Positive Structures in Lie Theory

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0.1

In late 19th century and early 20th century, a new branch of mathematics was born: Lie theory or the study of Lie groups and Lie algebras (Lie, Killing, E. Cartan, H. Weyl). It has become a central part of mathematics with applications everywhere. More recent developments in Lie theory are as follows.

- Analogues of simple Lie groups over any field (including finite fields where they explain most of the finite simple groups): Chevalley 1955;
- infinite dimensional versions of the simple Lie algebras/simple Lie groups: Kac and Moody 1967, Moody and Teo 1972;
- theory of quantum groups: Drinfeld and Jimbo 1985.

0.2

In Lie theory to any Cartan matrix one can associate a simply connected Lie group $G(\mathbf{C})$; Chevalley replaces $\mathbf{C}$ by any field $\mathbf{k}$ and gets a group $G(\mathbf{k})$. In [L94] we have defined the totally positive (TP) submonoid $G(\mathbf{R}_{>0})$ of $G(\mathbf{R})$ and its "upper triangular" part $U^+(\mathbf{R}_{>0})$. In this lecture we will review the TP-monoids $G(\mathbf{R}_{>0})$, $U^+(\mathbf{R}_{>0})$ attached to a Cartan matrix, which for simplicity is assumed to be simply-laced. In [L94] the nonsimply laced case is treated by reduction to the simply laced case.

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0.3

The total positivity theory in [L94] was a starting point for

- a solution of Arnold's problem for real flag manifolds, Rietsch 1997;
- the theory of cluster algebras, Fomin, Zelevinsky 2002;
- a theory of TP for the wonderful compactifications, He 2004;
- higher Teichmüller theory, Fock, Goncharov 2006;
- the use of the TP grassmannian in physics, Postnikov 2007, Arkani-Hamed, Trnka 2014;
- a theory of TP for the loop group of GL_n, Lam, Pylyavskyy 2012;
- a theory of TP for certain nonsplit real Lie groups, Guichard-Wienhard 2018;
- a new approach to certain aspects of quantum groups, Goncharov, Shen.

0.4

Schoenberg (1930) and Gantmacher-Krein (1935) (after initial contributions of Fekete and Polya (1912)) defined the notion of TP matrix in $GL_n(\mathbf{R})$: a matrix all of whose $s \times s$ minors are ≥ 0 for any s. Gantmacher and Krein showed that if for any s, all $s \times s$ minors of a matrix A are > 0 then the eigenvalues of A are real, distinct and > 0. For example, the Vandermonde matrix (A_{ij}), $A_{ij} = x_i^{j-1}$ with $x_1 < x_2 < \cdots < x_n$ real and > 0 is of this type. According to Polya and Szegö, the matrix (A_{ij}), $A_{ij} = \exp(x_i y_j)$ with $x_1 > x_2 > \cdots > x_n$, $y_1 > y_2 > \cdots > y_n$ real is also of this type.

The TP matrices in $GL_n(\mathbf{R})$ form a monoid under multiplication. This monoid is generated by diagonal matrices with > 0 entries on diagonal and by matrices which have 1 on diagonal and a single nonzero entry off diagonal which is >0 (Whitney, Loewner, 1950's). Our definition [L94] of the TP part of any $G(\mathbf{R})$ was inspired by the work of Whitney, Loewner.

However, to prove properties of the resulting monoid (such as the generalization of the Gantmacher-Krein theorem) I had to use the canonical bases in quantum groups (discovered in [L90]) and their positivity properties. The role of $s\times s$ minors is played in the general case by the canonical basis of [L90]. Unlike in [L94], here we define $G(\mathbf{R}_{>0})$ by generators and relations, independently of $G(\mathbf{R})$. Surprisingly, this definition of $G(\mathbf{R}_{>0})$ is simpler than that of $G(\mathbf{R})$ (see [ST]). From it one can recover the Chevalley groups $G(\mathbf{k})$ for any field $\mathbf{k}$. Namely, the relations between the generators of $G(\mathbf{R}_{>0})$ involve only rational functions with integer coefficients. They make sense over $\mathbf{k}$ and they give rise to a "birational form" of a semisimple group over $\mathbf{k}$. This is the quotient field of the coordinate ring of $G(\mathbf{k})$; then $G(\mathbf{k})$ itself appears as a subgroup of the automorphism group of this field. In this approach the form $G(\mathbf{R}_{>0})$ is the most basic, all other forms are deduced from it.

