Mathematical Encounters

To the memory of Mauricio M. Peixoto (april 15, 1921–april 28, 2019)

by Jorge Sotomayor*†

Abstract. This evocative essay focuses on mathematical activities witnessed by the author along 1962–1964 at IMPA. The list of research problems proposed in September 1962 by Mauricio Peixoto at the Seminar on the Qualitative Theory of Differential Equations is pointed out as a landmark for the beginning of the systematic research in the Qualitative Theory of Differential Equations and Dynamical Systems in Brazil.

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1. Letters, November 1961

Prof. Mauricio Peixoto
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Dear Prof. Peixoto,

My name is Jorge Sotomayor. I write following the advice of my teacher, Prof. José Tola[1], who has mentioned to you my interest in pursuing advanced mathematical studies in Brazil. He has told me of your request that I send you a report on my mathematical education, indicating, as much as possible, the level of knowledge that I have attained so far.

Presently I am finishing the third year at the Institute of Mathematics of San Marcos University, Lima. Towards December I will have finished Mathematical Analysis (a two-year course, which covers most of the book by T. Apostol). The same for Differential Geometry (a year course, which covers almost completely the book of T. Willmore) and Analytic Functions (a one-semester course, based on the book of Ahlfors and notes by Prof. Tola). Last semester I took a course on Modern Algebra, also taught by Tola, based on parts of van der Waerden and local lecture notes.

I mention only the courses to which I have devoted special attention, and in which I feel somewhat confident and fulfilled. I particularly enjoy the books of Apostol and Willmore and have worked a lot on them.

So far I have remained ignorant of the theoretical aspects of Differential Equations, having only taken last year a one–semester course on methods of solutions of ordinary ones. No other courses on ODEs or PDEs are taught here.

J. S.

Dear Sotomayor,

Next year, towards April, I will start at IMPA a series of courses and seminars on the Qualitative Theory of Differential Equations. In order that you may have a chance to catch up with the level of the other prospective participants, I suggest you do the following reading:

On *Differential Equations*: study *Coddington and Levinson*, Chapters 1, 3, 15 and 16. You might find more accessible the little book by *Hurewicz*, whose

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[1] https://es.wikipedia.org/wiki/José_Tola_Pasquel, distinguished Peruvian mathematician and educator.

contents almost coincide with the material listed above.

On *Topology*: read *Seifert and Threlfall*; you should try to reach Homology, the structure of surfaces, and the Euler Characteristic on manifolds. The little book by *Pontryagin* is also suitable for this.

I am writing now to Dr. Lélio Gama, Director of IMPA, recommending you for a fellowship. You will hear directly from him later.

M. P.

The fellowship was granted. This remarkable gesture of solidarity made possible my arrival to Rio de Janeiro on time for the beginning of the seminar on the Qualitative Theory of Differential Equations. To this end the reading suggestions in the letter above were faithfully followed, not without considerable struggle.

From December 1961 to February 1962, took place my mathematical preparation for the trip to Brazil. The outcome of this endeavor was the feeling of the existence of a Theory of Differential Equations founded, not in only in Calculus, but in Mathematical Analysis and in Geometry. The Poincaré–Bendixson Theorem was a remarkable sample. This initial sensation led me to fancy that things of great interest could dwell inside such Theory.

2. A List of Open Problems in ODEs, September 1962

Besides Peixoto's Seminar, other engaging mathematical activities were witnessed by the author at IMPA along the years 1962-64. In [28], on which this section is based, was given a panoramic view of them as well as of the preparatory topics presented at the Seminar, previous to Peixoto's lecture: *Open Problems in the Qualitative Theory of Ordinary Differential Equations,* delivered in September 1962.

The following problems were proposed and discussed:

1. *First order structurally stable systems.*

Consider the complement $\mathcal{B}$ of the set $\mathcal{S}$ of C^r-structurally stable vector fields on a two-dimensional manifold, relative to the set $\mathcal{X}$ of all vector fields on the manifold. Let $\mathcal{B}$ be endowed with the C^r topology. Characterize the set $\mathcal{S}_1$ of those vector fields that are structurally stable with respect to arbitrarily small perturbations inside $\mathcal{B} = \mathcal{X} \backslash \mathcal{S}$.

This problem goes back to a 1938 research announcement of A. A. Andronov and E. A. Leontovich [3], [4]. They gave a characterization of $\mathcal{S}_1$ for a compact disk in the plane. This initiates the systematic study of the bifurcations (qualitative changes) that occur in families of vector fields as they cross $\mathcal{B}$. In the research announcement—contained in a dense four pages note—they stated that the most stable bifurcations, regarding $\mathcal{B}$ with the induced topology, occur in $\mathcal{S}_1$, [20], [22].

