which is now a twisted tensor product of the second form in (10.1). The minus sign in $u = -t^*$ arises again from (11.14) and also reflects the sign difference between the two kinds of twisted tensor products, see again [**16**, Def. II.1.4].

The product on (11.17) is as described by Kadeishvili–Saneblidze [**17**, eq. (12)]. With our sign convention and in Sweedler notation it is of the form

$$(a \otimes b) \cdot (a' \otimes b') = \sum_{k \geqslant 0} \sum_{(b)} (-1)^k\, a\, E_k(u(b_{(1)}), \dots, u(b_{(k)}); a') \otimes b_{(k+1)}\, b' \tag{11.18}$$

for a, $a' \in C^*(X)$ and b, $b' \in C^*(F)$. The transposes

$$E_k = \bigl(E^k\bigr)^*\colon C^*(X)^{\otimes k} \otimes C^*(X) \to C^*(X) \tag{11.19}$$

are the structure maps of the hga $C^*(X)$, see Remark 3.2. Note that the sum over k in (11.18) is in fact only over $0 \leqslant k \leqslant |b| + |a'|$ because of the vanishing condition (3.4).

We summarize our discussion so far as follows.

Proposition 11.3. *Let $X \times_\tau F$ be a fibre bundle where both the fibre F and the structure group G have only finitely many non-degenerate simplices. It follows that the dga $C^*(X) \otimes_u C^*(F)$ with the product (11.18) is a model for $X \times_\tau F$. The quasi-isomorphism connecting this dga with $C^*(X \times_\tau F)$ is natural in X, G and F.*

We now look at the duals of the filtrations introduced in the previous section. Because the filtrations $\mathcal{F}(F)$ and $\mathcal{F}(X, F)$ are comultiplicative, the dual filtrations of $C^*(F) = C^0(F)$ and $C^*(X) \otimes_u C^*(F)$,

$$\mathcal{F}^{-p}(F) = \bigl\{\, \gamma \in C^0(F) \bigm| \gamma(m) = 0 \text{ for all } m \in \mathcal{F}_{-p-1}(F) \,\bigr\}, \tag{11.20}$$

$$\mathcal{F}^{-p}(X, F) = C^*(X) \otimes_u \mathcal{F}^{-p}(F) \tag{11.21}$$

are multiplicative. Specializing to field coefficients, we arrive at the following conclusion. It generalizes a result of Rüping–Stephan [**24**, Cor. 4.19] for finite p-groups and coefficients of prime characteristic p, see also [**24**, Rem. 4.20].

Proposition 11.4. *Let $\Bbbk$ be a field, and let G be a finite group such that the augmentation ideal $\mathfrak{a} \lhd \Bbbk[G]$ is nilpotent. There is a multiplicative spectral sequence $\mathcal{E}_r$ converging to $H^*(X \times_\tau F)$ whose first page is of the form*

$$\mathcal{E}_1^{p,q} = H^{p+q}(X) \otimes \operatorname{gr}^q(F)$$

with componentwise product, where $\operatorname{gr}^*(F)$ is the graded algebra associated to the filtration (11.20). The spectral sequence is natural in X, G and F.

12. The Serre spectral sequence

Theorem 1.3 allows for a short proof of the product structure in the cohomological Serre spectral sequence. The same applies to the comultiplicative structure in the homological setting considered by Chan [**4**, Thm. 1.2]. We assume throughout this section that $\Bbbk$ is a principal ideal domain.

Recall that if the homology $H(C)$ of a dgc C is free over $\Bbbk$, then it is a graded coalgebra with diagonal

$$H(C) \longrightarrow H(C \otimes C) \xrightarrow{\cong} H(C) \otimes H(C) \tag{12.1}$$

where the last map is the inverse of the Künneth isomorphism. (We have mentioned a special case of this already in (11.6).)

Proposition 12.1. *Let $E = X \times_\tau F$ be a twisted Cartesian product with the simplicial group G as structure group.*

(i) *Assume that $H(X)$ and $H(F)$ are free over $\Bbbk$ and that G_0 acts trivially on $H(F)$. The homological Serre spectral sequence is a spectral sequence of coalgebras with the componentwise coproduct on*

$$\mathcal{E}^2_{pq} = H_p(X) \otimes H_q(F),$$

converging to $H(E)$ as a coalgebra.

