Geometry Motivated by Physics

by Shing-Tung Yau*

It is a great honor to be invited for this important occasion. The first time I had met Prof. Yang was in the fall term of 1971 when he gave a series of public lectures at Stony Brook. I was deeply impressed by the intuitive power of physics as described by the great master. In the following years, I saw the impact of Yang–Mills theory on mathematics. It was exciting for a geometer to study the effect of physical intuition in our field and what we could do for physics in return. I spent the next thirty years exploring such an interaction. It was a fruitful research: from general relativity, to gauge theory and to string theory. The influence of Prof. Yang is very deep and I am grateful to him. In this note, I will explain such a personal experience. The choice of topic is therefore rather subjective.

The concept of space (or space-time) has evolved according to our ability in mathematics and our understanding of nature. In the ancient times, figures were constructed from line segments, planes and spheres (or quadrics like the parabola). Beautiful theorems in plane and solid geometry were proved by the Greeks. The most fundamental theorem is the Pythagorean theorem. Even modern geometry demands that this theorem holds infinitesimally. The fact that there are only five platonic solids had been a fascinating statement for mathematicians, philosophers and physicists (they are tetrahedron, octahedron, cube, icosahedron's, dodecahedron).

Although Archimedes had applied the concept of infinite process to geometry, it was not until the full development of Calculus (Newton, Leibniz) that we had the tools to study curved space. The foundation of calculus of variation due to Enter et al. also gave rise to many important geometric objects that are fundamental in modern geometry (e.g. minimal surfaces).

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Euler also introduced the important concept of Euler number which is the foundation of all topological invariants. While geodesics and various geometric quantities were introduced on surfaces in three space, the very fundamental concept of intrinsic curvature was first introduced by Gauss. The product of two principle curvatures depends only on the first fundamental form and is independent of isometric deformation of surfaces. The product is called Gauss curvature, and its introduction is the birth of modern geometry, and it also inspired the famous work of Riemann. (Apparently Gauss was interested in geometry because he was asked to survey land.)

C.F. Gauss said that: "I am becoming more and more convinced that the necessity of our geometry cannot be proved, at least not by human reason nor for human reason. Perhaps in another life we will be able to obtain insight into the nature of space which is now unattainable. Until then we must place geometry not in the same class with arithmetic which is purely *a priori*, but with mechanics."

From this quotation of Gauss, we can see that he was deeply excited by what should be called space.

With the continuous development of mathematics and inputs from physics, we are facing a similar situation. Our concept of space may not be adequate.

Later, Riemann formally introduced the concept of abstract Riemann geometry which is free from being a subspace of Euclidean space.

Tensor Calculus was then developed on abstract space. Noncommutativity of covariant differentiation

$$D_{\frac{\partial}{\partial x}} D_{\frac{\partial}{\partial y}} \neq D_{\frac{\partial}{\partial y}} D_{\frac{\partial}{\partial x}}$$

gives rise to the concept of curvature of a connection.

In his approach to understand the multivalued analytical function, the concept of Riemann surface was introduced by Riemann. In order to make holomorphic function single valued, the concept of uniformization was introduced.

This development of Riemann surface has had a deep influence on modern geometry. It is a fundamental tool to understand surface in three space and was developed into the theory of complex manifold and projective geometry. They are the fundamental algebraic curves in higher dimensional space.

When Einstein formulated the unification theory of gravity with special relativity, the background mathematics was Riemannian geometry. His equation is

$$R_{ij} - \frac{R}{2} g_{ij} = T_{ij}.$$

Ricci tensor is associated with matter while the full curvature tensor is related to full gravity behavior. This is a spectacular development. Geometry cannot be separated from physics anymore.

Any attempts to explicitly construct solutions of Einstein Equation, with physical boundary condition, end up with singularity. The spherical symmetric solution is found by Schwarzchild. It is the most important model for black hole. The basic question in general relativity is: Given a nonsingular set of generic data at time zero, what will the future space-time look like?

Penrose–Hawking proved that: If there is a trapped surface, singularity will develop. The theory of Penrose–Hawking is based on the technique of geometry. Only after this theory of "trapped surface", did we know singularity cannot be avoided. Penrose proposed the following fundamental question. For a generic nonsingular initial data set, the only possible future singularity is the type of black hole.