2. *The problem of the arc.*

Prove or disprove that a continuous curve (an arc) in the space $\mathcal{X}$ of vector fields of class C^r on the sphere can be arbitrarily well-approximated by a continuous curve that meets only finitely many *bifurcation points*; that is, points outside the set of structurally stable vector fields, at which qualitative changes occur.

Later research established that both share great complexity, which grows quickly with the dimension. This knowledge became apparent after the work of S. Smale [17] and also Newhouse [9] and Palis–Takens [10].

The understanding of the phenomenon of persistent accumulation of bifurcations implies that the problem of the arc as stated above has a negative answer. However, after removing the requirement of the approximation, Peixoto and S. Newhouse proved that every pair of structurally stable vector fields is connected by an arc that meets only finitely many bifurcation points. See [14].

3. *The classification problem.*

Use combinatorial invariants to classify the connected components of the open set of structurally stable vector fields. The essential difficulty of this problem is to determine when two structurally stable vector fields agree up to a homeomorphism that preserves their orbits and is isotopic to the identity.

Progress on this problem has been made by C. Gutierrez (1944–2008), W. de Melo (1947–2016), and by Peixoto [13].

4. *The existence of nontrivial minimal sets.*

Do invariant perfect sets (that is, sets that are nonempty, compact, and transversally totally discontinuous) exist for differential equations of class C^2 on orientable two-dimensional manifolds?

This problem goes back to H. Poincaré and A. Denjoy and was known to experts.

It was solved in the negative direction by A. J. Schwartz [16]. Peixoto presented this result from a preprint that he received in November 1962.

5. *Structurally stable second order differential equations.*

For equations of the form $x'' = f(x, x')$ (more precisely, for systems of the form $x' = y,\ y' = f(x, y)$), characterize structural stability, and prove the genericity of structural stability, in the spirit of Peixoto's results for vector fields on two-dimensional manifolds.

Problems 2 to 4 were assigned, in one-to-one correspondence, to the senior participants of the seminar. The first and last problems were held in reserve for a few months.

Peixoto's support led me to attack the first problem on his list.

The preparation for the presentation of the understanding the results, formulated for plane regions, by Andronov and Leontovich, [3], and the subsequent struggle to extend them to surfaces, that I made in May 1963 at Peixoto's seminar, led me to the hunch that the First and Second Problems were intimately interconnected.

The effort to provide a conceptual geometric synthesis for my extension of [3] to surfaces was further elaborated for my participation in a session of short communications delivered at the Fourth Brazilian Mathematics Colloquium, in July 1963, [5], [28], where all the people working under Peixoto's supervision made reports. Only Ivan Kupka, the most knowledgeable of the group, delivered a plenary lecture.

The critical appreciation of the *ensemble* of these presentations led me to the strong conviction that the First and Second Problems were part of the same problem and that their suitable synthesis should be presented in terms of infinite dimensional submanifolds and transversality in the Banach space of all vector fields on a surface, [28].

This was the beginning of a mathematical endeavor that would naturally led to the second one and, later, touched also the fifth one [24].

The results of the work carried out by the author, inspired by the geometrization of differential equations in Peixoto's papers [11] and [12], were presented in [20].

3. A Glimpse into Structural Stability

The concept of structural stability was established during the collaboration of the Russian mathematicians A. Andronov and L. Pontryagin that started in 1932 [15]. It first appeared in their research note published in 1937. Andronov (who was also a physicist) started a very important Russian school in Dynamical Systems. He left a remarkable mathematical heritage, highly respected both in Russia and in the West.

For a dynamic model—that is, a differential equation or system $x' = f(x)$—to faithfully represent a phenomenon of the physical world, it must have a certain degree of stability. Small perturbations, unavoidable in the recording of data and experimentation, should not affect its essential features. Mathematically this is expressed by the requirement that the *phase portrait* of the model, which is the geometric synthesis of the system, must be topologically unchanged by small perturbations. In other words, the phase portraits of f and $f + \Delta f$ must agree up to a homeomorphism of the form $I + \Delta I$, where I is the identity transformation of the phase space of the system and $||\Delta I||$ is small. A homeomorphism of the form $I + \Delta I$ is called an ϵ-*h*omeomorphism if $||\Delta I|| < \epsilon$; that is, it moves points at most ϵ units from their original positions.