(ii) *Assume that F is of finite type, that $H^*(X)$ or $H^*(F)$ is flat over $\Bbbk$ and that G_0 acts trivially on $H^*(F)$. The cohomological Serre spectral sequence is a spectral sequence of algebras with the componentwise product on*

$$\mathcal{E}_2^{pq} = H^p(X) \otimes H^q(F),$$

converging to $H^*(E)$ as an algebra.

Proof. By Theorem 1.3, the dgc $C(E)$ is quasi-isomorphic to $M = C(X) \otimes_t C(F)$ with the coproduct (3.17). We filter M by increasing degree in $C(X)$ and then $M \otimes M$ via the tensor product filtration. Let $\mathcal{E}^r$ be the associated spectral sequence converging to $H(M)$ and $\mathcal{F}^r$ the one converging to $H(M \otimes M)$.

Since G_0 acts trivially on $H(F)$, the definition (5.3) of Szczarba's twisting cochain tells us that this module is annihilated by $t(x)$ for any $x \in X_1$. Therefore,

$$\mathcal{E}^0_{pq} = C_p(X) \otimes C_q(F), \qquad d^0 = 1 \otimes d, \tag{12.2}$$

$$\mathcal{E}^1_{pq} = C_p(X) \otimes H_q(F), \qquad d^1 = d \otimes 1, \tag{12.3}$$

$$\mathcal{E}^2_{pq} = H_p(X) \otimes H_q(F) \tag{12.4}$$

and similarly

$$\mathcal{F}^1_{pq} = \bigoplus_{p_1+p_2=p} \bigoplus_{q_1+q_2=q} C_{p_1}(X) \otimes H_{q_1}(F) \otimes C_{p_2}(X) \otimes H_{q_2}(F), \tag{12.5}$$

$$\mathcal{F}^2_{pq} = \bigoplus_{p_1+p_2=p} \bigoplus_{q_1+q_2=q} H_{p_1}(X) \otimes H_{q_1}(F) \otimes H_{p_2}(X) \otimes H_{q_2}(F). \tag{12.6}$$

Inspection of the formula (3.17) shows that the coproduct is filtration-preserving and that the induced maps between the first and second pages of the spectral sequences are the componentwise diagonals: In the notation of Sections 6 and 9, summands corresponding to partitions $\boldsymbol{p}$ with $\ell_1(\boldsymbol{p}) > 0$ do not contribute, again by the annihilation property of t mentioned above, and among the remaining ones those with $\ell(\boldsymbol{p}) < p$ end up in a lower filtration degree. This proves the first part.

The transpose $\psi^*\colon C^*(E) \to M^*$ of ψ is a quasi-isomorphism of dgas. We filter M^* by the dual filtration, which leads to a spectral sequence $\mathcal{E}_r$ converging to $H^*(E)$. Since F is of finite type, we have

$$\mathcal{E}_0^{pq} = \bigl(C_p(X) \otimes C_q(F)\bigr)^*, \tag{12.7}$$

$$\mathcal{E}_1^{pq} = C^p(X) \otimes H^q(F) \tag{12.8}$$

by the cohomological Künneth theorem [**5**, Prop. VI.10.24, case II], hence

$$\mathcal{E}_2^{pq} = H^p(X) \otimes H^q(F) \tag{12.9}$$

by its homological counterpart [**5**, Thm. VI.9.13] and the assumption that $t(x)$ annihilates $H^*(F)$ for any $x \in X_1$. By the same argument as before, the products on (12.8) and (12.9) are componentwise. This concludes the proof. □

Appendix A. Comparison with Baues' diagonal

Baues [**1**, Sec. 1] has defined a diagonal on $\boldsymbol{\Omega}\, C(X)$ for any 1-reduced simplicial set X. In this appendix we compare his map with the diagonal (3.17) induced by the hgc structure of $C(X)$ (which of course is defined for any $X \neq \varnothing$). Up to sign, this has already been done by Quesney [**22**, Prop. 5.1].

Proposition A.1. *For a* 1*-reduced simplicial set X the diagonal* (3.17) *on* $\boldsymbol{\Omega}\, C(X)$ *is the same as Baues'.*

This implies that the diagonal (3.17) is also equal to the one constructed by Hess–Parent–Scott–Tonks via homological perturbation theory [**14**, Secs. 4 & 5].[7]

Proof. Let $x \in X$ be an n-simplex. The terms in Baues' formula for $\Delta\,\langle x\rangle$ [**1**, p. 334] are indexed by the subsets $b \subset \underline{n-1}$. It not difficult to see that in analogy with

[7]Hess–Parent–Scott–Tonks state that their recursively defined diagonal agrees with Baues'. This includes the sign of each summand (A.3), which is not made explicit for their own formula.

