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CONFIGURATION SPACES OF CLUSTERS AS E_d-ALGEBRAS

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Abstract

It is a classical result that configuration spaces of labelled particles in $\mathbb{R}^d$ are free E_d-algebras and that their d-fold bar construction is equivalent to the d-fold suspension of the labelling space.

In this paper, we study a variation of these spaces, namely configuration spaces of labelled *clusters* of particles. These configuration spaces are again E_d-algebras, and we give geometric models for their iterated bar construction in two different ways: one establishes a description of these configuration spaces of clusters as *cellular* E_1-algebras, and the other one uses an additional *verticality* constraint. In the last section, we apply these results in order to calculate the stable homology of certain *vertical* configuration spaces.

1. Introduction and overview

Let us start with the classical definition of configuration spaces: for a space E and a natural number $r \geqslant 0$, the *ordered configuration space of* r *particles in* E is defined to be

$$\tilde{C}_r(E) \coloneqq \{(z_1, \ldots, z_r) \in E^r;\ z_i \neq z_j \text{ for } i \neq j\}.$$

The r^{th} symmetric group $\mathfrak{S}_r$ acts freely on $\tilde{C}_r(E)$ by permuting coordinates, and we call the quotient $C_r(E) \coloneqq \tilde{C}_r(E)/\mathfrak{S}_r$ the *unordered configuration space of* r *particles in* E. For a based space X, we define the *labelled configuration space* $C(E; X)$ as the union of all $\tilde{C}_r(E) \times_{\mathfrak{S}_r} X^r$, quotiented by the relation that identifies $[z_1, \ldots, z_r; x_1, \ldots, x_r]$ with $[z_1, \ldots, \widehat{z_i}, \ldots, z_r; x_1, \ldots, \widehat{x_i}, \ldots, x_r]$ if x_i is the basepoint of X. Visually, each particle z_i carries a label $x_i \in X$, and if the label reaches the basepoint, then this particle vanishes.

For the case $E = \mathbb{R}^d$ with $d \geqslant 1$, the labelled configuration space $C(\mathbb{R}^d; X)$ is an E_d-algebra, more precisely: it admits an action of the little d-cubes operad $\mathcal{C}_d$ by inserting configurations into boxes [**14**]. It is even equivalent to the *free* E_d-algebra over X, and its d-fold bar construction is equivalent to the d-fold suspension $\Sigma^d X$, see [**19**].

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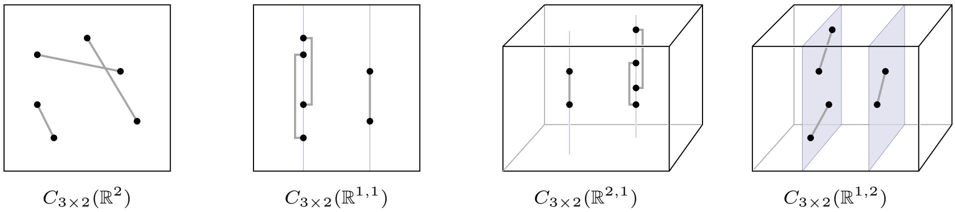

Figure 1: Several configuration spaces of three 2-clusters inside $\mathbb{R}^2$ and $\mathbb{R}^3$.

This paper studies variations of these labelled configuration spaces, which additionally carry the information that some of the particles 'belong together'—that is: they form a *cluster*—, and investigates their structure as E_d-algebras.

Definition 1.1. Let E be a space and let $k \geqslant 1$ be an integer. A *k-cluster* is an ordered configuration of k distinct particles in E. For each integer $r \geqslant 0$, let $\tilde{C}_{r\times k}(E)$ be the space of r disjoint k-clusters. Then $\mathfrak{S}_k \wr \mathfrak{S}_r$ acts on $\tilde{C}_{r\times k}(E)$ by permuting particles within the same cluster, and by permuting clusters. The quotient $C_{r\times k}(E)$ is called the *configuration space of r k-clusters in E.*

Intuitively, $C_{r\times k}(E)$ parametrises unordered collections of pairwise disjoint subsets of E, all of cardinality k, see the first case of Figure 1: it is a covering space of $C_{r\cdot k}(E)$. The above definition has a labelled counterpart (see Definition 2.7 for details):

Definition 1.2. For a well-based space X, we define the configuration space $C^k(E; X)$ of unordered k-clusters in E; each cluster carries a label inside X, and if the label reaches the basepoint, then the entire cluster vanishes.

Definitions 1.1 and 1.2 can be given in slightly higher generality, by allowing configuration of clusters with different sizes and balancing the internal ordering of a cluster with a given symmetric action on the labelling space: this is done in Section 2.

As one can easily see, the configuration space $C^k(\mathbb{R}^d; X)$ again admits the structure of an E_d-algebra by inserting configurations of clusters into boxes. One of our goals is to give a geometric interpretation of the d-fold bar construction of $C^k(\mathbb{R}^d; X)$. While this seems to be hard in general, we can give an answer in the case $d = 1$: In Section 3, we decompose $C^k(\mathbb{R}; X)$ into 'free components', i.e. we give an E_1-cellular decomposition in the sense of [**7**, **8**, **10**]. For this purpose, we define what it means for a collection of clusters to be *entangled.* This gives rise to an E_1-filtration $\mathcal{F}_\bullet C^k(\mathbb{R}; X)$ such that each $\mathcal{F}_w C^k(\mathbb{R}; X)$ arises from $\mathcal{F}_{w-1} C^k(\mathbb{R}; X)$ by attaching free E_1-algebras. Using that the bar construction turns E_1-cell attachments into usual cell attachments, we show:

Theorem A. *There is a weak equivalence $BC^k(\mathbb{R}; X) \simeq \Sigma \bigvee_e X^{\wedge \# e}$, where e ranges in a set of 'entanglement types' (Construction 3.2) and has a weight $\# e \geqslant 1$.*

This result is surprising for two reasons: first, it shows that the attaching maps for vanishing clusters simplify drastically when applying the bar construction, and second, it implies that if X path-connected, then $C^k(\mathbb{R}; X)$ is equivalent to a free E_1-algebra.

In addition to that, we can give a partial answer for the cases $d \geqslant 2$. To this aim, we introduce a slightly more general family of configuration spaces of clusters (see also Definition 4.1):

Definition 1.3. For an integer $0 \leqslant p \leqslant d$, and k and r as above, we define the subspace $C_{r\times k}(\mathbb{R}^{p,d-p}) \subseteq C_{r\times k}(\mathbb{R}^d)$, called *vertical configuration space*, where particles within the same cluster share their first p coordinates. For a based space X, we define the subspace $C^k(\mathbb{R}^{p,d-p};X) \subseteq C^k(\mathbb{R}^d;X)$ with the same constraint.

Several of these configuration spaces are depicted in Figure 1. In the case $p = d-1$, we require that all particles of the same cluster lie on a common vertical line; hence the terminology. It is not hard to see that $C^k(\mathbb{R}^{p,d-p};X) \subseteq C^k(\mathbb{R}^d;X)$ is an E_d-subalgebra, and it turns out that the first p delooping steps are manageable by a straightforward adaption of the methods from Segal's argument [19]:

Theorem B. *$B^pC^k(\mathbb{R}^{p,d-p};X) \simeq C^k(\mathbb{R}^{d-p};\Sigma^pX)$ as E_{d-p}-algebras.*

Informally, this means that the first p delooping steps 'resolve' the verticality constraint. This is perhaps not so surprising: in the E_d-algebra $C^k(\mathbb{R}^{p,d-p};X)$, clusters play the rôle of particles in the classical labelled configuration space, and from the perspective of the first p coordinates, they also behave as such. Theorems A and B are special cases of Theorems 3.4 and 4.5, respectively: those also cover the case of configuration spaces of clusters with different sizes and balanced labels.

Combining Theorems A and B, we obtain a model for the iterated bar construction of the E_{p+1}-algebra $C^k(\mathbb{R}^{p,1};X)$. This can be used to calculate the *stable* homology of these spaces: it is shown in [2] that adding a new cluster $C_{r\times k}(\mathbb{R}^{p,1}) \to C_{(r+1)\times k}(\mathbb{R}^{p,1})$ is homologically stable for $p \geqslant 1$. We determine the stable homology $H_\bullet(C_{\infty\times k}(\mathbb{R}^{p,1}))$ as follows: there is a distinguished entanglement type e_0 (see Construction 3.2) corresponding to a *single* k-cluster. To each finitely supported family $\lambda = (\lambda_e)_{e\neq e_0}$ of integers $\lambda_e \geqslant 0$, where e ranges in the set of all entanglement types of k-clusters, we assign a shifting parameter $s(\lambda)$ and a graded module $M_\bullet(\mathbb{R}^{p+1};\lambda[\infty])$, which is the (twisted) stable homology of a sequence of certain *coloured* configuration spaces [16], the stabilisation step given by adding particles of colour e_0, see Construction 5.3. We then show the following:

Theorem C. *For each $p,k \geqslant 1$, we have an isomorphism of graded modules*

$$H_\bullet(C_{\infty\times k}(\mathbb{R}^{p,1})) \cong \textstyle\bigoplus_\lambda M_{\bullet-p\cdot s(\lambda)}(\mathbb{R}^{p+1};\lambda[\infty]).$$

The corresponding *unstable* modules $M_\bullet(\mathbb{R}^{p+1};\lambda[n])$ already appeared in [2] to describe the homology of certain filtration quotients of $C_{r\times k}(\mathbb{R}^{p,1})$; however, it remained open if the associated spectral sequence collapses on its first page and if the extension problem is trivial. Theorem C tells us that this is at least stably the case.

Related work Configuration spaces of clusters have been studied from the perspective of homological stability [17, 22] and in relation to Hurwitz spaces [21]. Moreover, the 'clustering' of particles is useful to describe an enhancement of the little d-cubes operad $\mathcal{C}_d$ that acts on moduli spaces of manifolds with *multiple* boundary components; this is a leading principle in the author's PhD thesis [12].

Vertical configuration spaces, especially their higher homotopy groups and their homological stability, have been studied in [**2**, **9**, **13**, **18**]. They are also closely related to *fibrewise* configuration spaces, which appear in [**4**] in order to formulate an approximation theorem for configurations with twisted labels and labels in partial abelian monoids. Moreover, these spaces assemble into a coloured operad $\mathcal{V}_{p,d-p}$, which is similar to the extended Swiss cheese operad [**23**] and acts on moduli spaces of d-dimensional manifolds with p-dimensional foliations, see [**3**, **12**] for the case of surfaces with a 1-dimensional foliation.

OUTLOOK We still do not know what the iterated bar construction of the full E_d-algebra $C^k(\mathbb{R}^d; X)$ looks like for $d \geqslant 2$. One might try to enhance the E_d-cellular methods for the case $d = 1$ to the general case; however, we lack a good notion of higher-dimensional entanglement types. On the other hand, it would already be interesting to know if the E_d-algebra $C^k(\mathbb{R}^d; X)$ is equivalent to a free E_d-algebra if X is path-connected.

