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A COHOMOLOGICAL BUNDLE THEORY FOR PRESHEAF COHOMOLOGY

MIHAIL HURMUZOV

(*communicated by Brooke Shipley*)

Abstract

We develop a bundle theory of presheaves on small categories, based on similar work by Brent Everitt and Paul Turner. For a certain set of presheaves on posets, we produce a Leray-Serre type spectral sequence that gives a reduction property for the cohomology of the presheaf. This extends the usual cohomological reduction of posets with a unique maximum.

1. Introduction

In [2] Everitt and Turner develop a bundle theory for coloured posets. In their development, coloured posets act as a generalisation of Khovanov's 'cube' construction in his celebrated paper on the categorification of the Jones polynomial [4]. The notion of homology for coloured posets differs from the usual definition of presheaf homology, so it is desirable to rebuild the theory in a more versatile form. In this paper, we do away with coloured posets and move to the full generality of presheaves of modules on small categories. Additionally, we rework the arguments from the more natural *co*homological point of view. For the main theoretical result we impose the general assumption that the base $\mathbf{B}$ of our bundle $\xi\colon \mathbf{B} \to \mathbf{Sh}$ is a poset category and that for each $x \in \mathbf{B}$, the small category of $\xi(x)$ is also a poset category; we call such a bundle a *poset bundle of presheaves*:

Main Theorem. *Let $\xi\colon \mathbf{B} \to \mathbf{Sh}$ be a poset bundle of presheaves with $\mathbf{B}$ a recursively admissible finite poset, and $(\mathbf{E}_\xi, F_\xi)$ the associated total presheaf. Then there is a spectral sequence that converges to the cohomology of the total presheaf:*

$$E_2^{p,q} = H^p(\mathbf{B}; \mathcal{H}^q_{fib}) \Rightarrow H^\bullet(\mathbf{E}_\xi; F_\xi).$$

Bundles of presheaves and the total presheaf are defined in §2, while recursively admissible posets are defined in §6.

In the context of presheaf cohomology, the spectral sequence for a poset bundle of presheaves converges to the cohomology of the fiber at the maximum of the base. Thus, while the main theorem of [2] is able to model Khovanov homology, our key application is as follows.

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Main Application. *Let* $\mathbf{E}$ *and* $\mathbf{B}$ *be posets, with* $\mathbf{B}$ *recursively admissible. Suppose that* $\pi\colon \mathbf{E} \to \mathbf{B}$ *is an onto poset map such that for all* $x < y$ *in* $\mathbf{B}$*, the subposet* $\pi^{-1}(x) \cup \pi^{-1}(y)$ *of* $\mathbf{E}$ *is admissible for* $\pi^{-1}(x), \pi^{-1}(y)$*. Then*

$$H^{\bullet}(\mathbf{E}; F) \cong H^{\bullet}(\pi^{-1}(1); F),$$

for all $F \in \mathbf{Sh}(\mathbf{E})$*, where* 1 *is the unique maximum of* $\mathbf{B}$*.*

We proceed as follows. In §2 we define a category **Sh** of presheaves on small categories that features morphisms between objects reminiscent of the induced maps in [**1**, p. 4]. A bundle is then just a small category decorated with objects and morphisms of **Sh**. We also give a way to 'glue up' the elements of **Sh** in a bundle into a *total presheaf.* The main aim of the paper is to understand the relationship between this total presheaf and the bundle. In §3 we describe explicitly the cochain modules giving rise to the presheaf cohomology of a presheaf. This gives the concrete tools needed to establish the quasi-isomorphisms in the main theorem. Next, the general construction of a spectral sequence from a bicomplex provides the first step of the overarching argument – a spectral sequence, constructed from the bundle, that converges to a particular cohomology. The rest of the argument is establishing that this particular cohomology coincides (in some restricted cases) with the usual presheaf cohomology of the total presheaf.

The chain map ω that will witness this coincidence is defined in §5. Similarly to [**2**], it involves signed combinations of traversals of a grid, determined by a pair of sequences in the base small category and in one of the small categories over an object of the base. The cohomological viewpoint here necessitates our ω goes 'the opposite way' to that in [**2**]. The next section collects some technical tools and the definition of a recursively admissible poset – the restricted case in which the main theorem holds. The bulk of the work is in §7 and §8 with the establishment of two explicit quasi-isomorphisms, giving rise to two long exact sequence in the cohomologies of the total complex and of the presheaf. This is done by careful manipulation of spectral sequences and morphisms between them. Section §9 collects the results into a proof of the main theorem.

The final section §10 gives a recipe for turning a presheaf on a poset into the total presheaf of a bundle. If the base of that bundle is recursively admissible, then applying our main theorem completes the proof of the main application. We finish with an example of a repeated use of this application.

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2. The category Sh

For the rest of the paper, R is a commutative ring with 1.

We define a category **Sh** of presheaves on small categories. An object $(\mathbf{C}, F)$ of this category consists of a small category $\mathbf{C}$ and a presheaf F on $\mathbf{C}$. A **Sh**-morphism

$\gamma\colon (\mathbf{C}, F) \to (\mathbf{D}, G)$ is a pair of maps (γ_1, γ_2), where $\gamma_1\colon \mathbf{D} \to \mathbf{C}$ is a covariant functor and $\gamma_2\colon F\gamma_1 \to G$ is a natural transformation:

$$\begin{tikzcd}
\mathbf{C} \arrow[drr, "F"] \arrow[dr, phantom, "\Downarrow \gamma_2"] & & \\
 & & {}_R Mod \\
\mathbf{D} \arrow[uu, "\gamma_1"] \arrow[urr, "G"'] & &
\end{tikzcd} \quad \text{or} \quad \begin{tikzcd}
F(\gamma_1(x)) \arrow[r, "F(\gamma_1(y)\to\gamma_1(x))"] \arrow[d, "\gamma_{2x}"] \arrow[dr, phantom, "\circlearrowright"] & F(\gamma_1(y)) \arrow[d, "\gamma_{2y}"] \\
G(x) \arrow[r, "G(y\to x)"] & G(y)
\end{tikzcd}$$

The composition of two morphisms $\gamma\colon (\mathbf{C}, F) \to (\mathbf{D}, G)$ and $\delta\colon (\mathbf{D}, G) \to (\mathbf{E}, H)$ is then $(\gamma_1\delta_1, \delta_2\gamma_2)\colon (\mathbf{C}, F) \to (\mathbf{E}, H)$.

Definition 2.1. Let $\mathbf{B}$ be a small category. A *bundle of presheaves* with base $\mathbf{B}$ is a contravariant functor $\xi\colon \mathbf{B} \to \mathbf{Sh}$.

Example 2.2.

1. A *constant bundle* $\xi = \mathbf{B} \times (\mathbf{C}, F)$ is a bundle of presheaves with $\xi(x) = (\mathbf{C}, F)$ for all $x \in \mathbf{B}$ and $\xi(x \to y) = \mathrm{id}_{(\mathbf{C},F)}$ for all arrows $x \to y$.
2. A bundle of coloured posets with base $\mathbf{B}$ in the language of [2] is a covariant functor ζ from a poset $\mathbf{B}$ with a unique maximum to the category $\mathbf{CP}_R$ of coloured posets. Such a bundle of coloured posets gives rise to a bundle of presheaves $\xi\colon \mathbf{B}^{op} \to \mathbf{Sh}$, where if $\zeta(x) = (\mathbf{P}, F)$, then $\xi(x) = (\mathbf{P}^{op}, F)$.
3. If $\mathbf{P}$ and $\mathbf{Q}$ are posets, then an object $F \in \mathbf{Sh}(\mathbf{P} \times \mathbf{Q})$ determines a bundle of presheaves $\xi\colon \mathbf{P} \to \mathbf{Sh}$. For any $x \in \mathbf{P}$, denote by F_x the functor from the full subcategory $\{x\} \times \mathbf{Q}$ of $\mathbf{P} \times \mathbf{Q}$ that agrees with F. Then $\xi(x) = (\mathbf{Q}, F_x)$ for all $x \in \mathbf{P}$ and $\xi(x \to y) = (\mathrm{id}_{\mathbf{Q}}, F_{x\to y})$, where $F_{x\to y}|_z\colon F_y(y, z) \to F_x(x, z)$ agrees with F.
4. We can also model a group action on a presheaf $(\mathbf{C}, F)$. Let the category $\mathbf{C}_G$ have one object $\bullet$ and let the morphisms of $\mathbf{C}_G$ be given by G, with composition given by the group operation. Then a bundle of presheaves $\xi\colon \mathbf{C}_G \to \mathbf{Sh}$ with $\xi(\bullet) = (\mathbf{C}, F)$ describes the action of G on $(\mathbf{C}, F)$.

For clarity, if ξ is a bundle of presheaves with base $\mathbf{B}$ and $x \in \mathbf{B}$, then we will use the notation $\mathbf{E}_x$ for the small category that is the first coordinate of $\xi(x)$ and F_x for the second coordinate of $\xi(x)$. Also, if $y \in \mathbf{E}_x$, then $\pi(y) = x$, i.e. π indicates which fiber y comes from. Finally, consider the $\mathbf{Sh}$-morphism $\xi(x_1 \to x_2)$. For its first coordinate we write $\xi_1(x_1 \to x_2)\colon \mathbf{E}_{x_1} \to \mathbf{E}_{x_2}$ instead of $\xi(x_1 \to x_2)_1$; similarly $\xi_2(x_1 \to x_2)\colon F_{x_2}\xi_1(x_1 \to x_2) \to F_{x_1}$ instead of $\xi(x_1 \to x_2)_2$ for the second.

Definition 2.3. Let $\mathbf{B}$ be a small category and ξ a bundle of presheaves with base $\mathbf{B}$. The associated *total presheaf* $(\mathbf{E}_\xi, F_\xi)$ consists of a small category $\mathbf{E}_\xi$ and a presheaf $F_\xi\colon \mathbf{E}_\xi \to {}_R Mod$, defined as follows (also see Figure 1):

- As a small category, $\mathrm{Obj}(\mathbf{E}_\xi) = \bigsqcup_{x\in\mathbf{B}} \mathrm{Obj}(\mathbf{E}_x)$. The simple arrows of $\mathbf{E}_\xi$ are of two types. There is an arrow $y_1 \to y_2$ in $\mathbf{E}_\xi$ if
  a) $y_1, y_2 \in \mathbf{E}_x$ for some $x \in \mathbf{B}$ and $y_1 \to y_2$ is an arrow in $\mathbf{E}_x$;
  b) $x_1 \to x_2$ is a non-identity arrow in $\mathbf{B}$, y_1 and y_2 are objects of $\mathbf{E}_{x_1}$ and $\mathbf{E}_{x_2}$, respectively, and we have $\xi_1(x_1 \to x_2)(y_1) = y_2$.

The set of all arrows of $\mathbf{E}_\xi$ is the smallest set containing the simple arrows that is closed under composition, where

- for any $x \in \mathbf{B}$, composition of arrows of type a) from $\mathbf{E}_x$ is given by the composition in $\mathbf{E}_x$,
- composition of arrows of type b) (and identity arrows) is given by composition in $\mathbf{B}$.

Additionally, we impose the commutativity of squares: if $x_1 \to x_2$ is an arrow in $\mathbf{B}$ and $y_1 \to y_2$ is an arrow in $\mathbf{E}_{x_1}$, then the square below commutes in $\mathbf{E}_\xi$:

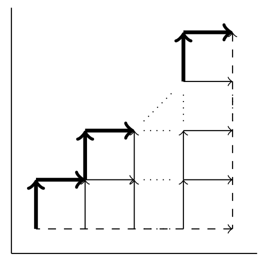
$$
\begin{tikzcd}
 & y_2 \arrow[r] & \xi_1(x_1 \to x_2)(y_2) \\
\mathbf{E}_\xi : & & \\
 & y_1 \arrow[r] \arrow[uu] & \xi_1(x_1 \to x_2)(y_1) \arrow[uu] \\
\mathbf{B} : & x_1 \arrow[r] & x_2
\end{tikzcd}
$$

- As a presheaf, F_ξ sends an object $y \in \mathbf{E}_\xi$ with $\pi(y) = x$ to $F_x(y)$. Arrows $y_1 \to y_2$ of type a) from some $\mathbf{E}_x$ are sent to the map $F_x(y_1 \to y_2)$; arrows $y_1 \to y_2$ of type b) with $\pi(y_1) = x_1, \pi(y_2) = x_2$ are sent to $\xi_2(x_1 \to x_2)_{y_1}$. Composition arrows are sent to the appropriate composition of the above maps.

$$
\begin{tikzcd}
 & F_1(y_2) \arrow[dd, dashed, "F_1(y_1 \to y_2)"'] & & F_2(z_2) \arrow[ll, dotted, "\gamma_2|_{y_2}"'] \arrow[dd, dashed, "F_2(z_1 \to z_2)"] \arrow[ddll, dash dot] \\
{}_R Mod : & & & \\
 & F_1(y_1) & & F_2(z_1) \arrow[ll, dotted, "\gamma_2|_{y_1}"'] \\
 & & \arrow[u, "F_\xi"'] & \\
 & y_2 \arrow[rr, dotted, "\gamma_1(y_2)=z_2"'] & & z_2 \\
\mathbf{E}_\xi : & & & \\
 & y_1 \arrow[uu, dashed, "y_1 \to y_2 \in \mathbf{C}_1"] \arrow[rr, dotted, "\gamma_1(y_1)=z_1"] \arrow[uurr, dash dot] & & z_1 \arrow[uu, dashed, "z_1 \to z_2 \in \mathbf{C}_2"'] \\
\mathbf{Sh} : & (\mathbf{C}_1, F_1) & & (\mathbf{C}_2, F_2) \arrow[ll, "\xi(x_1 \to x_2)=\gamma"'] \\
 & & \arrow[u, "\xi"'] & \\
\mathbf{B} : & x_1 \arrow[rr] & & x_2
\end{tikzcd}
$$

Figure 1: Constructing the total presheaf $(\mathbf{E}_\xi, F_\xi)$. Arrows of type a) are dashed, arrows of type b) are dotted, and composition arrows are dash-dotted.

a)



b)

Figure 2: An alternating sequence of a) and b) arrows and the resulting commutative grid.

