John Conway, My Thoughts After His Passing

by Robert L. Griess Jr.*

The mathematician John Conway passed away on April 11, 2020. In this note, I am recounting some of my own experiences with him and his work on finite groups. Since I knew Conway for more than 50 years, It seems natural for me to write this article in the first person.

I met him in January 1970, at the Cambridge University DPMMS (Department of Pure Mathematics and Mathematical Statistics). I spent about five months there at the invitation of my advisor John Thompson, while working on my University of Chicago thesis on Schur multipliers of the finite simple groups.

Just before 1970, there had been a recent and dramatic increase in the number of known finite simple groups. (At that time, there was no sense that the classification of finite simple groups would be completed.) Conway was responsible for discovering several of these groups via his study of the 24-dimensional Leech lattice [1, 2]. He was justly proud. He determined the structure of the remarkably large isometry group of the Leech lattice, including its order $2^{22}3^{9}5^{4}7^{2}11\cdot 13\cdot 23$. This group is usually notated Co_0. Results included discoveries of some new sporadic simple groups (Co_1, Co_2, Co_3) and re-discoveries of then-recently discovered sporadic groups (those of Hall-Janko, Higman-Sims, McLaughlin, Suzuki). Built into the Leech lattice theory were the five Mathieu groups, discovered in the 19th century. A single context captured twelve sporadic simple groups! His results led to a tremendous increase in his status as a mathematician, and he soon received many lecture invitations, including one to the 1970 International Congress of Mathematicians in Nice.

* Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043 USA
E-mail: rlg@umich.edu

Conway's office in DPMMS was near the lounge, which was full of puzzles and scratch paper. He was keen to discuss games and mathematics with whomever was available to listen. His Game of Life had attracted worldwide interest. His quick wit was on display in frequent conversations in the lounge and throughout the building.

Around that time and later, Conway was attracted to "special phenomena" which he observed in particular groups and tried to generalize. Some of the things he looked into seemed like offbeat momentary interests, but they were an expression of his broad curiosity and search for mathematical substance. One idea he had in the late 1960s involved what he called *trine groups*, groups generated by a conjugacy class of subgroups of order 3 so that any two generate a subgroup on a given short list. In his case, that list was something like $\mathbb{Z}_3 \times \mathbb{Z}_3, Alt_4, SL(2,3)$ and maybe also Alt_5 or $SL(2,5)$. He collaborated with the computational mathematician John McKay. A partial list of trine groups was obtained; many were subgroups of the big Conway group, the isometry group of the Leech lattice. Such a short-lived and exploratory project could now be viewed in the context of the later theories of Fischer, Timmesfeld, Aschbacher, et al. about finite groups generated by a conjugacy class of elementary abelian subgroups so that any two generate a group on a restricted list.

Conway was aware of ongoing developments in the classification of finite simple groups and was ready to jump in if something caught his interest. Sometime in the spring of 1973, he and David Wales created a computer construction of the Rudvalis group, about a year after hearing evidence that such a group might exist. They built a 28-dimensional representation of the double cover of the Rudvalis group, written over the scalars $\mathbb{Z}[\sqrt{-1}]$ [5]. Conway was, however, not part of the team working systematically to classify finite simple groups.

In November 1973, Bernd Fischer and I independently found evidence for a very large sporadic group that later came to be known as the Monster. In brief, existence was proved [8] and uniqueness was proved later [11]. I'll say more on this later. Around the early to mid-1970s, there was an ongoing stream of conjectures about new finite simple groups that might exist, though at the time there was no clear idea as to how many, and when genuine finite simple groups would be generated by these searches.

In the late 1960s, Simon Norton, a young math genius, arrived at at Cambridge University. Around the early 1970s, Conway recruited Norton to work on finite simple groups. In the late 70s, John Conway and Simon Norton developed *Monstrous Moonshine* [4]. It involved a putative sporadic simple group, the Monster. (At this time, it was unclear as to whether existence of the Monster would ever be settled because of its size, about 8×10^{53}.) Monstrous Moonshine theory was inspired by two ideas. The first was John McKay's surprising observation that 196884 (the first nontrivial coefficient of the elliptic modular function $j(z)$, for z in the upper half complex plane) equals 1+196883. In more detail, $j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots$, where $q = e^{2\pi iz}$. The number 196883 was expected to be the smallest degree of a nontrivial irreducible representation of the Monster. Secondly, John Thompson [13] proposed that there could be a graded space $V = \bigoplus_{n \geq -1} V_n$, where V_n is a finite dimensional module for the Monster, so that the formal series $\sum_{n \geq -1} dim(V_n)q^n$ equals $j(z) - 744$ and that the series $\sum_{n \geq -1} tr(g\,|_{V_n})q^n$ could be interesting for all g in the Monster. They were indeed.