A closely related question is: can quantum gravity "cure" singularity of space-time. String theory does provide some hint that space-time may need a new concept. However, we still need to answer the following: what is black hole singularity?

Let us now turn to another development in geometry. The development of Hamiltonian mechanics for celestial mechanics motivated the study of the behavior of geodesics for general Riemann manifold. Many important questions are being asked. For example, are there infinite number of closed geodesics on any compact manifold? For manifold with negative curvature, closed geodesics are closely related to the fundamental group of the manifold. The set of the lengths of closed geodesics is known to be related to the spectrum of the Laplacian. It has been very fruitful to study the spectrum of Laplacian from the point of view of semi-classical limit of quantum mechanics. The generalization to quantum field theory is remarkable.

An immediate generalization of geodesics is minimal surfaces. At the beginning, they are related to Plateau problems: surfaces that minimize areas with given boundaries.

The existence theory of minimal disk in a general Riemannian manifold was solved by Morrey whose method of regularity is fundamental for the theory of calculus of variations of two variables. When the minimal surface has no boundary, we can ask many questions similar to those for geodesics, however surfaces now have topology and they have to be taken into account in their calculations.

The question of how to count the number of minimal surfaces, their areas and their index has been useful in studying topology of three-dimensional manifolds. The study of stability of minimal surfaces gives rise to the study of black holes as developed by Schoen–Yau.

The calculus of variation developed by Morse, Morrey and others has been a powerful tool in geometry. Morse theory was used by R. Bott and S. Smale to solve an outstanding question in differential topology including periodicity theorem, and also to handle body decomposition theorems. Global theory of minimal surfaces has a lot of important applications. Some of these can be mentioned in the following. Sacks–Uhlenbeck demonstrated the existence of minimal sphere when the universal cover of the manifold is not contractible. Siu–Yau made use of this to prove the Frenkel conjecture that every compact Kähler manifold with positive curvature is HP". Schoen–Yau, Sacks–Uhlenbeck proved the existence of minimal surfaces with higher genus. Meeks–Yau, Meeks–Simon–Yau proved embeddedness of these surfaces. Thus questions on three-dimensional manifolds (e.g. Smith Conjecture) were solved when coupled with Thurston's theory as observed by C. Gordan. Schoen–Yau solved the positive mass conjecture based on minimal surface theory.

In another direction, the development of quantum mechanics strongly motivated the study of the spectrum of natural operators in geometry. The natural operators are Laplacian and Dirac operators. (For complex manifolds, they are fi operators as well.) One of the great achievements was the proof of index theorem by Atiyah–Singer which allows us to count solutions of global differential equations when a certain obstruction vanishes.

A spectrum of operators give rise to topological invariants of manifolds: Zeta function was constructed through the heat kernel. The work of Ray–Singer on the determinant of the natural operators has become fundamental for later works on quantum field theory on manifolds.

Coupled with index theory is the Hodge theory and the Bochner vanishing theorem. When the curvature has a favorable sign, the vanishing theorem has been very powerful to establish existence theorems. However vanishing theorem does depend on Hodge theory which was motivated by two important subjects: fluid dynamics on two-dimensional surfaces and the Maxwell equations.

Hodge theory with coefficients in arbitrary bundle is one of the most powerful tools in modern mathematics. In the hands of Hodge, Kodaira, Hirzebruch, Atiyah, Deligne, Calabi, Griffiths, Schmid, Sir and others, the theory was used to study the structure of algebraic manifolds, especially for computing dimensions of holomorphic solutions of linear systems on complex manifolds. The very fundamental question of Hodge on the algebraic cycle is based on Hodge theory.

Maxwell equations can be considered as Abelian gauge theory in contrast to the Yang–Mills theory: the non-Abelian gauge theory. The works of E. Cartan on structure equations and Whitney on immersion theory had provided strong motivation to study the theory of Fiber bundle and Vector bundles. Important invariants like Stiefel–Whitney characteristic classes were needed to understand the topology of such bundles. In the forties, Pontryajin and Chem introduced their classes through connections over the bundles.

The curvature form representation of Pontryajin and Chem classes have been fundamental in topology and geometry. The idea of minimizing L'-norm of curvature of connections on a given bundle did not appear in mathematics literature until 1954, when Yang–Mills published their work. At the same time, Calabi proposed to look at Kähler metric with minimal *L'-norm* on its curvature. But the analysis was difficult in those days.