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2. Basic constructions

In this section, we formally introduce the aforementioned configuration spaces of (labelled) clusters and make their E_d-algebra structure explicit. As already mentioned in the introduction, this can be done without much further effort in slightly higher generality, by allowing configurations of clusters with different sizes at the same time.

Definition 2.1. Let E be a space and $K = (k_1, \ldots, k_r)$ be a tuple of integers $k_i \geqslant 1$. We start with a reindexing and let $\tilde{C}_K(E) \coloneqq \tilde{C}_{k_1+\cdots+k_r}(E)$, but we denote its elements by tuples $(\vec{z}_1, \ldots, \vec{z}_r)$, where $\vec{z}_i = (z_{i,1}, \ldots, z_{i,k_i})$, and we call $\vec{z}_i$ a *cluster of size* k_i.

If we denote by $r(k) \geqslant 0$ the number of occurrences of k in the tuple K, then the group $\mathfrak{S}_K \coloneqq \prod_{k \geqslant 1} \mathfrak{S}_k \wr \mathfrak{S}_{r(k)}$ acts on $\tilde{C}_K(E)$ by permuting clusters of the same size and by permuting the internal ordering of each cluster. We call the quotient $C_K(E) \coloneqq \tilde{C}_K(E)/\mathfrak{S}_K$ the *configuration space of clusters in* E and denote elements in $C_K(E)$ as (unordered) sums $\sum_{i=1}^{r} [\vec{z}_i]$ of unordered clusters $[\vec{z}_i] = [z_{i,1}, \ldots, z_{i,k_i}] = \{z_{i,1}, \ldots, z_{i,k_i}\}$.

Example 2.2. Let $k \geqslant 1$ and $r \geqslant 0$ be integers. If $r \times k \coloneqq (k, \ldots, k)$ denotes the tuple of length r, then $C_{r\times k}(E)$ is exactly the configuration space from the introduction.

Labelled configuration spaces of clusters should generalise the classical notion of a labelled configuration space in the following way: we want to assign to each k-cluster a label inside a based space X_k and balance the internal ordering of each cluster with a given symmetric action on X_k. In order to make this definition precise, we first have to introduce an indexing category, which is a special case of the Grothendieck construction and generalises the notion of a wreath product $G \wr \mathfrak{S}_r = G^r \rtimes \mathfrak{S}_r$.

Definition 2.3. Let **Inj** be the small category with objects $\underline{r} := \{1, \dots, r\}$ for all non-negative integers $r \in \{0, 1, 2, \dots\}$, and with morphisms $\underline{r} \to \underline{r}'$ being all injective maps of finite sets. Then **Inj** is spanned by two sorts of maps: on the one hand *permutations* $\tau \in \mathfrak{S}_r$, and on the other hand, the *top cofaces* $d^r \colon \underline{r-1} \to \underline{r}$, where for each $1 \leqslant i \leqslant r$, we denote by d^i the unique strictly monotone function whose image does not contain the element $i \in \underline{r}$. Whenever we apply a contravariant functor to **Inj**, we write $d_i := (d^i)^*$.

Definition 2.4 (Tuples)**.** Let $K = (k_1, \dots, k_r)$ be a tuple of integers $k_i \geqslant 1$.

1. We denote by $|K| := k_1 + \dots + k_r$ the *size* and by $\#K := r$ the *length* of K.
2. For a sequence $\boldsymbol{X} := (X_k)_{k \geqslant 1}$ of objects in a complete category, we define $\boldsymbol{X}^K := X_{k_1} \times \dots \times X_{k_r}$.
3. For any injection $u \colon \underline{t} \hookrightarrow \underline{r}$ we define the *pullback* $u^*K := (k_{u(1)}, \dots, k_{u(t)})$.
4. $\mathfrak{S}_r$ acts on the set of r-tuples by pullback and we denote the orbit of K by $[K]$.

Definition 2.5 (Wreath products)**.** Let $\mathfrak{G} = (\mathfrak{G}_k)_{k \geqslant 1}$ be a sequence of discrete groups. We define the *wreath product* $\mathfrak{G} \wr \mathbf{Inj}$ as the following small category:

1. objects are tuples $K = (k_1, \dots, k_r)$ with $r \geqslant 0$ and $k_i \geqslant 1$ an integer;
2. morphisms $K \to L$ are pairs $(u, \boldsymbol{g})$, where $\boldsymbol{g} \in \mathfrak{G}^K$ and $u \colon \underline{r} \hookrightarrow \underline{s}$ with $K = u^*L$;
3. composition is given by $(v, \boldsymbol{h}) \circ (u, \boldsymbol{g}) := (v \circ u, u^*\boldsymbol{h} \cdot \boldsymbol{g})$.

Construction 2.6. Let $\mathfrak{G} = (\mathfrak{G}_k)_{k \geqslant 1}$ be a sequence of discrete groups and $\boldsymbol{X} = (X_k)_{k \geqslant 1}$ be a *based $\mathfrak{G}$-sequence*, i.e. a sequence of based spaces, together with basepoint-preserving actions of $\mathfrak{G}_k$ on X_k. Then we obtain a functor to the category of topological spaces

$$\boldsymbol{X}^{-} \colon \mathfrak{G} \wr \mathbf{Inj} \longrightarrow \mathbf{Top}, \quad K \mapsto \boldsymbol{X}^K = X_{k_1} \times \dots \times X_{k_r}$$

as follows: for each injective map $u \colon \underline{r} \hookrightarrow \underline{s}$, each fibre has at most one element and we put $u_*(x_1, \dots, x_r) := (x_{u^{-1}(1)}, \dots, x_{u^{-1}(s)})$, where we define $x_\varnothing$ to be the basepoint. Moreover, $\mathfrak{G}^K$ acts on $\boldsymbol{X}^K$ component-wise and we put $(u, \boldsymbol{g})_*(\boldsymbol{x}) := u_*(\boldsymbol{g} \cdot \boldsymbol{x})$ for $\boldsymbol{x} \in \boldsymbol{X}^K$.

Definition 2.7. Consider the sequence $\mathfrak{S} := (\mathfrak{S}_k)_{k \geqslant 1}$ of symmetric groups. Then, for each space E, the family of spaces $\tilde{C}_K(E)$ constitutes a functor $(\mathfrak{S} \wr \mathbf{Inj})^{\mathrm{op}} \longrightarrow \mathbf{Top}$ by permuting clusters of the same size, by permuting the internal ordering of each cluster, and by declaring that for each $1 \leqslant i \leqslant r$, the face map $d_i \colon \tilde{C}_K(E) \to \tilde{C}_{d_i K}(E)$ forgets the i^{th} cluster. If we are additionally given a based symmetric sequence $\boldsymbol{X} = (X_k)_{k \geqslant 1}$, then we define the *configuration space of labelled clusters* $C(E;\boldsymbol{X})$ to be the coend

$$C(E;\boldsymbol{X}) \coloneqq \int^{K\in\mathfrak{S}\wr\mathbf{Inj}} \tilde{C}_K(E)\times \boldsymbol{X}^K$$
$$= \operatorname{coeq}\left(\coprod_{K,L} \tilde{C}_L(E)\times(\mathfrak{S}\wr\mathbf{Inj})\binom{K}{L}\times\boldsymbol{X}^K \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \coprod_K \tilde{C}_K(E)\times\boldsymbol{X}^K \right),$$

where $(\mathfrak{S}\wr\mathbf{Inj})\binom{K}{L}$ is the set of morphisms $w\colon K\to L$ and where $\alpha(\boldsymbol{z},w,\boldsymbol{x})=(w^*\boldsymbol{z},\boldsymbol{x})$ and $\beta(\boldsymbol{z},w,\boldsymbol{x})=(\boldsymbol{z},w_*\boldsymbol{x})$. Each tuple $(\vec{z}_1,\dots,\vec{z}_r,x_1,\dots,x_r)$ in $\tilde{C}_K(E)\times\boldsymbol{X}^K$ represents a configuration $\sum_i \vec{z}_i\otimes x_i$ in $C(E;\boldsymbol{X})$, where $\sigma^*\vec{z}_i\otimes x_i=\vec{z}_i\otimes\sigma_*x_i$ for each $\sigma\in\mathfrak{S}_{k_i}$.

Remark 2.8. It is perhaps surprising how many different variations of these spaces can be produced by a suitable choice of the labelling sequence:

1. If all X_k carry a trivial $\mathfrak{S}_k$-action, then $C(E;\boldsymbol{X})$ contains unordered collections of labelled and internally unordered clusters. For example, let $\underline{\mathbb{S}}^0$ be the sequence with the 0-sphere $\mathbb{S}^0$, together with trivial $\mathfrak{S}_k$-actions, in each degree. Then we have
$$C(E;\underline{\mathbb{S}}^0)\cong\textstyle\coprod_{[K]} C_K(E).$$
2. For $k\geqslant 1$ and a based space X with a based $\mathfrak{S}_k$-action, let $X[k]\coloneqq(X_l)_{l\geqslant 1}$ be the sequence with $X_k\coloneqq X$ and $X_l\coloneqq *$ for $l\neq k$. Then $C(E;X[k])$ contains only configurations where all clusters have size k. If the $\mathfrak{S}_k$-action is trivial, then $C(E;X[k])$ is exactly the space $C^k(E;X)$ from the introduction. In particular, we have
$$C(E;\mathbb{S}^0[k])=C^k(E;\mathbb{S}^0)\cong\textstyle\coprod_{r\geqslant 0} C_{r\times k}(E).$$
3. If we define $\mathfrak{S}_+$ to be the based symmetric sequence with $(\mathfrak{S}_+)_k\coloneqq\{*\}\sqcup\mathfrak{S}_k$, together with the left translation, then $C(E;\mathfrak{S}_+)$ contains *unordered* collections of unlabelled, but internally *ordered* clusters.
4. For a based space X, let $X^\wedge$ be the based symmetric sequence with $(X^\wedge)_k\coloneqq X^{\wedge k}$, with $\mathfrak{S}_k$ acting by coordinate permutation. Then $C(E;X^\wedge)$ contains configurations of clusters where each *particle* inside a cluster carries a label in X, and if *one* of these labels reaches the basepoint, then the entire cluster vanishes.

We finish this section by formally defining the action of $\mathcal{C}_d$ on $C(\mathbb{R}^d;\boldsymbol{X})$ by inserting configurations of labelled clusters into boxes as in Figure 2.