Proposition 2.4. *Any composition arrow f in $\mathbf{E}_\xi$ is equal to gh, for some type a) arrow g and some type b) arrow h.*

Since compositions of arrows of type a) or b) are still arrows of the same type, a composition arrow in E is an alternating sequence of arrows of type a) and b). If f starts with a type a) arrow and ends with a type b), then consider the diagram in Figure 2. If f is the composition of bolded arrows, we can construct the commutative square grid below and to the right using the commutativity of squares in Definition 2.3. This implies that f is equal to the composition of dashed arrows, as required. And if f starts with a type b) or ends with a type a), the last step subsumes those with the other horizontal or vertical dashed arrows, again giving a resulting composition of gh for some h of type b) and some g of type a).

Proposition 2.5. *The pair $(\mathbf{E}_\xi, F_\xi)$ above is an object of* $\mathbf{Sh}$.

It is easily checked that $\mathbf{E}_\xi$ is a small category. Furthermore, since the action of F_ξ on composition arrows is defined as the composition of actions on simple arrows, functoriality of F_ξ follows from the functoriality of ξ and F_x for all $x \in \mathbf{B}$.

Remark 2.6. The commutativity of squares imposed on $\mathbf{E}_\xi$ in Definition 2.3 enables us to prove Proposition 2.4 at the category level. Indeed as in [**2**], a similar proposition necessarily holds at the level of the presheaf, since the module homomorphisms at type b) arrows come from the natural transformations $\xi_2(x_1 \to x_2)$ and so the relevant squares commute. We prefer pushing the commutativity to the category $\mathbf{E}_\xi$, because of certain later arguments (e.g. Lemma 6.4).

3. Cochain modules of $(\mathbf{C}, F)$

A version of the exposition in this section can be found in [**6**]. A presheaf F on $\mathbf{C}$ acts as a covariant functor on the poset of simplices of the nerve $N\mathbf{C}$ of $\mathbf{C}$. For

$\sigma = x_0 \to x_1 \to \cdots \to x_i$, $\tau = x_{i_0} \to \cdots \to x_{i_k}$, we set $F(\sigma) = F(x_0)$, and

$$F(\tau \subseteq \sigma) = F(x_0 \to x_{i_0}) = F(x_{i_0}) \to F(x_0),$$

where the arrow $x_0 \to x_{i_0}$ is given by the appropriate composition of arrows in σ. The module for the k-cochains ($k \geqslant 0$) is

$$\mathcal{S}^k(N\mathbf{C}; F) = \prod_\sigma F(\sigma),$$

where the product ranges over all k-simplices $\sigma = x_0 \to \cdots \to x_k$. For $k < 0$, we set $\mathcal{S}^k(N\mathbf{C}; F) = 0$. The differential $d^k \colon \mathcal{S}^{k-1}(N\mathbf{C}; F) \to \mathcal{S}^k(N\mathbf{C}; F)$ for $k > 0$ is given by

$$(d^k u)|_\sigma = \sum_{j=0}^{k} (-1)^j F(\sigma_j \subseteq \sigma)(u|_{\sigma_j}),$$

where $\sigma = x_0 \to x_1 \to \cdots \to x_k$, $u \in \mathcal{S}^{k-1}(N\mathbf{C}; F)$, and $\sigma_j = x_0 \to \cdots \to \hat{x}_j \to \cdots \to x_k$. For $k \leqslant 0$, $d^k = 0$. It is easily seen that $\mathcal{S}^\bullet(N\mathbf{C}; F)$ is a chain complex. For notational brevity, we will also write it as $\mathcal{S}^\bullet(\mathbf{C}; F)$.

Given a $\mathbf{Sh}$-morphism $\gamma \colon (\mathbf{C}, F) \to (\mathbf{D}, G)$, there is an induced map on chains $\gamma^\bullet : \mathcal{S}^\bullet(\mathbf{C}; F) \to \mathcal{S}^\bullet(\mathbf{D}; G)$ defined by

$$\gamma^\bullet u|_\sigma = \gamma_{2x_0}(u|_{\gamma_1(\sigma)}).$$

Lemma 3.1. *The induced map $\gamma^\bullet$ is a well-defined chain map.*

This follows from an easy calculation, using the naturality of γ_2.

We have thus defined a covariant functor $\mathcal{S}^\bullet \colon \mathbf{Sh} \to \mathbf{Ch}_R$, from pairs of small categories and presheaves to chain complexes over R. In particular, we have a covariant functor $\mathcal{S}^q \colon \mathbf{Sh} \to {}_R Mod$ for each $q \in \mathbb{Z}$. Since homology is a functor from chain complexes to graded R-modules, we also have a covariant functor $H^\bullet \mathcal{S}^\bullet \colon \mathbf{Sh} \to \mathbf{Gr}\,{}_R Mod$. In particular, we have a covariant functor $H^q \mathcal{S}^\bullet \colon \mathbf{Sh} \to {}_R Mod$ for each $q \in \mathbb{Z}$.

Given a bundle $\xi \colon \mathbf{B} \to \mathbf{Sh}$, the above gives us two presheaves on $\mathbf{B}$. For any $q \in \mathbb{Z}$ the *q-cochain presheaf* of $\mathbf{B}$ is the presheaf $\mathcal{S}^q \colon \mathbf{B} \to {}_R Mod$, i.e. the composition

$$\mathbf{B} \xrightarrow{\xi} \mathbf{Sh} \xrightarrow{\mathcal{S}^q} {}_R Mod.$$

Similarly, the *homology of the fibres* presheaf of $\mathbf{B}$ is the presheaf $\mathcal{H}^q_{fib} \colon \mathbf{B} \to {}_R Mod$, i.e. the composition

$$\mathbf{B} \xrightarrow{\xi} \mathbf{Sh} \xrightarrow{\mathcal{S}^\bullet} \mathbf{Ch}_R \xrightarrow{\mathcal{H}^q_{fib}} {}_R Mod.$$

Explicitly, if $x \in \mathbf{B}$, then $\mathcal{H}^q_{fib}(x) = H^q(\mathbf{E}_x; F_x)$.

4. The bicomplex $\mathcal{S}^p(\mathbf{B}; \mathcal{S}^q)$

We want to construct a bicomplex (adapting [2, §2]) by taking the p-cochain presheaf of $(\mathbf{B}, \mathcal{S}^q)$.

Let $\xi\colon \mathbf{B} \to \mathbf{Sh}$ be a bundle of presheaves and suppose $x \to y$ is an arrow in $\mathbf{B}$. We have the commutative square

$$\begin{tikzcd}
\mathcal{S}^{q-1}(\mathbf{E}_x; F_x) \arrow[r, "d"] & \mathcal{S}^{q}(\mathbf{E}_x; F_x) \\
\mathcal{S}^{q-1}(\mathbf{E}_y; F_y) \arrow[r, "d"] \arrow[u] & \mathcal{S}^{q}(\mathbf{E}_y; F_y) \arrow[u]
\end{tikzcd}$$

where the vertical maps are the chain map from Lemma 3.1 induced by $\xi(x \to y)$. In particular, the differential d induces a $\mathbf{Sh}$-morphism $\gamma\colon (\mathbf{B}, \mathcal{S}^{q-1}) \to (\mathbf{B}, \mathcal{S}^{q})$, where γ_1 is the identity functor and γ_2 are the differentials at each object of $\mathbf{B}$. This gives us the induced map

$$\gamma^\bullet\colon \mathcal{S}^\bullet(\mathbf{B}; \mathcal{S}^{q-1}) \to \mathcal{S}^\bullet(\mathbf{B}; \mathcal{S}^{q}).$$

Applying this for all $q \in \mathbb{Z}$ gives a grid of commutative squares of the form:

$$\begin{tikzcd}
\mathcal{S}^{p-1}(\mathbf{B}; \mathcal{S}^{q}) \arrow[r] & \mathcal{S}^{p}(\mathbf{B}; \mathcal{S}^{q}) \\
\mathcal{S}^{p-1}(\mathbf{B}; \mathcal{S}^{q-1}) \arrow[r] \arrow[u] & \mathcal{S}^{p}(\mathbf{B}; \mathcal{S}^{q-1}) \arrow[u]
\end{tikzcd}$$

To make the squares anti-commute instead, we apply the usual 'Jedi sign trick', i.e. we include a factor of -1 in every other horizontal map. We will be concerned with this bicomplex in particular in later chapters, so we will sometimes refer to it as just $\mathcal{K}_\xi^{p,q}$. Explicitly, we have

$$\mathcal{K}_\xi^{p,q} = \mathcal{S}^p(\mathbf{B}; \mathcal{S}^q);$$

if we denote

$$\sigma = x_0 \to \ldots \to x_p \in N\mathbf{B} \text{ and } \tau = y_0 \to \ldots \to y_q \in N\mathbf{E}_{x_0},$$

then the vertical differential $d^v\colon \mathcal{S}^p(\mathbf{B}; \mathcal{S}^{q-1}) \to \mathcal{S}^p(\mathbf{B}; \mathcal{S}^q)$ is defined by

$$(d^v u)|_{\sigma,\tau} = F_{x_0}(y_0 \to y_1)(u|_{\sigma,\tau_0}) + \sum_{j=1}^{q} (-1)^j (u|_{\sigma,\tau_j})$$

and the horizontal differential $d^h\colon \mathcal{S}^{p-1}(\mathbf{B}; \mathcal{S}^q) \to \mathcal{S}^p(\mathbf{B}; \mathcal{S}^q)$ is defined by

$$(d^h u)|_{\sigma,\tau} = (-1)^{p+q} \left(\gamma_{y_0}(u|_{\sigma_0,\gamma_1(\tau)}) + \sum_{i=1}^{p} (-1)^i (u|_{\sigma_j,\tau}) \right),$$

where $\xi_2(x_0 \to x_1) = \gamma$.

We can place the modules $\mathcal{K}_\xi^{p,q}$ on the E_0 page of a spectral sequence and use the vertical maps as the differentials on that page. We can further use the quotients of the horizontal maps for the differentials on the E_1 page. This is sometimes referred to as a Leray-Serre style spectral sequence (e.g. [**7**, §5]).

Proposition 4.1. *The E_2 page of the spectral sequence defined above has*

$$E_2^{p,q} = H^p(\mathbf{B}; \mathcal{H}^q_{fib}).$$

Proof. Note that the differentials on the E_2 page are of degree $(2,-1)$. Consider the following diagram

$$\begin{tikzcd}
\xi \arrow[r, mapsto, "\mathcal{S}^\bullet"] & (\mathbf{B}, \mathcal{S}^\bullet) \arrow[rr, mapsto, "\mathcal{S}^p"] \arrow[drr, mapsto, bend right, "H^\bullet"] & & \mathcal{S}^p(\mathbf{B}; \mathcal{S}^\bullet) \arrow[r, mapsto, "H^\bullet"] & H^\bullet(\mathcal{S}^p(\mathbf{B}; \mathcal{S}^\bullet)) \\
& & (\mathbf{B}, \mathcal{H}^\bullet_{fib}) \arrow[rr, mapsto, "\mathcal{S}^p"] & & \mathcal{S}^p(\mathbf{B}; \mathcal{H}^\bullet_{fib})
\end{tikzcd}$$

The top path is how we get the modules in a given column on the E_1 page – we take vertical homology of a column in E_0. On the other hand, taking horizontal homology of rows formed by $\mathcal{S}^p(\mathbf{B}; \mathcal{H}^q_{fib})$ clearly gives the required modules $H^p(\mathbf{B}; \mathcal{H}^q_{fib})$. It is then enough to show that the two graded modules at the ends of the two paths are equal for each $p \in \mathbb{Z}$. This follows directly from cohomology commuting with the direct product. □

Now, there is a total complex associated to $\mathcal{K}^{p,q}_\xi$. We will denote it as $T^\bullet_\xi$. Explicitly,

$$T^n_\xi := \prod_{p+q=n} \mathcal{K}^{p,q}_\xi,$$

with $d = d^h + d^v$. Then, from the general construction of a spectral sequence from a bicomplex (see [7]) and from the above proposition, we have the presheaf cohomological version of [2, Proposition 2.2]:

Proposition 4.2. *If $\xi : \mathbf{B} \to \mathbf{Sh}$ is a bundle of presheaves, then there is a spectral sequence*

$$E^{p,q}_2 = H^p(B; \mathcal{H}^q_{fib}) \Longrightarrow H^\bullet(T^\bullet_\xi).$$

5. Grid traversals

For a given bundle of presheaves ξ, we define a chain map $\omega\colon \mathcal{S}^\bullet(\mathbf{E}_\xi; F_\xi) \to T^\bullet_\xi$, where $\mathcal{S}^\bullet(\mathbf{E}_\xi; F_\xi)$ is the chain complex constructed in §3 on the total presheaf of ξ (recall Definition 2.3), and $T^\bullet_\xi$ is the total complex associated to the bicomplex $\mathcal{K}^{\bullet,\bullet}_\xi$ constructed in §4. If $\sigma = x_0 \to \cdots \to x_p \in N\mathbf{B}$ and $\tau = y_0 \to \cdots \to y_q \in N\mathbf{E}_{x_0}$, then to each pair (σ, τ) we will associate a (signed) combination of all traversals of a particular grid in $\mathbf{E}_\xi$.