0.5

We now describe the content of various sections. In §1 we define a positive structure on a set. Such structures have appeared in [L90], [L94] in connection with various objects in Lie theory. In §2 we define the monoid $U^+(\mathbf{R}_{>0})$. In §3 we define the monoid $G(\mathbf{R}_{>0})$. In §4 we use this monoid to recover the Chevalley groups over a field. In §5 we define the non-negative part of a flag manifold.

1. Positive Structures

1.1

The TP monoid can be defined not only over $\mathbf{R}_{>0}$ but over a structure K in which addition, multiplication, division (but no substraction) are defined. In [L94] three such K were considered.

(i) $K=\mathbf{R}_{>0}$;

(ii) $K=\mathbf{R}(t)_{>0}$, the set of $f\in\mathbf{R}(t)$ of form $f=t^e f_0/f_1$ for some f_0,f_1 in $\mathbf{R}[t]$ with constant term in $\mathbf{R}_{>0}, e\in\mathbf{Z}$ (t is an indeterminate);

(iii) $K=\mathbf{Z}$.

In case (i) and (ii), K is contained in a field $\mathbf{R}$ or $\mathbf{R}(t)$ and the sum and product are induced from that field. In case (iii) we consider a new sum $(a,b)\mapsto \min(a,b)$ and a new product $(a,b)\mapsto a+b$. A 4th case is

(iv) $K=\{1\}$

with $1+1=1, 1\times 1=1$.

In each case K is a semifield (a terminology of Berenstein, Fomin, Zelevinsky 1996): a set with two operations, $+$, $\times$, which is an abelian group with respect to $\times$, an abelian semigroup with respect to $+$ and in which $(a+b)c=ac+bc$ for all a,b,c. We fix a semifield K. There is an obvious semifield homomorphism $K\to\{1\}$. We denote by (1) the unit element of K with respect to $\times$.

1.2

In [L94] we observed that there is a semifield homomorphism $\alpha:\mathbf{R}(t)_{>0}\to\mathbf{Z}$ given by $t^e f_0/f_1\mapsto e$ which connects geometrical objects over $\mathbf{R}(t)_{>0}$ with piecewise linear objects involving only integers. I believe that this was the first time that such a connection (today known as tropicalization) was used in relation to Lie theory.

1.3

For any $m\in\mathbf{Z}_{>0}$ let $\mathcal{P}_m$ be set of all nonzero polynomials in m indeterminates $X_1,X_2,\dots,X_m$ with coefficients in $\mathbf{N}$.

A function $(a_1,a_2,\dots,a_m)\mapsto(a'_1,a'_2,\dots,a'_m)$ from K^m to K^m is said to be *admissible* if for any s we have $a'_s=P_s(a_1,a_2,\dots,a_m)/Q_s(a_1,a_2,\dots,a_m)$ where P_s,Q_s are in $\mathcal{P}_m$. (This ratio makes sense since K is a semifield.) In the case where $K=\mathbf{Z}$, such a function is piecewise-linear. If $m=0$, the unique map $K^0\to K^0$ is considered to be admissible (K^0 is a point.)

1.4

A *positive structure* on a set X consists of a family of bijections $f_j:K^m\xrightarrow{\sim}X$ (with $m\geq 0$ fixed) indexed by j in a finite set $\mathcal{J}$, such that $f_{j'}^{-1}f_j:K^m\to K^m$ is admissible for any j,j' in $\mathcal{J}$; the bijections f_j are said to be the *coordinate charts* of the positive structure. The results of [L94], [L97], [L98], can be interpreted as saying that various objects in Lie theory admit natural positive structures.

