# Volume Minimization and Obstructions to Solving Some Problems in Kähler Geometry

**by Akito Futaki[*] and Hajime Ono[†]**

**Abstract.** There is an obstruction to the existence of Kähler-Einstein metrics which is used to define the GIT weight for K-stability, and it has been extended to various geometric problems. This survey paper considers such extended obstructions to the existence problem of Kähler-Ricci solitons, Sasaki-Einstein metrics and (conformally) Einstein-Maxwell Kähler metrics. These three cases have a common feature that the obstructions are parametrized by a space of vector fields. We see, in these three cases, the obstructions are obtained as the derivative of suitable volume functionals. This tells us for which vector fields we should try to solve the existence problems.

## 1. Introduction

The existence of a Kähler-Einstein metric on a compact complex manifold $M$ has been known since 1970's in the case when $c_1(M) < 0$ by Aubin [3] and Yau [51] where the Kähler class is the canonical class $K_M$, and in the case when $c_1(M) = 0$ by Yau [51] where the Kähler class is arbitrary positive $(1,1)$-class. In the remaining case when $c_1(M) > 0$, i.e. in the case when $M$ is a Fano manifold, the existence of a Kähler-Einstein metric is characterized by a condition called the K-stability by the recent works of Chen-Donaldson-Sun [11] and Tian [47]. The K-stability is a condition in geometric invariant theory where the GIT weight, called the Donaldson-Futaki invariant [19], is defined extending an obstruction, now called the classical Futaki invariant, obtained in [25], [26]. The latter is defined for smooth compact Kähler manifolds and is an obstruction to admit a constant scalar curvature Kähler metrics (cscK metrics for short). Note that for a Kähler form in the anti-canonical class on a Fano manifold, being a cscK metrics is equivalent to being a Kähler-Einstein metric. On the other hand, the Donaldson-Futaki invariant is defined for possibly singular central fibers of $\mathbf{C}^*$-equivariant degenerations, called the test configurations, and a polarized Kähler manifold $(M,L)$ is said to be K-stable if the Donaldson-Futaki invariant of the central fiber is non-negative for any test configurations and if the equality holds exactly when the test configuration is product. Note that for the product configurations the Donaldson-Futaki invariant coincides with the classical Futaki invariant. The Fano case is the core of the conjecture known as the Yau-Tian-Donaldson conjecture stating that a polarized Kähler manifold $(M,L)$ should admit a cscK metric with its Kähler form in $c_1(L)$ if and only if $(M,L)$ is K-stable. In the Kähler-Einstein problem for the Fano case we take $L = K_M^{-1}$. The Yau-Tian-Donaldson conjecture for cscK problem with general polarizations is still remaining unsolved. There are many variants of the Yau-Tian-Donaldson conjecture. For example, K-stability characterizations for Kähler-Ricci solitons and Sasaki-Einstein metrics have been obtained respectively in [16] and [15]. It is usually difficult to check whether a manifold is

[*] Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan
E-mail: afutaki@ms.u-tokyo.ac.jp
[†] Department of Mathematics, Saitama University, 255 Shimo-Okubo, Sakura-Ku, Saitama 380-8570, Japan
E-mail: hono@rimath.saitama-u.ac.jp

K-stable since there are infinitely many test configurations. However, in the cases with large symmetry groups checking K-stability can be easier, see [31], [17], [18]. For alternate proofs for the Yau-Tian-Donaldson conjecture for the Fano case, other important contributions, recent further developments and applications, the reader is referred to the two survey papers of Donaldson [21], [22].

The present survey paper focuses on extensions of the classical Futaki invariants for Kähler-Ricci solitons, Sasaki-Einstein metrics and Einstein-Maxwell Kähler metrics. Existence problems for these three types of metrics have a common feature that they depend on the choice of a holomorphic Killing vector field, and accordingly their obstructions have parameter space consisting of holomorphic Killing vector fields in an appropriate Lie algebra. The Ricci solitons are self-similar solutions of the Ricci flow and important object in the study of singularity formations of the Ricci flow. On a compact Kähler manifold, a Kähler-Ricci soliton is a Kähler metric $g$ satisfying

$$\mathrm{Ric}_g = g + L_{\mathrm{grad} f}\, g \tag{1}$$

which is equivalent to

$$\rho_g = \omega_g + i\partial\overline{\partial} f$$

where $f$ is a Hamiltonian function for a holomorphic Killing vector field $X$, i.e. $X = J\mathrm{grad} f$, and $\rho_g$ and $\omega_g$ are respectively the Ricci form and the Kähler form of $g$. Since $\rho_g/2\pi$ represents the first Chern class, if a Kähler-Ricci soliton exists, the compact manifold $M$ is necessarily a Fano manifold. Note also a Killing vector field on a compact Kähler manifold is necessarily holomorphic. Given a Killing vector field $X$ we consider the toral group $T$ obtained by taking the closure of the flow generated by $X$, and ask if there is a $T$-invariant Kähler-Ricci soliton $g$ satisfying (1) with $X = J\mathrm{grad} f$. This problem is reduced to solving a Monge-Ampère type equation. However, Tian and Zhu [48] showed that there is an obstruction $\mathrm{Fut}_X$ to solving (1). Thus if one choose an $X$ with non-vanishing $\mathrm{Fut}_X$ then one can never get a solution to the Monge-Ampère equation. Tian and Zhu [48] showed that there is a twisted volume functional $\mathrm{Vol}$ on the space of $X$ such that the derivative at $X$ of $\mathrm{Vol}$ is equal to $\mathrm{Fut}_X$:

$$d\mathrm{Vol}_X = \mathrm{Fut}_X. \tag{2}$$

They further showed that the volume functional is proper and convex on the space of $X$. Since holomorphic Killing vector fields on a compact Kähler manifold constitute a finite dimensional vector space the volume functional has a unique minimum on the space of $X$. This gives the right choice to solve the equation (1).

Sasaki-Einstein metrics caught considerable attention in mathematical physics through its role in the AdS/CFT correspondence, and the volume minimization is the key to find Sasaki-Einstein metrics. The Sasakian structure on an odd dimensional manifold $S$ is by definition a Riemannian structure on $S$ such that its Riemannian cone $C(S)$ has a Kähler structure. Fixing a complex structure on the cone $C(S)$, the deformation of the Sasakian structure on $S$ is given by the deformation of the cone structure on $C(S)$, namely the deformation of the radial function $r$. The Reeb vector field is then given by $Jr\partial/\partial r$. To each Reeb vector field one can assign a Sasakian structure on $S$. Thus one can define the volume functional $\mathrm{Vol}$ on the space of Reeb vector fields. The volume depends only on the Reeb vector field and is independent of the choice of the Sasakian structure with the given Reeb vector field. This fact is similar to the fact in Kähler geometry that the volume depends only on the Kähler class and is independent of the choice of the Kähler form in the given Kähler class. The space of Reeb vector fields is the inside of the dual cone to the moment map image of the Kähler cone $C(S)$, and the volume functional $\mathrm{Vol}$ is a homogeneous function on this space. Thus we may consider a slice which gives a bounded domain sitting inside the dual cone. On the other hand the Sasaki-Einstein condition is equivalent to the Kähler cone $C(S)$ being Ricci-flat, and is also equivalent to the local transverse geometry of the Reeb flow being Kähler-Einstein with positive scalar curvature. One can then associate to each Reeb vector field $\xi$ an obstruction $\mathrm{Fut}_\xi$ similarly to the Fano Kähler-Einstein problem [30], [7]. Martelli-Sparks-Yau [43] show for transversely Fano Sasakian manifolds

$$d\mathrm{Vol}_\xi = \mathrm{Fut}_\xi. \tag{3}$$

In the case when $S$ is toric Sasakian, meaning when the cone $C(S)$ is toric Kähler, Martelli-Sparks-Yau further show that $\mathrm{Vol}$ is a proper convex function on the slice in the dual cone consisting of the Reeb vector fields for which the volume functional $\mathrm{Vol}$ is defined. Thus there is a unique minimum $\xi$, and it is shown in [30], for any transversely Fano toric Sasakian manifold, there is a Sasaki-Einstein metric with the choice of the unique minimum $\xi$ as the Sasakian structure.