Andronov and Pontryagin stated a characterization of structurally stable systems on a disk in the plane. This work was supported by the analysis of numerous concrete models of mechanical systems and electrical circuits, performed by Andronov and his associates [2]. The concept of structural stability, initially called *robustness*, represents a remarkable evolution of the continuation method of Poincaré.

When the American mathematician S. Lefschetz translated the writings of Andronov and his collaborators from Russian to English [1], he changed the name of the concept to the more descriptive one it has today. He also stimulated H. F. DeBaggis to work on a proof of the main result as stated by Andronov and Pontryagin.

Peixoto improved the results of the Russian pioneers in several directions.

For example, he introduced the space $\mathcal{X}$ of all vector fields, and he established the openness and genericity of structurally stable vector fields on the plane and on orientable surfaces. He also removed the ϵ-homeomorphism requirement from the original definition, proving that it is equivalent to the existence of any homeomorphism. This was a substantial improvement of the Andronov-Pontryagin theory, which was formulated for a disk in the plane [11].

The transition from the plane to surfaces, as in Peixoto's work, takes us from classical ODEs to the modern theory of Dynamical Systems, from Andronov and Pontryagin to D. V. Anosov and S. Smale.

It has also raised delicate problems—for instance, the *closing lemma*—that have challenged mathematicians for decades [8].

In [18] S. Smale regards Peixoto's structural stability theorem as the prototypical example and fundamental model to follow for global analysis.

In Sotomayor [29] the reader can find biographical data about Mauricio Peixoto.

4. Other Encounters, Some Reaching a More Distant Past

Once a general surface, not just the plane, is regarded as the natural domain for the global analysis of differential equations, it is natural to inquire into the stability properties of the differential equations of Classical Differential Geometry that are naturally attached to curved surfaces. Examples are given by the differential equations defining the lines of principal curvature, see [27], and the asymptotic curves, see [26].

This shift in perspective assigned a new significance to questions that are suggested by attentive readings after fortunate encounters with G. Monge, C. Dupin, and G. Darboux (see [25], [26], and [27]).

Elaboration of the some ideas of Classical Differential Geometry were recast in geometric language and in a new perspective, under the influence of ideas such as Genericity, Structural Stability and Bifurcation Theory, coming from Differential Equations and Dynamical Systems. A partial view of the results obtained can be found in [25], [26], and [27]. A recent historical essay on this subject can be found in Garcia–Sotomayor [30].

The encounters with Peixoto, epistolar, on September, 1961 and, in presence, at his lecture on the list of ODE problems in September 1962, were the beginning of several other equally fruitful ones.

I will mention only two of them: Steve Smale (1930–), Berkeley, 1966–1967 and Renée Thom (1923–2004), Bures sur Ivette September–December, 1972. The deep contributions of these outstanding mathematicians are still in the process of assimilation.

I attribute to the influence of Smale my initial incursion in n-dimensional bifurcations [19], about which I made a presentation toward November 1987 in his inspiring and challenging Dynamical Systems Seminar.

To Thom I owe my initial interest in the Theory of Singularities [21].

5. The Beginning of the Brazilian School in Dynamical Systems

The Open Problems session in September 1962 was the high point of the seminar that Peixoto inaugurated at IMPA in April of that year.

The evocative essay *A List of Problems on ODEs* [28] sets Peixoto's lecture reported above in the center of the stage for the mathematical research activities at IMPA—its courses, seminars, and visitors—during the years 1962–1964.

Peixoto's endeavors as Research Director and his seminar on the Qualitative Theory of Differential Equations launched the first effective effort in Brazil to stimulate research in the field.

This landmark in the history of Brazilian mathematics constitutes the beginning of the Brazilian school of Dynamical Systems.

After the starting step given by Peixoto, several successive generations in Dynamical Systems, in diversified research directions came in, spreading in Brazil and abroad.

The achievement of Arthur Avila, recipient of the Fields Medal 2014, must be pointed out as a remarkable landmark on this field [7].

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References

[1] A. A. Andronov and C. E. Chaikin, Theory of Oscillations, English edition by S. Lefschetz, Princeton University Press, 1949.

[2] A. A. Andronov et al., Theory of Oscillators, Translated by F. Immirzi, Addison Wesley, New York, 1966. Reprinted by Dover, 1987.

[3] A. A. Andronov, E. Leontovich et al., *Sur la theorie de la variation de la structure qualitative de la division du plan en trajectoires,* Dokl. Akad. Nauk., **21** (1938), 427–430

[4] A. A. Andronov, E. Leontovich et al., Theory of bifurcations of dynamic systems in the plane, Israel. Prog. Scient. Trans., Jerusalem, 1973.