It was not until 1970s that the technique of nonlinear partial differential equations was more mature to deal with problems that arose from general relativity, gauge theory and string theory.

A deep understanding of linear theory is key to nonlinear theory. Good estimates for harmonic functions and eigenfunctions were developed by Cheng and myself. Applications to questions on affine geometry, Minkowski problem and maximal hypersurfaces were developed. Semi-linear elliptic equations were developed on the theory of conformal deformation of matrices by Yamabe, Trudinger, Aubin and Schoen.

The theory of quasi-linear equations was developed based on the understanding

of regularity of minimal surfaces and harmonic maps. Works of Morrey, Nirenberg, De Giorgi, Federer–Fleiming, Almgren, Allard, Uhlenbeck, Simon, Schoen, Hamilton, and myself contributed to the theory in 1970s.

The development of Yang–Mills theory in mathematics started from the interaction of Yang with J. Simon who realized the Yang–Mills theory was about the theory of curvature of connections on fiber bundles.

In the mid-severities, most efforts on the mathematics of Yang–Mills theory were concentrated on the self-dual equations over the four sphere. Note that the energy of Yang–Mills field can be shown to be not less than a suitable topological invariant of the bundle. The field is called self-dual or anti-self-dual if the inequality becomes equality. They played a fundamental role in physics. They are the so–called BPS states which are very stable states. The works of Atiyah–Singer–Hitchin initiated the application of index theorem to calculate the dimension of the moduli of self-dual Yang–Mills field. Then it was realized that Penrose twistor theory can be used to find all the solutions of the self-dual solutions of Yang–Mills field over the four sphere (Atiyah–Drinfeld–Hitchin–Manin).

In the late twenties, Karen Uhlenbeck had developed the general elliptic theory of Yang–Mills theory. It is fundamental in understanding the compactification of the moduli space of Yang–Mills field.

C. Taubes found a remarkable way to construct self-dual solutions using singular perturbation. Based on these works, in 1983, Donaldson was able to study the global moduli space of self-dual solutions on a reasonable general four manifold. He then used the topology of the moduli space to create invariants of the four–dimensional manifolds. This remarkable development gave the first fundamental breakthrough in four-dimensional topology, giving a counter-example to Smale's theory of h-cobordism theorem in this dimension.

The Donaldson theory was later realized by Witten to be related to supersymmetric Yang–Mills theory, the theory developed by Seiberg and Witten (around 1995) was found to give topological invariants of four manifolds. It is believed to be closely related to the Donaldson invariants. The Seiberg–Witten invariants were then applied to solve several important questions on geometry and topology. For example, algebraic surface of the general type cannot be diffeomorphic to rational surfaces. The Thom conjecture that the only embedded surface in a homology class represented by an algebraic curve must have genus not less than the genus of the algebraic curve, was proved by Kronheimer and Mrowka.

A very remarkable development was due to Taubes in relating the Seiberg–Witten invariants to the existence of Pseudo holomorphic curves in symplectic manifolds.

The supersymmetric Yang–Mills theory has many other important consequences in the modern development of string theory and in algebraic geometry including the understanding of counting algebraic curves in Calabi–Yau manifolds.

We shall also describe later the theory of Hamilton Yang–Mills connections on complex manifolds. The fact that anti-self-dual connection gives rise to holomorphic bundle was in fact observed by Prof. Yang. The theory developed by Donaldson–Uhlenbeck–Yau has important applications to algebraic geometry and string theory. We shall come back to this later.

In 1976, I solved the Calabi Conjecture on proving existence (and uniqueness) of Kähler–Einstein metrics. It includes the important case of Kähler Metric with zero Ricci curvature on manifolds with zero first Chem class.

To understand the significance of latter metrics, recall a most important class of solutions of Einstein equation that is Vacuua.

Traditionally, there are the following ways to find solutions:

1. Assume large groups of symmetries, either in Riemannian or Lorentzian case. The most important example is Schwarzchild metric, which has maximal symmetries.
2. Assume supersymmetries, i.e. assume there are parallel spinors.

The holomony group is then reduced to subgroups of $O(\mathrm{n})$. Besides locally symmetric space, the most important spaces that have special holomony groups are $SU(n)$, G_2 or $\mathrm{Spin}(7)$.