Conformally Kähler Einstein-Maxwell metrics are relatively newer subject. The Einstein-Maxwell equation has been studied in general relativity in real dimension 4. In [34], LeBrun showed that, on a compact Kähler surface $(M,g)$, if there is a positive smooth function $f$ with $J\mathrm{grad} f$ being a Killing vector field such that the Hermitian metric $\tilde{g} = f^{-2}g$ has constant scalar curvature then $\tilde{g}$ corresponds to a solution of the Einstein-Maxwell equation. Thus, fixing a holomorphic Killing vector field $K$ and a Kähler class $\Omega$, to find a Kähler form $\omega_g \in \Omega$ such that $\tilde{g} = f_K^{-2} g$ has constant scalar curvature is a problem in Kähler geometry, where $f_K$ is the Hamiltonian function of $K$ with respect to $\omega$. In fact, if $K = 0$ then the problem is exactly the same as the Yau-Tian-Donaldson conjecture. Apostolov and Maschler [2] further set the problem into the Donaldson-Fujiki picture, and formulated an extension $\mathrm{Fut}_K$ of the classical Futaki invariant parametrized by $K$. In [2], such $\tilde{g}$ is called a conformally Kähler, Einstein-Maxwell metric. But we consider the problem of finding $(g, f_K)$ with $\omega_g$ in a fixed Kähler class, and therefore it is more convenient to call such $g$ a (conformally) Einstein-Maxwell Kähler metric, or even preferably omitting the word "conformally". We then showed in [28] that the derivative at $K$ of a suitably defined volume functional Vol on the space of $K$ satisfies

$$d\mathrm{Vol}_K = \mathrm{Fut}_K. \tag{4}$$

However the volume functional is neither convex nor proper in general, and can have several critical points.

In all these three cases, the critical points correspond to the cases when the classical Futaki invariant vanishes. However, this may not be enough to have a solution, but the K-stability may be the next issue.

In section 2, 3 and 4 we give more details on Kähler-Ricci solitons, Einstein-Maxwell Kähler metrics and Sasaki-Einstein metrics respectively.

## 2. Kähler-Ricci Solitons

In this section, we see how a holomorphic Killing vector field which admits a Kähler-Ricci soliton is determined through the idea of volume minimization [48].

Let $M$ be an $m$-dimensional Fano manifold. A Kähler metric $g$ on $M$ with the Kähler form $\omega_g \in 2\pi c_1(M)$ is called a *Kähler-Ricci soliton* if there exists a holomorphic vector field $X$ on $M$ such that

$$\rho_g - \omega_g = L_X \omega_g \tag{5}$$

holds, where $\rho_g$ denotes the Ricci form of $g$ and $L_X$ is the Lie derivative along $X$. In particular, if $X = 0$, $g$ is a Kähler-Einstein metric. Since $\rho_g$ and $\omega_g$ represent $2\pi c_1(M)$, there exists a real-valued smooth function $h_g$ such that

$$\rho_g - \omega_g = i\partial\bar{\partial} h_g. \tag{6}$$

On the other hand, for any holomorphic vector field $X$, the $(0,1)$-form $\iota_X \omega_g$ is $\bar{\partial}$-closed. Therefore, by the Hodge theorem, there exists a unique complex-valued smooth function $\theta_X(g)$ such that

$$\iota_X \omega_g = i\bar{\partial}\theta_X(g), \quad \int_M e^{\theta_X(g)} \omega_g^m = \int_M \omega_g^m. \tag{7}$$

Hence we have

$$L_X \omega_g = i\partial\bar{\partial}\theta_X(g). \tag{8}$$

By (5), (6) and (8), a Kähler metric $g$ is a Kähler-Ricci soliton with respect to a holomorphic vector field $X$ if and only if $h_g - \theta_X(g)$ is constant.

It is difficult to determine $h_g - \theta_X(g)$ explicitly. However, Tian and Zhu [48] proved that the integral of $v(h_g - \theta_X(g))e^{\theta_X(g)}$ is independent of the choice of $g$, where $v$ is a holomorphic vector fields, and it defines a holomorphic invariant.

**Theorem 2.1** ([48]). *Let $\mathfrak{h}(M)$ be the Lie algebra which consists of all holomorphic vector fields on $M$. For a Kähler form $\omega_g \in 2\pi c_1(M)$ and $X \in \mathfrak{h}(M)$, we define a linear function $\mathrm{Fut}_X$ on $\mathfrak{h}(M)$ as*

$$\mathrm{Fut}_X(v) = \int_M v(h_g - \theta_X(g))e^{\theta_X(g)} \omega_g^m, \quad v \in \mathfrak{h}(M). \tag{9}$$

*Then $\mathrm{Fut}_X$ is independent of the choice of $\omega_g \in 2\pi c_1(M)$.*

*If $M$ admits a Kähler-Ricci soliton with respect to $X \in \mathfrak{h}(M)$, then $\mathrm{Fut}_X$ vanishes identically on $\mathfrak{h}(M)$.*

Note here that when $X = 0$, this holomorphic invariant coincides with the Futaki invariant, which is an obstruction to the existence of Kähler-Einstein metrics in $c_1(M)$ [25].

We next see that the invariant $\mathrm{Fut}_X$ can be obtained as the first variation of some function on $\mathfrak{h}(M)$ [48]. Such characterization of the holomorphic invariant plays a key role in §3 and §4.

Let $X \in \mathfrak{h}(M)$. We renormalize the function $\theta_X(g)$ defined by (7) to $\tilde{\theta}_X(g)$ by adding a constant such that

$$\int_M \tilde{\theta}_X(g) e^{h_g} \omega_g^m = 0. \tag{10}$$

**Proposition 2.2** ([48]). *Let a function $f$ on $\mathfrak{h}(M)$ be given by*

$$f(Z) = \int_M e^{\tilde{\theta}_Z(g)} \omega_g^m. \tag{11}$$

*Then $f(Z)$ is independent of the choice of Kähler metrics with the Kähler class $2\pi c_1(M)$. Moreover the differential of $f$ at $X$ in the direction of $v \in \mathfrak{h}(M)$ is a constant multiple of $\mathrm{Fut}_X(v)$.*

By this proposition, if there exists a Kähler-Ricci soliton with respect to a holomorphic vector field $X$, it is a critical point of $f$.