2. The Monoid $U^+(K)$

2.1 The Cartan Matrix

We fix a finite graph; it is a pair consisting of two finite sets I,H and a map which to each $h\in H$ associates a two-element subset $[h]$ of I. The Cartan matrix $A=(i:j)_{i,j\in I}$ is given by $i:i=2$ for all $i\in I$ while if i,j in I are distinct then $i:j$ is -1 times the number of $h\in H$ such that $[h]=\{i,j\}$.

An example of a Cartan matrix is:

$$I=\{i,j\},\ A=\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}.$$

We fix a Cartan matrix A. For applications to Lie theory A is assumed to be positive definite. But several of the subsequent definitions make sense without this assumption.

We attach to A and a field $\mathbf{k}$ a group $G(\mathbf{k})$. When A is positive definite, $G(\mathbf{k})$ is the group of $\mathbf{k}$-points of a simply connected semisimple split algebraic group of type A over $\mathbf{k}$. Without the assumption that A is positive definite, the analogous group $G(\mathbf{k})$ (with $\mathbf{k}$ of characteristic 0) has been defined in [MT], [Ma], [Ti].

We will associate to A and K a monoid $G(K)$ and a submonoid $U^+(K)$ of $G(K)$. In the case where $K = \mathbf{R}_{>0}$ (resp. $K = \mathbf{R}(t)_{>0}$), $G(K)$ and $U^+(K)$ can be viewed as submonoids of $G(\mathbf{k})$ where $\mathbf{k} = \mathbf{R}$ (resp. $\mathbf{k} = \mathbf{R}(t)$). In the case where $K = \mathbf{R}_{>0}, \mathbf{k} = \mathbf{R}, G(\mathbf{R}) = SL_n(\mathbf{R})$, $U^+(K)$ is the monoid of TP matrices in $G(\mathbf{R})$ which are upper triangular with 1 on diagonal. We first define $U^+(K)$.

2.2

Let $U^+(K)$ be the monoid (with 1) with generators i^a with $i \in I$, $a \in K$ and relations

$i^a i^b = i^{a+b}$ for $i \in I$, a,b in K;
$i^a j^b i^c = j^{bc/(a+c)} i^{a+c} j^{ab/(a+c)}$ for $i,j \in I$ with $i:j = -1$, a,b,c in K;
$i^a j^b = j^b i^a$ for $i,j \in I$ with $i:j = 0$, a,b in K.

(In the case where $K = \mathbf{Z}$, relations of the type considered above involve piecewise-linear functions; they first appeared in [L90] in connection with the parametrization of the canonical basis.) The definition of $U^+(K)$ is reminiscent of the definition of the Coxeter group attached to A. In the case where $K = \mathbf{Z}$ and A is positive definite the definition of $U^+(K)$ given above first appeared in [L94, 9.11].

2.3

When $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $K = \mathbf{R}_{>0}$, we can identify $U^+(K)$ with the submonoid of $SL_3(\mathbf{R})$ generated by

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

with a,b in $\mathbf{R}_{>0}$.

2.4

Let W be the Coxeter group attached to A. It has generators i with $i \in I$ and relations $ii = 1$ for $i \in I$; $iji = jij$ for $i,j \in I$, $i:j = -1$; $ij = ji$ for $i,j \in I$, $i:j = 0$. Let $\mathcal{O}_w$ be the set of reduced expressions of w that is the set of sequences $(i_1, i_2, \ldots, i_m)$ in I such that $i_1 i_2 \ldots i_m = w$ in $U^+(\{1\})$ where m is minimum. We write $m = |w|$ (= length of w).