To form this grid, we lay out σ and τ (see Figure 3) and complete the grid using the morphisms $\xi(x_i \to x_{i+1})$ – on the figure we denote $y_{i+1,j} = \xi_1(x_i \to x_{i+1})(y_{i,j})$ and $y_{0,j} = y_j$.

A *grid traversal* $z \in N\mathbf{E}_\xi$ of the grid of (σ, τ) is a chain of length $(p+q)$ of arrows in the grid. In particular, each arrow in z is either

$$\xi_1(x_0 \to x_i)(y_j \to y_{j+1}) \text{ or } y_{i,j} \to \xi_1(x_i \to x_{i+1})(y_{i,j}).$$

Note that these correspond to type a) and type b) in Definition 2.3.

For each grid traversal z of the grid of (σ, τ), define

$$m(z) = \#\{\text{squares in the grid below and to the right of } z\}.$$

Furthermore, define $\varsigma(q) = \left\lceil \dfrac{q}{2} \right\rceil = \min\left\{ n \in \mathbb{Z} \mid n \geqslant \dfrac{q}{2} \right\}$.

$$
\begin{array}{ccccccccc}
 & & \tau & & & & & & \\
 & & \shortparallel & & & & & & \\
 & & y_{0,q} & \longrightarrow & y_{1,q} & \longrightarrow & \cdots & \longrightarrow & y_{p,q} \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
 & & \vdots & & \vdots & & & & \vdots \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
 & & y_{0,0} & \longrightarrow & y_{1,0} & \longrightarrow & \cdots & \longrightarrow & y_{p,0} \\
 & & & & & & & & \\
\sigma & = & x_0 & \longrightarrow & x_1 & \longrightarrow & \cdots & \longrightarrow & x_p
\end{array}
$$

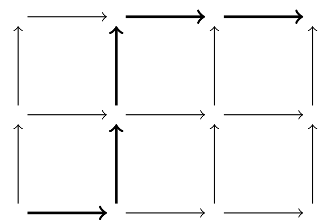


Figure 3: The grid of (σ, τ) (left) and an example grid traversal (right).

We can now define the chain map. The map $\omega\colon \mathcal{S}^\bullet(\mathbf{E}_\xi; F_\xi) \to T^\bullet_\xi$ is defined, for any $u \in \mathcal{S}^\bullet(\mathbf{E}_\xi; F_\xi)$, by

$$(\omega u)|_{\sigma,\tau} = (-1)^{\varsigma(q)} \sum_z (-1)^{m(z)} u|_z,$$

where the sum is taken over all traversals z of the grid of (σ, τ).

Proposition 5.1. *The map ω defined above is a chain map.*

Proof. The argument here is analogous to the argument showing that a similar map is a chain map in [2, Proposition 5.2]. ☐

Note the difference between grid traversals here and in [2] (where they are referred to as *multi-sequences*) – in [2] the traversals stop one step short of reaching the 'top-right' object in the grid, which is dictated by the 'truncated' nature of poset homology. Apart from this, the setup here is very similar, with $\varsigma(q)$ here a rephrasing of $\alpha(q)$ in [2].

6. Technical tools

Up to this point, for a bundle of presheaves $\xi\colon \mathbf{B} \to \mathbf{Sh}$, we have constructed the total presheaf $(\mathbf{E}_\xi, F_\xi)$ and its simplicial complex $\mathcal{S}^\bullet(\mathbf{E}_\xi; F_\xi)$, as well as the bicomplex $\mathcal{K}^{\bullet,\bullet}_\xi$ and its total complex $T^\bullet_\xi$. We know that the spectral sequence of the bicomplex converges to $H^\bullet T^\bullet_\xi$, but we would like to identify cases where it converges to the cohomology of the total presheaf.

Definition 6.1. A bundle of presheaves $\xi\colon \mathbf{B} \to \mathbf{Sh}$ is a *poset bundle of presheaves* if both $\mathbf{B}$ and $\mathbf{E}_x$ for all $x \in \mathbf{B}$ are finite posets.

From this point on, all small categories in sight are assumed to be finite posets. If $x, y \in \mathbf{B}$, we say that y *covers* x (denoted $x \prec y$) if, whenever $z \in \mathbf{B}$ is such that $x \leqslant z \leqslant y$, we have $z = x$ or $z = y$. We also say that $\mathbf{B}$ *has a* 0 (or is a poset with 0) if $\mathbf{B}$ has a unique minimal element $0 \in \mathbf{B}$.

Now, for an element $x \in \mathbf{B}$, define $\mathbf{B}_{\geqslant x}$ and $\mathbf{B}_{\not\geqslant x}$ to be the full subcategories of $\mathbf{B}$ with

$$\operatorname{Obj} \mathbf{B}_{\geqslant x} := \{z \in \operatorname{Obj} \mathbf{B} \mid x \leqslant z\} \text{ and } \operatorname{Obj} \mathbf{B}_{\not\geqslant x} := \operatorname{Obj} \mathbf{B} \setminus \operatorname{Obj} \mathbf{B}_{\geqslant x}.$$

Note that both $\mathbf{B}_{\geqslant x}$ and $\mathbf{B}_{\not\geqslant x}$ inherit the poset structure of $\mathbf{B}$. We will occasionally omit Obj when we refer to the objects of a poset category if the meaning is clear from context.

The key property we will exploit is the following.

Definition 6.2. Assume $\mathbf{B}$ is a poset.

1. Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be full subposets of $\mathbf{B}$. We call $\mathbf{B}$ *admissible for* $\mathbf{B}_1, \mathbf{B}_2$ if
   - $\mathbf{B}_1 \cap \mathbf{B}_2 = \emptyset$,
   - $\mathbf{B}_1 \cup \mathbf{B}_2 = \mathbf{B}$,
   - there are no $x \in \mathbf{B}_2$ and $y \in \mathbf{B}_1$ with $x \leqslant y$, and
   - for all $x \in \mathbf{B}_1$, the full subposet $\{y \in \mathbf{B}_2 \mid x \leqslant y\} \subseteq \mathbf{B}_2$ is non-empty and has a unique minimum.
2. We call $\mathbf{B}$ *admissible for* $x \in \mathbf{B}$ if $\mathbf{B}$ is admissible for $\mathbf{B}_{\not\geqslant x}, \mathbf{B}_{\geqslant x}$. Note that the first three requirements of admissibility are automatically satisfied for $\mathbf{B}_{\not\geqslant x}, \mathbf{B}_{\geqslant x}$ (see bottom of Figure 6). We also denote the poset in the last requirement by

$$\mathbf{B}_{\geqslant x}^{\geqslant y} := \{z \in \mathbf{B}_{\geqslant x} \mid y \leqslant z\} = \mathbf{B}_{\geqslant x} \cap \mathbf{B}_{\geqslant y}.$$

3. We call $\mathbf{B}$ *recursively admissible* if $\mathbf{B}$ has a 0 and either
   - $\mathbf{B}$ is Boolean of rank 1, or
   - $\mathbf{B}$ is admissible for some $x \succ 0$ and both $\mathbf{B}_{\geqslant x}$ and $\mathbf{B}_{\not\geqslant x}$ are recursively admissible.

Example 6.3.

- The Boolean lattices $\mathbb{B}_n$ are recursively admissible (Figure 4).
- In the homological setup of [**2**], the Bruhat posets of type $I_2(m)$ are specially admissible (see [**2**, Example 3.7]). In the language of this paper they are just admissible (Figure 5).
- Let $\mathbb{B}_n^-$ be the Boolean lattice of rank n with its maximum removed. Let $\mathbb{B}_n^+$ be Boolean lattice of rank n with another maximum added; more precisely, it is the poset with objects

$$\operatorname{Obj} \mathbb{B}_n^+ = \operatorname{Obj} \mathbb{B}_n \cup \{1^+\},$$

  such that if $x_1, x_2 \in \mathbb{B}_n$, then $x_1 \leqslant x_2$ in $\mathbb{B}_n^+$ if and only if $x_1 \leqslant x_2$ in $\mathbb{B}_n$; and $x \leqslant 1^+$ for all $x \in \mathbb{B}_n \setminus \{1\}$ (where 1 is the maximum of $\mathbb{B}_n$). The posets $\mathbb{B}_n^\pm$ are non-admissible for all n – if they were, then the maximum x of $\mathbf{B}_1$ would be covered by exactly one element of $\mathbf{B}_2$, but there are no such elements x in $\mathbb{B}_n^\pm$.

If we have a poset bundle of presheaves $\xi \colon \mathbf{B} \to \mathbf{Sh}$ and a subcategory $\mathbf{C}$ of $\mathbf{B}$, we can restrict the bundle ξ to $\mathbf{C}$ to obtain another bundle $\xi_{\mathbf{C}} \colon \mathbf{C} \to \mathbf{Sh}$ with total presheaf $(\mathbf{E}_{\xi_{\mathbf{C}}}; F_{\xi_{\mathbf{C}}})$. When the bundle ξ is clear from context, we will just use $(\mathbf{E}_{\mathbf{C}}; F_{\mathbf{C}})$. Note that we use $(\mathbf{E}_x; F_x)$ for the presheaf $\xi(x)$ when x is an object of $\mathbf{B}$, which (almost) coincides with $(\mathbf{E}_{\mathbf{C}}; F_{\mathbf{C}})$ when $\mathbf{C}$ is the subcategory of $\mathbf{B}$ consisting only of x and its identity arrow.

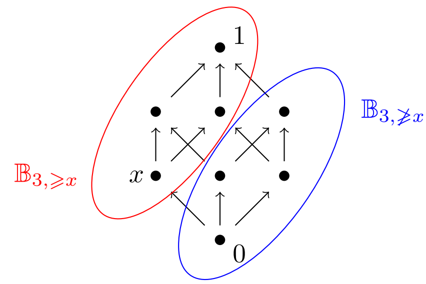


Figure 4: The poset $\mathbb{B}_3$ is admissible for x.

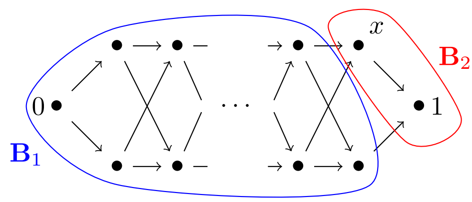


Figure 5: The poset $I_2(m)$ is admissible for $\mathbf{B}_1, \mathbf{B}_2$.

The next lemma shows how admissibility of $\mathbf{B}$ extends to $\mathbf{E}_\xi$.

Lemma 6.4. *Let $\mathbf{B}$ be admissible for some $x \in \mathbf{B}$ and $\xi\colon \mathbf{B} \to \mathbf{Sh}$ be a poset bundle of presheaves with total presheaf $(\mathbf{E}_\xi; F_\xi)$. Then $\mathbf{E}_\xi$ is admissible for $\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}, \mathbf{E}_{\mathbf{B}_{\geqslant x}}$.*

Proof. It is immediate that $\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}$ and $\mathbf{E}_{\mathbf{B}_{\geqslant x}}$ are disjoint, that $\mathbf{E}_{\mathbf{B}_{\not\geqslant x}} \cup \mathbf{E}_{\mathbf{B}_{\geqslant x}} = \mathbf{E}_\xi$, and that there is no arrow from an object of $\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}$ to an object of $\mathbf{E}_{\mathbf{B}_{\geqslant x}}$. It remains to show that for all $w \in \mathbf{E}_{\mathbf{B}_{\not\geqslant x}}$, the subposet $\{z \in \mathbf{E}_{\mathbf{B}_{\geqslant x}} \mid w \leqslant z\}$ has a unique minimal element.