Let $\mathrm{Aut}^0(M)$ be the identity component of the holomorphic automorphism group of $M$ and $K$ a maximal compact subgroup. Then the Chevalley decomposition allows us to write $\mathrm{Aut}^0(M)$ as a semi-direct product

$$\mathrm{Aut}^0(M) = \mathrm{Aut}_r(M) \ltimes R_u, \tag{12}$$

where $\mathrm{Aut}_r(M)$ is a reductive algebraic subgroup of $\mathrm{Aut}^0(M)$ which is the complexification of $K$, and $R_u$ is the unipotent radical of $\mathrm{Aut}^0(M)$. Let $\mathfrak{h}_r(M)$ and $\mathfrak{h}_u(M)$ be the Lie algebras of $\mathrm{Aut}_r(M)$ and $R_u$ respectively. From the decomposition (12), we obtain

$$\mathfrak{h}(M) = \mathfrak{h}_r(M) + \mathfrak{h}_u(M). \tag{13}$$

**Proposition 2.3** ([48]). *Let* $\mathrm{Vol}$ *be the restriction of* $f$ *to* $\mathfrak{h}_r(M)$. *Then* $\mathrm{Vol}$ *is a convex, proper real-valued function. Hence there exists a unique minimum point* $X_0 \in \mathfrak{h}_r(M)$ *of* $\mathrm{Vol}$.

By Proposition 2.2, $\mathrm{Fut}_{X_0}$ vanishes identically on $\mathfrak{h}_r(M)$. This minimum $X_0$ is the right choice to solve the Kähler-Ricci soliton equation. Note here that, to combine Proposition 2.3 with the result of Saito [46], $\mathrm{Fut}_{X_0}$ vanishes identically on $\mathfrak{h}(M)$.

For toric Fano manifold, we can calculate $X_0$ as follows [50]. Let $M$ be an $m$-dimensional toric Fano manifold with the Kähler class $c_1(M)$ and $\Delta_M \subset \mathbf{R}^m$ the corresponding moment polytope. It is well-known that $\Delta_M$ is an $m$-dimensional reflexive Delzant polytope.

Let $T$ be the maximal torus of $\mathrm{Aut}(M)$ and $\mathfrak{h}_0(M)$ its Lie algebra. $T$ is isomorphic to the $m$-dimensional algebraic torus $(\mathbf{C}^\times)^m$ and $\mathfrak{h}_0(M)$ is the maximal Abelian Lie subalgebra of $\mathfrak{h}(M)$. If we take the affine logarithm coordinates $(w_1,\dots,w_m) = (x_1 + i\theta_1,\dots,x_m + i\theta_m)$ on $T \cong \mathbf{R}^m \times (S^1)^m$, $\mathfrak{h}_0(M)$ is spanned by the basis $\{\frac{\partial}{\partial w_1},\dots \frac{\partial}{\partial w_m}\}$. Since $X_0 \in \mathfrak{h}_0(M)$, $X_0$ can be expressed in the form

$$X_0 = \sum_{i=1}^{m} c_i \frac{\partial}{\partial w_i}. \tag{14}$$

**Proposition 2.4** ([50]). *The constants* $c_1,\dots,c_m$ *in* (14) *are given by the following conditions:*

$$\int_{\Delta_M} y_i \exp\left\{\sum_{l=1}^{m} c_l y_l\right\} dy = 0, \quad i = 1,\dots,m. \tag{15}$$

## 3. Einstein-Maxwell Kähler Geometry

In this section, we first introduce the notion of conformally Kähler, Einstein-Maxwell (cKEM for short) metrics defined by Apostolov-Maschler [2] and give non Kähler examples of cKEM metrics in any dimension. We then define an obstruction to the existence of cKEM metrics called cKEM-Futaki invariant and consider it from the view point of volume minimization. At the end of this section we give some results of computations on toric surfaces.

Let $(M,J)$ be a compact Kähler manifold. We call a Hermitian metric $\tilde{g}$ on $(M,J)$ a *conformally Kähler, Einstein-Maxwell metric* if it satisfies the following three conditions:

(a) There exists a positive smooth function $f$ on $M$ such that $g = f^2\tilde{g}$ is Kähler.

(b) The Hamiltonian vector field $K = J\mathrm{grad}_g f$ is Killing for both $g$ and $\tilde{g}$.

(c) $s_{\tilde{g}}$ is constant.

As we mentioned in the Introduction, we call the Kähler metric $g$ in (a) an *Einstein-Maxwell Kähler metric*.

By the definition above, cscK metrics are cKEM metrics. However we consider them as trivial cKEM metrics.

The notion of cKEM metrics were introduced by Apostolov-Maschler in [2] as a generalization of strongly Hermitian solutions of the Einstein-Maxwell equation. We review some results by LeBrun on strongly Hermitian solutions, see [34], [35].

Let $M$ be a compact manifold. A pair $(g,F)$ of a Riemannian metric $g$ and a real 2-form $F$ is called a solution of the *Einstein-Maxwell equation* if it satisfies

$$dF = 0,\ d *_g F = 0,\ [\mathrm{Ric}_g + F\circ F]_0 = 0,$$

where $(F\circ F)_{jk} = F_j{}^{\ell} F_{\ell k}$ and $[\ ]_0$ denotes the trace free part. This equation is the Euler-Lagrange equation of the following functional which is studied in general relativity:

$$(g,F) \mapsto \int_M \left(s_g + |F|_g^2\right) dv_g.$$

LeBrun investigated Einstein-Maxwell equation when $M$ is a complex surface in detail, especially he introduced the notion of strongly Hermitian solutions: Let $(g,F)$ be a solution of Einstein-Maxwell equation on a complex surface $(M,J)$. It is called a *strongly Hermitian solution* if it satisfies

$$\mathrm{Ric}_g(J\cdot,J\cdot) = \mathrm{Ric}_g(\cdot,\cdot),\quad F(J\cdot,J\cdot) = F(\cdot,\cdot).$$

LeBrun [34] pointed out that the metric component of a strongly Hermitian solution is a cKEM metric. Conversely, he also showed that for a cKEM metric $\tilde{g}$, one obtains a strongly Hermitian solution

$$(\tilde{g}, \omega_g + \frac{1}{2} f^{-2}[\rho_{\tilde{g}}]_0).$$

We next give some examples of cKEM metrics other than cscK metrics. Typical known examples are conformally Kähler, Einstein metrics by Page [45] on the one point blow up of $\mathbf{CP}^2$, by Chen-LeBrun-Weber [12] on the two point blow up of $\mathbf{CP}^2$, by Apostolov-Calderbank-Gauduchon [1] on 4-orbifolds and by Bérard-Bergery [4] on $\mathbf{CP}^1$-bundle over Fano Kähler-Einstein manifolds. Non-Einstein cKEM examples are constructed by LeBrun [34], [35] showing that there are ambitoric examples on $\mathbf{CP}^1\times\mathbf{CP}^1$ and the one point blow up of $\mathbf{CP}^2$, and by Koca-Tønnesen-Friedman [32] on ruled surfaces of higher genus. The authors extended LeBrun's construction on $\mathbf{CP}^1\times\mathbf{CP}^1$ to $\mathbf{CP}^1\times M$ where $M$ is a compact cscK manifold of arbitrary dimensions as follows [28].