When the manifolds is Kähler (holomony group $= U(n)$), the problem reduces to solve the complex Monge–Ampere equation

$$\det\left(g_{ij} + \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j}\right) = e^F \det(g_{ij}).$$

For a compact Kähler manifold, with first Chem class zero, there is a unique Ricci flat metric in each Kähler class. When there is cosmology constant ci, the equation is

$$\det\left(g_{ij} + \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j}\right) = e^{-\alpha u} e^F \det(g_{ij})$$

when $\alpha < 0$, it is much easier, when $\alpha > 0$, there are obstructions and is not completely understood. In the latter case, I conjectured twenty years ago its existence should come from the stability of the projective structure of the projective manifold. Simon Donaldson has recently made important progress on my conjecture.

My motivation to link stability of algebraic structures to solutions of elliptic equations comes from two sources:

1. In 1976, I proved the following Chem number in equations for algebraic surfaces with general type

$$3c_2(M) \geq c_1^2(M).$$

   The proof was based on the existence of Kähler–Einstein metric with negative constant. Miyaoka was able to prove the same inequality using the idea of Bogomolov, this idea was related to the theory of stability of bundle.
2. Based on the above mentioned work, I was motivated to study Yang–Mills connections over stable holomorphic bundles. This was proved by Donaldson for algebraic surfaces and Uhlenbeck–Yau for general Kähler manifolds.
3. In the theory of Ricci flow to change metric, the asymptotic behavior is clearly related to stability.

This theory of DUY is now an important piece of (2,0) theory in string models. Both the metric and the bundle theory contribute significantly to algebraic geometry and string theory. For string theory compactification, based on Kluza–Klein theory, both the Ricci feat metric and the Hermitian Yang–Mills contribute to the vacuua. For algebraic geometry, they are the building blocks for algebraic structures. It is remarkable that while we did not provide solutions of the metric and the connections in closed form, computation based on sophisticated algebraic geometry is possible.

Let me now describe ways to construct Calabi–Yau manifolds (Kähler manifolds with zero first Chem class).

(I gave a list of methods of construction in the Argone lab conference in 1983.)

1. Kummer Constructions, or more generally, orbifold construction.
2. Complete intersections of hypersurfaces in Fano manifolds, in particular, in the product of weighted projective spaces. The most important example was constructed by me in 1983.

$$\sum x_i^3 = 0, \quad \sum y_i^3 = 0, \quad \sum x_i y_i = 0$$

   in $CP^3 \times CP^3$.

   It admits an action of group of order 3, the Euler number is -6. This manifold is significant for building physical models as it gives three families of Fermions and it has nontrivial Wilson lines. Later Tian observed that my construction could be extended to more examples. But B. Greene observed they are deformations of the above manifold. Jun Li and I are able to deform the tangent bundle plus trivial bundle to stabilize *SU(4)* bundle. This allows interpretation for Heterotic String Theory.

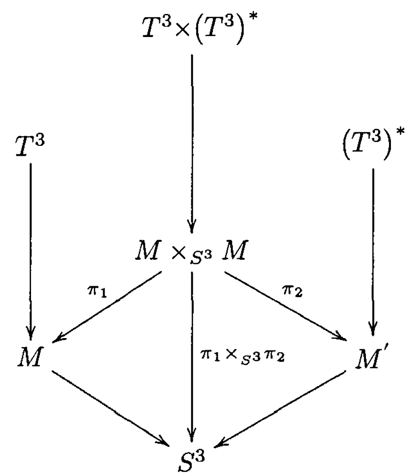
3. Hypersurface in Toric variety. The first such construction was due to Roan–Yau. Now it has become a standard construction. Batyrev observed mirror constructions of such manifolds in terms of duality of polyhedrons.
4. Branch cover construction. For example, take a quantic in CP^3. We can take branch cover of CP^3 along it to obtain quantic in CP^4. In general, any hypersurface can be associated with Calabi–Yau manifolds in high dimension.

Similar questions can be asked for noncompact complete Ricci flat manifolds. Vafa and his coauthors have considered this class of manifolds as local models and have been very successful in the computation of their instanton construction. How to understand the moduli space of such metrics? When the manifold is topologically finite, I conjectured in 1978 that M can be compactified to a Kähler manifold whose exceptional set in defined by an anticanonical divisor. The converse is basically true. When the anticanonical divisor is nonsingular, it is relatively easy and has been written up later. It was applied to construct semi-flat Ricci flat manifolds in my paper with Greene, Shapley and Vafa on the construction of Cosmic String.