formula (3.7) Baues' diagonal is of the form

$$\Delta\langle x\rangle = \langle x\rangle \otimes 1 + \sum_{k=0}^{\infty} \tilde{\mathbf{E}}^k(x) \tag{A.1}$$

for certain functions

$$\tilde{\mathbf{E}}^k \colon C(X) \to \mathbf{\Omega}_k\, C(X) \otimes \mathbf{\Omega}_1\, C(X). \tag{A.2}$$

Moreover, each non-zero summand appearing in $\tilde{\mathbf{E}}^k(x)$ can be written as

$$\pm\big\langle\, x_1^{\boldsymbol{p}} \,\big|\, \dots \,\big|\, x_k^{\boldsymbol{p}} \,\big\rangle \otimes \langle x_{k+1}^{\boldsymbol{p}}\rangle \tag{A.3}$$

for the unique interval cut $\boldsymbol{p}$ of $[n]$ associated to e_k such that $x_{k+1}^{\boldsymbol{p}}$ contains the vertices indexed by b plus 0 and n. Hence, up to sign, we get the claimed identity

$$\mathbf{s}^{\otimes(k+1)}\, \tilde{\mathbf{E}}^k(x) = \mathbf{s}^{\otimes(k+1)}\, \mathbf{E}^k(x) = (-1)^k AW_{e_k}(x). \tag{A.4}$$

It remains to verify the sign, where we proceed by induction on k. The case $k=0$ is trivial because $AW_{(1)}$ is the identity map and $\tilde{\mathbf{E}}^0 = \mathbf{s}^{-1}$ the inverse of the suspension map $\mathbf{s}$.

For $k>1$ we compare the signs associated to an interval cut

$$\boldsymbol{p} : p_0 \xrightarrow{k+1} p_1 \xrightarrow{1} p_2 \xrightarrow{k+1} \cdots \xrightarrow{k+1} p_{2k-1} \xrightarrow{k} p_{2k} \xrightarrow{k+1} p_{2k+1} \tag{A.5}$$

for the surjection e_k with those for the interval cut

$$\boldsymbol{p}' \colon p_0 \xrightarrow{k} p_1 \xrightarrow{1} p_2 \xrightarrow{k} \cdots \xrightarrow{k} p_{2k-1} \tag{A.6}$$

for e_{k-1}. We compute the exponents of all the signs involved, always modulo 2. The exponents of the permutation signs differ by

$$\begin{aligned} \operatorname{perm}(\boldsymbol{p}) - \operatorname{perm}(\boldsymbol{p}') &\equiv (p_{2k} - p_{2k-1})\Bigg(p_1 + 1 + \sum_{i=1}^{k-1}(p_{2i+1} - p_{2i} + 1)\Bigg) \\ &\equiv (p_{2k} - p_{2k-1})\Bigg(\sum_{i=1}^{2k-1} p_i + k\Bigg) \end{aligned} \tag{A.7}$$

since we have to move the interval corresponding to $e_k(2k) = k$ before all preceding (inner) intervals corresponding to $e_k(1) = e_k(3) = \cdots = e_k(2k-1) = k+1$. The exponents of the position signs change by p_{2k-1} because of the additional inner interval for $e_k(2k-1) = k+1$.

The sign for the summand (A.3) is the sign of the shuffle[8] $(\underline{n-1} \setminus b, b)$. Hence, by passing from $k-1$ to k, the exponent of this sign changes by

$$\begin{aligned} &(p_{2k} - p_{2k-1} - 1)\Bigg(p_1 + \sum_{i=1}^{k-1}(p_{2i+1} - p_{2i} + 1)\Bigg) \\ &\qquad\qquad \equiv (p_{2k} - p_{2k-1} - 1)\Bigg(\sum_{i=1}^{2k-1} p_i + k + 1\Bigg) \end{aligned} \tag{A.8}$$

[8]Strictly speaking, this is not a shuffle in the sense of Section 2.4 as $0 \notin \underline{n-1} = \{1,\dots,n-1\}$.

because we have to move all elements in the interior of the k-th interval before all previous values occurring in b, that is, all vertices in $x^{\boldsymbol{p}}_{k+1}$ with indices strictly between 0 and p_{2k-1}.