Let $g_1$ be an $S^1$-invariant metric on $\mathbf{CP}^1$ with $\mathrm{Vol}(\mathbf{CP}^1, g_1) = 2\pi$ and $g_2$ a Kähler metric with $s_{g_2} = c$ on an $(m-1)$-dimensional compact complex manifold $M$. The $S^1$-invariant metric $g_1$ can be written in the action-angle coordinates $(t,\theta) \in (a, a+1) \times (0, 2\pi]$ as

$$g_1 = \frac{dt^2}{\Psi(t)} + \Psi(t)d\theta^2$$

for some smooth function $\Psi(t)$ which satisfies the following boundary condition:

$$(16) \qquad \begin{aligned} &\Psi(a) = \Psi(a+1) = 0, \\ &\Psi'(a) = -\Psi'(a+1) = 2, \ \Psi > 0 \text{ on } (a, a+1). \end{aligned}$$

Then we see that the constant scalar curvature equation $s_{\tilde{g}} = d$ for the metric $\tilde{g} = (g_1 + g_2)/t^2$ on $\mathbf{CP}^1 \times M$ reduces to the following ODE:

$$(17) \qquad t^2\Psi'' - 2(2m-1)t\Psi' + 2m(2m-1)\Psi = ct^2 - d.$$

**Theorem 3.1** ([28]). *Let $c > 8m-8$. Then there exist $a > 0$ and $d > 0$ such that there exists a unique solution $\Psi$ of the ODE (17) which satisfies the condition (16). As a result, for any Kähler metric $g_2$ with $s_{g_2} = c$ on an $(m-1)$-dimensional compact complex manifold $M$,*

$$\tilde{g} = \frac{1}{t^2}\left(\frac{dt^2}{\Psi(t)} + \Psi(t)d\theta^2 + g_2\right)$$

*is an $S^1$-invariant cKEM metric on $\mathbf{CP}^1 \times M$.*

We next consider the existence problem of cKEM metrics. Let $(M, J)$ be a compact complex manifold of $\dim_{\mathbf{C}} M = m$. We fix a compact subgroup $G \subset \mathrm{Aut}(M,J)$, a Kähler class $\Omega$, $K \in \mathfrak{g}$ and a sufficiently large $a \in \mathbf{R}$. Denote by $\mathcal{K}_\Omega^G$ the space of $G$-invariant Kähler metrics $g$ with $\omega_g \in \Omega$. For $g \in \mathcal{K}_\Omega^G$, there exists a unique function $f_{K,a,g} \in C^\infty(M)$ satisfying the following two conditions:

$$(18) \qquad \iota_K \omega_g = -df_{K,a,g}, \ \int_M f_{K,a,g}\frac{\omega_g^m}{m} = a.$$

Note here that, for fixed $(K,a)$, $\min\{f_{K,a,g}(x) \,|\, x \in M\}$ is independent of $g \in \mathcal{K}_\Omega^G$, see [2]. So if we choose $a$ sufficiently large, $f_{K,a,g}$ is positive for any $g \in \mathcal{K}_\Omega^G$. Then we can ask the following existence problem; does there exists a Kähler metric $g$ in $\mathcal{K}_\Omega^G$ such that $\tilde{g}_{K,a} = f_{K,a,g}^{-2} g$ is a cKEM metric?

When $K = 0$, this is just the existence problem of cscK metrics in $\mathcal{K}_\Omega^G$. As a generalization of the Futaki invariant [25], [26], Apostolov-Maschler [2] defined the following integral invariant for non-zero $K$.

**Theorem 3.2** ([2]). *The linear function*

$$(19) \qquad \mathrm{Fut}_{\Omega,K,a}^G : \mathfrak{g} \to \mathbf{R}, \ \ \mathrm{Fut}_{\Omega,K,a}^G(H) := \int_M \frac{s_{\tilde{g}_{K,a}} - c_{\Omega,K,a}}{f_{K,a,g}^{2m+1}} f_{H,b,g}\frac{\omega_g^m}{m!},$$

*is independent of the choice of Kähler metric $g \in \mathcal{K}_\Omega^G$ and $b \in \mathbf{R}$. Here*

$$(20) \qquad c_{\Omega,K,a} := \frac{\displaystyle\int_M s_{\tilde{g}_{K,a}} f_{K,a,g}^{-2m-1}\frac{\omega_g^m}{m!}}{\displaystyle\int_M f_{K,a,g}^{-2m-1}\frac{\omega_g^m}{m!}}.$$

*is a constant which is independent of the choice of $g \in \mathcal{K}_\Omega^G$. If there exists a Kähler metric $g \in \mathcal{K}_\Omega^G$ such that $\tilde{g}_{K,a}$ is a cKEM metric, then $\mathrm{Fut}_{\Omega,K,a}^G$ vanishes identically.*

We call this linear function $\mathrm{Fut}_{\Omega,K,a}^G$ as the *cKEM-Futaki invariant* for $(K,a)$. We notice here that cKEM-Futaki invariant is parametrized by the pair $(K,a)$. This situation bears resemblance to the holomorphic invariant (9) which is an obstruction to the existence of Kähler-Ricci solitons. In fact, we now see that the cKEM-Futaki invariant can be characterized as the first variation of the volume function. To that end, we recall that constant scalar curvature Riemannian metrics can be characterized as follows. Let $M$ be a compact manifold with $n = \dim M \geq 3$ and $Riem(M)$ the set consists of all Riemannian metrics on $M$. The scalar curvature $s_{g_0}$ of a Riemannian metric $g_0 \in Riem(M)$ is constant if and only if $g_0$ is a critical point of the following normalized Einstein-Hilbert functional on the conformal class of $g_0$:

$$(21) \qquad EH(g) := \frac{\displaystyle\int_M s_g dv_g}{(\mathrm{Vol}(M,g))^{\frac{n-2}{n}}}$$

In our case, this functional gives the "integral" of the cKEM-Futaki invariant!

**Proposition 3.3** ([2]). *For a fixed $(K,a)$, $EH(\tilde{g}_{K,a})$ is independent of the choice of $g \in \mathcal{K}_\Omega^G$.*

As a consequence, if there exists $g \in \mathcal{K}_\Omega^G$ such that $\tilde{g}_{K,a}$ is a cKEM metric, then the pair $(K,a)$ is a critical point of the function

$$(22) \qquad (K,a) \mapsto EH(K,a) := EH(\tilde{g}_{K,a}).$$

The set of pairs

$$\mathcal{P}_\Omega^G := \{(K,a) \in \mathfrak{g} \times \mathbf{R} \,|\, f_{K,a,g} > 0, \ g \in \mathcal{K}_\Omega^G\}$$

is a cone in the finite dimensional real vector space $\mathfrak{g} \times \mathbf{R}$. Since the normalized Einstein-Hilbert functional is scale invariant, the function $EH$ on $\mathcal{P}_\Omega^G$ reduces to the function on the quotient space $\mathcal{P}_\Omega^G/\mathbf{R}_+$. If we choose representatives normalized as follows, $EH$ can be represented as a power of the volume function. We define a constant $d_{\Omega,K,a}$ by

$$(23) \qquad d_{\Omega,K,a} := \frac{\displaystyle\int_M s_{\tilde{g}_{K,a}} dv_{\tilde{g}_{K,a}}}{\mathrm{Vol}(M, \tilde{g}_{K,a})} = \frac{\displaystyle\int_M s_{\tilde{g}_{K,a}} f_{K,a,g}^{-2m}\frac{\omega_g^m}{m!}}{\displaystyle\int_M f_{K,a,g}^{-2m}\frac{\omega_g^m}{m!}}.$$