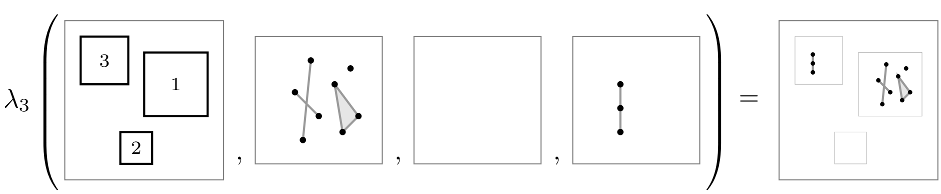


Figure 2: An instance of $\lambda_3\colon\mathcal{C}_2(3)\times C(\mathbb{R}^2;\underline{\mathbb{S}}^0)^3\to C(\mathbb{R}^2;\underline{\mathbb{S}}^0)$

Construction 2.9. Let $\boldsymbol{X} = (X_k)_{k\geqslant 1}$ be a symmetric sequence as before and let $d \geqslant 1$. Then $C(\mathbb{R}^d;\boldsymbol{X})$ admits the structure of a $\mathcal{C}_d$-algebra: recall that operations in $\mathcal{C}_d(s)$ are tuples $(c_1,\dots,c_s)$ of rectilinear embeddings $c_j\colon [0;1]^d \hookrightarrow [0;1]^d$ with pairwise disjoint image. If pick an identification $\mathbb{R} \cong (0;1)$, and thus, by coordinate-wise application, also $\mathbb{R}^d \cong (0;1)^d$, then our structure maps $\lambda_s\colon \mathcal{C}_d(s)\times C(\mathbb{R}^d;\boldsymbol{X})^s \to C(\mathbb{R}^d;\boldsymbol{X})$ are given by

$$\lambda_s\left((c_1,\dots,c_s), \sum_{i=1}^{r_1} \vec{z}_{1,i}\otimes x_{1,i},\dots,\sum_{i=1}^{r_s}\vec{z}_{s,i}\otimes x_{s,i}\right) \coloneqq \sum_{j=1}^{s}\sum_{i=1}^{r_j} c_j(\vec{z}_{j,i})\otimes x_{j,i}.$$

3. Cellular decompositions of clustered configuration spaces

In this section, we study the homotopy type of the bar construction $BC(\mathbb{R};\boldsymbol{X})$. After having introduced the necessary combinatorics, we start by discussing the instructive example of $C(\mathbb{R};\underline{\mathbb{S}}^0)$, and then establish a strategy for the general case.

Definition 3.1. For each integer $n \geqslant 0$, a *partition* of $\{1,\dots,n\}$ is a tuple $\xi = (S_1,\dots,S_r)$ of non-empty subsets $S_i \subseteq \{1,\dots,n\}$ such that:

1. the collection $\{S_1,\dots,S_r\}$ is a partition of $\{1,\dots,n\}$;
2. the entries are ordered by their minimum, i.e. $\min(S_1) < \dots < \min(S_r)$.

We write $|\xi| \coloneqq n$ and $K(\xi) \coloneqq (\#S_1,\dots,\#S_r)$. Let Ξ be the set of all partitions for all n.

Construction 3.2. We have a product $\Xi\times\Xi\to\Xi$ by *stacking* partitions: more precisely, for two partitions $\xi = (S_1,\dots,S_r)$ and $\xi' = (S'_1,\dots,S'_{r'})$, we let

$$\xi \sqcup \xi' \coloneqq (S_1,\dots,S_r,|\xi|+S'_1,\dots,|\xi|+S'_{r'}).$$

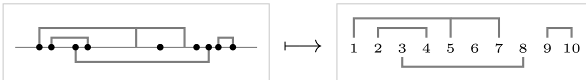
Thus, Ξ becomes a monoid with neutral element the empty partition $\varnothing$. This monoid is free: we call a partition $e\in\Xi$ *indecomposable*, or an *entanglement type*, if it is neither empty nor the product of two non-empty partitions, and we denote[1] the subset of them by $\mathbb{E}\subseteq\Xi$. Then the monoid Ξ is freely generated by $\mathbb{E}$. This generating set $\mathbb{E}$ is graded: for an entanglement type $e = (S_1,\dots,S_w)$ we let $\#e \coloneqq w$ be its *weight.*

Example 3.3. We have a map $\chi\colon C(\mathbb{R};\underline{\mathbb{S}}^0)\to\Xi$ of E_1-algebras given by identifying, for each $\sum_i[\vec{z}_i]\in C_K(\mathbb{R})$, the set $\bigcup_i[\vec{z}_i]\subseteq\mathbb{R}$ with $\{1,\dots,|K|\}$ in a monotone way, and regard clusters as entries of the partition, see Figure 3.

This map admits a section s by including $\{1,\dots,|K|\}$ into $\mathbb{R}$, and the composition $s\circ\chi$ is homotopic to the identity by linear interpolation. Thus, χ is an equivalence of E_1-algebras. Since Ξ is a freely generated by $\mathbb{E}$, we get $BC(\mathbb{R};\underline{\mathbb{S}}^0)\simeq\bigvee_{e\in\mathbb{E}}\mathbb{S}^1$.

In the case of general labelling sequences $\boldsymbol{X} = (X_k)_{k\geqslant 1}$ with non-isolated basepoints, we have to deal with the phenomenon that new clusters can suddenly arise or vanish when a label leaves or enters the basepoint, respectively. In order to gain control 'near' the basepoint, we will have to assume that each X_k is *well-based*, i.e. the basepoint inclusion $*\hookrightarrow X_k$ is a cofibration in the Quillen model structure of spaces (it is not necessary to consider cofibrations of $\mathfrak{S}_k$-spaces; see Remark 3.6 for a conceptual reason.)

[1] As a typographical mnemonic, Ξ looks like a 'decomposable' version of $\mathbb{E}$.



$= (\{1,5,7\},\{2,4\},\{3,8\},\{6\},\{9,10\})$

Figure 3: An instance of $\chi\colon C(\mathbb{R};\mathbb{S}^0)\to\Xi$

Theorem 3.4. *Let $\boldsymbol{X}=(X_k)_{k\geqslant 1}$ be a sequence of well-based spaces (with arbitrary based $\mathfrak{S}_k$-actions on X_k). Then we have a weak equivalence, abbreviating $\boldsymbol{X}^{\wedge K}:=X_{k_1}\wedge\cdots\wedge X_{k_r}$,*

$$BC(\mathbb{R};\boldsymbol{X})\simeq\Sigma\bigvee\nolimits_{e\in\mathbb{E}}\boldsymbol{X}^{\wedge K(e)}.$$

Example 3.5. In many special cases, Theorem 3.4 has an easier shape:

1. For $\boldsymbol{X}=\mathbb{S}^0$, this is precisely Example 3.3.
2. Given $k\geqslant 1$ and a well-based space X, endowed with the trivial $\mathfrak{S}_k$-action, the case of $\boldsymbol{X}:=X[k]$ recovers Theorem A. Note that only entanglement types e with $K(e)=r\times k$ for some r are relevant here, since $X[k]^{\wedge K(e)}=*$ otherwise.
3. If X is a well-based space, then the case of $\boldsymbol{X}:=X[1]$ recovers Segal's result: if $K\neq(1,\dots,1)$, then $X[1]^{\wedge K}=*$, but there is only one entanglement type involving only singletons, namely $(\{1\})$. Thus, we get $BC(\mathbb{R};X)=BC(\mathbb{R};X[1])\simeq\Sigma X$.

Remark 3.6. The reader should not be surprised by the fact that the symmetric actions on $\boldsymbol{X}$ do not appear on the right side—they are also irrelevant for the left side: for each tuple $K=(k_1,\dots,k_r)$, the action of $\prod_i\mathfrak{S}_{k_i}$ on $\tilde{C}_K(\mathbb{R})$ induces a free action on π_0, so we can alternatively restrict to the subspace $\tilde{C}^{<}_K(\mathbb{R})$ containing configurations $(\vec{z}_1,\dots,\vec{z}_r)$ of clusters where each cluster $\vec{z}_i=(z_{i,1},\dots,z_{i,k_i})$ satisfies $z_{i,1}<\cdots<z_{i,k_i}$ in $\mathbb{R}$: if we write $\mathbf{1}:=(1)_{k\geqslant 0}$ for the sequence of trivial groups, then we get the equivalent description

$$C(\mathbb{R};\boldsymbol{X})\cong\int^{K\in\mathbf{1}\wr\mathbf{Inj}}\tilde{C}^{<}_K(\mathbb{R})\times\boldsymbol{X}^K.$$

We prove Theorem 3.4 by decomposing $C(\mathbb{R};\boldsymbol{X})$ into *free* E_1-algebras as follows:

Construction 3.7. For each integer $w\geqslant 0$, let $\mathcal{F}_w\Xi\subseteq\Xi$ be the submonoid generated by all entanglement types of weight at most w. This gives rise to a filtration, which is exhaustive since $e\in\mathcal{F}_{\#e}\Xi$.

Using the map $\chi\colon\coprod_{[K]}C_K(\mathbb{R})\to\Xi$ from Example 3.3, we construct an exhaustive filtration of $C(\mathbb{R};\boldsymbol{X})$ for each based symmetric sequence $\boldsymbol{X}$ by defining

$$\mathcal{F}_wC(\mathbb{R};\boldsymbol{X}):=\left\{\sum_i\vec{z}_i\otimes x_i;\text{ all }x_i\neq *\text{ and }\chi\left(\sum_i[\vec{z}_i]\right)\in\mathcal{F}_w\Xi\right\}.$$

Since χ is a map of E_1-algebras, each $\mathcal{F}_wC(\mathbb{R};\boldsymbol{X})$ is an E_1-subalgebra of $C(\mathbb{R};\boldsymbol{X})$, and since the bar construction commutes with filtered colimits, we recover $BC(\mathbb{R};\boldsymbol{X})$ as the direct limit of the spaces $B\mathcal{F}_\bullet C(\mathbb{R};\boldsymbol{X})$.

Visually, given a labelled configuration inside $C(\mathbb{R};\boldsymbol{X})$, two clusters with non-trivial label are 'entangled' if their convex hulls on the real line intersect, see Figure 3, and each equivalence class with respect to this relation determines an entanglement type (in Figure 3, there are two equivalence classes, with weights 4 and 1). Then $\mathcal{F}_w C(\mathbb{R}; \boldsymbol{X})$ contains all configurations for which only entanglement types of weight at most w occur.

The main part of the proof of Theorem 3.4 is to see that $\mathcal{F}_w C(\mathbb{R}; \boldsymbol{X})$ is equivalent to an E_1-algebra that arises from $\mathcal{F}_{w-1} C(\mathbb{R}; \boldsymbol{X})$ by attaching a free E_1-algebra. Let us first establish the notion of an E_1-cell attachment, which is inspired by [7].

Construction 3.8. If $\mathcal{O}$ is an operad with $\mathcal{O}(0) = \{*\}$, then each $\mathcal{O}$-algebra has an underlying based space. The forgetful functor U to based spaces has a left adjoint, called F. Explicitly, FX is given by quotienting $\coprod_{r \geqslant 0} \mathcal{O}(r) \times_{\mathfrak{S}_r} X^r$ by the basepoint relations from Section 1.