The theory of Conway and Norton was conceived just as two feelings were developing within the finite group theory community during the late 1970s. The first sentiment held that the classification of finite simple groups *might* be within reach since an end game had been envisioned [6]. The second was that finite simple group theory had, for years, been getting increasingly deep and thereby (unintentionally) distancing itself from the rest of mathematics. Conway and Norton's Monstrous Moonshine theory asserted extensive and dramatic connections between a putative Monster and the theory of modular forms. Roughly speaking, it was based on their discovery of a near-bijective correspondence between conjugacy classes of the Monster and the known set of genus 0 function fields on the upper half complex plane. The series indicated in the previous paragraph played an important role. Suddenly, there were rich possibilities for studying connections between two important areas of mathematics. Mathematicians from outside finite group theory took interest. A few years later, the Monster was proved to exist (see below) [8]. Work on Monstrous Moonshine and related matters continues to this day.

In early 1980s, I announced construction of the Monster, a very large sporadic simple group, of order $2^{46}3^{20}5^{9}7^{6}11^{2}13^{3}17{\cdot}19{\cdot}23{\cdot}39{\cdot}31{\cdot}41{\cdot}47{\cdot}59{\cdot}71$, about 8×10^{53} [8]. Its existence proof was one of the final steps in completing the classification of finite simple groups. Actually, I constructed both a commutative, nonassociative algebra of dimension $196883 = 47{\cdot}59{\cdot}71$ and a group of algebra automorphisms. The idea to use such an algebra was inspired by Norton's study of invariants of the Monster on a putative irreducible module R of dimension 196883. Norton conjectured that R was self-dual and had an invariant symmetric tensor of degree 3. This means that R would have the structure of a commutative algebra for which the Monster would act as algebra automorphisms. Conway and Norton dialogued regularly about such an algebra and similar ones associated with smaller finite groups. To the best of my knowledge, no useful axioms for such algebras were ever made firm.

I studied many aspects of a possible algebra structure on a 196883 dimensional irreducible module. Defining the algebra structure on a vector space of dimension 196883 and suitable generators for a finite group with the right properties was quite hard. Given such an algebra structure on this irreducible module, there was a multi-parameter family of ways to define an algebra of dimension 196884 with identity and the same automorphism group. Eventually, a fairly natural choice arose. It involved promoting a 299-dimensional subalgebra of the original algebra to a 300-dimensional algebra isomorphic to the Jordan algebra of 24×24 symmetric matrices.

Many people examined my proof. Jacques Tits made many improvements and studies, both on the 196883-dimensional algebra and a determination of the automorphism group. (See for example, [14, 15] and other articles.) In 1984, Conway offered a more efficient construction of the Monster and an associated 196884-dimensional algebra [3]. He used the Parker loop, which is a Moufang loop (a kind of nonassociative group) built from the binary Golay code of dimension 12 over the field of integers mod 2. It has 2^{13} elements. The loop enabled Conway to write down compact formulas for the algebra multiplication and show existence of an "extra automorphism," a point that was particularly difficult in [8]. His formulas were compatible with my original formulas. Such a loop was not known to exist until the early 1980s, several years after the original construction of the Monster [7, 8]. For background on related loops, see [9, 10]. There is now a shorter existence proof of the Monster with relatively little computation using vertex algebra theory, which was done by Ching-Hung

Lam and myself [12]. Uniqueness of the monster was finally proved in [11]. The symbol $\mathbb{M}$ is now common notation for the Monster.