By the argument in [2], $d_{\Omega,K,a}$ is independent of the choice of $g \in \mathcal{K}_\Omega^G$. Note here that, for general $(K,a) \in \mathcal{P}_\Omega^G$,

$c_{\Omega,K,a} \neq d_{\Omega,K,a}$. However if there exists a cKEM metric $\tilde{g}_{K,a}$ then $c_{\Omega,K,a} = d_{\Omega,K,a}$. Hence $c_{\Omega,K,a} - d_{\Omega,K,a}$ gives an obstruction to the existence of cKEM metric $\tilde{g}_{K,a}$. If we set

$$\tilde{\mathcal{P}}^G_\Omega(\gamma) := \{(K,a) \in \mathcal{P}^G_\Omega \,|\, d_{\Omega,k,a} = \gamma\}$$

for a constant $\gamma$, then

$$EH(K,a) = \gamma \mathrm{Vol}(K,a)^{\frac{1}{m}} := \gamma \mathrm{Vol}(\tilde{g}_{K,a})^{\frac{1}{m}} \tag{24}$$

on $\tilde{\mathcal{P}}^G_\Omega(\gamma)$. By the first variation formula of the normalized Einstein-Hilbert functional (cf. [5]), we have

$$\begin{aligned}(25)\quad & \frac{d}{dt}_{|t=0} EH(K+tH,a) \\ &= \frac{2-2m}{\mathrm{Vol}(K,a)^{\frac{m-1}{m}}} \int_M \left( \frac{s_{\tilde{g}_{K,a}} - d_{\Omega,K,a}}{f^{2m+1}_{K,a,g}} \right) f_{H,0,g} \frac{\omega_g^m}{m!}\end{aligned}$$

and

$$\begin{aligned}(26)\quad & \frac{d}{dt}_{|t=0} EH(K,a+tb) \\ &= \frac{2-2m}{\mathrm{Vol}(K,a)^{\frac{m-1}{m}}} (c_{\Omega,K,a} - d_{\Omega,K,a}) \int_M \frac{1}{f^{2m+1}_{K,a,g}} \frac{\omega_g^m}{m!}.\end{aligned}$$

Therefore cKEM metrics have the following volume minimizing property.

**Theorem 3.4** ([28]). *Suppose that there exists a Kähler metric $g \in \mathcal{K}^G_\Omega$ such that $\tilde{g}_{K,a}$ is a cKEM metric for $(K,a) \in \tilde{\mathcal{P}}^G_\Omega(\gamma)$. Then $(K,a)$ is a critical point of the volume function $\mathrm{Vol} : \tilde{\mathcal{P}}^G_\Omega(\gamma) \to \mathbf{R}$. Furter, $(K,a)$ is a critical point of $\mathrm{Vol}$ if and only if $\mathrm{Fut}^G_{\Omega,K,a} \equiv 0$.*

For example, let $(M,J,g)$ be an $m$-dimensional compact toric Kähler manifold. We denote by $\Delta \subset \mathbf{R}^m$ the moment polytope. Then we see that

$$EH(K,a) = \frac{4\pi}{(m!)^{\frac{1}{m}}} \frac{\displaystyle\int_{\partial\Delta} \frac{1}{f^{2m-2}_{K,a}} d\sigma}{\left( \displaystyle\int_\Delta \frac{1}{f^{2m}_{K,a}} d\mu \right)^{\frac{m-1}{m}}} \tag{27}$$

for

$$(K,a) \in \mathcal{P}^{T^m}_\Delta \simeq \{ f_{K,a}(\mu) := \sum_{i=1}^m K_i\mu_i + a \,|\, f_{K,a} > 0 \text{ on } \Delta\} \tag{28}$$

(cf. [2] or [28].) Therefore, when $m = 2$, we want to know the critical points of

$$EH(a,b,c)^2 = 8\pi^2 \frac{\left( \displaystyle\int_{\partial\Delta} \frac{1}{(a\mu_1 + b\mu_2 + c)^2} d\sigma \right)^2}{\displaystyle\int_\Delta \frac{1}{(a\mu_1 + b\mu_2 + c)^4} d\mu} \tag{29}$$

with $a\mu_1 + b\mu_2 + c$ is positive on $\Delta$. For $\mathbf{CP}^2, \mathbf{CP}^1 \times \mathbf{CP}^1$ and the one point blow up of $\mathbf{CP}^2$, we summarize results of computations.

• $M = \mathbf{CP}^2$ : In this case, up to scale and translations, $\Delta$ is the convex hull of the three points $(0,0), (1,0)$ and $(0,1)$. The critical point of the function $EH$ on $\mathcal{P}^{T^2}_\Delta / \mathbf{R}_+$ is only $[(0,0,1)]$.

• $M = \mathbf{CP}^1 \times \mathbf{CP}^1$ : Let $\Delta_p$ be the convex hull of $(0,0), (p,0), (p,1)$ and $(0,1)$, where $p \geq 1$.

When $1 \leq p \leq 2$, $EH$ has the unique critical point $[(0,0,1)]$.

On the other hand, when $p > 2$, there exist three critical points

$$[(0,0,1)], \quad \left[ \left( \pm 1, 0, \frac{1}{2} \left( \frac{p^{\frac{3}{2}}}{\sqrt{p-2}} \mp p \right) \right) \right].$$

We emphasize that this result shows that the volume function is not convex unlike the case of Kähler-Ricci solitons and of Sasaki-Einstein metrics, see §2 and §4.

• $M =$ one point blow up of $\mathbf{CP}^2$ :

Let $\Delta_p$ be the convex hull of $(0,0), (p,0), (p,1-p)$ and $(0,1)$, where $0 < p < 1$.

For $0 < p < 1$,

$$\left[ \left( 1, 0, \frac{p(1-\sqrt{1-p})}{2\sqrt{1-p} + p - 2} \right) \right] \tag{30}$$

is a critical point of $EH$.

When $\frac{8}{9} < p < 1$ there are the following two more critical points

$$\left[ \left( -1, 0, \frac{p(3p \pm \sqrt{9p^2 - 8p})}{2(p \pm \sqrt{9p^2 - 8p})} \right) \right]. \tag{31}$$

Let $0 < \alpha < \beta < 1$ be the real roots of

$$F(p) := p^4 - 4p^3 + 16p^2 - 16p + 4 = 0.$$

When $0 < p < \alpha$, there are the following two critical points

$$\left[ \left( p^2 - 4p + 2 \pm \sqrt{F(p)}, \pm 2\sqrt{F(p)}, p^2 + 2p - 2 \mp \sqrt{F(p)} \right) \right]. \tag{32}$$

An extension of Lichnerowicz-Matsushima theorem asserting the reductiveness of the automorphism group on a cKEM manifold is obtained in [29] and [33].