For a map $\imath\colon A \to Y$ of based spaces, an $\mathcal{O}$-algebra M, along with a based map $g\colon A \to UM$, we define the *$\mathcal{O}$-cell attachment* $M \sqcup^{\mathcal{O}}_A Y$ as the pushout of $\mathcal{O}$-algebras

$$\begin{array}{ccc} FA & \xrightarrow{\bar{g}} & M \\ {\scriptstyle F\imath}\downarrow & \ulcorner & \downarrow \\ FY & \longrightarrow & M \sqcup^{\mathcal{O}}_A Y, \end{array}$$

where $\bar{g}$ is the adjoint of g. If $T := UF$ denotes the monad associated with $\mathcal{O}$, then $M \sqcup^{\mathcal{O}}_A Y$ is the reflexive coequaliser (in $\mathcal{O}$-algebras, as well as in based spaces) of

$$F(TUM \sqcup_A Y) \rightrightarrows F(UM \sqcup_A Y). \tag{1}$$

Here $UM \sqcup_A Y$ and $TUM \sqcup_A Y$ are pushouts of based spaces, the first arrow of Equation (1) is induced by the action $TUM \to UM$, the second arrow is given by applying F to the inclusion $TUM \sqcup_A Y \to T(UM \sqcup_A M)$ and composing with the counit $FT = FUF \Rightarrow F$, and the degeneracy is induced by the unit $UM \to TUM$, see [7, § 6.1].

Example 3.9. Let us unravel the above construction in two cases:

1. Restricting to one model, an E_1-*algebra* is the same as an algebra over $\mathcal{C}_1$. If M is an E_1-algebra and $\imath\colon A \to Y$ and $f\colon A \to UM$ are based maps, then points in $M \sqcup^{E_1}_A Y$ are given by configurations of disjoint subintervals
$$c_1, \ldots, c_s\colon [0;1] \hookrightarrow [0;1],$$
each carrying a label in $UM \sqcup_A Y$, quotiented by the usual basepoint relation; and additionally, if c_i is labelled by $\lambda_t(c'_1, \ldots, c'_t; m_1, \ldots, m_t) \in M$ with $(c'_1, \ldots, c'_t) \in \mathcal{C}_1(t)$, $m_1, \ldots, m_t \in M$, and $\lambda_t\colon \mathcal{C}_1(t) \times M^t \to M$ being the $\mathcal{C}_1$-action, then the configuration is identified with the one where the interval c_i is replaced by the intervals $c_i \circ c'_1, \ldots, c_i \circ c'_t$, carrying the labels $m_1, \ldots, m_t$, respectively.
2. Algebras over the *associative operad* are the same as topological monoids. If M is a topological monoid and $\imath\colon A \to Y$ and $f\colon A \to UM$ are based maps, then points in $M \sqcup^{\mathrm{Mon}}_A Y$ are given by *strings* $\zeta_1 \cdots \zeta_r$ with $\zeta_i \in UM \sqcup_A Y$. If ζ_i is the basepoint, then it can be omitted from the string, and if $\zeta_i, \zeta_{i+1} \in UM$, then the substring $\zeta_i \zeta_{i+1}$ can be replaced by the single letter that equals the actual product $\zeta_i \cdot \zeta_{i+1}$.

Remark 3.10. For our purposes, it is convenient to have a homotopically better behaved construction: the reflexive pair Equation (1) is part of an entire simplicial $\mathcal{O}$-algebra

$$P^{\mathcal{O}}_\bullet(M, A, Y)\colon [n] \mapsto F(T^n UM \sqcup_A Y).$$

Its geometric realisation is denoted by $M \cup^{\mathcal{O}}_A Y$ and called the *derived $\mathcal{O}$-cell attachment*, see [11, § 3.1], [7, § 8.3.6]. We have an augmentation $P^{\mathcal{O}}_\bullet(M, A, Y) \to M \sqcup^{\mathcal{O}}_A Y$, inducing a map $M \cup^{\mathcal{O}}_A Y \to M \sqcup^{\mathcal{O}}_A Y$ of $\mathcal{O}$-algebras. Therefore, maps out of the derived attachment into another $\mathcal{O}$-algebra can equally well be declared on M and Y.

If $\imath\colon A \to Y$ is a cofibration between well-based spaces (in the Quillen model structure of spaces) and if M is cofibrant (in the projective model structure on $\mathcal{O}$-algebras), then the above map $M \cup^{\mathcal{O}}_A Y \to M \sqcup^{\mathcal{O}}_A Y$ is a weak equivalence, compare [7, § 8.2]: this reflects the fact that under these conditions, the *actual* pushout is a *homotopy* pushout.

In the case of E_1-algebras, it follows[2] from [11, Prop. 98] that the bar construction $B(M \cup^{E_1}_A Y)$ arises from BM by attaching ΣY along the map $\Sigma A \to \Sigma UM \to BM$, i.e. the bar construction turns derived E_1-attachments into suspended attachments.

After this general interlude, let us come back to the configuration spaces $C(\mathbb{R}; \boldsymbol{X})$.

Definition 3.11. Let $\boldsymbol{X} = (X_k)_{k\geqslant 1}$ be a sequence of well-based spaces and $K = (k_1, \dots, k_r)$ be a tuple of positive integers. Then we define

$$\boldsymbol{X}^{\Delta K} \coloneqq \{(x_1, \dots, x_r) \in \boldsymbol{X}^K;\ x_i = *_{k_i} \text{ for some } i\} \subseteq \boldsymbol{X}^K$$

as the subspace of *degenerated tuples*, with basepoint $(*_{k_1}, \dots, *_{k_r})$. Note that since each X_k is assumed to be well-based, $\boldsymbol{X}^{\Delta K} \hookrightarrow \boldsymbol{X}^K$ is a cofibration of well-based spaces.

Construction 3.12. For each entanglement type e of weight w, we have a based map $\tilde{f}_e\colon \boldsymbol{X}^{K(e)} \to \mathcal{F}_w C(\mathbb{R}; \boldsymbol{X})$ defined as follows: if we write $e = (S_1, \dots, S_w)$ and include the set $\{1, \dots, |e|\}$ canonically into $\mathbb{R}$, then each subset S_i, together with the order inherited from $\mathbb{R}$, can be regarded as an ordered cluster $\vec{z}_i$, and we put

$$\tilde{f}_e(x_1, \dots, x_w) \coloneqq \sum_{i=1}^{w} \vec{z}_i \otimes x_i \in \mathcal{F}_w C(\mathbb{R}; \boldsymbol{X}).$$

If $(x_1, \dots, x_w)$ lies in the subspace $\boldsymbol{X}^{\Delta K(e)}$, then the labelled configuration $\tilde{f}_e(x_1, \dots, x_w)$ has at most $w - 1$ non-trivial clusters, and thus, the restriction f_e of $\tilde{f}_e$ to $\boldsymbol{X}^{\Delta K(e)}$ lands in the filtration component $\mathcal{F}_{w-1} C(\mathbb{R}; \boldsymbol{X})$. If we use the bouquet $\bigvee_{\#e=w} f_e$ to E_1-attach $\bigvee_{\#e=w} \boldsymbol{X}^{K(e)}$ to $\mathcal{F}_{w-1} C(\mathbb{R}; \boldsymbol{X})$ in the derived sense, then the extensions $\tilde{f}_e$ declare, via the universal property, a map of E_1-algebras under $\mathcal{F}_{w-1} C(\mathbb{R}; \boldsymbol{X})$ of the form

$$\varphi_w\colon \mathcal{F}_{w-1} C(\mathbb{R}; \boldsymbol{X}) \cup^{E_1}_{\bigvee_e \boldsymbol{X}^{\Delta K(e)}} \bigvee_{\#e=w} \boldsymbol{X}^{K(e)} \longrightarrow \mathcal{F}_w C(\mathbb{R}; \boldsymbol{X}). \tag{2}$$

Lemma 3.13. *The map φ_w is an equivalence of E_1-algebras.*

[2] To be precise, [11, Prop. 98] only treats the case where (Y, A) is a disc $(\mathbb{D}^n, \mathbb{S}^{n-1})$. However, the proof goes through for the general case without any modifications.

This shows that, up to equivalence, $C(\mathbb{R};\boldsymbol{X})$ can inductively be built by attaching free E_1-algebras. We first prove Theorem 3.4 using Lemma 3.13, and then prove the Lemma.

Proof of Theorem 3.4. Let us abbreviate $C := C(\mathbb{R};\boldsymbol{X})$. Using that each X_k is well-based, the inclusions $U\mathcal{F}_{w-1}C \hookrightarrow U\mathcal{F}_w C$, and as a consequence also the inclusions $B\mathcal{F}_{w-1}C \hookrightarrow B\mathcal{F}_w C$, are (Hurewicz) cofibrations of spaces. Therefore, BC is equivalent to the *homotopy* colimit over the filtration components $B\mathcal{F}_\bullet C$. Since $B\mathcal{F}_0 C$ is just a point, it suffices to show that the induced map $B\mathcal{F}_{w-1}C \to B\mathcal{F}_w C$ is equivalent to the inclusion into the bouquet $B\mathcal{F}_{w-1}C \hookrightarrow B\mathcal{F}_{w-1}C \vee \Sigma \bigvee_{\#e=w} \boldsymbol{X}^{\wedge K(e)}$ for each w.

This equivalence is established in two steps: first, we use the equivalence φ_w from Lemma 3.13. Again, since each X_k is well-based, the induced map $B\varphi_w$ is a weak equivalence of based spaces (the map $B_\bullet(\Sigma, T^{E_1}, \varphi_w)$ among the two-sided bar constructions is a levelwise equivalence of proper simplicial spaces). As the bar construction turns E_1-attachments into suspended attachments (see Remark 3.10), we get a homotopy pushout

$$\begin{array}{ccc} \Sigma \bigvee_{\#e=w} \boldsymbol{X}^{\Delta K(e)} & \longrightarrow & B\mathcal{F}_{w-1}C \\ \big\downarrow & \ulcorner & \big\downarrow \\ \Sigma \bigvee_{\#e=w} \boldsymbol{X}^{K(e)} & \longrightarrow & B\mathcal{F}_w C. \end{array}$$

Second, we consider the left vertical map: by elementary homotopy theory, the cofibre sequence $\boldsymbol{X}^{\Delta K(e)} \to \boldsymbol{X}^{K(e)} \to \boldsymbol{X}^{\wedge K(e)}$ splits after a single suspension for each e. Thus, each of the summands in the left vertical map above is equivalent to the wedge inclusion $\Sigma\boldsymbol{X}^{\Delta K(e)} \hookrightarrow \Sigma\boldsymbol{X}^{\Delta K(e)} \vee \Sigma\boldsymbol{X}^{\wedge K(e)}$. As the attaching map is, for each e, defined on the *first* of the two wedge summands, the attachment is the same as adding the *second* one. □

Proof of Lemma 3.13. Recall that we have to show that the map φ_w from Equation (2) is a weak equivalence. First, we simplify our notation: as before, we write $C := C(\mathbb{R};\boldsymbol{X})$; and additionally, let $\tilde{f}_w := \bigvee_{\#e=w} \tilde{f}_e$, $f_w := \bigvee_{\#e=w} f_e$, $\boldsymbol{X}_w := \bigvee_{\#e=w} \boldsymbol{X}^{K(e)}$, and $\boldsymbol{X}_w^\Delta := \bigvee_{\#e=w} \boldsymbol{X}^{\Delta K(e)}$.