When $K = \{1\}$, $U^+(K)$ is the monoid (with 1) with generators i^1 with $i \in I$ and relations $i^1 i^1 = i^1$ for $i \in I$; $i^1 j^1 i^1 = j^1 i^1 j^1$ for $i,j \in I$, $i:j = -1$; $i^1 j^1 = j^1 i^1$ for $i,j \in I$, $i:j = 0$. By a lemma of Iwahori and Matsumoto we have can identify (as sets) $W = U^+(\{1\})$ by $w = i_1 i_2 \ldots i_m \leftrightarrow i_1^1 i_2^1 \ldots i_m^1$ for any $(i_1, i_2, \ldots, i_m) \in \mathcal{O}_w$. This bijection is not compatible with the monoid structures.

2.5

The semifield homomorphism $K \to \{1\}$ induces a map of monoids $U^+(K) \to U^+(\{1\})$. Let $U_w^+(K)$ be the fibre over $w \in U^+(\{1\})$. We have $U^+(K) = \sqcup_{w \in W} U_w^+(K)$.

We now fix $w \in W$. It turns out that the set $U_w^+(K)$ can be parametrized by K^m, in fact in many ways, indexed by $\mathcal{O}_w$. For $\mathbf{i} = (i_1, i_2, \ldots, i_m) \in \mathcal{O}_w$ we define $\phi_{\mathbf{i}} : K^m \to U_w^+(K)$ by

$$\phi_{\mathbf{i}}(a_1, a_2, \ldots, a_m) = i_1^{a_1} i_2^{a_2} \ldots i_m^{a_m}.$$

This is a bijection. Now $U_w^+(K)$ together with the bijections $\phi_{\mathbf{i}} : K^m \to U_w^+(K)$ is an example of a positive structure. (We will see later other such structures.)

2.6

Let $w \in W, m = |w|$. In the case $K = \mathbf{Z}$, $U_w^+(\mathbf{N}) := \phi_{\mathbf{i}}(\mathbf{N}^m) \subset U_w^+(\mathbf{Z})$ is independent of $\mathbf{i} \in \mathcal{O}_w$. We set $U^+(\mathbf{N}) = \sqcup_{w \in W} U_w^+(\mathbf{N})$; this is a subset of $U^+(\mathbf{Z})$.

When W is finite, let w_I be the element of maximal length of W. Let $\nu = |w_I|$. Now $U_{w_I}^+(\mathbf{N})$ was interpreted in [L90] as an indexing set for the canonical basis of the plus part of a quantized enveloping algebra. A similar interpretation holds for $U_w^+(\mathbf{N})$ when W is not necessarily finite and w is arbitrary, using [L96, 8.2].

3. The Monoid $G(K)$

3.1

In order to define the monoid $G(K)$ we consider besides I, two other copies $-I = \{-i; i \in I\}$, $\underline{I} = \{\underline{i}; i \in I\}$ of I, in obvious bijection with I. For $\epsilon = \pm 1$, $i \in I$ we write $\epsilon i = i$ if $\epsilon = 1$, $\epsilon i = -i$ if $\epsilon = -1$.

Let $G(K)$ be the monoid (with 1) with generators $i^a, (-i)^a, \underline{i}^a$ with $i \in I, a \in K$ and the relations below.

(i) $(\epsilon i)^a (\epsilon i)^b = (\epsilon i)^{a+b}$ for $i \in I$, $\epsilon = \pm 1$, a,b in K;
(ii) $(\epsilon i)^a (\epsilon j)^b (\epsilon i)^c = (\epsilon j)^{bc/(a+c)} (\epsilon i)^{a+c} (\epsilon j)^{ab/(a+c)}$ for i,j in I with $i:j = -1$, $\epsilon = \pm 1$, a,b,c in K;
(iii) $(\epsilon i)^a (\epsilon j)^b = (\epsilon j)^b (\epsilon i)^a$ for i,j in I with $i:j = 0$, $\epsilon = \pm 1$, a,b in K;
(iv) $(\epsilon i)^a (-\epsilon i)^b = (-\epsilon i)^{b/(1+ab)} \underline{i}^{(1+ab)^\epsilon} (\epsilon i)^{a/(1+ab)}$ for $i \in I$, $\epsilon = \pm 1$, a,b in K;
(v) $\underline{i}^a \underline{i}^b = \underline{i}^{ab}$, $\underline{i}^{(1)} = 1$ for $i \in I$, a,b in K;
(vi) $\underline{i}^a \underline{j}^b = \underline{j}^b \underline{i}^a$ for i,j in I, a,b in K;
(vii) $\underline{j}^a (\epsilon i)^b = (\epsilon i)^{a^{\epsilon(i:j)} b} \underline{j}^a$ for i,j in I, $\epsilon = \pm 1$, a,b in K;
(viii) $(\epsilon i)^a (-\epsilon j)^b = (-\epsilon j)^b (\epsilon i)^a$ for $i \neq j$ in I, $\epsilon = \pm 1$, a,b in K.