## 4. Sasakian Geometry

A Sasakian structure is often referred to as an odd dimensional analogue of the Kähler structure. It roughly consists of a contact structure, a Riemannian structure compatible with the contact structure and an almost complex structure on the contact bundle. There are many equivalent definitions, but the following definition is the most simple and rigorous one. In Riemannian point view, a Sasakian manifold is a Riemannian manifold $(S,g)$ whose cone manifold $(C(S),\bar{g})$

with $C(S) \cong S \times \mathbf{R}^+$ and $\overline{g} = dr^2 + r^2 g$ is Kähler, where $r$ is the standard coordinate on $\mathbf{R}^+$. In this paper we always assume $S$ is closed and connected. From the definition, $S$ is odd-dimensional and we put $\dim S = 2m+1$, and thus $\dim_{\mathbf{C}} C(S) = m+1$. $S$ is identified with the submanifold $\{r=1\} \subset C(S)$. The Kähler form on $C(S)$ is given by $i\partial\overline{\partial}r^2$. From this we see that, fixing the holomorphic structure on $C(S)$, the Sasakian structure is determined by the radial function $r$ since the Riemannian structure is induced from the Kähler structure of $C(S)$. We consider the deformations of the Sasakian structure on $S$ fixing the complex structure $J$ on $C(S)$.

On the other hand, $S$ also inherits a contact structure with the contact form

$$\eta = (i(\overline{\partial} - \partial)\log r)|_{r=1}.$$

It is well known [6] that the Sasakian structure is determined by the transverse Kähler structure of the flow generated by the Reeb vector field $\xi$ of $\eta$. The Reeb vector field $\xi$ is obtained by restricting the vector field $\tilde{\xi} := J(r\frac{\partial}{\partial r})$ on $C(S)$ to $S = \{r=1\} \subset C(S)$. This is a standard fact known as the "Kähler sandwich": The Sasakian structure is equivalently given by the Kähler structure on the cone or given by the transverse Kähler structure on the local orbit spaces of the Reeb flow, see [6] for the detail. From this we see that the Sasakian structure can be deformed by the deformation of the choice of Reeb vector field in the Lie algebra $\mathrm{Lie}(T_\xi)$ of the torus $T_\xi$ obtained by taking the closure of the flow generated by $\xi$ since the deformed Reeb flow still has transverse Kähler structure. Then by choosing a rational point in $\mathrm{Lie}(T_\xi)$ we obtain a Reeb vector field obtained as an $S^1$-action on an ample line bundle over an orbifold. Thus $C(S)$ has an affine algebraic variety $\mathcal{A}$ with only one singular point at the apex as an underlying space.

Let $G$ be the group of biholomorphisms of $\mathcal{A} = C(S)$ preserving the cone structure, that is, $\mathrm{Lie}(G)$ consists of the real parts of holomorphic vector fields on $\mathcal{A}$ commuting with $r\frac{\partial}{\partial r}$. Let $T$ be the maximal torus of $G$ containing $T_\xi$. Note here that it is a standard fact that $r\frac{\partial}{\partial r}$ preserves $J$ and that $\tilde{\xi} - iJ\tilde{\xi}$ is a holomorphic vector field. The deformation space of $T$-invariant Sasakian structures containing the Sasakian structure of $S$, or equivalently $T$-invariant Kähler cone structures on $\mathcal{A}$, is given by the space $\mathcal{R}$ of $T$-invariant smooth positive functions $r : \mathcal{A} \to \mathbf{R}$ such that $i\partial\overline{\partial}r^2$ is positive $(1,1)$-form:

$$\mathcal{R} := \{r : \mathcal{A} \to \mathbf{R} \mid T\text{-invariant},\ i\partial\overline{\partial}r^2 > 0\}.$$

Since the Reeb vector field $\tilde{\xi} = Jr\frac{\partial}{\partial r}$ is the real part of a holomorphic Killing vector field and $T$ is the maximal torus in $G$, $Jr\frac{\partial}{\partial r}$ is in $\mathrm{Lie}(T)$ for any $r \in \mathcal{R}$. The set of all Reeb vector fields corresponding to $r \in \mathcal{R}$ is the dual cone of the cone obtained as the moment map image of $C(S)$, and is called the *Sasaki cone*. We define the volume functional $\mathrm{Vol} : \mathcal{R} \to \mathbf{R}$ by

$$\mathrm{Vol}(r) = \mathrm{vol}(S_r) \tag{33}$$

where $\mathrm{vol}(S_r)$ denotes the volume of the Sasakian manifold $S_r = \{r=1\}$ for $r \in \mathcal{R}$. Let $r(t)_{-\epsilon<t<\epsilon}$ be a one parameter family in $\mathcal{R}$ with $r(0) = r$, and put $Y := \frac{d}{dt}|_{t=0}\tilde{\xi}(t)$ where $\tilde{\xi}(t) = Jr(t)\frac{\partial}{\partial r(t)}$. Then the first variaton of $\mathrm{Vol}(r)$ is given by

$$\frac{d\mathrm{Vol}(r(t))}{dt}|_{t=0} = -4(m+1)\int_{S_r} \eta(Y)\,dvol_r \tag{34}$$

where $dvol_r$ is the volume element of $S_r$, see [30], Proposition 8.3, or [43], Appendix C1. The second variation is given by

$$\frac{d}{dt}|_{t=0}\left(-4(m+1)\int_{S_{r(t)}} \eta(X)\,dvol_{r(t)}\right) = 4(m+1)(2m+4)\int_{S_r} \eta(X)\eta(Y)\,dvol_r, \tag{35}$$

see [30], Proposition 8.4, or [43], Appendix C2. The second variation formula shows that the volume functional is convex.

A Sasakian manifold $S$ is called a Sasaki-Einstein manifold if it is an Einstein manifold as a Riemannian manifold. This occurs exactly when $C(S)$ is a Ricci-flat Kähler cone (i.e. Calabi-Yau cone). From the view point of the Kähler sandwich, this occurs exactly when the transverse Kähler structure of the Reeb flow is Kähler-Einstein with positive scalar curvature. A typical example is the $(2m+1)$-dimensional standard sphere which is Sasaki-Einstein. In this case, the cone is $\mathbf{C}^{m+1}$ which is Ricci-flat Kähler, the Reeb flow is the standard $S^1$-action, and the orbit space is the complex projective space which is a Kähler-Einstein manifold of positive scalar curvature.

When the Reeb flow generates an $S^1$-action the quotient space is a Fano orbifold. For general Sasakian structures the complex geometry of the local orbit spaces are described as "basic" geometry. For example, we have the basic $\partial$-operator $\partial_B$, the basic $\overline{\partial}$-operator $\overline{\partial}_B$, the basic Dolbeault cohomology $H^*_{\overline{\partial}_B}$, the basic Kähler metric $g_B$, the basic Kähler form $\omega_B$, and the basic Ricci form $\rho_B$, the basic first Chern class $c_1^B$ and etc. With these notations, the Sasaki-Einstein equation becomes

$$\rho_B = (2m+2)\omega_B.$$

Thus a necessary condition for the existence of a Sasaki-Einstein metric is that the basic first Chern class is represented by a positive multiple of the basic Kähler class:

$$2\pi c_1^B = (2m+2)[\omega_B]$$

in $H^2_{\bar\partial_B}$. This last condition is equivalent to the topological condition that $c_1(D)=0$ and that $c_1^B>0$ where $D$ denotes the contact structure determined by the contact form $\eta$, see [30], Proposition 4.3. We say in this paper that $S$ is transversely Fano if $c_1(D)=0$ and $c_1^B>0$.