Now the proof strategy is to 'discard' contractible information on both sides of φ_w by introducing a 'thin' version D of C, which is even a topological monoid, and which comes with a filtration $\mathcal{F}_\bullet D$ by submonoids. The proof then proceeds as follows:

1. Construct the topological monoid D and its filtration $\mathcal{F}_\bullet D$, and construct equivalences $\rho_\bullet\colon \mathcal{F}_\bullet C \to \mathcal{F}_\bullet D$ of E_1-algebras, which commute with the inclusions.
2. We use $\rho_{w-1} f_w\colon \boldsymbol{X}_w^\Delta \to \mathcal{F}_{w-1}D$ to attach $\boldsymbol{X}_w$ to $\mathcal{F}_{w-1}D$. Show that the induced map $\rho_{w-1} \cup^{E_1}_{\boldsymbol{X}_w^\Delta} \boldsymbol{X}_w\colon \mathcal{F}_{w-1}C \cup^{E_1}_{\boldsymbol{X}_w^\Delta} \boldsymbol{X}_w \to \mathcal{F}_{w-1}D \cup^{E_1}_{\boldsymbol{X}_w^\Delta} \boldsymbol{X}_w$ is an equivalence.
3. Let $\alpha\colon T^{E_1} \Rightarrow T^{\mathrm{Mon}}$ be the transformation of monads. Show that the induced map $\mathcal{F}_{w-1}D \cup^{\alpha}_{\boldsymbol{X}_w^\Delta} \boldsymbol{X}_w\colon \mathcal{F}_{w-1}D \cup^{E_1}_{\boldsymbol{X}_w^\Delta} \boldsymbol{X}_w \to \mathcal{F}_{w-1}D \cup^{\mathrm{Mon}}_{\boldsymbol{X}_w^\Delta} \boldsymbol{X}_w$ is an equivalence.
4. Show that the map $\psi_w\colon \mathcal{F}_{w-1}D \cup^{\mathrm{Mon}}_{\boldsymbol{X}_w^\Delta} \boldsymbol{X}_w \to \mathcal{F}_w D$ that is, via the universal property, induced by $\rho_w \tilde{f}_w\colon \boldsymbol{X}_w \to \mathcal{F}_w D$ is an equivalence.

Since φ_w is induced by $\tilde{f}_w \colon \boldsymbol{X}_w \to \mathcal{F}_w C$ and ψ_w is induced by $\rho_w \tilde{f}_w \colon \boldsymbol{X}_w \to \mathcal{F}_w D$, the above maps assemble into a commutative square

$$\begin{array}{ccc} \mathcal{F}_{w-1}C \cup^{E_1}_{\boldsymbol{X}_w^{\vartriangle}} \boldsymbol{X}_w & \xrightarrow{\varphi_w} & \mathcal{F}_w C \\ {\scriptstyle \rho_{w-1}\cup^{E_1}_{\boldsymbol{X}_w^{\vartriangle}}\boldsymbol{X}_w}\downarrow\simeq & & \\ \mathcal{F}_{w-1}D \cup^{E_1}_{\boldsymbol{X}_w^{\vartriangle}} \boldsymbol{X}_w & & \simeq \downarrow \rho_w \\ {\scriptstyle \mathcal{F}_{w-1}D\cup^{\alpha}_{\boldsymbol{X}_w^{\vartriangle}}\boldsymbol{X}_w}\downarrow\simeq & & \\ \mathcal{F}_{w-1}D \cup^{\mathrm{Mon}}_{\boldsymbol{X}_w^{\vartriangle}} \boldsymbol{X}_w & \xrightarrow[\simeq]{\psi_w} & \mathcal{F}_w D. \end{array} \tag{3}$$

It then follows from the 2-out-of-3-property that the map φ_w in question is a weak equivalence, which finishes the proof. Let us go through the steps 1–4:

Step 1 Replacing $\tilde{C}_K(\mathbb{R})$ by its set of path components, we define

$$D \coloneqq \int^K \pi_0 \tilde{C}_K(\mathbb{R}) \times \boldsymbol{X}^K.$$

Then elements in D are equivalence classes $[\xi; x_1, \dots, x_r]$, where $\xi = (S_1, \dots, S_r)$ is a partition and where $x_i \in X_{\#S_i}$; and if x_i is the basepoint, then $[\xi; x_1, \dots, x_r]$ is identified with $[d_i\xi; x_1, \dots, \widehat{x_i}, \dots, x_r]$, where $d_i\xi$ arises from ξ by removing S_i and relabelling the remaining subsets. Defining

$$[\xi; x_1, \dots, x_r] \cdot [\xi'; x_1', \dots, x_{r'}'] \coloneqq [\xi \sqcup \xi'; x_1, \dots, x_r, x_1', \dots, x_{r'}'],$$

D becomes a topological monoid, in particular an E_1-algebra. Moreover, D is filtered by submonoids $\mathcal{F}_w D \subseteq D$ containing only points that can be represented by $(\xi; x_1, \dots, x_r)$ where no x_i is a basepoint and $\xi \in \mathcal{F}_w \Xi$. We have, for each w, a map $\rho_w \colon \mathcal{F}_w C \to \mathcal{F}_w D$ induced by the canonical projections $\tilde{C}_K(\mathbb{R}) \to \pi_0 \tilde{C}_K(\mathbb{R})$. This clearly is a morphism of E_1-algebras, and it commutes with the filtration in the sense that the (co-)restriction of ρ_w to the $(w-1)^{\text{st}}$ filtration level is precisely ρ_{w-1}. We show that each ρ_w is a homotopy equivalence: since each X_k is well-based, we find, for each $k \geqslant 1$, a map $u_k \colon X_k \to [0;1]$ satisfying $u_k^{-1}(0) = \{*_k\}$. These can be used to construct a section s_w of ρ_w by sending $[\xi; x_1, \dots, x_r]$ to a labelled configuration in $\mathbb{R}$, employing the unique inclusion $\nu \colon \{1, \dots, |K|\} \hookrightarrow \mathbb{R}$ with $\nu(1) = 0$ and $\nu(j+1) - \nu(j) = u_{k_i}(x_i)$ for $j \in S_i$. Finally, the composition $s_w \circ \rho_w$ is homotopic to the identity by linear interpolation.[3]

Step 2 Since the monad $T \coloneqq T^{E_1}$ preserves well-based objects and equivalences between them, and since $\boldsymbol{X}_w^{\vartriangle} \hookrightarrow \boldsymbol{X}_w$ is a cofibration, the maps $T^\bullet \rho_{w-1} \sqcup_{\boldsymbol{X}_w^{\vartriangle}} \boldsymbol{X}_w$ are equivalences, and the same applies to $T(T^\bullet \rho_{w-1} \sqcup_{\boldsymbol{X}_w^{\vartriangle}} \boldsymbol{X}_w)$. This shows that $UP_\bullet^{E_1}(\rho_{w-1}, \boldsymbol{X}_w^{\vartriangle}, \boldsymbol{X}_w)$ is a levelwise equivalence; finally, we use that the simplicial spaces on both sides are proper as the unit of the monad is a cofibration.

Step 3 We use that $\alpha \colon T^{E_1} Y \to T^{\mathrm{Mon}} Y$ is a homotopy equivalence if Y is well-based; this is just a variation of the above argument. This shows that the induced

[3] This is the usual argument showing that for a well-based space X, the classical labelled configuration space $C(\mathbb{R}; X)$ is equivalent to the reduced James product over X.

map $UP^{E_1}_\bullet(\mathcal{F}_{w-1}D, \boldsymbol{X}^\Delta_w, \boldsymbol{X}_w) \to UP^{\mathrm{Mon}}_\bullet(\mathcal{F}_{w-1}D, \boldsymbol{X}^\Delta_w, \boldsymbol{X}_w)$ is a levelwise equivalence. Finally, we use again that both simplicial spaces are proper to obtain the substatement.

Step 4 We have to show that ψ_w is a weak equivalence. To do so, we show that the map $\psi'_w\colon \mathcal{F}_{w-1}D \sqcup^{\mathrm{Mon}}_{\boldsymbol{X}^\Delta_w} \boldsymbol{X}_w \to \mathcal{F}_w D$ from the *strict* pushout is an *isomorphism*. Since $\mathcal{F}_0 D = *$, this inductively shows that $\mathcal{F}_{w-1}D$ is cofibrant in the projective model structure. As $\boldsymbol{X}^\Delta_w \hookrightarrow \boldsymbol{X}_w$ is a cofibration of well-based spaces,

$$\mathcal{F}_{w-1}D \cup^{\mathrm{Mon}}_{\boldsymbol{X}^\Delta_w} \boldsymbol{X}_w \to \mathcal{F}_{w-1}D \sqcup^{\mathrm{Mon}}_{\boldsymbol{X}^\Delta_w} \boldsymbol{X}_w$$

is a weak equivalence, which then finishes the proof. To show that ψ'_w is an isomorphism, recall that points in $\mathcal{F}_{w-1}D \sqcup^{\mathrm{Mon}}_{\boldsymbol{X}^\Delta_w} \boldsymbol{X}_w$ are strings $\zeta_1 \cdots \zeta_s$ with letters ζ_i in the space $U\mathcal{F}_{w-1}D \sqcup_{\boldsymbol{X}^\Delta_w} \boldsymbol{X}_w$, identified by the relations from Example 3.9. Then the inverse of ψ'_w is given as follows: each point $m \in \mathcal{F}_w D$ can be written as $[\xi; x_1, \ldots, x_r]$ such that no x_i is the respective basepoint. We can decompose $\xi = e_1 \sqcup \cdots \sqcup e_s$ into entanglement types, i.e. $m = [e_1; \boldsymbol{x}_1] \cdots [e_s; \boldsymbol{x}_s]$ with $\boldsymbol{x}_i \coloneqq (x_{w_1+\cdots+w_{i-1}+1}, \ldots, x_{w_1+\cdots+w_i})$ for $w_i \coloneqq \# e_i$. If $w_i \leqslant w-1$, then the factor $[e_i; \boldsymbol{x}_i]$ already lies in $U\mathcal{F}_{w-1}D$, and if $w_i = w$, then $[e_i, \boldsymbol{x}_i]$ can be regarded as an element in $\boldsymbol{X}_w$. In this way, m determines a string with letters in $U\mathcal{F}_{w-1}D \sqcup_{\boldsymbol{X}^\Delta_w} \boldsymbol{X}_w$ as above. One easily checks that this assignment factors through the relations for $\mathcal{F}_w D$ and indeed forms an inverse of φ'_w. □