3.2

When $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $K = \mathbf{R}_{>0}$, we can identify $G(K)$ with the submonoid of $SL_3(\mathbf{R})$ generated by:

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & d & 1 \end{pmatrix},$$
$$\begin{pmatrix} e & 0 & 0 \\ 0 & (1/e) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & (1/f) \end{pmatrix},$$

with a,b,c,d,e,f in $\mathbf{R}_{>0}$.

3.3

The assignment $i^a \mapsto i^a$ (with $i \in I, a \in K$) defines a monoid isomorphism of $U^+(K)$ onto a submonoid of $G(K)$; when $K = \{1\}$, we denote by $w \in G(\{1\})$ the image of $w \in U(\{1\})$ under this imbedding. The assignment $i^a \mapsto (-i)^a$ (with $i \in I, a \in K$) defines a monoid isomorphism of $U^+(K)$ onto a submonoid of $G(K)$; when $K = \{1\}$, we denote by $-w \in G(\{1\})$ the image of $w \in U(\{1\})$ under this imbedding. The map $W \times W \to G(\{1\})$, $(w,w') \mapsto w(-w')$ is a bijection of sets (not of monoids).

3.4

Tits has said that W ought to be regarded as the Chevalley group $G(\mathbf{k})$ where $\mathbf{k}$ is the (non-existent) field with one element. But $G(\{1\})$ is defined for the semifield $\{1\}$. The bijections $W \xrightarrow{\sim} U^+(\{1\})$, $W \times W \xrightarrow{\sim} G(\{1\})$ almost realizes the wish of Tits.

3.5

For general K, the semifield homomorphism $K \to \{1\}$ induces a monoid homomorphism $G(K) \to G(\{1\})$. Let $G_{w,-w'}(K)$ be the fibre over $w(-w')$ of this homomorphism. We have $G(K) = \sqcup_{(w,w') \in W \times W} G_{w,-w'}(K)$. We now fix $(w,w') \in W \times W$. Let $M = |w| + |w'| + r$. It turns out that the set $G_{w,-w'}(K)$ can be parametrized by K^M, in fact in many ways, indexed by a certain finite set $\mathcal{O}_{w,-w'}$. Let $\mathcal{O}_{-w'}$ be the set of sequences $(-i_1, -i_2, \dots, -i_{|w'|})$ in $-I$ such that $(i_1, i_2, \dots, i_{|w'|}) \in \mathcal{O}_{w'}$. Let $\mathcal{O}_{w,-w'}$ be the set of sequences $(h_1, h_2, \dots, h_M)$ in $I \sqcup (-I) \sqcup \underline{I}$ such that the subsequence consisting of symbols in I is in $\mathcal{O}_w$, the subsequence consisting of symbols in $-I$ is in $\mathcal{O}_{-w'}$, the subsequence consisting of symbols in $\underline{I}$ contains each symbol $\underline{i}$ (with $i \in I$) exactly once.