Since $w \in \mathbf{E}_{\mathbf{B}_{\not\geqslant x}}$, w is an element of a particular $\mathbf{E}_y$ for some $y \in \mathbf{B}_{\not\geqslant x}$. By the admissibility of $\mathbf{B}$, that means that the poset $\mathbf{B}^{\geqslant y}_{\geqslant x}$ has a unique minimum, say v. Then $y \leqslant v$ and thus there is an arrow $y \to v$ in $\mathbf{B}$. Denote the presheaf morphism given by this arrow as γ. By the construction of the total presheaf, we have that $w \leqslant \gamma_1(w)$.

Suppose $w \leqslant z$ for some $z \in \mathbf{E}_{\mathbf{B}_{\geqslant x}}$ and suppose $z \in \mathbf{E}_u$, $u \in \mathbf{B}_{\geqslant x}$. Then by our argument in Proposition 2.4 we have a $z_0 \in \mathbf{E}_u$ with $w \leqslant z_0 \leqslant z$ and an arrow $y \to u$ giving rise to a presheaf morphism γ'. Thus u is in $\mathbf{B}^{\geqslant y}_{\geqslant x}$, not just $\mathbf{B}_{\geqslant x}$. Since v is the minimal element of $\mathbf{B}^{\geqslant y}_{\geqslant x}$, we have that $v \leqslant u$. But there is a unique arrow $y \to u$, so γ'_1 factors through $\mathbf{E}_v$ and the presheaf morphism given by $v \to u$ maps $\gamma_1(w)$ to z_0. This means that $\gamma_1(w) \leqslant z_0 \leqslant z$, therefore $\gamma_1(w)$ is the minimum of the set $\{z \in \mathbf{E}_{\mathbf{B}_{\geqslant x}} \mid w \leqslant z\}$. Refer to Figure 6 for the relevant objects. ☐

For constant bundles over certain posets we have a calculation of the cohomology of the total complex. We recall the following facts about morphisms of spectral

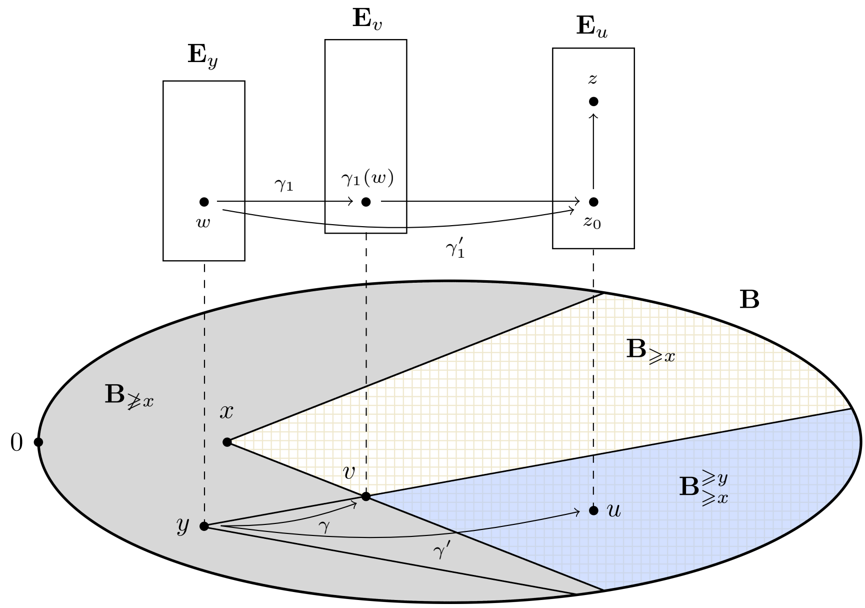


Figure 6: The poset $\mathbf{B}$ with $y \in \mathbf{B}_{\ngeqslant x}$ and the fibers over y, v and u.

sequences (see [5]). If E, E' are spectral sequences constructed from filtrations (or bicomplexes), then a morphism of filtrations (or of bicomplexes) induces a morphism $E \to E'$. A spectral sequence E is *bounded below* if for each degree n there is an integer $s = s(n)$ such that $E_0^{p,q} = 0$ when $p < s$ and $p + q = n$. If E, E' are bounded below spectral sequences and $f\colon E \to E'$ is a morphism, such that for some r the homomorphisms $f_r^{p,q}\colon E_r^{p,q} \to E_r'^{p,q}$ are isomorphisms for each p, q, then the maps $f_\infty^{p,q}\colon E_\infty^{p,q} \to E_\infty'^{p,q}$ are also isomorphisms. The above is an adapted version of the *mapping lemma* [5, 7].

Proposition 6.5. *Suppose $\mathbf{B}$ is a poset, $x \in \mathbf{B}$ is a unique minimum, and $(\mathbf{C}, F)$ is an object of* **Sh**. *If $\xi = \mathbf{B} \times (\mathbf{C}, F)$ is a constant bundle (recall Example 2.2), then there is a chain map $\varphi^\bullet\colon \mathcal{S}^\bullet(\mathbf{C}; F) \to T_\xi^\bullet$ such that the induced map on cohomology $\varphi^\bullet\colon H^\bullet(\mathbf{C}; F) \to H^\bullet T_\xi^\bullet$ is an isomorphism.*

Proof. It is straightforward to see why $\mathcal{S}^\bullet(\mathbf{C}; F)$ is quasi-isomorphic to $T_\xi^\bullet$. The E_2 page of the spectral sequence for ξ has

$$E_2^{p,q} = H^p(\mathbf{B}, \Delta H^q(\mathbf{C}; F)).$$

Since the right-hand side is the cohomology of a constant presheaf, the only non-zero positions on the E_2 page are in the column $p = 0$; so the sequence collapses and we can read off $H^\bullet T_\xi^\bullet$. Explicitly,

$$H^p(\mathbf{B}; \Delta H^q(\mathbf{C}; F)) = \begin{cases} H^q(\mathbf{C}; F), & \text{if } p = 0, \\ 0, & \text{otherwise.} \end{cases}$$

So $H^\bullet(\mathbf{C};F) \cong H^\bullet T_\xi^\bullet$. It is still useful to describe the explicit quasi-isomorphism; we will use a version of this explicit chain map in the proof of Proposition 7.2.

First consider the constant presheaf $(\mathbf{P}, \Delta A)$, where $\mathbf{P}$ is a poset with a unique minimum. Recall that

$$H^n(\mathbf{P};\Delta A) \cong \begin{cases} A, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

We now construct an explicit map for the isomorphism above. So let $u \in \mathcal{S}^0(\mathbf{P};\Delta A)$ be such that $du = 0$. Since we have a unique minimum 0, for any $x \in \mathbf{P}$, there is an arrow $0 \leqslant x$ in $\mathbf{P}$. Then $0 = du|_{0\leqslant x} = u|_x - u|_0$, so $u|_x = u|_0$ for all $x \in \mathbf{P}$. Denote such a constant element of $\mathcal{S}^0(\mathbf{P};\Delta A)$ by u_a if $u_a|_x = a \in A$ for all $x \in \mathbf{P}$. So the isomorphism we are looking for is $\theta\colon A \to H^0(\mathbf{P},\Delta A) : a \mapsto u_a$.

Now consider the (trivial) chain complex $\iota^\bullet(A)$ defined by

$$\iota^n(A) = \begin{cases} A, & \text{if } n = 0, \\ 0, & \text{otherwise,} \end{cases}$$

and $d^n_{\iota^\bullet(A)} = 0$ for all n. Define the map $\psi^\bullet\colon \iota^\bullet(A) \to \mathcal{S}^\bullet(\mathbf{P};\Delta A)$ as

$$\psi^n = \begin{cases} \theta, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & A \arrow[r] \arrow[d, "\theta"] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & \mathcal{S}^0(\mathbf{P};\Delta A) \arrow[r] & \mathcal{S}^1(\mathbf{P};\Delta A) \arrow[r] & \cdots
\end{tikzcd}$$

To see this is a chain map, note that $\theta(a) \in \ker(d^0)$, so $d\theta = 0$. All other squares commute since all compositions are the 0 map.

Crucially, $\psi^\bullet$ is a quasi-isomorphism. This is because $H^0\iota^\bullet(A) = A$ and by construction θ induces the isomorphism $H^0\iota^\bullet(A) \to H^0\mathcal{S}^\bullet(\mathbf{P};\Delta A)$. Note that the map $-\psi^\bullet$ is also a quasi-isomorphism, since $-\theta$ induces $-\operatorname{id}\colon A \to A$ in homology.

Returning to the case of the constant bundle $\xi = \mathbf{B} \times (\mathbf{P}, F)$, we can now define $\varphi^n\colon \mathcal{S}^n(\mathbf{C};F) \to T^n_\xi$ by

$$\varphi u|_{\sigma,\tau} = \begin{cases} u|_\tau, & \text{if } \operatorname{length}(\sigma) = 0, \\ 0, & \text{otherwise.} \end{cases}$$

A routine check shows that $\varphi^\bullet$ is a chain map.

We define a bicomplex $\mathcal{L}^{\bullet,\bullet}$ by

$$\mathcal{L}^{p,q} = \begin{cases} \mathcal{S}^q(\mathbf{C};F) & \text{if } p = 0, \\ 0, & \text{otherwise} \end{cases}$$

and we let $d^h_{\mathcal{L}} = 0$, $d^v_{\mathcal{L}} = 0$ on the non-zero columns, and $d^v_{\mathcal{L}} = d_{\mathcal{S}^\bullet(\mathbf{C};F)}$ on the 0-th column.

Recall the bicomplex $\mathcal{K}^{p,q}_\xi = \mathcal{S}^p(\mathbf{B};\mathcal{S}^q)$ defined in Section 4. We want to show that φ induces a morphism of these two bicomplexes. To that effect, we need three facts:

1. First, it is clear that $\varphi(\mathcal{S}^q(\mathbf{C};F)) \subseteq \mathcal{S}^0(\mathbf{B};\mathcal{S}^q)$.

2. Second, we need φ to induce a chain map on the vertical complexes. This is the zero map for $p \neq 0$. Consider the diagram fragment

$$\begin{tikzcd} \mathcal{S}^{q+1}(\mathbf{C}; F) \arrow[r, "\varphi"] & \mathcal{S}^0(\mathbf{B}; \mathcal{S}^{q+1}) \\ \mathcal{S}^q(\mathbf{C}; F) \arrow[r, "\varphi"] \arrow[u, "d"] & \mathcal{S}^0(\mathbf{B}; \mathcal{S}^q) \arrow[u, "d^v"'] \end{tikzcd}$$

We want to show $d^v\varphi = \varphi d$. Let $u \in \mathcal{S}^0(\mathbf{B}; \mathcal{S}^{q+1})$, $x \in \mathbf{B}$, $y_0 \leqslant \cdots \leqslant y_{q+1} \in \mathbf{C}$.

$$\begin{aligned} d^v\varphi u|_{x,y_0\leqslant\cdots\leqslant y_{q+1}} &= \sum_{i=0}^{q+1} \varphi u|_{x,y_0\leqslant\cdots\leqslant\hat{y}_i\leqslant\cdots\leqslant y_{q+1}} = \sum_{i=0}^{q+1} u|_{x,y_0\leqslant\cdots\leqslant\hat{y}_i\leqslant\cdots\leqslant y_{q+1}} \\ &= du|_{y_0\leqslant\cdots\leqslant y_{q+1}} = \varphi du|_{x,y_0\leqslant\cdots\leqslant y_{q+1}}. \end{aligned}$$

Therefore φ induces a chain map on vertical complexes.

3. Finally, we need φ to induce chain maps on horizontal complexes. Consider the diagram

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & \mathcal{S}^q(\mathbf{C}; F) \arrow[r] \arrow[d, "\varphi"] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\ \cdots \arrow[r] & 0 \arrow[r] & \mathcal{S}^0(\mathbf{B}; \mathcal{S}^q) \arrow[r] & \mathcal{S}^1(\mathbf{B}; \mathcal{S}^q) \arrow[r] & \cdots \end{tikzcd}$$

This is just an instance of the map ψ with $A = \mathcal{S}^q(\mathbf{C}; F)$.

Now consider the two spectral sequences E and E' associated to the bicomplexes $\mathcal{L}^{\bullet,\bullet}$ and $\mathcal{K}_\xi^{\bullet,\bullet}$, respectively. The morphism of bicomplexes φ induces a morphism $E \to E'$ of spectral sequences. Note also that both E and E' are bounded below. We have $E_1^{p,q} = 0$ if $p \neq 0$ and $E_1^{0,q} = H^q(\mathbf{C}; F)$, while ${E'_1}^{p,q} = \mathcal{S}^p(\mathbf{B}; \mathcal{H}^q_{fib})$.

As with a constant bundle, the induced maps φ are quasi-isomorphisms on the horizontal complexes. This means that φ induces isomorphisms on the second pages of E and E'. By the mapping lemma, we have an induced isomorphism $\varphi\colon E_\infty^{p,q} \to E_\infty'^{p,q}$.

By the above, the construction of the total complex of a bicomplex, and Proposition 4.2, we can conclude that φ gives an isomorphism $\varphi\colon H^\bullet(\mathbf{C}; F) \to H^\bullet T_\xi^\bullet$. ☐

7. Long exact sequence in the cohomology of the total complex

This and the following section are functionally similar to [**2**, §4].