Corollary 3.14. *Let $\boldsymbol{X} = (X_k)_{k\geqslant 1}$ be a well-based sequence such that each X_k is path-connected. Then $C(\mathbb{R}; \boldsymbol{X})$ is equivalent to a free E_1-algebra.*

Proof. Since each X_k is path-connected, the E_1-algebra $C(\mathbb{R}; \boldsymbol{X})$ is path-connected as well. Therefore, the canonical map $C(\mathbb{R}; \boldsymbol{X}) \to \Omega BC(\mathbb{R}; \boldsymbol{X})$ is an equivalence. Now we use that by Theorem 3.4, $BC(\mathbb{R}; \boldsymbol{X})$ is equivalent to $\Sigma \bigvee_e \boldsymbol{X}^{\wedge K(e)}$. Since $\bigvee_e \boldsymbol{X}^{\wedge K(e)}$ is path-connected, this establishes an equivalence of E_1-algebras

$$C(\mathbb{R}; \boldsymbol{X}) \simeq \Omega BC(\mathbb{R}; \boldsymbol{X}) \simeq \Omega\Sigma \bigvee\nolimits_e \boldsymbol{X}^{\wedge K(e)} \simeq F^{E_1}(\bigvee\nolimits_e \boldsymbol{X}^{\wedge K(e)}).$$ □

4. Iterated bar constructions of vertical configuration spaces

While we understood the bar construction of the E_1-algebra $C(\mathbb{R}; \boldsymbol{X})$ in the previous section, the iterated bar construction of the E_d-algebra $C(\mathbb{R}^d; \boldsymbol{X})$ still has no geometric interpretation for $d \geqslant 2$. In this section, we give a partial answer by introducing a family of subalgebras $C(\mathbb{R}^{p,d-p}; \boldsymbol{X}) \subseteq C(\mathbb{R}^d; \boldsymbol{X})$ and studying their p-fold bar construction.

As already motivated in the introduction, these subalgebras are constructed by imposing a certain 'verticality' condition on particles within the same cluster. Let us start by making this definition precise.

Definition 4.1. Let $\pi\colon E \to B$ be a map of spaces. A cluster $\vec{z} = (z_1, \ldots, z_k)$ in E is called *π-vertical*, if all particles $z_1, \ldots, z_k$ lie in the same fibre. For each tuple $K = (k_1, \ldots, k_r)$, we let $\tilde{C}^\pi_K(E) \subseteq \tilde{C}_K(E)$ be the subspace of all $(\vec{z}_1, \ldots, \vec{z}_r)$ such that each $\vec{z}_i$ is π-vertical. Then the action of $\mathfrak{S}_K$ on $\tilde{C}_K(E)$ restricts to $\tilde{C}^\pi_K(E)$ and we define $C^\pi_K(E)$ as the quotient. We call these spaces *ordered* and *unordered vertical configuration spaces*, respectively.

The spaces $\tilde{C}^\pi_K(E)$ assemble into a functor $(\mathfrak{S} \wr \mathbf{Inj})^{\mathrm{op}} \longrightarrow \mathbf{Top}$ by permuting and omitting clusters as before. For a based symmetric sequence $\boldsymbol{X} = (X_k)_{k\geqslant 1}$, we define

$C^\pi(E;\boldsymbol{X}) \coloneqq \int^K \tilde{C}^\pi_K(E) \times \boldsymbol{X}^K$. In other words, $C^\pi(E;\boldsymbol{X}) \subseteq C(E;\boldsymbol{X})$ is the subspace of labelled configurations where each cluster is π-vertical.

Example 4.2. For each $0 \leqslant p \leqslant d$, we consider the projection $\pi\colon \mathbb{R}^d \to \mathbb{R}^p$ to the first p coordinates, and define—for a tuple K or a sequence $\boldsymbol{X}$, respectively—the spaces

$$C_K(\mathbb{R}^{p,d-p}) \coloneqq C^\pi_K(\mathbb{R}^d),$$
$$C(\mathbb{R}^{p,d-p};\boldsymbol{X}) \coloneqq C^\pi(\mathbb{R}^d;\boldsymbol{X}).$$

These are exactly the spaces depicted in Figure 1 from the introduction. Note that the subspace $C(\mathbb{R}^{p,d-p};\boldsymbol{X}) \subseteq C(\mathbb{R}^d;\boldsymbol{X})$ is even an E_d-subalgebra: this follows directly from the observation that for each little cube $c\colon [0;1]^d \hookrightarrow [0;1]^d$ and a vertical cluster $\vec{z}$, the rescaled cluster $c(\vec{z})$ is again vertical.

Restricting the action of $\mathcal{C}_d$ to its first p coordinates, we can ask for the p-fold bar construction of $C(\mathbb{R}^{p,d-p};\boldsymbol{X})$, which still is an E_{d-p}-algebra. In order to formulate our result, we need two more definitions:

Definition 4.3. We call a based symmetric sequence $\boldsymbol{X} = (X_k)_{k\geqslant 1}$ *equivariantly well-based* if each $* \hookrightarrow X_k$ is a cofibration in the projective model structure on $\mathfrak{S}_k$-spaces.

Definition 4.4. For a based symmetric sequence $\boldsymbol{X} = (X_k)_{k\geqslant 1}$, we define $\Sigma\boldsymbol{X}$ to be the sequence with $(\Sigma\boldsymbol{X})_k = \Sigma X_k$, together with the induced $\mathfrak{S}_k$-actions.

Theorem 4.5. *If $\boldsymbol{X}$ is equivariantly well-based, then there is an equivalence of E_{d-p}-algebras*

$$B^p C(\mathbb{R}^{p,d-p};\boldsymbol{X}) \simeq C(\mathbb{R}^{d-p};\Sigma^p\boldsymbol{X}).$$

This equivalence is again a generalisation of Segal's result [**19**]: for each well-based space X, the labelled configuration space $C(\mathbb{R}^{p,d-p};X[1])$ is isomorphic to $C(\mathbb{R}^d;X)$, since all clusters have only a single particle, and hence Theorem 4.5 boils down to the well-known equivalence

$$B^p C(\mathbb{R}^d;X) \simeq C(\mathbb{R}^{d-p};\Sigma^p X)$$

of E_{d-p}-algebras. In the case of $\boldsymbol{X} = X[k]$ for some well-based space X with trivial $\mathfrak{S}_k$-action, we recover Theorem B.

The proof of Theorem 4.5 is nothing but a straightforward generalisation of Segal's proof, using at all stages that inside $C(\mathbb{R}^{p,d-p};\boldsymbol{X})$, clusters look as *single* particles from the perspective of the first p coordinates.

Proof. We strongly encourage the reader to compare the following proof to Segal's original one [**19**], as we shortened many arguments that can be copied verbatim. Throughout the proof, let us abbreviate $q \coloneqq d - p$.

Step 1 We first translate our statement into the language of [**19**] by considering a rectification of $C_{p,q}(\boldsymbol{X}) \coloneqq C(\mathbb{R}^{p,q};\boldsymbol{X})$, which is even a true monoid: let $\mathrm{pr}_1\colon \mathbb{R}^d \to \mathbb{R}$

be the projection to the first coordinate; then we define the *support* of

$$c = \sum_i \vec{z}_i \otimes x_i \in C_{p,q}(\boldsymbol{X})$$

as $\operatorname{supp}(c) \coloneqq \bigcup_i \operatorname{pr}_1(\vec{z}_i) \subseteq \mathbb{R}$ and let

$$C'_{p,q}(\boldsymbol{X}) \coloneqq \Big\{(t,c) \in \mathbb{R}_{\geqslant 0} \times C_{p,q}(\boldsymbol{X});\ \operatorname{supp}(c) \subseteq (0;t)\Big\}.$$

By putting $(t,c)\cdot(t',c') = (t+t', c+T_t c')$, the space $C'_{p,q}(\boldsymbol{X})$ becomes a topological monoid: here T_t is translation by $(t,0,\dots,0)$. Note that $C'_{p,q}(\boldsymbol{X})$ is the 'Moore' rectification $RC_{p,q}(\boldsymbol{X})$ that appears in [5, Prop. 1.9]: its bar construction $BC'_{p,q}(\boldsymbol{X})$ is, as an E_{d-1}-algebra, equivalent to the bar construction $BC_{p,q}(\boldsymbol{X})$. On the other hand, it follows from [6, Cor. 7.9] that $BC'_{p,q}(\boldsymbol{X})$ can be calculated by the usual nerve construction for topological monoids (rather than the two-sided bar construction), which is a clustered version of the simplicial space that Segal studied. We show the analogue of [19, Prop. 2.1]: *for each $p \geqslant 1$, we have an equivalence $BC'_{p,q}(\boldsymbol{X}) \simeq C_{p-1,q}(\Sigma\boldsymbol{X})$ of E_{d-1}-algebras.* Then the statement follows by induction.

Step 2 We consider the *partial* abelian monoid $D_{p-1,q}(\boldsymbol{X})$, whose underlying space[4] is $C_{p-1,q}(\boldsymbol{X})$, but where—instead of the E_1-multiplication—we call two labelled configurations summable if they are disjoint; in that case, the sum is their union. Recall that the classifying space of a partial monoid M is the geometric realisation of its nerve $N_\bullet M$, where $N_n M \subseteq M^n$ contains composable n-tuples. Exactly as in [19, Prop. 2.3], we obtain an isomorphism of E_{d-1}-algebras $BD_{p-1,q}(\boldsymbol{X}) \cong C_{p-1,q}(\Sigma\boldsymbol{X})$ by amalgamating the levelwise maps $\varphi_n\colon N_n D_{p-1,q}(\boldsymbol{X}) \times \Delta^n \to C_{p-1,q}(\Sigma\boldsymbol{X})$ with (writing $\Sigma X_k = X_k \wedge S^1$)

$$\varphi_n\left(\sum_{i=1}^{r_1} \vec{z}_{1,i} \otimes x_{1,i}, \dots, \sum_{i=1}^{r_n} \vec{z}_{n,i} \otimes x_{n,i}; t_1 \leqslant \dots \leqslant t_n\right) = \sum_{j=1}^{n}\sum_{i=1}^{r_j} \vec{z}_{j,i} \otimes (x_{j,i} \wedge t_j).$$

Step 3 We have a second projection $\operatorname{pr}_2\colon \mathbb{R}^d \to \mathbb{R}^{d-1}$ and we call

$$\sum_i \vec{z}_i \otimes x_i \in C_{p,q}(\boldsymbol{X})$$

with $x_i \neq *$ *projectable* if the restriction $\operatorname{pr}_2|_{\bigcup_i [\vec{z}_i]}$ is injective. Let $C''_{p,q}(\boldsymbol{X}) \subseteq C'_{p,q}(\boldsymbol{X})$ be the subspace of pairs (t,c) with projectable c. Then $C''_{p,q}(\boldsymbol{X})$ is a *partial* submonoid with respect to the concatenation, where projectable configurations can be multiplied if their product is again projectable. Moreover, we have a map of partial monoids $C''_{p,q}(\boldsymbol{X}) \to D_{p-1,q}(\boldsymbol{X})$ by projecting, see Figure 4. As in [19], the induced maps $N_\bullet C''_{p,q}(\boldsymbol{X}) \to N_\bullet D_{p-1,q}(\boldsymbol{X})$ between the spaces of composable tuples are homotopy equivalences and, since $\boldsymbol{X}$ was assumed to be equivariantly well-based, our simplicial spaces are proper, so we have a homotopy equivalence among the classifying spaces $BC''_{p,q}(\boldsymbol{X}) \to BD_{p-1,q}(\boldsymbol{X})$.