For $\mathbf{h} = (h_1, h_2, \dots, h_M) \in \mathcal{O}_{w,-w'}$ we define $\psi_{\mathbf{h}} : K^M \to G_{w,-w'}(K)$ by

$$\psi_{\mathbf{h}}(a_1, a_2, \dots, a_M) = h_1^{a_1} h_2^{a_2} \dots h_M^{a_M}.$$

This is a bijection. The bijections $\psi_{\mathbf{h}} : K^M \to G_{w,-w'}(K)$ (for various $\mathbf{h} \in \mathcal{O}_{w,-w'}$) define a positive structure on $G_{w,-w'}(K)$.

In the case where $K = \mathbf{R}_{>0}$ or $K = \mathbf{R}(t)_{>0}$, the statements above are proved by using Bruhat decomposition in the group $G(\mathbf{k})$ considered in 2.1 with $\mathbf{k} = \mathbf{R}$ or $\mathbf{R}(t)$. (When W is finite this is done in [L19]. See also the proof of [L94, Lemma 2.3] and [L94, 2.7].) The case where $K = \mathbf{Z}$ follows from the case where $K = \mathbf{R}(t)_{>0}$, using $\alpha : \mathbf{R}(t)_{>0} \to \mathbf{Z}$ in 1.2.

4. Chevalley Groups

4.1

In this section we assume that $K = \mathbf{R}_{>0}$ and that $I \neq \emptyset$. Let $\mathbf{k}_0$ be a field and let $\mathbf{k}$ be an algebraic closure of $\mathbf{k}_0$.

Let $w \in W, w' \in W$. Let $M = |w| + |w'| + r$. For $\mathbf{h}, \mathbf{h}'$ in $\mathcal{O}_{w,-w'}$, $\psi_{\mathbf{h}'}^{-1}\psi_{\mathbf{h}} : K^M \to K^M$ (see 3.5) is of the form $(a_1, a_2, \dots, a_M) \mapsto (a'_1, a'_2, \dots, a'_M)$ where $a'_s = (P_{\mathbf{h}}^{\mathbf{h}'})_s(a_1, a_2, \dots, a_M)/(Q_{\mathbf{h}}^{\mathbf{h}'})_s(a_1, a_2, \dots, a_M)$ and each of $(P_{\mathbf{h}}^{\mathbf{h}'})_s, (Q_{\mathbf{h}}^{\mathbf{h}'})_s$ is a nonzero polynomial in $\mathbf{N}[X_1, X_2, \dots, X_M]$ (independent of K) such that the g.c.d. of its $\neq 0$ coeff. is 1.

Applying the obvious ring homomorphism $\mathbf{Z} \to \mathbf{k}_0$ to the coefficients of these polynomials we obtain $\neq 0$ polynomials $(\bar{P}_{\mathbf{h}}^{\mathbf{h}'})_s$, $(\bar{Q}_{\mathbf{h}}^{\mathbf{h}'})_s$ in $\mathbf{k}_0[X_1, X_2, \dots, X_M]$. We define a rational map $\bar{\psi}_{\mathbf{h}}^{\mathbf{h}'} : \mathbf{k}^M \to \mathbf{k}^M$ by

$$(z_1, z_2, \dots, z_M) \mapsto (z'_1, z'_2, \dots, z'_m),$$
$$z'_s = (\bar{P}_{\mathbf{h}}^{\mathbf{h}'})_s(z_1, z_2, \dots, z_M)/(\bar{Q}_{\mathbf{h}}^{\mathbf{h}'})_s(z_1, z_2, \dots, z_M)$$

This is a birational isomorphism. It induces an automorphism $[\psi_{\mathbf{h}}^{\mathbf{h}'}]$ of the quotient field $[\mathbf{k}^M]$ of the coordinate ring of $\mathbf{k}^M$. We have $[\psi_{\mathbf{h}}^{\mathbf{h}'}][\psi_{\mathbf{h}'}^{\mathbf{h}''}] = [\psi_{\mathbf{h}}^{\mathbf{h}''}]$ for any $\mathbf{h}, \mathbf{h}', \mathbf{h}''$. Hence there is a well defined field $[G_{w,-w'}(\mathbf{k})]$ containing $\mathbf{k}$ with $\mathbf{k}$-linear field isomorphisms $[\psi_{\mathbf{h}}] : [G_{w,-w'}(\mathbf{k})] \to [\mathbf{k}^M]$ (for $\mathbf{h} \in \mathcal{O}_{w,-w'}$) such that