When the anticanonical divisor is simple (normal crossing) the problem of parameterizing these metrics is very interesting. A very important case of our construction is that of algebraic manifold M with anticanonical divisor D to be nonsingular Calabi–Yau hypersurface (so that the normal bundle is trivial). The metric I constructed on $M \backslash D$ is cylindrical along D and can be glued along a tabular neighborhood of D to form a new CY manifold. Conversely for a large class of CY manifolds, we can pull it apart along a CY hypersurface. Gukov and I have been looking for the physics of such manifolds.

A very important understanding of Calabi–Yau manifold comes from the concept of mirror construction. It can be considered as a symmetry on the category of Calabi–Yau manifolds. This was suggested by Lerche–Vafa, and Dixon. Later, GreenwPlesser demonstrated its existence in the case of quantic. It realizes the duality between II_A and II_B theory from string theory. The duality allows one to compute the number of instantons arisen in II_A theory. These are the holomorphic curves in Calabi–Yau manifolds. The computation of the number of holomorphic curves in algebraic manifolds dates back to the nineteenth century. It was called enumerative geometry. Input from mirror symmetry settles this "kind" of old problem. It was initialized by Candalas et al. and finally solved by Liu–Lian–Yau (and Givental independently in case the manifold is more special). The work of Liu–Lian–Yau–Givental was a rigorous mathematical piece of work that settles the old problem. However, it did not give a true understanding of mirror symmetry. In 1995, Strominger–Yau–Zaslow proposed that Calabi–Yau manifold should have a foliation (singular) of special Lagrangian torus. The mirror manifold should be obtained by taking duality along the leaves of the foliation.

The SYZ construction is based on the newly developed M-theory. Therefore the geometric construction has support from the intuition of physics. The complicated question of singularities involved in such construction is expected to be solvable. The SYZ construction can be seen from the following diagram.

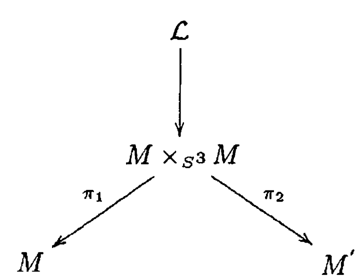


The above diagram allows us to transfer objects from M to M' and vice-versa. This generalized Fourier–Mukai transform was extensively studied in algebraic geometry.

From the SYZ construction, we know that special Lagrangian Cycles from M should move to stable holomorphic bundles over M and coupling should be preserved (Leung–Yau–Zaslow).

The construction should explain most of the questions in mirror symmetry. A very important one is how to map odd dimensional cohomology of a Calabi–Yau manifold to the even dimensional cohomology of its mirror.

I proposed the following diagram.



where $\mathcal{L}$ is the Poincare bundle defined fiberwisely on $T^3 \times (T^3)^*$ which produces a line bundle over the fiber product $M \times_{S^3} M'$. The map from odd cohomology of M to even cohomology of M' should be the composite of the following maps.

$$H^3(M) \to H^3(M \times_{S^3} M')$$
$$\omega \to \pi_1^* \omega$$
$$H^3(M \times_{S^3} M') \to H^*(M \times_{S^3} M') \to H^{even}(M')$$
$$\omega \to \omega \wedge e^{c_1(\mathcal{L})} \to (\pi_2)_*(\omega \wedge e^{c_1(\mathcal{L})}).$$

This map should exhibit the mirror symmetry in the cohomology level.

Many interesting questions arose in the SYZ construction. Most of the geometric quantities including complex structure and Ricci flat metrics will require quantum correction from disk instantons. (There is a background semi-flat Ricci flat metric given by Cosmic String construction.) How to compute such instantons are nontrivial. Works by C.C. Liu, Katz, Kefeng Liu, C.H. Liu, J. Zhou, myself and others are making progress over these important questions.

Around the same time as the development of SYZ, Kontzerich proposed homological mirror conjecture which suggested that derived category of M should be the same as the Fukaya category of M'. A lot of activities were initiated, especially by Bridgeland, Orlov, Kawamata, Thomas and others.

The concept of mirror symmetry was initiated by string theory. It does solve many outstanding questions in algebraic geometry that is otherwise difficult to understand. The ideas inspired by string theory in mathematics has been able to unify naturally many diverse concepts in mathematics. It helps to solve classical problems. It is inconceivable that nature will fool us by leading us so deeply into the core of mathematics.