[4] In contrast to Segal, we decided to introduce a new letter D for this to avoid confusion when speaking of its bar construction.

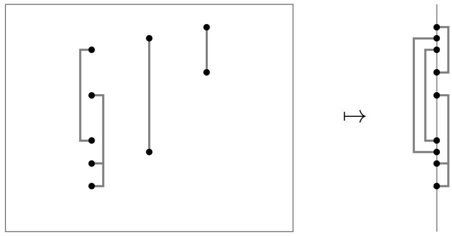


Figure 4: An instance of the map $C''_{p,q}(\boldsymbol{X}) \to D_{p-1,q}(\boldsymbol{X})$

Step 4 In the last step, which is a bit lengthy and which we outsource into Lemma 4.6, we show that the inclusion $C''_{p,q}(\boldsymbol{X}) \subseteq C'_{p,q}(\boldsymbol{X})$ of (partial) monoids induces a homotopy equivalence among classifying spaces: this is the analogue of [19, Prop. 2.4]. We therefore end up with a zig-zag of homotopy equivalences

$$BC'_{p,q}(\boldsymbol{X}) \xleftarrow{\simeq} BC''_{p,q}(\boldsymbol{X}) \xrightarrow{\simeq} BD_{p-1,q}(\boldsymbol{X}) \xrightarrow{\cong} C_{p-1,q}(\Sigma\boldsymbol{X}).$$

Since all three maps leave the remaining $d-1$ coordinates unchanged, they are morphisms of E_{d-1}-algebras, so $BC'_{p,q}(\boldsymbol{X})$ and $C_{p-1,q}(\Sigma\boldsymbol{X})$ are equivalent as E_{d-1}-algebras. □

We are left to show Lemma 4.6. Even though the proof is both technical and very similar to Segal's one, we decided to spell out some details, as they show at which stages the verticality constraint is used.

Lemma 4.6. *The inclusion $C''_{p,q}(\boldsymbol{X}) \subseteq C'_{p,q}(\boldsymbol{X})$ induces an equivalence on classifying spaces.*

Proof. There is an equivalent description of BM for a (partial) monoid M: consider the topological category $\mathbf{C}(M)$ with object space M, and arrows $m \to m'$ being pairs $(m_1, m_2) \in M \times M$ with $m_1 \cdot m \cdot m_2 = m'$. Then $BM \cong |\mathbf{C}(M)|$, see [19, Prop. 2.5].

Let Q be the space of triples (a, b, c) with $a \leqslant 0 \leqslant b$ and $c = \sum_i \vec{z}_i \otimes x_i \in C_{p,q}(\boldsymbol{X})$ with support in $(a; b)$. We give Q a partial order as follows: For each interval $L \subseteq \mathbb{R}$ and $c \in C_{p,q}(\boldsymbol{X})$ whose support avoids ∂L, we define $c|_L$ as the subconfiguration that comprises all $\vec{z}_i \otimes x_i$ satisfying $\mathrm{pr}_1(\vec{z}_i) \in L$. Here we use that $\mathrm{pr}_1(\vec{z}_i)$ is a single value in $\mathbb{R}$ by the verticality condition. Now we let $(a, b, c) \leqslant (a', b', c')$ if $[a; b] \subseteq [a'; b']$ and $c = c'|_{[a;b]}$, see Figure 5. We get a functor $\pi \colon Q \to \mathbf{C}(C'_{p,q}(\boldsymbol{X}))$, $\pi(a, b, c) := (b - a, T_{-a}c)$ and we can copy [19, Lem. 2.6] verbatim to show that $|\pi|$ is shrinkable, i.e. it has a section s such that $s \circ |\pi| \simeq \mathrm{id}$ by a homotopy $h_\bullet$ with $|\pi| \circ h_t = |\pi|$ for all t. Let $P \subseteq Q$ be the subspace of all (a, b, c) with projectable c. Then $\pi(P) = C''_{p,q}(\boldsymbol{X})$, so it is enough to show that $|P| \to |Q|$ is a homotopy equivalence. To do so, we use [19, Prop. 2.7]:

Proposition. *Let Q be a good*[5] *ordered space such that:*

Q1. *For $\nu_1, \nu_2, \nu \in Q$ with $\nu_1, \nu_2 \leqslant \nu$ there exists* $\inf(\nu_1, \nu_2)$,

[5] A *good ordered space* is an ordered space Q such that its nerve is a good simplicial space. A topological monoid $(M, 1)$ is good if 1 has a contractible neighbourhood, see [19, A2].

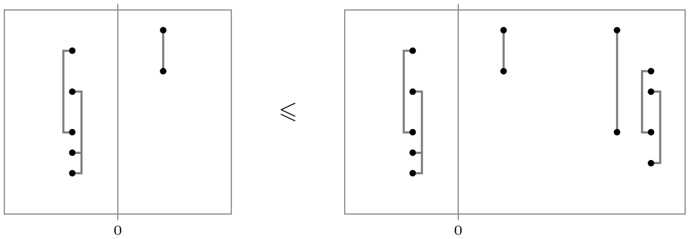


Figure 5: The left configuration is *smaller* than the right one since it is a restriction of the latter.

Q2. *Wherever defined,* $(\nu_1, \nu_2) \mapsto \inf(\nu_1, \nu_2)$ *is continuous.*

Moreover let $Q' \subseteq Q$ *be open such that:*

Q3. *For* $\nu' \in Q'$ *and* $\nu \leqslant \nu'$*, we have* $\nu \in Q'$*,*

Q4. *There is a numerable open cover* $(W_i)_{i \in I}$ *and* $w_i \colon W_i \to Q'$ *with* $w_i(\nu) \leqslant \nu$*.*

Then $|Q'| \to |Q|$ *is a homotopy equivalence.*

As in [**19**, A2], our special Q is good; and additionally, the assumptions Q1 and Q2 are satisfied by the explicit construction of our order.

Since $\boldsymbol{X}$ is equivariantly well-based, there are contractible and $\mathfrak{S}_k$-invariant neighbourhoods $U_k \subseteq X_k$ around the respective basepoints $*_k$, and equivariant homotopies moving U_k into $*_k$. We 'thicken' P to an open subset $Q' \subseteq Q$ containing all configurations that are projectable once we ignore clusters labelled in some U_k; we call these configurations *almost projectable*. Then $\imath \colon |P| \to |Q'|$ is a homotopy equivalence, with retraction ρ given by forgetting clusters labelled in some U_k, the homotopy $\imath \circ \rho \simeq \mathrm{id}_{|Q'|}$ induced by the homotopies from above. Moreover, Q' satisfies Q3 since restrictions of projectables are still projectable. As a cover, we define, for each $n \geqslant 1$,

$$W_n \coloneqq \Big\{(a, b, c);\ c|_{[-\frac{1}{n};\frac{1}{n}]} \text{ is almost projectable}\Big\}.$$

Then $W_n \subseteq Q$ is open, $(W_n)_{n \geqslant 1}$ is numerable, and since each c has only finitely many clusters, each c admits a $n > 0$ such that $c|_{[-\frac{1}{n};\frac{1}{n}]}$ projects to at most one point in $(-\frac{1}{n}; \frac{1}{n})$; hence the restriction has to be projectable: therefore, $(W_n)_{n \geqslant 1}$ is exhaustive. Finally, the maps $w_n \colon W_n \to Q'$ with $w_n(a, b, c) \coloneqq (\max(a, -\frac{1}{n}), \min(b, \frac{1}{n}), c|_{[-\frac{1}{n};\frac{1}{n}]})$ satisfy Q4. ☐

Combining Theorem 3.4 and Theorem 4.5, we obtain the following result:

Corollary 4.7. *Let* $\boldsymbol{X}$ *be equivariantly well-based. Then we have an equivalence*

$$B^{p+1} C(\mathbb{R}^{p,1}; \boldsymbol{X}) \simeq \Sigma^{p+1} \bigvee\nolimits_{e \in \mathbb{E}} \Sigma^{p \cdot (\# e - 1)} \boldsymbol{X}^{\wedge K(e)}.$$

5. Stable homology of vertical configuration spaces

We want to use the previously established homotopical results for an explicit homological statement about vertical configuration spaces: throughout this section, let $p \geqslant 1$ and $k \geqslant 1$. By inserting a new k-cluster on the far right side, we have stabilising maps $C_{r\times k}(\mathbb{R}^{p,1}) \to C_{(r+1)\times k}(\mathbb{R}^{p,1})$, see Figure 6. Extending work of [13, 17, 22], it is shown in [2, Thm. 4.3] that the induced map in $H_\bullet(-;\mathbb{Z})$ is split injective, and an isomorphism if $\bullet \leqslant \frac{r}{2}$. We give a description of the stable homology $H_\bullet(C_{\infty\times k}(\mathbb{R}^{p,1}))$.

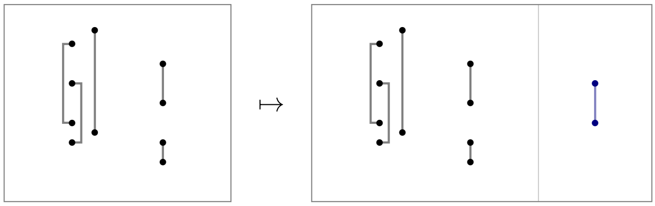


Figure 6: The stabilisation $C_{5\times 2}(\mathbb{R}^{1,1}) \to C_{6\times 2}(\mathbb{R}^{1,1})$

Construction 5.1 (Coloured configuration spaces)**.** Let I be an index set and $\alpha = (\alpha_i)_{i\in I}$ be a finitely supported family of non-negative integers (i.e. $\alpha_i \neq 0$ for only finitely many $i \in I$). For each space E, the group $\prod_{i\in I} \mathfrak{S}_{\alpha_i}$ acts freely on $\tilde{C}_{|\alpha|}(E)$, and we define the *coloured configuration space*

$$C^\alpha(E) := \tilde{C}_{|\alpha|}(E)/\textstyle\prod_{i\in I} \mathfrak{S}_{\alpha_i}.$$

This definition is rather similar to the one of the clustered configuration space $C_\alpha(E)$ from Definition 2.1, but we quotient out a bit less: intuitively, a point in $C^\alpha(E)$ is a disjoint configuration of unordered *coloured* particles, exactly α_i particles of colour i. Coloured configuration spaces have been studied in [16].