Let $\xi$ be the Reeb vector field on a Sasakian manifold $S$. A smooth function $f$ on $S$ is said to be basic if $\xi f=0$. A basic function is obtained locally by pulling back a smooth function on the local orbit space of the Reeb flow. A holomorphic vector field $Y$ in $\mathrm{Lie}(G)$ descends to a complex vector field on $S$ and also to a complex vector field on each local orbit space of the Reeb flow, both of which we also denote by the same letter $Y$. Then $Y$ is written on the local orbit space of the Reeb flow, which is Kähler, as

$$Y=\mathrm{grad}'_{g_B}u=g_B^{i\bar j}\frac{\partial u}{\partial\overline{z^j}}\frac{\partial}{\partial z^i} \tag{36}$$

where $z^1,\cdots,z^m$ are local holomorphic coordinates and $g_B$ is the transverse Kähler metric on the local orbit space of the Reeb flow. There is a real valued basic function $F_B$ such that

$$\rho_B-(2m+2)\omega_B=i\partial_B\overline{\partial}_BF_B. \tag{37}$$

Just as in the case of Fano manifolds (cf. [27], Theorem 2.4.3), there is an isomorphism between $\mathrm{Lie}(G)$ and the space $\Lambda_{2m+2}$ of eigenfunctions $u$ of the elliptic operator $\Delta_B^F$ defined by

$$\Delta_B^Fu:=\Delta_Bu-\nabla^iu\nabla_iF_B \tag{38}$$

where $\Delta_B=\overline{\partial}_B^*\overline{\partial}_B$ is the transverse $\overline{\partial}_B$-Laplacian and $\nabla$ denotes the Levi-Civita connection of the transverse Kähler structure, see [30], Theorem 5.1. Noting $\eta(Y)$ in (34) is basic, if $\eta(Y)=u$ in $\Lambda_{2m+2}$, then the right hand side of (34) is equal to

$$-2\int_S(2m+2)u\ dvol=-2\int_S(\Delta_Bu-\nabla^iu\nabla_iF_B)dvol \tag{39}$$

$$=2\int_S(\mathrm{grad}'_{g_B}u)F_B\ dvl. \tag{40}$$

The right hand side is equal to $\mathrm{Fut}_\xi$ where $\xi$ is the Reeb vector field which is determined by the Sasakian structure of $S$. This proves the volume minimization formula (3).

A Sasakian manifold $(S,g)$ is said to be toric if the Kähler cone manifold $C(S)$ is toric, namely $\dim_{\mathbf{C}}G=m+1$. When $S$ is toric and transversely Fano, Martelli-Sparks-Yau [43] showed that the volume functional is proper on the space $\Sigma$ of Reeb vector fields of charge $n$, which is a slice in the Sasaki cone, i.e. the dual cone of the moment map image of $C(S)$. Since the volume functional is convex by (35), there is a unique minimum on $\Sigma$ at which $\mathrm{Fut}_\xi$ vanishes. In [30] it is shown that for this minimum $\xi$ there is a Sasaki-Einstein metric. Uniqueness assertion is also shown in [13]. To sum up the following holds.

**Theorem 4.1** ([30], [13]). *Let $(S,g)$ be a compact toric Sasakian manifold with $c_1^B>0$ and $c_1(D)=0$. Then there exists a Sasaki-Einstein metric. Further, the identity component of the automorphism group for the transverse holomorphic structure acts transitively on the space of all Sasaki-Einstein metrics.*

In Kähler geometry, the Yau-Tian-Donaldson conjecture relates the existence problem of constant scalar curvature Kähler (cscK for short) metrics to K-stability. Similarly in Sasakian geometry, the existence problem of constant scalar curvature Sasaki (cscS for short) metrics is related to K-stability, see [14], [15], [49], [10] for example.

The cscS metrics are critical points of the Einstein-Hilbert functional $H:\mathcal{R}\to\mathbf{R}$ defined by

$$H(r)=\frac{\mathrm{TS}(r)^{m+1}}{\mathrm{Vol}(r)^m} \tag{41}$$

where $\mathrm{TS}(r)$ denotes the total scalar curvature of $S_r$. In the transversely Fano case, $\mathrm{TS}(r)=\mathrm{Vol}(r)$ and the Einstein-Hilbert functional coincides with the volume functional. For general Sasakian manifolds, i.e. for Sasakian manifolds which are not necessarily transversely Fano, it is known that the convexity fails for the Einstein-Hilbert functional, and there can be several critical points, see Legendre [37], and also [9]. This fact has resemblance in the study of Einstein-Maxwell Kähler metrics as can be seen in the ambitoric examples by LeBrun [35] on the one-point-blow-up of $\mathbf{CP}^2$. But it is shown by Boyer-Huang-Legendre [8] that all of the volume functional, the total scalar curvature and the Einstein-Hilbert functional are proper in that they tend to $+\infty$ as the Reeb vector field tends to the boundary of the Sasaki cone. This was shown by using the Duistermaat-Heckman formula.

The idea of volume minimization for Sasaki-Einstein metrics has been extended and applied to algebraic geometry. Odaka [44] considered generalizations of the normalized volume functional and Donaldson-Futaki invariant obtained as the derivative of the volume functional. Odaka observed the decrease of the Donaldson-Futaki invariant along the minimal model program using the concavity of the volume functional. Li [38], [39] considered normalized volume functional on the space of valuations on Fano manifolds and characterized K-semistability in terms of volume minimization. Note that when a Sasakian manifold is the circle bundle of an ample line bundle $L$ over $M$, then the Reeb vector field defines a valuation of the ring $\oplus_{k=0}^\infty H^0(M,L^k)$. In view of this, to define volume functional for valuations is natural. The normalization corresponds to the restriction of the Reeb vector fields to the ones with charge $n$. On the other hand the Gromov-Hausdorff limit of a sequence of Kähler-Einstein manifolds is homeomorphic to a normal algebraic variety and admits a Kähler-Einstein metric in the sense of pluripotential theory [23]. The tangent cone at a singular point admits a Ricci-flat cone structure, and thus it is a cone over a Sasaki-Einstein manifold on the regular set. Li-Xu [42] applies the volume minimization to show an algebraic nature of those tangent cones, answering to a question of Donaldson-Sun [24]. See also [40], [41].

## Acknowledgment

The first author would like to thank Yau Mathematical Sciences Center at Tsinghua University for its hospitality where this survey paper was completed.

## References

[1] V. Apostolov, D. M. J. Calderbank, and P. Gauduchon: Ambitoric geometry II: extremal toric surfaces and Einstein 4-orbifolds, Ann. Sci. Éc. Norm. Supér. (4) 48 (2015), no. 5, 1075–1112.

[2] V. Apostolov and G. Maschler: Conformally Kähler, Einstein-Maxwell geometry, arXiv preprint arXiv:1512.06391.

[3] T. Aubin: Equations du type de Monge-Ampère sur les variétés kählériennes compactes, C. R. Acad. Sci. Paris, 283 (1976), 119–121.

[4] L. Bérard-Bergery, Lionel: Sur de nouvelles variétés riemanniennes d'Einstein. (French) [Some new Einstein Riemannian manifolds] Institut Élie Cartan, 6, 1–60, Inst. Élie Cartan, 6, Univ. Nancy, Nancy, 1982.