If we have a poset bundle $\xi\colon \mathbf{B} \to \mathbf{Sh}$ and a subcategory $\mathbf{C}$ of $\mathbf{B}$, then we will denote the chain complex $T^\bullet_{\xi_\mathbf{C}}$ (recall §4) by just $T^\bullet_\mathbf{C}$. Below we headline the main result of this section and leave the proof until we have built up the required machinery.

Theorem 7.1. *Let $\xi\colon \mathbf{B} \to \mathbf{Sh}$ be a poset bundle of presheaves with $\mathbf{B}$ an admissible poset for $x \succ 0$. Then there is a long exact sequence*

$$\cdots \to H^{n-1}T^\bullet_{\mathbf{B}_{\not\geqslant x}} \to H^nT^\bullet_\xi \to H^nT^\bullet_{\mathbf{B}_{\geqslant x}} \oplus H^nT^\bullet_{\mathbf{B}_{\not\geqslant x}} \to H^nT^\bullet_{\mathbf{B}_{\not\geqslant x}} \to H^{n+1}T^\bullet_\xi \to \cdots$$

We will need to leverage the admissibility condition in the theorem to establish the connection between the total complex of the whole presheaf and those of the two smaller parts $\mathbf{B}_{\geqslant x}$ and $\mathbf{B}_{\not\geqslant x}$, determined by the element $x \succ 0$. Recall that we assume all the $\mathbf{E}_y$ are posets.

Where possible, we will use x's to refer to objects in $\mathbf{B}_{\not\geqslant x}$ and z's to refer to objects of $\mathbf{B}_{\geqslant x}$. We can write down explicitly what T^n_ξ, $T^n_{\mathbf{B}_{\geqslant x}}$, and $T^n_{\mathbf{B}_{\not\geqslant x}}$ are:

$$T^n_\xi = \bigoplus_{p+q=n} \prod_{\substack{x_0 \leqslant \cdots \leqslant x_p \in \mathbf{B} \\ y_0 \leqslant \cdots \leqslant y_q \in \mathbf{E}_{x_0}}} F_{x_0}(y_0), \qquad T^n_{\mathbf{B}_{\geqslant x}} = \bigoplus_{p+q=n} \prod_{\substack{z_0 \leqslant \cdots \leqslant z_p \in \mathbf{B}_{\geqslant x} \\ y_0 \leqslant \cdots \leqslant y_q \in \mathbf{E}_{z_0}}} F_{z_0}(y_0),$$

$$T^n_{\mathbf{B}_{\not\geqslant x}} = \bigoplus_{p+q=n} \prod_{\substack{x_0 \leqslant \cdots \leqslant x_p \in \mathbf{B}_{\not\geqslant x} \\ y_0 \leqslant \cdots \leqslant y_q \in \mathbf{E}_{x_0}}} F_{x_0}(y_0).$$

Define the quotient map

$$\rho\colon T^n_\xi \to T^n_{\mathbf{B}_{\geqslant x}} \oplus T^n_{\mathbf{B}_{\not\geqslant x}}$$

by setting to 0 any coordinate corresponding to a sequence $x_0 \leqslant \cdots \leqslant x_p \in \mathbf{B}$ that has objects in both $\mathbf{B}_{\geqslant x}$ and $\mathbf{B}_{\not\geqslant x}$. Explicitly, if $u \in T^{p+q}_\xi$, $\sigma = x_0 \leqslant \cdots \leqslant x_p \in \mathbf{B}_{\geqslant x}$ or $\mathbf{B}_{\not\geqslant x}$, and $\tau \in \mathbf{E}_{x_0}$, then $\rho u|_{\sigma,\tau} = u|_{\sigma,\tau}$.

To see that ρ is a chain map, let $\{x_0, \ldots, x_p\} \subseteq \mathbf{B}_{\geqslant x}$. We have

$$\begin{aligned} \rho du|_{\sigma,\tau} = du|_{\sigma,\tau} &= \sum_{i=0}^{p} (-1)^i u|_{\sigma_i,\tau} + (-1)^{p+q} \sum_{j=0}^{q} (-1)^j u|_{\sigma,\tau_j} \\ &= \sum_{i=0}^{p} (-1)^i \rho u|_{\sigma_i,\tau} + (-1)^{p+q} \sum_{j=0}^{q} (-1)^j \rho u|_{\sigma,\tau_j} = d\rho u|_{\sigma,\tau}. \end{aligned}$$

The calculation is analogous if $\{x_0, \ldots, x_p\} \subseteq \mathbf{B}_{\not\geqslant x}$. Therefore ρ is a chain map. It is also clearly surjective, so we have a short exact sequence

$$0 \to M^\bullet \to T^\bullet_\xi \to T^\bullet_{\mathbf{B}_{\geqslant x}} \oplus T^\bullet_{\mathbf{B}_{\not\geqslant x}} \to 0$$

for a particular chain complex $M^\bullet$. We describe $M^\bullet$ explicitly:

$$M^n = \bigoplus_{p+q=n} \prod_{x_0 \leqslant \cdots \leqslant x_p} \prod_{y_0 \leqslant \cdots \leqslant y_q \in \mathbf{E}_{x_0}} F_{x_0}(y_0),$$

where $x_0 \in \mathbf{B}_{\not\geqslant x}, x_p \in \mathbf{B}_{\geqslant x}$. We can rewrite $M^\bullet$ to pay attention to how many of the x_i's are in $\mathbf{B}_{\not\geqslant x}$ and how many are in $\mathbf{B}_{\geqslant x}$:

$$M^n = \bigoplus_{s+t+q=n} \prod_{x_0 \leqslant \cdots \leqslant x_s \leqslant z_0 \leqslant \cdots \leqslant z_{t-1}} \prod_{y_0 \leqslant \cdots \leqslant y_q \in \mathbf{E}_{x_0}} F_{x_0}(y_0),$$

where $x_i \in \mathbf{B}_{\not\geqslant x}, z_i \in \mathbf{B}_{\geqslant x}, s \geqslant 0, t \geqslant 1$.

Proposition 7.2. *Let $\xi\colon \mathbf{B} \to \mathbf{Sh}$ be a poset bundle of presheaves with $\mathbf{B}$ an admissible poset for $x \succ 0$. If $M^\bullet$ is as above, there is a chain map $\varphi_1\colon T^{n-1}_{\mathbf{B}_{\not\geqslant x}} \to M^n$ that induces an isomorphism in cohomology.*

Proof. In an attempt to keep the notation less cluttered, write $K^n = T^{n-1}_{\mathbf{B}_{\not\geqslant x}}$.

We define the chain map $\varphi_1 \colon K^n \to M^n$, which will extend to a morphism of filtered complexes. By showing that φ_1 induces isomorphisms on the first pages of the two spectral sequences associated to the two filtrations, the Mapping Lemma implies that φ_1 is a quasi-isomorphism.

Let $\sigma = x_0 \leqslant \cdots \leqslant x_s \leqslant z_0 \leqslant \cdots \leqslant z_{t-1}$ be a sequence in $\mathbf{B}$ with $x_i \in \mathbf{B}_{\not\geqslant x}$ and let $z_i \in \mathbf{B}_{\geqslant x}$, $s \geqslant 0$, $t \geqslant 1$. Denote $\sigma' = x_0 \leqslant \cdots \leqslant x_s$. Also let $\tau = y_0 \leqslant \cdots \leqslant y_q$ be a sequence in $\mathbf{E}_{x_0}$. Now if $s+t+q = n$, we define $\varphi_1 \colon K^n \to M^n$ by

$$\varphi_1 u|_{\sigma,\tau} = \begin{cases} (-1)^q u|_{\sigma',\tau} & \text{if } t = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively, φ_1 acts like the map φ in Proposition 6.5 on the portion of $M^\bullet$ that matches $T^\bullet_{\mathbf{B}_{\not\geqslant x}}$. A routine check shows that φ_1 is a chain map.

Now we define filtrations of $M^\bullet$ and $K^\bullet$:

$$\mathcal{F}^p M^n = \{u \in M^n : u|_{\sigma,\tau} \neq 0 \Rightarrow \sigma = x_0 \leqslant \cdots \leqslant x_s \leqslant z_0 \leqslant \cdots \leqslant z_{t-1} \text{ with } s \geqslant p\},$$

$$\mathcal{J}^p K^n = \{u \in K^n : u|_{\sigma,\tau} \neq 0 \Rightarrow \sigma = x_0 \leqslant \cdots \leqslant x_s \text{ with } s \geqslant p\}.$$

We want to use the Mapping Lemma for these two filtrations, so the next step is establishing all the assumptions of the lemma. We prove them for $\mathcal{F}$ with the arguments for $\mathcal{J}$ being analogous.

($\mathcal{F}$ is a filtration) It is clear from the definition of $\mathcal{F}$ that $\mathcal{F}^{p+1}M^n \subseteq \mathcal{F}^p M^n$ for each p and n. Remains to show that $\mathcal{F}^p M^\bullet$ is a cochain complex for each p. Let $\sigma = x_0 \leqslant \cdots \leqslant x_s \leqslant z_0 \leqslant \cdots \leqslant z_{t-1}$ with $s < p$ and $u \in \mathcal{F}^p M^n$. Then for any sequence $\tau \in \mathbf{E}_{x_0}$ (of appropriate length q) we have

$$du|_{\sigma,\tau} = \sum_{i=0}^{s} (-1)^i u|_{\sigma_i,\tau} + (-1)^{s+1} \sum_{k=0}^{t-1} (-1)^k u|_{\sigma_{s+k},\tau} + (-1)^{s+t+q} \sum_{\ell=0}^{q} (-1)^\ell u|_{\sigma,\tau_\ell}.$$

The summands in the first sum correspond to x-sequences of length $s-1 < p$, while the summands in the other two sums correspond to x-sequences of length $s < p$. All those coordinates are 0 in $u \in \mathcal{F}^p M^n$, so d induces a differential on $\mathcal{F}^p M^\bullet$.

($\mathcal{F}$ is convergent below) Observe that $\mathcal{F}^0 M^n = M^n$, since M^n does not have any coordinates corresponding to sequences in $\mathbf{B}$ not containing elements of $\mathbf{B}_{\not\geqslant x}$.

($\mathcal{F}$ is bounded above) We have $\mathcal{F}^n M^n = 0$, since we need $s+t+q = n$ and $t \geqslant 1$.

(φ_1 is a morphism of filtrations) Let $u \in \mathcal{J}^p K^n$. Set $\sigma = x_0 \leqslant \cdots \leqslant x_s \leqslant z$ and $\tau = y_0 \leqslant \cdots \leqslant y_q$. First suppose $s+q+1 \neq n$. The potentially non-zero coordinates of $\varphi_1 u|_{\sigma,\tau}$ correspond to sequences of combined length satisfying $s+q \neq n-1$, so they are also 0. Now suppose $s < p$. Again, the potentially non-zero coordinates of $\varphi_1 u|_{\sigma,\tau}$ correspond to x-sequences of length $s < p$, so are also 0. Thus $\varphi_1(\mathcal{J}^p K^n) \subseteq \mathcal{F}^p M^n$.
To see that φ_1 induces chain maps $\mathcal{J}^p K^\bullet \to \mathcal{F}^p M^\bullet$ for every p, note that we already know that $d\varphi_1 = \varphi_1 d$ and that $\varphi_1(\mathcal{J}^p K^n) \subseteq \mathcal{F}^p M^n$.

Let E, E' be the spectral sequences associated to the filtrations $\mathcal{F}, \mathcal{J}$, respectively.