A *parity map* is an assignment $t\colon I \to \mathbb{Z}_2$: it merely divides I into 'odd' and 'even' colours. For each parity map and each finitely supported tuple $\alpha = (\alpha_i)_{i\in I}$, we have a sign function $\prod_i \mathfrak{S}_{\alpha_i} \to \{\pm 1\}$ sending $(\sigma_i)_{i\in I}$ to the product of signs $\prod_i \mathrm{sg}(\sigma_i)^{t(i)}$. Via the canonical projection $\pi_1(C^\alpha(E)) \to \prod_i \mathfrak{S}_{\alpha_i}$, this gives rise to a local system ε^α on $C^\alpha(E)$. If the parity map is clear from the context, we write

$$M_\bullet(E;\alpha) := H_\bullet(C^\alpha(E);\varepsilon^\alpha).$$

Although Construction 5.1 might seem unrelated at first glance, the modules $M_\bullet(E;\alpha)$ are useful to describe the homology of vertical configuration spaces:

Definition 5.2. Let $\mathbb{E}[k] \subseteq \mathbb{E}$ be the subset of entanglement types $e = (S_1, \ldots, S_w)$ such that each S_i is of cardinality k. Fixing a dimension $p \geqslant 1$, the parity of e is defined to be $p\cdot(\#e - 1)$, that is: an entanglement type is *even* if it has odd weight or if p is even.

For each finite tuple $\alpha = (\alpha_e)_{e\in\mathbb{E}[k]}$, we define $r(\alpha) := \sum_e \alpha_e \cdot \#e$ and $s(\alpha) := r(\alpha) - |\alpha|$. Intuitively, α tells us how often which entanglement type can be seen, $r(\alpha)$ tells us how many *clusters* are involved, and $s(\alpha)$ measures the difference between the number of clusters and the number of entanglement types.

In [2, § 4], we introduce a filtration $\mathcal{F}_\bullet C_{r\times k}(\mathbb{R}^{p,1})$, and in [2, Prop. 4.19], we establish an isomorphism of graded abelian groups

$$H_\bullet(\mathcal{F}_s C_{r\times k}(\mathbb{R}^{p,1}), \mathcal{F}_{s-1} C_{r\times k}(\mathbb{R}^{p,1})) \cong \bigoplus_{(r(\alpha),s(\alpha))=(r,s)} M_{\bullet - p\cdot s}(\mathbb{R}^{p+1};\alpha) \tag{4}$$

as follows: given a coloured configuration, we 'insert', at each particle in $\mathbb{R}^{p+1}$ of colour

e, a standard configuration that realises the entanglement type e along a vertical line. The degree shift and the sign system is caused—via the Thom isomorphism—by small perturbations of the clusters, tracking all possibilities how to 'break' an entanglement.

However, we could not determine if the Leray spectral sequence associated with the above filtration collapses on its first page and if the extension problem is trivial [**2**, Outl. 4.22]: this would imply that $H_\bullet(C_{r\times k}(\mathbb{R}^{p,1})) \cong \bigoplus_{r(\alpha)=r} M_{\bullet - p\cdot s(\alpha)}(\mathbb{R}^{p+1};\alpha)$. We show that this is at least stably the case.

Construction 5.3 (Stabilisation)**.** Let I be an index set as before, and we pick a distinguished colour $i_0 \in I$. If $\lambda = (\lambda_i)_{i\in I\smallsetminus\{i_0\}}$ is a finitely supported tuple of integers $\lambda_i \geqslant 0$ and $n \geqslant |\lambda|$ is an integer, then we let $\lambda[n]$ be the I-indexed tuple that additionally contains the entry $\lambda_{i_0} = n - |\lambda|$.

Adding a point of colour i_0 on the far right side gives rise to a stabilisation map $C^{\lambda[n]}(\mathbb{R}^d) \to C^{\lambda[n+1]}(\mathbb{R}^d)$ among coloured configuration spaces. For each parity map, the local system $\varepsilon^{\lambda[n+1]}$ restricts to $\varepsilon^{\lambda[n]}$ along the stabilisation map, and for the case in which i_0 has even parity, it is shown in [**2**, Lem. 4.21] that the induced map $M_\bullet(\mathbb{R}^d;\lambda[n]) \to M_\bullet(\mathbb{R}^d;\lambda[n+1])$ is split injective, and bijective for $\bullet \leqslant \frac{n-|\lambda|}{2}$: this is a signed version of [**16**, Cor. C]. We define the stable module

$$M_\bullet(\mathbb{R}^d;\lambda[\infty]) \coloneqq \varinjlim_n M_\bullet(\mathbb{R}^d;\lambda[n]).$$

Example 5.4. There is a single entanglement type $e_0 = (\{1,\dots,k\}) \in \mathbb{E}[k]$ of weight 1; it clearly has even parity for each $p \geqslant 1$. Adding a cluster $C_{r\times k}(\mathbb{R}^{p,1}) \to C_{(r+1)\times k}(\mathbb{R}^{p,1})$ preserves the aforementioned filtration from [**2**], and translates via Equation (4) to the stabilisations $C^{\lambda[n]}(\mathbb{R}^{p+1}) \to C^{\lambda[n+1]}(\mathbb{R}^{p+1})$ by adding a particle of colour e_0. This was the key ingredient for the proof of homological stability [**2**, Thm. 4.3]. Finally, note that $s(\lambda[n])$ from Definition 5.2 is independent of n, so we can just write $s(\lambda)$.

Theorem C. *For each $p \geqslant 1$, we have an isomorphism of graded abelian groups*

$$H_\bullet(C_{\infty\times k}(\mathbb{R}^{p,1})) \cong \bigoplus\nolimits_\lambda M_{\bullet - p\cdot s(\lambda)}(\mathbb{R}^{p+1};\lambda[\infty]),$$

where λ ranges in the set of finitely supported tuples indexed by $\mathbb{E}[k] \smallsetminus \{e_0\}$.

Before proving the theorem, we note that for $k = 1$, both sides are clearly the same: as the verticality condition becomes empty, we have $C_{\infty\times k}(\mathbb{R}^{p,1}) = C_\infty(\mathbb{R}^{p+1})$. On the other hand, since $\mathbb{E}[1]$ contains *only* the distinguished entanglement type e_0, the only possible λ is the empty tuple, and in this case, $s(\lambda) = 0$ and $M_\bullet(\mathbb{R}^{p+1};\lambda[\infty])$ is just the stable homology of the sequence of spaces $C_n(\mathbb{R}^{p+1})$.

Proof. As before, let $\mathbb{S}^0[k]$ be the based symmetric sequence whose k^{th} space is the 0-sphere $\mathbb{S}^0$, and whose remaining constituents are trivial. As in Remark 2.8, the labelled vertical configuration space $C(\mathbb{R}^{p,1};\mathbb{S}^0[k])$ is isomorphic to $\coprod_r C_{r\times k}(\mathbb{R}^{p,1})$. Since $p \geqslant 1$, $C(\mathbb{R}^{p,1};\mathbb{S}^0[k])$ is at least an E_2-algebra, in particular H-commutative. Hence the group completion theorem [**15**, Prop. 1] applies and we calculate the stable homology as

$$H_\bullet\big(C_{\infty\times k}(\mathbb{R}^{p,1})\big) \cong H_\bullet\Big(\Omega_0^{p+1} B^{p+1} C(\mathbb{R}^{p,1};\mathbb{S}^0[k])\Big),$$

where Ω_0 denotes the path component of the constant loop. Using Corollary 4.7, we obtain $B^{p+1}C(\mathbb{R}^{p,1};\mathbb{S}^0[k]) \simeq \Sigma^{p+1}\bigvee_e \mathbb{S}^{p\cdot(\#e-1)}$, where e ranges in $\mathbb{E}[k]$. Now we use that this space can be desuspended $p+1$ times, i.e. we calculate the stable homology of a *free* E_{p+1}-algebra: the bouquet $\bigvee_e \mathbb{S}^{p\cdot(\#e-1)}$ has two path components, namely $\bigvee_{\#e\geqslant 2} \mathbb{S}^{p\cdot(\#e-1)}$, which also contains the basepoint, and $\{e_0\}$. If we denote by $C_m \subseteq C(\mathbb{R}^{p+1}; \bigvee_e \mathbb{S}^{p\cdot(\#e-1)})$ the component of configurations with exactly m particles labelled by e_0, then we have stabilisations $C_m \to C_{m+1}$ by adding a particle with label e_0, and we denote its colimit by C_∞. By applying the group completion theorem once again, we obtain

$$H_\bullet(C_{\infty\times k}(\mathbb{R}^{p,1})) \cong H_\bullet\Big(\Omega_0^{p+1}\Sigma^{p+1}\bigvee_e \mathbb{S}^{p\cdot(\#e-1)}\Big) \cong H_\bullet(C_\infty).$$

The space C_m admits a stable splitting $\Sigma^\infty_+ C_m \simeq \Sigma^\infty \bigvee_\alpha D^\alpha$ in the spirit of [20], where α ranges in tuples with $\alpha_{e_0} = m$, and D^α is the subspace of configurations that have, for each e, at most α_e particles with labels in the sphere corresponding to e, quotiented by the subspace of configurations where at least one of these labels is the basepoint.

As in [1, § 2.6], D^α is the Thom space of a disc bundle over $C^\alpha(\mathbb{R}^{p+1})$ (whose sign system is exactly ε^α), so we get a Thom isomorphism $\tilde{H}_\bullet(D^\alpha) \cong M_{\bullet-p\cdot s(\alpha)}(\mathbb{R}^{p+1};\alpha)$. Altogether, we have $H_\bullet(C_m) \cong \bigoplus_\alpha M_{\bullet-p\cdot s(\alpha)}(\mathbb{R}^{p+1};\alpha)$, where α ranges in tuples with $\alpha_{e_0} = m$. Under this identification, the stabilisation maps $H_\bullet(C_m) \to H_\bullet(C_{m+1})$ split as the sum of stabilising maps $M_{\bullet-p\cdot s(\lambda)}(\mathbb{R}^{p+1};\lambda[n]) \to M_{\bullet-p\cdot s(\lambda)}(\mathbb{R}^{p+1};\lambda[n+1])$, indexed by all λ and with $n = m + |\lambda|$. This proves the claim. □

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