[5] A. Besse: Einstein manifolds, Springer-Verlag, Berlin-Heidelberg-Tokyo, 1987.

[6] C. P. Boyer and K. Galicki: Sasakian geometry. Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008.

[7] C. P. Boyer, K. Galicki, and S. R. Simanca: Canonical Sasakian metrics, Comm. Math. Phys. 279 (2008), no. 3, 705–733.

[8] C. P. Boyer, H. Huang, and E. Legendre: An application of the Duistertmaat–Heckman Theorem and its extensions in Sasaki Geometry, arXiv preprint arXiv:1708.03006 (2017).

[9] C. P. Boyer, H. Huang, E. Legendre, and C. W. Tønnesen-Friedman: The Einstein-Hilbert Functional and the Sasaki-Futaki Invariant, International Mathematics Research Notices 2017, no. 7, 1942–1974.

[10] C. P. Boyer and C. van Coevering: Relative K-stability and Extremal Sasaki metrics, arXiv preprint arXiv:1608.06184 (2016).

[11] X. X. Chen, S. K. Donaldson, and S. Sun: Kähler-Einstein metric on Fano manifolds. III: limits with cone angle approaches $2\pi$ and completion of the main proof, J. Amer. Math. Soc. 28 (2015), 235–278.

[12] X.-X. Chen, C. Lebrun, and B. Weber: On conformally Kähler, Einstein manifolds, J. Amer. Math. Soc. 21 (2008), no. 4, 1137–1168.

[13] K. Cho, A. Futaki, and H. Ono: Uniqueness and examples of toric Sasaki-Einstein metrics, Comm. Math. Phys. 277 (2008), 439–458.

[14] T. Collins and G. Székelyhidi: K-semistability for irregular Sasakian manifolds, arXiv preprint arXiv: 1204.2230.

[15] T. Collins and G. Székelyhidi: Sasaki-Einstein metrics and K-stability, arXiv preprint arXiv:1512.07213.

[16] V. Datar and G. Székelyhidi: Kähler-Einstein metrics along the smooth continuity method, Geom. Funct. Anal. 26 (2016), no. 4, 975–1010.

[17] T. Delcroix: Kähler-Einstein metrics on group compactifications, Geom. Funct. Anal. 27 (2017), no. 1, 78–129.

[18] T. Delcroix: K-stability of Fano spherical varieties, arXiv preprint arXiv:1608.01852.

[19] S. K. Donaldson: Scalar curvature and stability of toric varieties, J. Differential Geometry 62 (2002), 289–349.

[20] S. K. Donaldson: Constant scalar curvature metrics on toric surfaces, Geom. Funct. Anal. 19 (2009), no. 1, 83–136.

[21] S. Donaldson: Stability of algebraic varieties and Kähler geometry, arXiv preprint arXiv:1702.05745.

[22] S. Donaldson: Kähler-Einstein metrics and algebraic geometry, arXiv preprint arXiv:1702.05748.

[23] S. Donaldson and S. Sun: Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry, Acta Mathematica 213 (2014), 63–106.

[24] S. Donaldson and S. Sun: Gromov-Hausdorff limits of Kähler manifolds and algebraic geometry, II, arXiv preprint arXiv:1507.05082 (2015).

[25] A. Futaki: An obstruction to the existence of Einstein Kähler metrics, Invent. Math. 73 (1983), 437–443.

[26] A. Futaki: On compact Kähler manifolds of constant scalar curvature, Proc. Japan Acad., Ser. A 59 (1983), 401–402.

[27] A. Futaki: Kähler-Einstein metrics and integral invariants, Lecture Notes in Math., vol. 1314, Springer-Verlag, Berline-Heidelberg-New York, (1988).

[28] A. Futaki and H. Ono: Volume minimization and Conformally Kähler, Einstein-Maxwell geometry, preprint, arXiv:1706.07953.

[29] A. Futaki and H. Ono: Conformally Einstein-Maxwell Kähler metrics and the structure of the automorphisms, preprint, arXiv:1708.01958.

[30] A. Futaki, H. Ono, and G. Wang: Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds, Journal of Differential Geometry 83 (2009), 585–635.

[31] N. Ilten and H. Süss: K-stability for Fano manifolds with torus action of complexity 1, Duke Math. J. 166 (2017), no. 1, 177–204.

[32] C. Koca and C. W. Tønnesen-Friedman: Strongly Hermitian Einstein- Maxwell solutions on ruled surfaces, Ann. Glob. Annl. Geom. 50 (2016), 29–46.

[33] A. Lahdili: Automorphisms and deformations of conformally Kähler, Einstein-Maxwell metrics, arXiv:1708.01507.

[34] C. LeBrun: The Einstein-Maxwell equations, Kähler metrics, and Hermitian geometry, J. Geom. Phys. 91 (2015), 163–171.

[35] C. LeBrun: The Einstein-Maxwell equations and conformally Kähler geometry, Commun. Math. Phys. 344 (2016), 621–653.

[36] E. Legendre: Toric geometry of convex quadrilaterals, J. Symplectic Geom. 9 (2011), no. 3, 343–385.

[37] E. Legendre: Existence and non-uniqueness of constant scalar curvature toric Sasaki metrics, Compositio Math. 147 (2011), 1613–1634.

[38] C. Li: Minimizing normalized volumes of valuations, arXiv preprint arXiv:1511.08164 (2015).

[39] C. Li: K-semistability is equivariant volume minimization, arXiv preprint arXiv:1512.07205 (2015).

[40] C. Li and Y. Liu: Kähler-Einstein metrics and volume minimization, arXiv preprint arXiv:1602.05094 (2016).
[41] C. Li and C. Xu: Stability of Valuations and Kollár Components, arXiv preprint arXiv:1604.05398 (2016).
[42] C. Li and C. Xu: Stability of Valuations: Higher Rational Rank, arXiv preprint arXiv:1707.05561 (2017).
[43] D. Martelli, J. Sparks, and S.-T. Yau: Sasaki-Einstein manifolds and volume minimisation, Comm. Math. Phys. 280 (2008), no. 3, 611–673.
[44] Y. Odaka: Invariants of varieties and singularities inspired by Kähler-Einstein problems, Proc. Japan Acad., Ser A 91 (2015), 50–55.
[45] D. Page: A compact rotating gravitational instanton, Phys. Lett. B 79 (1978), 235–238.
[46] S. Saito: On the vanishing of the holomorphic invariants for Kähler-Ricci solitons, Proc. Japan Acad., Ser. A 90 (2014), 57–59.
[47] G. Tian: K-Stability and Kähler-Einstein Metrics, Comm. Pure Appl. Math. 68 (2015), no. 7, 1085–1156.
[48] G. Tian and X.-H. Zhu: A new holomorphic invariant and uniqueness of Kähler-Ricci solitons, Comment. Math. Helv. 77 (2002), 297–325.
[49] C. van Coevering and C. Tipler: Deformations of constant scalar curvature Sasakian metrics and K-stability, Int. Math. Res. Not. IMRN 2015, no. 22, 11566–11604.
[50] X.-J. Wang and X. Zhu: Kähler-Ricci solitons on toric manifolds with positive first Chern class, Adv. Math. 188 (2004), no. 1, 87–103.
[51] S.-T.Yau: On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation I, Comm. Pure Appl. Math. 31 (1978), 339–441.