We have

$$E_0^{p,q} = \frac{\mathcal{F}^p M^{p+q}}{\mathcal{F}^{p+1} M^{p+q}} = \{u \in M^{p+q} : u|_{\sigma,\tau} \neq 0 \Rightarrow \sigma = x_0 \leqslant \cdots \leqslant x_p \leqslant z_0 \leqslant \cdots \leqslant z_{t-1}\},$$

$$E_0'^{p,q} = \frac{\mathcal{J}^p K^{p+q}}{\mathcal{J}^{p+1} K^{p+q}} = \{u \in K^{p+q} : u|_{\sigma,\tau} \neq 0 \Rightarrow \sigma = x_0 \leqslant \cdots \leqslant x_p\}.$$

The vertical differentials in E_0 are given by

$$\begin{aligned} du|_{x_0 \leqslant \cdots \leqslant x_p \leqslant z_0 \leqslant \cdots \leqslant z_{t-1}, y_0 \leqslant \cdots \leqslant y_{q-t}} &= \\ = (-1)^{p+1} \sum_{i=0}^{t-1} (-1)^i u|_{x_0 \leqslant \cdots \leqslant x_p \leqslant z_0 \leqslant \cdots \leqslant \hat{z}_i \leqslant \cdots \leqslant z_{t-1}, y_0 \leqslant \cdots \leqslant y_{q-t}} &+ \\ + (-1)^{p+q} \sum_{\ell=0}^{q-t} (-1)^\ell u|_{x_0 \leqslant \cdots \leqslant x_p \leqslant z_0 \leqslant \cdots \leqslant z_{t-1}, y_0 \leqslant \cdots \leqslant \hat{y}_\ell \leqslant \cdots \leqslant y_{q-t}}& \end{aligned}$$

and the vertical differentials in E_0' are given by

$$du|_{x_0 \leqslant \cdots \leqslant x_p, y_0 \leqslant \cdots \leqslant y_q} = (-1)^{p+q} \sum_{\ell=0}^{q} (-1)^\ell u|_{x_0 \leqslant \cdots \leqslant x_p, y_0 \leqslant \cdots \leqslant \hat{y}_\ell \leqslant \cdots \leqslant y_q}.$$

Using the notation from Definition 6.2 we can thus rewrite

$$E_0^{p,\bullet} = \prod_{x_0 \leqslant \cdots \leqslant x_p} (-1)^{p+1} T^{\bullet -1}_{\mathbf{B}^{\geqslant x_p}_{\ngeqslant x} \times (\mathbf{E}_{x_0}, F_{x_0})} \quad \text{and} \quad E_0'^{p,\bullet} = \prod_{x_0 \leqslant \cdots \leqslant x_p} (-1)^{p+q} \mathcal{S}^{\bullet - 1}(\mathbf{E}_{x_0}; F_{x_0}).$$

Now note that φ_1 acts as the product over all p-long x-sequences in $\mathbf{B}_{\ngeqslant x}$ of the maps in Proposition 6.5, since $\mathbf{B}$ is an admissible poset and thus the subposet $\mathbf{B}^{\geqslant x_p}_{\ngeqslant x}$ has a unique minimum. This means that $\varphi_1 \colon E_0'^{p,\bullet} \to E_0^{p,\bullet}$ is a quasi-isomorphism and thus

$$E_1'^{p,q} = H^p(E_0'^{p,\bullet}) \overset{\varphi_1^\bullet}{\cong} H^p(E_0^{p,\bullet}) = E_1^{p,q}.$$

The Mapping Lemma then implies that

$$\varphi_1^\bullet \colon H^{n-1} T^\bullet_{\mathbf{B}_{\ngeqslant x}} \cong H^n(M^\bullet). \qquad \square$$

We can now easily complete the proof of the theorem, headlined at the start of this section.

Proof of Theorem 7.1. We have the short exact sequence from before

$$0 \to M^\bullet \to T^\bullet_\xi \to T^n_{\mathbf{B}_{\geqslant x}} \oplus T^n_{\mathbf{B}_{\ngeqslant x}} \to 0,$$

from which we get a long exact sequence in homology

$$\begin{aligned} \cdots \to H^{n-1} T^\bullet_{\mathbf{B}_{\geqslant x}} \oplus H^{n-1} T^\bullet_{\mathbf{B}_{\ngeqslant x}} \to H^n M^\bullet \to H^n T^\bullet_\xi \to H^n T^\bullet_{\mathbf{B}_{\geqslant x}} \oplus H^n T^\bullet_{\mathbf{B}_{\ngeqslant x}} \to& \\ \to H^{n+1} M^\bullet \to \cdots& \end{aligned}$$

Replacing the occurrences of $H^n M^\bullet$ with $H^{n-1} T^\bullet_{\mathbf{B}_{\ngeqslant x}}$ and the maps around those occurrences with the appropriate compositions with $\varphi_1^\bullet$ and $\varphi_1^{\bullet -1}$ gives the required long exact sequence. $\square$

8. Long exact sequence in presheaf cohomology

We now repeat this procedure for the cochain complex of the total presheaf $(\mathbf{E}_\xi, F_\xi)$. The story is fairly similar to that of the previous section, so we are a little briefer. Again, we headline the main result, with the proof delayed until the end of the section.

Theorem 8.1. *Let $\xi\colon \mathbf{B} \to \mathbf{Sh}$ be a poset bundle of presheaves with $\mathbf{B}$ an admissible poset. Then there is a long exact sequence*

$$\begin{aligned}\cdots \to H^{n-1}(\mathbf{E}_{\mathbf{B}_{\not\geqslant x}};F_{\mathbf{B}_{\not\geqslant x}}) \to H^n(\mathbf{E}_\xi; F_\xi) \to& \\ \to H^n(\mathbf{E}_{\mathbf{B}_{\geqslant x}}; F_{\mathbf{B}_{\geqslant x}}) \oplus H^n(\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}; F_{\mathbf{B}_{\not\geqslant x}}) &\to \cdots\end{aligned}$$

Where possible, we will use x's to refer to objects in $\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}$ and z's to refer to objects of $\mathbf{E}_{\mathbf{B}_{\geqslant x}}$. We can write down explicitly:

$$\mathcal{S}^n(\mathbf{E}_\xi; F_\xi) = \prod_{x_0 \leqslant \cdots \leqslant x_n \in \mathbf{E}_\xi} F_\xi(x_0), \quad \mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\geqslant x}}; F_{\mathbf{B}_{\geqslant x}}) = \prod_{z_0 \leqslant \cdots \leqslant z_n \in \mathbf{E}_{\mathbf{B}_{\geqslant x}}} F_\xi(x_0),$$

$$\mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}; F_{\mathbf{B}_{\not\geqslant x}}) = \prod_{x_0 \leqslant \cdots \leqslant x_n \in \mathbf{E}_{\mathbf{B}_{\not\geqslant x}}} F_\xi(x_0).$$

Define another quotient map

$$\rho\colon \mathcal{S}^n(\mathbf{E}_\xi; F_\xi) \to \mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\geqslant x}}; F_{\mathbf{B}_{\geqslant x}}) \oplus \mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}; F_{\mathbf{B}_{\not\geqslant x}})$$

by setting to 0 any coordinate corresponding to a sequence $x_0 \leqslant \cdots \leqslant x_n$ in $\mathbf{E}_\xi$ that has objects in both $\mathbf{E}_{\mathbf{B}_{\geqslant x}}$ and $\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}$. This is a chain map by an analogous argument to the one for the quotient before Proposition 7.2.

The map ρ is clearly surjective, so we have a short exact sequence

$$0 \to N^\bullet \to \mathcal{S}^\bullet(\mathbf{E}_\xi; F_\xi) \to \mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\geqslant x}}; F_{\mathbf{B}_{\geqslant x}}) \oplus \mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}; F_{\mathbf{B}_{\not\geqslant x}}) \to 0$$

for a particular chain complex $N^\bullet$.

We describe $N^\bullet$ explicitly:

$$N^n = \prod_{x_0 \leqslant \cdots \leqslant x_p \leqslant z_0 \leqslant \cdots \leqslant z_{n-p-1}} F_\xi(x_0),$$

where $x_i \in \mathbf{E}_{\mathbf{B}_{\not\geqslant x}}, z_i \in \mathbf{E}_{\mathbf{B}_{\geqslant x}}, p \geqslant 0, n-p \geqslant 1$.

Proposition 8.2. *Let $\xi\colon \mathbf{B} \to \mathbf{Sh}$ be a poset bundle of presheaves with $\mathbf{B}$ an admissible poset for $x \succ 0$. If $N^\bullet$ is as above, there is a chain map*

$$\varphi_2\colon \mathcal{S}^{n-1}(\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}; F_{\mathbf{B}_{\not\geqslant x}}) \to N^n$$

that induces an isomorphism in cohomology.

Proof. We define a filtration $\mathcal{J}$ of $N^\bullet$:

$$\mathcal{J}^p N^n = \{u \in N^n : u|_\sigma \neq 0 \Rightarrow \sigma = x_0 \leqslant \cdots \leqslant x_s \leqslant z_0 \leqslant \cdots \leqslant z_{n-s-1}, \text{ with } s \geqslant p\}.$$

The proof that this is a filtration is analogous to the proofs of the filtrations from Proposition 7.2.

Let E be the spectral sequence associated to the filtration $\mathcal{J}$ of N. We have

$$E_0^{p+q} = \frac{\mathcal{J}^p N^{p+q}}{\mathcal{J}^{p+1} N^{p+q}} = \{u \in B^n : u|_\sigma \neq 0 \Rightarrow \sigma = x_0 \leqslant \cdots \leqslant x_p \leqslant z_0 \leqslant \cdots \leqslant z_{q-1}\}.$$

The vertical differentials in E_0 are given by

$$du|_{x_0 \leqslant \cdots \leqslant x_p \leqslant z_0 \leqslant \cdots \leqslant z_{q-1}} = (-1)^{p+1} \sum_{i=0}^{q-1} (-1)^i u|_{x_0 \leqslant \cdots \leqslant x_p \leqslant z_0 \leqslant \cdots \leqslant \hat{z}_i \leqslant \cdots \leqslant z_{q-1}}.$$

We can thus write

$$E_0^{p,\bullet} = \prod_{x_0 \leqslant \cdots \leqslant x_p} (-1)^{p+1} \mathcal{S}^{\bullet - 1}(\{z \in \mathbf{E}_{\mathbf{B}_{\geqslant x}} \mid z \geqslant x_p\}, \Delta F_\xi(x_0)).$$

But the $\mathcal{S}$ complex on the right is of a poset with a constant presheaf. By Lemma 6.4 the underlying poset has a unique minimum, so

$$E_1^{p,q} = H^q E_0^{p,\bullet} = \begin{cases} \prod_{x_0 \leqslant \cdots \leqslant x_p} (-1)^{p+1} F_\xi(x_0) & \text{if } q = 1, \\ 0 & \text{otherwise.} \end{cases}$$
$$= \begin{cases} (-1)^n \mathcal{S}^{n-1}(\mathbf{E}_{\mathbf{B}_{\ngeqslant x}}; F_{\mathbf{B}_{\ngeqslant x}}) & \text{if } q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

So on the E_1 page we have the single $q = 1$ row

$$\cdots \to (-1)^n \mathcal{S}^{n-1}(\mathbf{E}_{\mathbf{B}_{\ngeqslant x}}; F_{\mathbf{B}_{\ngeqslant x}}) \to (-1)^{n+1} \mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\ngeqslant x}}; F_{\mathbf{B}_{\ngeqslant x}}) \to \cdots .$$

The differential on this page is induced by the differential

$$du|_{x_0 \leqslant \cdots \leqslant x_p \leqslant z_0 \leqslant \cdots \leqslant z_{q-1}} = \sum_{i=0}^{p} (-1)^i u|_{x_0 \leqslant \cdots \leqslant \hat{x}_i \leqslant \cdots \leqslant x_p \leqslant z_0 \leqslant \cdots \leqslant z_{q-1}},$$

which, since it keeps the z-sequence constant, induces the following differential on the above row on the E_1 page:

$$du|_{x_0 \leqslant \cdots \leqslant x_p} = \sum_{i=0}^{p} (-1)^i u|_{x_0 \leqslant \cdots \leqslant \hat{x}_i \leqslant \cdots \leqslant x_p}.$$

Since $d(-d) = (-d)d = 0$, $\ker(-d) = \ker d$, and $\operatorname{im}(-d) = \operatorname{im} d$, we have that the E_2 page is

$$E_2^{p,q} \cong \begin{cases} H^{p+q-1} \mathcal{S}^\bullet(\mathbf{E}_{\mathbf{B}_{\ngeqslant x}}; F_{\mathbf{B}_{\ngeqslant x}}) & \text{if } q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $E_2^{p,q} \cong E_\infty^{p,q}$ and so

$$E \Rightarrow H^{n-1} \mathcal{S}^\bullet(\mathbf{E}_{\mathbf{B}_{\ngeqslant x}}; F_{\mathbf{B}_{\ngeqslant x}}) \cong N^n.$$

In particular, this isomorphism is witnessed by a similar quasi-isomorphism to that in Proposition 7.2, namely $\varphi_2 : \mathcal{S}^{n-1}(\mathbf{E}_{\mathbf{B}_{\ngeqslant x}}; F_{\mathbf{B}_{\ngeqslant x}}) \to N^n$ defined by

$$\varphi_2 u|_{x_0 \leqslant \cdots \leqslant x_n} = \begin{cases} u|_{x_0 \leqslant \cdots \leqslant x_{n-1}} & \text{if } x_{n-1} \in \mathbf{E}_{\mathbf{B}_{\ngeqslant x}}, x_n \in \mathbf{E}_{\mathbf{B}_{\geqslant x}}, \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

We can now, again, easily prove the headlined theorem.