[5] Atas do Quarto Colóquio Brasileiro de Matemática, Poços de Caldas, 1–12 de julho de 1963. IMPA, Conselho Nacional de Pesquisas, 1965, 83 p.

[6] R. Garcia and J. Sotomayor, Differential Equations of Classical Geometry, a Qualitative Theory, 27^{th} Coloq. Bras. Mat., IMPA, Rio deJaneiro, 2009.

[7] E. Ghys, *The Work of A. Avila,* Proceedings ICM, 2014. https://www.imj-prg.fr/IMG/pdf/avila.pdf

[8] C. Gutierrez, *On the closing lemma,* Comp. and Appl. Math., **20** (2001), 179–186.

[9] S. Newhouse, *The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms,* Publ. Math. I.H.E.S., **50** (1979), 101–151.

[10] J. Palis and F. Takens, Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations, Fractal dimensions and infinitely many attractors. Cambridge Univ. Press, 1993.

[11] M. Peixoto, *Structural Stability on two-dimensional manifolds,* Topology, **1** (1962), 101–120.

[12] M. Peixoto, *On an approximation theorem of Kupka and Smale,* Jour. Diff. Equat., **3** (1967), 214–227.

[13] M. Peixoto, Dynamical Systems, Proceedings of a Symposium held at Salvador, Bahia, in 1971, Academic Press 1973.

[14] M. Peixoto and S. Newhouse, *There is a simple arc joining any two Morse-Smale flows,* Asterisque, **31** (1976), 15–41.

[15] L. S. Pontryagin, *A short autobiography.* Russ. Math. Surveys, **33**, n. o6 (1978).

[16] A. J. Schwartz, *A generalization of Poincaré-Bendixson Theorem to closed two-dimensional manifolds,* Amer. Jour. Math. **85** (1963), 453–458.

[17] S. Smale, *Differentiable Dynamical Systems*, Bulletin Amer. Math. Soc., **76** (1967), 747–817.

[18] S. Smale, *What is Global Analysis?,* Amer. Math. Month., **76** (1969), 4–9.

[19] J. Sotomayor, *Generic bifurcations of dynamical systems*, in [13].

[20] J. Sotomayor, *Generic one parameter families of vector fields on two dimensional manifolds,* Publ. Math. I.H.E.S., **43** (1974), 5–46.

[21] J. Sotomayor, Singularidades das Aplicações Diferenciáveis, III Escola Latino Americana de Matemática, Rio de Janeiro, IMPA, 1976.

[22] J. Sotomayor, Curvas definidas por Equações Diferencias no Plano, $13^{(}$th) Coloq. Bras. Mat, 1981.

[23] J. Sotomayor, *La mesure de l'ensemble de bifurcation des champs de vecteurs dans le plan,* Comptes Rendus de la Academie de Sciences de Paris (1981), 405–407.

[24] J. Sotomayor, Structurally stable second order differential equations. Differential equations (São Paulo, 1981), 284–301, Lecture Notes in Math., **957**, Springer, Berlin-New York, 1982.
[25] J. Sotomayor, *O Elipsóide de Monge,* Matemática Universitária, **15** (1993). Translated into Spanish and updated in Materials Matemàtiques, Univ. Aut. Barcelona, 2007: http://mat.uab.cat/matmat/PDFv2007/v2007n01.pdf.
[26] J. Sotomayor and C. Gutierrez, Lines of Curvature and Umbilic Points on Surfaces, 18$^{\text{th}}$ Brazilian Mathematics Colloquium, Lecture Notes, IMPA, Rio de Janeiro, 1991.
[27] J. Sotomayor, C. Gutierrez, Lines of Curvature Umbilic Points and Caratheodory Conjecture, *Resenhas do IME-USP*, **3** (1998), 2.
[28] J. Sotomayor, *On a list of ordinary differential equations problems,* São Paulo Journal of Mathematical Sciences, 13(1), (2019), 177–194. DOI 10.1007/s40863-018-0110-3
[29] J. Sotomayor, *A few words on Mauricio M. Peixoto on his* 80th *birthday,* Comput. Appl. Math., **20**, no. 1 (2001), 3–9.
[30] J. Sotomayor and R. Garcia, *Historical comments on Monge's ellipsoid and the configurations of lines of curvature on surfaces,* Antiq. Math. **10** (2016), 169–182.