Still modulo 2, the changes in the exponents add up to

$$\sum_{i=1}^{2k} p_i + k + 1 \equiv \sum_{i=1}^{k} (p_{2i} - p_{2i-1} + 1) + 1 \equiv \bigl| \bigl\langle x^{\boldsymbol{p}}_1 | \dots | x^{\boldsymbol{p}}_k \bigr\rangle \bigr| + 1. \tag{A.9}$$

This is exactly the exponent of the sign change we get when we pass from $k-1$ to k in (A.4). The sign exponent $|\langle x^{\boldsymbol{p}}_1 | \dots | x^{\boldsymbol{p}}_k \rangle|$ arises because we have to move the additional suspension operator past the element $\langle x^{\boldsymbol{p}}_1 | \dots | x^{\boldsymbol{p}}_k \rangle$. Another minus sign comes from the increased exponent on the right-hand side of (A.4). This completes the proof. ☐

Appendix B. Szczarba operators and degeneracy maps

Apparently, neither in Szczarba's paper [**26**] nor elsewhere in the literature one can find a proof that Szczarba's twisting cochain (5.3) and his twisted shuffle map (5.16) are actually well-defined on normalized chain complexes. The purpose of this appendix is to close this gap.

Recall from [**26**, eq. (3.1)] and [**15**, eq. (6)] that the simplicial operators

$$D_{\boldsymbol{i},k} \colon X_m \to X_{m+k} \tag{B.1}$$

for $0 \leqslant k \leqslant n$, $\boldsymbol{i} \in S_n$ and $m \geqslant n-k$ are recursively defined by

$$D_{\varnothing,0} = \mathrm{id} \qquad \text{and} \qquad D_{\boldsymbol{i},k} = \begin{cases} D'_{\boldsymbol{i}',k}\, s_0\, \partial_{i_1-k} & \text{if } k < i_1, \\ D'_{\boldsymbol{i}',k} & \text{if } k = i_1, \\ D'_{\boldsymbol{i}',k-1}\, s_0 & \text{if } k > i_1 \end{cases} \tag{B.2}$$

for $n \geqslant 1$ where $\boldsymbol{i}' = (i_2, \dots, i_n)$. Here D' denotes the derived operator of a simplicial operator D, compare [**26**, p. 199] or [**15**, p. 1863].

For $n \geqslant 1$ we introduce a map

$$\Phi \colon S_n \times [n] \to S_{n-1} \times [n-1], \qquad (\boldsymbol{i}, p) \mapsto (\boldsymbol{j}, q) \tag{B.3}$$

recursively via

$$\begin{cases} \boldsymbol{j} = (i_1 - 1, \boldsymbol{j}'), & q = q' + 1 \quad \text{if } p < i_1, \qquad (\boldsymbol{j}', q') \coloneqq \Phi(\boldsymbol{i}', p), \\ \boldsymbol{j} = \boldsymbol{i}', & q = 0 \quad \text{if } p = i_1 \text{ or } i_1 + 1, \\ \boldsymbol{j} = (i_1, \boldsymbol{j}'), & q = q' + 1 \quad \text{if } p > i_1 + 1, \quad (\boldsymbol{j}', q') \coloneqq \Phi(\boldsymbol{i}', p-1) \end{cases} \tag{B.4}$$

where again $\boldsymbol{i}' = (i_2, \dots, i_n)$. Note that the base case $n = 1$ is completely covered by the second line above since $i_1 = 0$ in that case.

Lemma B.1. *Let $n \geqslant 1$, $\boldsymbol{i} \in S_n$ and $p \in [n]$, and set $(\boldsymbol{j}, q) = \Phi(\boldsymbol{i}, p)$.*

(i) *For any $0 \leqslant k < p$ and any simplex x of dimension $m \geqslant n-k-1$ we have*

$$D_{\boldsymbol{i},k}\, s_{p-1-k}\, x = s_q\, D_{\boldsymbol{j},k}\, x.$$

(ii) *For any $p < k \leqslant n$ and any simplex x of dimension $m \geqslant n-k$ we have*

$$D_{\boldsymbol{i},k}\, x = s_q\, D_{\boldsymbol{j},k-1}\, x.$$

Proof. These are direct verifications by induction on n, based on the definitions of $D_{\boldsymbol{i},k}$ and Φ. The base cases are $\boldsymbol{i} = (0)$, $k = 0$, $p = 1$ and $\boldsymbol{i} = (0)$, $k = 1$, $p = 0$, respectively. In the induction step of the first formula, one distinguishes the cases $k < i_1$ (with the subcases $i_1 < p - 1$, $i_1 \in \{p - 1, p\}$ and $i_1 > p$), $k = i_1$ (with the subcases $i_1 < p - 1$ and $i_1 = p - 1$) and $k > i_1$. For the second formula one has the cases $k < i_1$, $k = i_1$ and $k > i_1$ (with the subcases $p < i_1$, $p \in \{i_1, i_1 + 1\}$ as well as $p > i_1 + 1$).