Proof of Theorem 8.1. We have the short exact sequence from before

$$0 \to N^\bullet \to \mathcal{S}^\bullet(\mathbf{E}_\xi; F_\xi) \to \mathcal{S}^\bullet(\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}; F_{\mathbf{B}_{\not\geqslant x}}) \oplus \mathcal{S}^\bullet(\mathbf{E}_{\mathbf{B}_{\geqslant x}}; F_{\mathbf{B}_{\geqslant x}}) \to 0,$$

from which we get a long exact sequence in homology

$$\cdots \to H^{n-1}(\mathbf{E}_{\mathbf{B}_{\geqslant x}}; F_{\mathbf{B}_{\geqslant x}}) \oplus H^{n-1}(\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}; F_{\mathbf{B}_{\not\geqslant x}}) \to H^n N^\bullet \to H^n(\mathbf{E}_\xi; F_\xi) \to$$
$$\to H^n(\mathbf{E}_{\mathbf{B}_{\geqslant x}}; F_{\mathbf{B}_{\geqslant x}}) \oplus H^n(\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}; F_{\mathbf{B}_{\not\geqslant x}}) \to H^{n+1} N^\bullet \to \cdots$$

Replacing the occurrences of $H^n N^\bullet$ with $H^{n-1}(\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}; F_{\mathbf{B}_{\not\geqslant x}})$ and the maps around those occurrences with the appropriate compositions with $\varphi_2^\bullet$ and $\varphi_2^{\bullet\,-1}$ gives the required long exact sequence. □

9. The bicomplex and the total presheaf

We have all the necessary prerequisites to prove the main theorem:

Theorem 9.1. *Let $\xi\colon \mathbf{B} \to \mathbf{Sh}$ be a poset bundle of presheaves with $\mathbf{B}$ a recursively admissible finite poset, and $(\mathbf{E}_\xi; F_\xi)$ the associated total presheaf. Then there is a spectral sequence*

$$E_2^{p,q} = H^p(\mathbf{B}; \mathcal{H}^q_{fib}) \Rightarrow H^\bullet(\mathbf{E}_\xi; F_\xi).$$

Proof. Proposition 4.2 gives us

$$E_2^{p,q} = H^p(\mathbf{B}; \mathcal{H}^q_{fib}) \Rightarrow H^\bullet T_\xi^\bullet,$$

so it is enough to show that $H^\bullet T_\xi^\bullet \cong H^\bullet(\mathbf{E}_\xi, F_\xi)$. We will do this by induction on the size of $\mathbf{B}$. Recall the chain map $\omega\colon \mathcal{S}^\bullet(\mathbf{E}_\xi; F_\xi) \to T_\xi^\bullet$ from Section 5:

$$\omega u|_{\sigma,\tau} = (-1)^{\varsigma(q)} \sum_z (-1)^{m(z)} u|_z,$$

where the sum is taken over all traversals z of the grid of (σ, τ). We have two short exact sequences from Theorems 7.2 and 8.2. The map ω gives a morphism of these short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & N^n \arrow[r, "\varepsilon"] \arrow[d, "\omega'"] & \mathcal{S}^n(\mathbf{E}_\xi; F_\xi) \arrow[r, "\pi"] \arrow[d, "\omega"] & \mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}; F_{\mathbf{B}_{\not\geqslant x}}) \oplus \mathcal{S}^n(\mathbf{E}_{\mathbf{B}_{\geqslant x}}; F_{\mathbf{B}_{\geqslant x}}) \arrow[r] \arrow[d, "\omega \oplus \omega"] & 0 \\
0 \arrow[r] & M^n \arrow[r, "\varepsilon"] & T_\xi^n \arrow[r, "\pi"] & T^n_{\mathbf{B}_{\not\geqslant x}} \oplus T^n_{\mathbf{B}_{\geqslant x}} \arrow[r] & 0
\end{tikzcd}$$

where the maps ε are the injections and the maps π the projections of the respective modules. The map ω' is the restriction of ω to the subcomplexes N^n and M^n. We need to check the commutativity of the two squares.

(Left square) The maps ε are just injections, so we have

$$\varepsilon\omega u|_{\sigma,\tau} = \omega u|_{\sigma,\tau} = (-1)^{\varsigma(q)} \sum_z (-1)^{m(z)} u|_z = (-1)^{\varsigma(q)} \sum_z (-1)^{m(z)} \varepsilon u|_z = \omega\varepsilon u|_{\sigma,\tau}.$$

$$\begin{tikzcd}
H^{n+1}N^\bullet \arrow[r, "\omega'^\bullet"] & H^{n+1}M^\bullet \\
H^n(\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}; F_{\mathbf{B}_{\not\geqslant x}}) \oplus H^n(\mathbf{E}_{\mathbf{B}_{\geqslant x}}; F_{\mathbf{B}_{\geqslant x}}) \arrow[u, "\delta"] \arrow[r, "\omega^\bullet \oplus \omega^\bullet"] & H^n T^\bullet_{\mathbf{B}_{\not\geqslant x}} \oplus H^n T^\bullet_{\mathbf{B}_{\geqslant x}} \arrow[u, "\delta"'] \\
H^n(\mathbf{E}_\xi; F_\xi) \arrow[u, "\pi^\bullet"] \arrow[r, "\omega^\bullet"] & H^n T^\bullet_\xi \arrow[u, "\pi^\bullet"'] \\
H^n N^\bullet \arrow[u, "\varepsilon^\bullet"] \arrow[r, "\omega'^\bullet"] & H^n M^\bullet \arrow[u, "\varepsilon^\bullet"'] \\
H^{n-1}(\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}; F_{\mathbf{B}_{\not\geqslant x}}) \oplus H^{n-1}(\mathbf{E}_{\mathbf{B}_{\geqslant x}}; F_{\mathbf{B}_{\geqslant x}}) \arrow[u, "\delta"] \arrow[r, "\omega^\bullet \oplus \omega^\bullet"] & H^{n-1} T^\bullet_{\mathbf{B}_{\not\geqslant x}} \oplus H^{n-1} T^\bullet_{\mathbf{B}_{\geqslant x}} \arrow[u, "\delta"']
\end{tikzcd}$$

Figure 7: A portion of the commutative diagram given by the morphism of short exact sequences.

(Right square) Similarly, the maps π are projections, so

$$\pi\omega u|_{\sigma,\tau} = \omega u|_{\sigma,\tau} = (-1)^{\varsigma(q)} \sum_z (-1)^{m(z)} u|_z = (-1)^{\varsigma(q)} \sum_z (-1)^{m(z)} \pi u|_z = \omega\pi u|_{\sigma,\tau}.$$

The naturality of the homology functor then gives a morphism of long exact sequences, which contains the commutative diagram in Figure 7.

Recall from Propositions 7.2 and 8.2 the quasi-isomorphisms

$$\varphi_1 \colon T^{n-1}_{\mathbf{B}_{\not\geqslant x}} \to M^n \text{ and } \varphi_2 \colon \mathcal{S}^{n-1}(\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}; F_{\mathbf{B}_{\not\geqslant x}}) \to M^n.$$

Claim. *The following diagram commutes*

$$\begin{tikzcd}
\mathcal{S}^{n-1}(\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}; F_{\mathbf{B}_{\not\geqslant x}}) \arrow[r, "\varphi_2"] \arrow[d, "\omega"] & N^n \arrow[d, "\omega'"] \\
T^{n-1}_{\mathbf{B}_{\not\geqslant x}} \arrow[r, "\varphi_1"] & M^n
\end{tikzcd}$$

Proof of claim. Let $u \in \mathcal{S}^{n-1}(\mathbf{E}_{\mathbf{B}_{\not\geqslant x}}; F_{\mathbf{B}_{\not\geqslant x}})$. Suppose

$$\sigma = x_0 \leqslant \cdots \leqslant x_s \leqslant z_0 \leqslant \cdots \leqslant z_{t-1},\ \tau = y_0 \leqslant \cdots \leqslant y_q$$

with $s + t + q = n$. If $t > 1$, it is clear that $\varphi_1\omega u|_{\sigma,\tau} = 0 = \omega'\varphi_2 u|_{\sigma,\tau}$, since each summand of $\omega'\varphi_2 u|_{\sigma,\tau}$ is 0 under φ_2. If $t = 1$, let $\sigma' = x_0 \leqslant \cdots \leqslant x_s$. Then we have

$$\omega'\varphi_2 u|_{\sigma,\tau} = (-1)^{\varsigma(q)} \sum_{z'} (-1)^{m(z')} \varphi_2 u|_{z'},$$

where the sum is taken over the traversals z' of (σ, τ).

Pick a traversal z' of (σ, τ). We zoom in on the top right of the grid of (σ, τ):

$$\begin{tikzcd} \cdots & y_1' \arrow[r] & y_2' \\ & \iddots & y_0' \arrow[u] \\ & & \vdots \end{tikzcd}$$

Note that $y_0', y_2' \in \mathbf{E}_{z_0}$. If z' passes through y_0', then $\varphi_2 u|_{z'} = 0$. If z' passes through y_1', then $\varphi_2 u|_{z'} = u|_z$, for a particular traversal z of (σ', τ). Moreover, in this second case there are exactly q many squares in the rightmost column that are in the count for $m(z')$, so $m(z') = q + m(z)$. Therefore we have

$$\begin{aligned} \omega' \varphi_2 u|_{\sigma,\tau} &= (-1)^{\varsigma(q)} \sum_{z'} (-1)^{m(z')} \varphi_2 u|_{z'} = (-1)^{\varsigma(q)} \sum_{z} (-1)^{m(z)+q} u|_z \\ &= (-1)^q (-1)^{\varsigma(q)} \sum_z (-1)^{m(z)} u|_z = (-1)^q \omega u_{\sigma',\tau} = \varphi_1 \omega u|_{\sigma,\tau}. \qquad \square \end{aligned}$$

We can then replace the occurrences of $N^\bullet$ and $M^\bullet$ in Figure 7 with, respectively, $\mathcal{S}^{\bullet-1}(\mathbf{E}_{\mathbf{B}\ngeqslant x}; F_{\mathbf{B}\ngeqslant x})$ and $T^{\bullet-1}_{\mathbf{B}\ngeqslant x}$, adjusting the incoming and outgoing maps as the appropriate compositions with $\varphi_1^\bullet$ and $\varphi_2^\bullet$. In the resulting commutative diagram, the two columns are exact since, by Propositions 7.2 and 8.2, the maps $\varphi_1^\bullet$ and $\varphi_2^\bullet$ are isomorphisms. The squares commute by the commutativity of the diagram from the morphism of long exact sequences and the claim.

We finish the proof by induction on the size of $\mathbf{B}$. If $|\operatorname{Obj}\mathbf{B}| = 1$, then

$$T_\xi^n = \mathcal{S}^0(\mathbf{B}; \mathcal{S}^n) = \prod_{x \in \mathbf{B}} \mathcal{S}^n(\mathbf{E}_x; F_x) = \mathcal{S}^n(\mathbf{E}_\xi; F_\xi),$$

and $\omega = (-1)^{\varsigma(q)} \operatorname{id}$, so ω is a quasi-isomorphism.

If $\omega : \mathcal{S}^n(\mathbf{E}_\xi; F_\xi) \to T_\xi^n$ is a quasi-isomorphism for $|\operatorname{Obj}\mathbf{B}| < i$, then we can form the commutative diagram in Figure 7 for $|\operatorname{Obj}\mathbf{B}| = i$, with $N^\bullet$ and $M^\bullet$ replaced as discussed above. Each row other than the middle one contains an instance of the inductive hypothesis, since both $\mathbf{B}_{\ngeqslant x}$ and $\mathbf{B}_{\geqslant x}$ have fewer objects than $\mathbf{B}$; and $\mathbf{B}$ is recursively admissible. Therefore, by the Five Lemma, the middle row is an isomorphism and thus ω is a quasi-isomorphism. This completes the induction and the proof of the theorem. $\square$

10. A reduction property for presheaf cohomology

The statement of Theorem 9.1 closely resembles that of [**2**, Theorem 5.1]. Despite this, the reframing of the result in terms of presheaf cohomology, as opposed to coloured poset homology, leads to applications that are quite different from those of the coloured poset version. The key difference, explored in this section, is that while the theorem in [**2**] models complex interactions between the homologies of the fibers of a bundle of coloured posets (seen in the application to Khovanov homology), the main theorem of this paper implies that if $\xi \colon \mathbf{B} \to \mathbf{Sh}$ is a poset bundle of presheaves with $\mathbf{B}$ recursively admissible, then it is only the cohomology of the presheaf at the maximum of $\mathbf{B}$ that contributes to the cohomology of the total presheaf of ξ.

By the end of this chapter, we will be able to conclude that, for example, the cohomology of a presheaf on the poset in Figure 8 can only be non-zero in degrees 0 and 1.

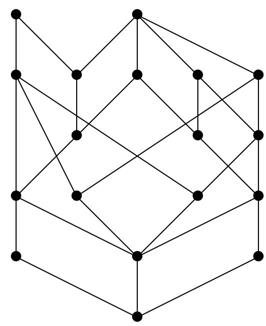


Figure 8: The cohomology of any presheaf on this poset is zero in all degrees $\neq 0, 1$. Convention is that arrows go up.

It turns out that the restriction to recursively admissible posets means that we only deal with posets with 1.

Proposition 10.1. *Let $\mathbf{B}$ be a recursively admissible poset. Then $\mathbf{B}$ has a unique maximum.*

This follows from the recursive definition (Definition 6.2): the poset $\mathbf{B}$ is either Boolean of rank 1, so it has a unique maximum, or all its maximums are contained in $\mathbf{B}_{\geqslant x}$ for some $x \succ 0$, since $\mathbf{B}_{\geqslant x}^{\geqslant y} \neq \emptyset$ for all $y \in \mathbf{B}_{\ngeqslant x}$.