While CY manifolds give a conformally invariant sigma model, a very interesting question is how to move from a general sigma model to a conformally invariant theory. This is provided by renormalization group flow.

In fact, twenty years ago, this was studied by Richard Hamilton. The equation

$$\frac{dg_{ij}}{dt} = -2R_{ij}$$

where g_{ij} is the metric and R_{ij} is its Ricci tensor.

The Hamilton equation has led to an important understanding of geometric

structures on manifolds. I proposed to Hamilton to provide a proof of the geometrization program of Thurston based on this flow. He was able to carry out this program in much more detail. Some new impacts were given by Perelman recently. In a closely related development, the dynamics of a surface moving by its mean curvature have shown some exciting development. Huisken and others have made important contributions. This will contribute to applied mathematics, geometry and physics.

The whole impact of geometry is moving dynamically, touching important parts of physics and engineering.

References

[1] B. Andreas, G. Curio, D.H. Ruiperez and S.-T. Yau, *Fourier-Mukai Transform and Mirror Symmetry for D-Branes on Elliptic Calabi-Yau*, arXiv:math/0012196.
[2] B. Andreas, G. Curio, D.H. Ruiperez and S.-T. Yau, *Fibrewise T-Duality for D-Branes on Elliptic Calabi-Yau*, arXiv:hep-th/0101129.
[3] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Yu. I. Manin, *Phys. Lett.* A65, 185 (1978).
[4] P. Candelas, X. de la Ossa, P. Green and L. Parkes, *Nucl. Phys.* B359, 21 (1991).
[5] S.K. Donaldson, *J. Differ. Geom.* 18, 269 (1983).
[6] S.K. Donaldson, *Proc. Lond. Math. Soc.* 60, 1 (1985).
[7] A. Givental, Equivariant Gromov–Witten invariants, *Int. Math. Res. Not.* 13, 613–663 (1996).
[8] B. Greene, A. Shapere, C. Vafa and S.-T. Yau, *Nucl. Phys.* B337, 1 (1990).
[9] R. Hamilton, *J. Differ. Geom.* 17, 255 (1982).
[10] S. Katz and C. Liu, *Adv. Theor. Math. Phys.* 5, 1 (2001).
[11] P.B. Kronheimer and T.S. Mrowka, *Math. Res. Lett.* 1, 797 (1994).
[12] B. Lian, K. Liu and S.-T. Yau, *Asian J. Math.* 1, 729 (1997).
[13] C. Liu, K. Liu and J. Zhou, *On a Proof of a Conjecture of Marino–Vafa on Hodge Integrals*, arXiv:math/0306257.
[14] S. Li and S.-T. Yau, *Mathematical Aspects of String Theory*, (World Scientific Publishing, Singapore, 1987), p. 560.
[15] N. Leung, S.-T. Yau and E. Zaslow, *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds* (Cambridge, MA, 1999), p. 209; *AMS/IP Stud. Adv. Math.* 23 (Amer. Math. Soc., Providence, RI, 2001).
[16] S.-S. Roan and S.-T. Yau, *Acta Math. Sin. (N.S.)* 3, 256 (1987).
[17] R. Schoen and S.-T. Yau, *Phys. Rev. Lett.* 42, 547 (1979).
[18] R.Schoen and S.-T. Yau, *Commun. Math. Phys.* 65, 45 (1979).
[19] A. Strominger, S.-T. Yau and E. Zaslow, *Nucl. Phys.* B479, 243 (1996).
[20] C. Taubes, *J. Differ. Geom.* 17, 139 (1982).
[21] C. Taubes, *Seiberg Witten and Gromov Invariants for Symplectic 4-Manifolds*, First Intl. Press Lect. Ser. Vol. 2 (Internationl Press).
[22] G. Tian and S.-T. Yau, *Mathematical Aspects of String Theory* (World Scientific, Singapore, 1987), p. 543.
[23] G. Tian and S.-T. Yau, *J. Am. Math. Soc.* 3, 579 (1990).
[24] K. Uhlenbeck and S.-T. Yau, *Commun. Pure Appl. Math.* 39, 257 (1986).
[25] E. Witten, *Math. Res. Lett.* 1, 769 (1994).
[26] S.-T. Yau, *Commun. Pure Appl. Math.* 31, 39 (1978).