Conway was a popular lecturer in introductory math courses, in talks to math educators (such as the presentation he made at Canadian Math Society meeting in Victoria, Vancouver Island, about 1991), and to students at all levels. He regularly gave vivid demonstrations with physical objects (pink rubber gloves, outsized playing cards, wooden puzzles) and liked to engage audience members. A few years ago, at his University of Michigan undergraduate colloquium, he asked audience members to the front of the room to move ribbons in braid-group-style, over and under neighboring ribbons. He loved sharing the pleasures of mathematics.

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References

[1] MR0237634 Conway, J. H. A perfect group of order 8,315,553,613,086,720,000 and the sporadic simple groups. Proc. Nat. Acad. Sci. U.S.A. 61 (1968), 398–400. (Reviewer: J. E. McLaughlin) 20.27

[2] MR0248216 Conway, J. H. A group of order 8,315,553,613,086,720,000. Bull. London Math. Soc. 1 (1969), 79–88. (Reviewer: W. J. Wong) 20.29

[3] MR0782233 Conway, J. H. A simple construction for the Fischer-Griess monster group. Invent. Math. 79 (1985), no. 3, 513–540. (Reviewer: Geoffrey Mason) 20D08

[4] MR0554399 Conway, J. H.; Norton, S. P. Monstrous moonshine. Bull. London Math. Soc. 11 (1979), no. 3, 308–339. (Reviewer: Stephen D. Smith) 20D08 (10D12)

[5] MR0335620 Conway, J. H.; Wales, D. B. Construction of the Rudvalis group of order 145,926,144,000. J. Algebra 27 (1973), 538–548. (Reviewer: Henry S. Leonard Jr.) 20D05

[6] MR0690900 (84g:20025) Gorenstein, D.; Lyons, R. The local structure of finite groups of characteristic 2 type. Mem. Amer. Math. Soc. 42 (1983), no. 276, vii+731 pp. 20D05

[7] MR0605419 Griess, Robert L., Jr. A construction of F_1 as automorphisms of a 196,883-dimensional algebra. Proc. Nat. Acad. Sci. U.S.A. 78 (1981), no. 2, part 1, 686–691. (Reviewer: Stephen D. Smith) 20D08 (17D99)

[8] MR0671653 Griess, Robert L., Jr. The friendly giant. Invent. Math. 69 (1982), no. 1, 1–102. (Reviewer: George Glauberman) 20D08

[9] MR0839580 Griess, Robert L., Jr. Code loops. J. Algebra 100 (1986), no. 1, 224–234. (Reviewer: Karl H. Robinson) 20N05 (94B05)

[10] MR0885104 Griess, Robert L., Jr. Sporadic groups, code loops and nonvanishing cohomology. Proceedings of the Northwestern conference on cohomology of groups (Evanston, Ill., 1985). J. Pure Appl. Algebra 44 (1987), no. 1–3, 191–214. (Reviewer: David Benson) 20D08 (20J06 20N05)

[11] MR1025167 (90j:20030) Griess, Robert L., Jr. (1-MI); Meierfrankenfeld, Ulrich (1-MIS); Segev, Yoav (IL-BGUN) A uniqueness proof for the Monster. Ann. of Math. (2) 130 (1989), no. 3, 567–602. (Reviewer: Gernot Stroth) 20D08

[12] MR2975262 Griess, Robert L., Jr.; Lam, Ching Hung A new existence proof of the monster by VOA theory. Michigan Math. J. 61 (2012), no. 3, 555–573. (Reviewer: Ozren Perše) 20D08 (17B69)

[13] MR0554402 Thompson, J. G. Some numerology between the Fischer-Griess Monster and the elliptic modular function. Bull. London Math. Soc. 11 (1979), no. 3, 352–353. (Reviewer: Stephen D. Smith) 20D08 (10D12)

[14] MR0768989 Tits, J. On R. Griess' "friendly giant". Invent. Math. 78 (1984), no. 3, 491–499. (Reviewer: Stephen D. Smith) 20D08

[15] MR0768956 Tits, J. Le Monstre (d'après R. Griess, B. Fischer et al.). (French) [The Monster (following R. Griess, B. Fischer et al.)] Seminar Bourbaki, Vol. 1983/84. Astérisque No. 121–122 (1985), 105–122. (Reviewer: P. Fong) 20D08