The admissibility property provides a kind of 'factorisation' for posets into bundles. The simplest way to do this is to turn an admissible poset into a bundle over a Boolean lattice of rank 1 $\mathbb{B}_1$. Note that Boolean lattices are recursively admissible, so we can later apply Theorem 9.1.

Lemma 10.2. *Let $\mathbf{E}$ be an admissible poset for $\mathbf{E}', \mathbf{E}''$ and $(\mathbf{E}, F) \in \mathbf{Sh}$. Then there is a poset bundle of presheaves $\xi \colon \mathbb{B}_1 \to \mathbf{Sh}$ such that $(\mathbf{E}_\xi, F_\xi) = (\mathbf{E}, F)$ (recall the construction of the total presheaf $(\mathbf{E}_\xi, F_\xi)$, Definition 2.3).*

Proof. We need to specify $\xi(0)$, $\xi(1)$, and $\xi(0 \leqslant 1)$.

- $\xi(0) = (\mathbf{E}', F)$,
- $\xi(1) = (\mathbf{E}'', F)$,
- the presheaf morphism $\gamma = \xi(0 \leqslant 1)$ consists of a covariant functor (a poset map in this setting) $\gamma_1 \colon \mathbf{E}' \to \mathbf{E}''$ and a natural transformation $\gamma_2 \colon F\gamma_1 \to F$:
  - Let $\gamma_1(x)$ be the unique minimum of $\{y \in \mathbf{E}'' \mid x \leqslant y\}$. Then if $x \leqslant x'$ in $\mathbf{E}'$, we have $\{y \in \mathbf{E}'' \mid x \leqslant y\} \supseteq \{y \in \mathbf{E}'' \mid x' \leqslant y\}$ and so $\gamma_1(x) \leqslant \gamma_1(x')$.
  - Since $x \leqslant \gamma_1(x)$, we have a morphism $F(x) \leftarrow F(\gamma_1(x))$ from $(\mathbf{E}, F)$. Set $\gamma_{2,x}$ to be this morphism.

Remains to show that $(\mathbf{E}, F) = (\mathbf{E}_\xi, F_\xi)$. It is enough to show that $\mathbf{E} = \mathbf{E}_\xi$ by the construction of F_ξ. If $x \leqslant y$ in $\mathbf{E}$ and either $x, y \in \mathbf{E}'$ or $x, y \in \mathbf{E}''$, then clearly $x \leqslant y$ in $\mathbf{E}_\xi$ (as an arrow of type a)). Suppose $x \leqslant y$ in $\mathbf{E}$ and $x \in \mathbf{E}'$, $y \in \mathbf{E}''$. Then $x \leqslant \gamma_1(x) \leqslant y$, so $x \leqslant y$ in $\mathbf{E}_\xi$. Conversely, the set of arrows in $\mathbf{E}_\xi$ is generated by inequalities that hold in $\mathbf{E}$. Therefore, $x \leqslant y$ in $\mathbf{E}$ if and only if $x \leqslant y$ in $\mathbf{E}_\xi$. □

We can also 'factorise' a poset into a bundle over a more complicated base.

Proposition 10.3. *Let $\mathbf{E}$ and $\mathbf{B}$ be posets, let $(\mathbf{E}, F) \in \mathbf{Sh}$, and let $\pi\colon \mathbf{E} \to \mathbf{B}$ be an onto poset map, such that for all $x < y$ in $\mathbf{B}$, the subposet $\pi^{-1}(x) \cup \pi^{-1}(y)$ of $\mathbf{E}$ is admissible for $\pi^{-1}(x), \pi^{-1}(y)$. Then there is a poset bundle of presheaves $\xi\colon \mathbf{B} \to \mathbf{Sh}$ such that $(\mathbf{E}, F) = (\mathbf{E}_\xi, F_\xi)$.*

Proof. Following the approach from the previous proposition, set $\xi(x) = (\pi^{-1}(x), F)$ and if $x < y$ in $\mathbf{B}$, then $\xi_1(x < y)$ sends $z \in \pi^{-1}(x)$ to the minimum of the subposet $\{w \in \pi^{-1}(y) \mid z \leqslant w\}$.

Now suppose $z < w$ in $\mathbf{E}$ and $z \in \pi^{-1}(x)$, $w \in \pi^{-1}(y)$. Since π is a poset map, $x < y$ in $\mathbf{B}$ and $z < \xi_1(x < y)(z) \leqslant w$ in $\mathbf{E}_\xi$.

If $z < w$ in $\mathbf{E}_\xi$ is an arrow of type b) or a composition arrow, then by Proposition 2.4 there is a $v \in \pi^{-1}(\pi(w))$, such that $z < v < w$ in $\mathbf{E}_\xi$, where $z < v$ and $v < w$ are arrows of type b) and a), respectively. But both those arrows exist in $\mathbf{E}$, so $z < w$ in $\mathbf{E}$. □

The following is a consequence of recursively admissible posets' having a unique maximum (or final object).

Proposition 10.4. *Let $\mathbf{B}$ be a recursively admissible poset and let $\xi\colon \mathbf{B} \to \mathbf{Sh}$ be a poset bundle of presheaves. If $1 \in \mathbf{B}$ is the unique maximal object, then*

$$H^\bullet(\mathbf{E}_\xi, F_\xi) \cong H^\bullet(\xi(1)).$$

Proof. Let E be the ξ's spectral sequence. We know that $E_2^{p,q} = H^p(\mathbf{B}; \mathcal{H}^q_{fib})$. Now, $\mathbf{B}$ has a unique maximum 1 (Proposition 10.1), so the functors $\varprojlim_{\mathbf{B}}$ and the 'evaluation at 1' functor $\_(1)\colon \mathbf{Sh}(\mathbf{B}) \to {}_R Mod$ are naturally isomorphic. But we know that evaluation functors are exact. Therefore

$$H^p(\mathbf{B}; \mathcal{H}^q_{fib}) = \begin{cases} H^q(\xi(1)), & \text{if } p = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The spectral sequence collapses and we get $H^n(T_\xi^\bullet) \cong H^n(\xi(1))$. Since $\mathbf{B}$ is recursively admissible, Theorem 9.1 applies, so $H^\bullet(\xi(1)) \cong H^\bullet(T_\xi^\bullet) \cong H^\bullet(\mathbf{E}_\xi, F_\xi)$. □

We can now package the discussion into our main application.

Theorem 10.5. *Let $\mathbf{E}$ be a poset and $\mathbf{B}$ be a recursively admissible poset. Suppose that $\pi\colon \mathbf{E} \to \mathbf{B}$ is an onto poset map such that for all $x < y$ in $\mathbf{B}$, the subposet $\pi^{-1}(x) \cup \pi^{-1}(y)$ of $\mathbf{E}$ is admissible for $\pi^{-1}(x), \pi^{-1}(y)$. Then*

$$H^\bullet(\mathbf{E}; F) \cong H^\bullet(\pi^{-1}(1); F)$$

for all $F \in \mathbf{Sh}(\mathbf{E})$, where 1 is the unique maximum of $\mathbf{B}$.

Remark 10.6. The above recipe can be applied repeatedly. Indeed, one can imagine cases where a poset $\mathbf{E}$ is admissible for $\mathbf{E}_1, \mathbf{E}_2$, and $\mathbf{E}_2$ is admissible for $\mathbf{E}_3, \mathbf{E}_4$, but $\mathbf{E}_1$ is not admissible, so the poset map $\pi : \mathbf{E} \to \mathbb{B}_2$ required for the above theorem does not exist. Despite this, we can apply the theorem twice with $\mathbf{B} = \mathbb{B}_1$ and deduce that $H^\bullet(\mathbf{E}; F) \cong H^\bullet(\mathbf{E}_4; F)$, for any $F \in \mathbf{Sh}(\mathbf{E})$.

Conversely, if the required poset map $\pi \colon \mathbf{E} \to \mathbf{B}$ exists for some recursively admissible $\mathbf{B}$, we can instead repeatedly apply Theorem 10.5 for $\mathbb{B}_1$, at each step applying the recursive definition. The upshot is that replacing the recursively admissible $\mathbf{B}$ with the concrete $\mathbb{B}_1$ in the above theorem results in an equivalent statement.

Example 10.7. We can now examine the explicit poset given at the start of the chapter (with arrowheads omitted, but always pointing up). Let $\mathbf{E}$ be the poset in Figure 8 and choose an $F \in \mathbf{Sh}(\mathbf{E})$. First, $\mathbf{E}$ is admissible for $\mathbf{E}_1, \mathbf{E}_2$ by inspection of the left-hand side diagram in Figure 9. The right-hand side shows a reduction with $\mathbf{B} = \mathbb{B}_2$.

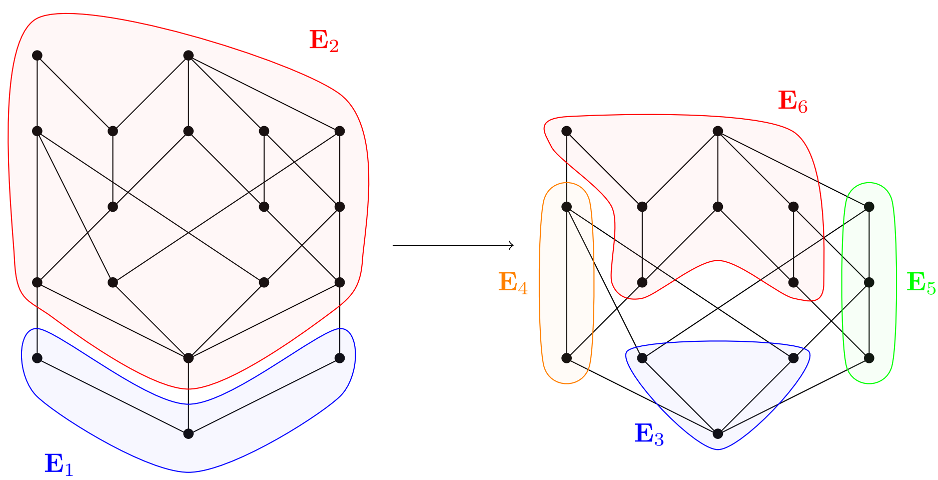


Figure 9: The first two reductions of the poset $\mathbf{E}$.

Another two applications of Theorem 10.5 with $\mathbf{B} = \mathbb{B}_1$ reduce the poset even further (Figure 10).

We thus have that $H^\bullet(\mathbf{E}; F) \cong H^\bullet(\mathbf{E}_7; F)$. To see that the cohomology of $(\mathbf{E}_7, F)$ is zero for all degrees $\geqslant 2$, we can use the chain complex $\mathcal{T}^\bullet(\mathbf{E}_7; F) := \mathcal{S}^\bullet(\mathbf{E}_7; F)/D^\bullet$, where $D^\bullet$ is the subcomplex consisting of the degenerate simplices in $\mathbf{E}_7$, i.e. the simplices that involve an identity arrow. This new chain complex $\mathcal{T}^\bullet$ is homotopy equivalent to $\mathcal{S}^\bullet$ (see [**1**, p. 138]) and since it only involves non-degenerate simplices, its cohomology is clearly trivial at degrees $\geqslant 2$.

There is also a more general example that we can apply our theorem to.

Proposition 10.8. *Let $\mathbf{E}$ be a poset and let $x \in \mathbf{E}$ be a total point, i.e. for all $y \in \mathbf{E}$, either $x \leqslant y$ or $y \leqslant x$. Then $H^\bullet(\mathbf{E}; F) \cong H^\bullet(\mathbf{E}_{\geqslant x}; F)$ for any $F \in \mathbf{Sh}(\mathbf{E})$.*

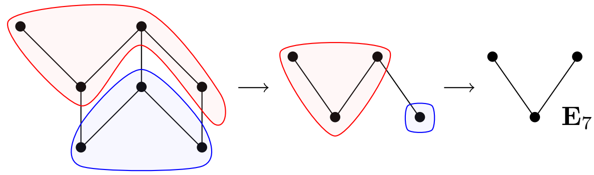


Figure 10: Further reduction of the poset $\mathbf{E}$.

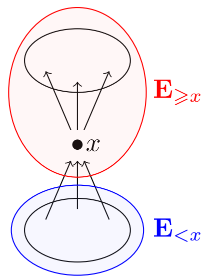


Figure 11: A decomposition of a poset with a total point.

Proof. If $\mathbf{E}_{<x} = \emptyset$, then $\mathbf{E} = \mathbf{E}_{\geqslant x}$ and the statement of the proposition is trivial. Otherwise, consider the subposets $\mathbf{E}_{\geqslant x}$ and $\mathbf{E}_{<x}$ (see Figure 11). For any $y \in \mathbf{E}_{<x}$, we have $\min \mathbf{E}_{\geqslant x}^{\geqslant y} = x$ and so $\mathbf{E}$ is admissible for $\mathbf{E}_{<x}, \mathbf{E}_{\geqslant x}$. Applying Theorem 10.5 gives the required result. $\square$

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Mihail Hurmuzov mihail@hurmuzov.com