For instance, for $n > 1$, $k < i_1$ and $i_1 > p$ we have

$$\begin{aligned} \mathrm{Sz}_{\boldsymbol{i}}\ s_p\, x &= D'_{\boldsymbol{i}',k}\, s_0\, \partial_{i_1-k}\, s_{p-1-k}\, x = D'_{\boldsymbol{i}',k}\, s_0\, s_{p-1-k}\, \partial_{i_1-k-1}\, x \\ &= D'_{\boldsymbol{i}',k}\, s_{p-k}\, s_0\, \partial_{i_1-k-1}\, x = \big(D_{\boldsymbol{i}',k}\, s_{p-1-k}\big)'\, s_0\, \partial_{i_1-k-1}\, x \\ &= \big(s_{q'}\, D_{\boldsymbol{j}',k}\big)'\, s_0\, \partial_{i_1-k-1}\, x \end{aligned} \tag{B.5}$$

by induction, where $(\boldsymbol{j}', q') = \Phi(\boldsymbol{i}', p)$. Then $\boldsymbol{j} = (i_1 - 1, \boldsymbol{j}')$ and $q = q' + 1$, hence

$$= s_{q'+1}\, D'_{\boldsymbol{j}',k}\, s_0\, \partial_{i_1-1-k}\, x = s_q\, D'_{\boldsymbol{j},k}\, x$$

since $k < p \leqslant i_1 - 1$. □

Proposition B.2. *Let $0 \leqslant p \leqslant n$, and let x be an n-simplex.*

(i) *For $\boldsymbol{i} \in S_n$ and $(\boldsymbol{j}, q) = \Phi(\boldsymbol{i}, p)$ we have*

$$\mathrm{Sz}_{\boldsymbol{i}}\ s_p\, x = s_q\ \mathrm{Sz}_{\boldsymbol{j}}\ x.$$

(ii) *For $\boldsymbol{i} \in S_{n+1}$ and $(\boldsymbol{j}, q) = \Phi(\boldsymbol{i}, p + 1)$ we have*

$$\widehat{\mathrm{S}}\mathrm{z}_{\boldsymbol{i}}\ s_p\, x = s_q\ \widehat{\mathrm{S}}\mathrm{z}_{\boldsymbol{j}}\ x.$$

Proof. These formulas follow from Lemma B.1 and the identities (2.17) and (2.18). For example, we have

$$\begin{aligned} \mathrm{Sz}_{\boldsymbol{i}}\ s_p\, x &= D_{\boldsymbol{i},0}\, \sigma(s_p\, x)\, D_{\boldsymbol{i},1}\, \sigma(\partial_0\, s_p\, x) \cdots D_{\boldsymbol{i},n}\, \sigma((\partial_0)^n s_p\, x) \\ &= D_{\boldsymbol{i},0}\, s_{p-1}\, \sigma(x) \cdots D_{\boldsymbol{i},p-1}\, s_0\, \sigma((\partial_0)^{p-1}\, x) D_{\boldsymbol{i},p}\, \sigma(s_0\, (\partial_0)^p\, x) \\ &\qquad \cdot D_{\boldsymbol{i},p+1}\, \sigma((\partial_0)^p\, x) \cdots D_{\boldsymbol{i},n}\, \sigma((\partial_0)^{n-1}\, x) \\ &= s_q\, D_{\boldsymbol{j},0}\, \sigma(x) \cdots s_q\, D_{\boldsymbol{j},p-1}\, \sigma((\partial_0)^{p-1}\, x) \cdot 1 \\ &\qquad \cdot s_q\, D_{\boldsymbol{j},p}\, \sigma((\partial_0)^p\, x) \cdots s_q\, D_{\boldsymbol{j},n-1}\, \sigma((\partial_0)^{n-1}\, x) \\ &= s_q\ \mathrm{Sz}_{\boldsymbol{j}}\, x. \end{aligned} \tag{B.6}$$

□

Corollary B.3. *Szczarba's twisting cochain t and the twisted shuffle map ψ descend to the normalized chain complexes.*

Proof. This is a consequence of the formulas just established and, for the twisting cochain t, the identity $t(s_0\, x) = \sigma(s_0\, x) - 1 = 0$ for any 0-simplex x. □

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