$$[\psi_{\mathbf{h}}^{\mathbf{h}'}] = [\psi_{\mathbf{h}}][\psi_{\mathbf{h}'}]^{-1} : [\mathbf{k}^M] \to [\mathbf{k}^M] \text{ for all } \mathbf{h}, \mathbf{h}'.$$

4.2

We now assume that W is finite. Let w_I, ν be as in 2.6. Let $M = 2\nu + r$. Let $i \in I, \epsilon = \pm 1, z \in \mathbf{k}_0$. We can choose $\mathbf{h} = (h_1, h_2, \dots, h_M) \in \mathcal{O}_{\omega,-\omega}$ such that $h_1 = \epsilon i$. The isomorphism $\mathbf{k}^M \to \mathbf{k}^M$, $(z_1, z_2, \dots, z_M) \mapsto (z_1 - z, z_2, \dots, z_m)$ induces a field isomorphism $\tau_z : [\mathbf{k}^M] \to [\mathbf{k}^M]$. Let $\mathbf{A}$ be the group of all $\mathbf{k}$-linear field automorphisms of $[G_{\omega,-\omega}(\mathbf{k})]$. We define $(\epsilon i)^z \in \mathbf{A}$ as the composition

$$[G_{\omega,-\omega}(\mathbf{k})] \xrightarrow{[\psi_{\mathbf{h}}]} [\mathbf{k}^M] \xrightarrow{\tau_z} [\mathbf{k}^M] \xrightarrow{[\psi_{\mathbf{h}}]^{-1}} [G_{\omega,-\omega}(\mathbf{k})].$$

Now $(\epsilon i)^z$ is independent of the choice of $\mathbf{h}$. Let $G(\mathbf{k}_0)$ be the subgroup of $\mathbf{A}$ generated by $(\epsilon i)^z$ for various $i \in I, \epsilon = \pm 1, z \in \mathbf{k}_0$. Then $G(\mathbf{k}_0)$ is the Chevalley group associated to $\mathbf{k}_0$ and our Cartan matrix.

5. Flag Manifolds

5.1

In this section W is not assumed to be finite. We assume that K is $\mathbf{R}_{>0}$. Let $G(\mathbf{R})$ be the group considered in 2.1. Let V be an $\mathbf{R}$-vector space which is an irreducible highest weight integrable representation of $G(\mathbf{R})$ whose highest weight takes the value 1 at any simple coroot. Let η be a highest weight vector of V. Let $\mathbf{B}$ be the canonical basis of V (see [L93, 11.10]) containing η. Let P be the set of lines in the $\mathbf{R}$-vector space V. Let $P_{\geq 0}$ be the set of all $L \in P$ such that for some $x \in L - \{0\}$ all coordinates of x with respect to the basis $\mathbf{B}$ are ≥ 0. The flag manifold $\mathcal{B}$ of $G(\mathbf{R})$ is defined as the subset of P consisting of lines in the $G(\mathbf{R})$-orbit of the line spanned by η. We define $\mathcal{B}(K) = \mathcal{B} \cap P_{\geq 0}$. By a positivity property [L93, 22.1.7] of $\mathbf{B}$ (stated in the simply laced case but whose proof remains valid in our case), the obvious $G(\mathbf{R})$-action on $\mathcal{B}$ restricts to a $G(K)$-action on $\mathcal{B}(K)$. (As mentioned in 2.1, $G(K)$ can be viewed as a submonoid of $G(\mathbf{R})$.) When W is finite, $\mathcal{B}(K)$ is the same as the subset $\mathcal{B}_{\geq 0}$ defined in [L94, §8]. (This follows from [L94, 8.17].)

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