My Life and Times with the Sporadic Simple Groups*

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Abstract.
Five sporadic simple groups were proposed in the 19th century and 21 additional ones arose during the period 1965–1975. Since the early 1950s, there has been much thought about the nature of finite simple groups and how sporadic groups are placed in mathematics. While in mathematics graduate school at The University of Chicago, I became fascinated with the unfolding story of sporadic simple groups. It involved multiple theories, detective work and experiments. In this article, I shall describe some of the people, important ideas and evolution of thinking about sporadic simple groups. Most should be accessible to a general mathematical audience.

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1. Introduction

I shall discuss how our discoveries of the 26 sporadic simple groups evolved, with emphasis on what I myself experienced or heard from witnesses. Writing this account in the first person seems natural. My passion about finite group theory began to grow in the late 1960s.

The five Mathieu groups were not part of natural infinite families, like the alternating groups, or classical matrix groups like $PSL(n,q)$. The earliest use of the term "sporadic group" may be the second edition (1911) of Burnside's book [27] (note N) where he comments about the Mathieu groups:

> *"These apparently sporadic simple groups would probably repay a closer examination than they have yet received".*

The first edition of Burnside's book (1897) [26] and Dickson's Linear Groups (1901) [55] both mention the Mathieu groups, but do not use the term "sporadic group".

It is worth mentioning that Burnside felt that groups of odd order were likely to be solvable [27] (note M). This was confirmed by the celebrated theorem of Walter Feit and John Thompson in 1963 [70]. A consequence is that a nonabelian finite simple group has order divisible by 2 and so contains involutions (an *involution* is an element of order 2).

The term *sporadic simple group* has come to mean a nonabelian finite simple group which is not a group of Lie type or an alternating group. The term *sporadic group* usually means a sporadic simple group but in practice it has meant a group G which contains a normal *quasisimple subgroup* Q (meaning, Q is perfect and $Q/Z(Q)$ is a finite simple group) so that $Q/Z(Q)$ is sporadic and $C_G(Q) = Z(Q)$. Such a G is an upwards extension of Q by a subgroup of the outer automorphism group of Q.

It is a consequence of the CFSG (classification of finite simple groups) that there are no sporadic groups other than the 26 known by the early 1980s. A list will be described soon.

At the beginning of the CFSG era, which started in the early 1950s, the meaning of sporadic was less precise. Then, it seemed *a priori* possible that, besides the alternating groups and groups of Lie type, there might be an infinite series of previously unknown simple groups, which *could* have been called sporadic. In the 1960s, researchers entertained pleasant fantasies about finding infinitely many new finite simple groups. I remember speculations about series of graphs or integral lattices with big automorphism groups.

In the group theory community, "discovery" of a finite simple group meant finding strong evidence for its existence. Proof of existence usually came later, frequently done by someone other than the discoverer. Standards were fairly high. When the term discovery was used publicly by an established group theorist, there was not, to my knowledge, an eventual proof of nonexistence. More cautious terms like "possible discovery" were used for preliminary studies, some of which did lead to a contradiction. The term "putative simple group" seemed appropriate for a serious candidate not yet proven to exist. Peter Neumann's use of this term around early 1974 is my earliest recollection. John Conway used it in his 1970 ICM lecture [41]. Walter Feit's extensive 1970 ICM survey [71] presents his view of finite simple group theory at that time, using the term "potential" simple group (in reference to the Lyons group) but not the term "putative".

Discoveries of previously unknown simple groups were strongly connected to the ongoing classification of the finite simple groups. The program to classify finite simple groups came to life in the early 1950s. Its resolution was announced around the early 1980s, I believe, at the American Mathematical Society annual meeting during January, 1981, in San Francisco. At a finite group theory special session in this meeting, Daniel Gorenstein made the announcement that the CFSG was essentially complete. Ron Solomon confirms my memory about this. Speakers at special sessions customarily provide an abstract in advance to the Notices of the American Mathematical Society. For the Gorenstein announcement, there was no abstract provided and I am not aware of any written record of the event.

The announcement was optimistic. Unresolved issues arose, including a substantial one. They were identified and eventually dealt with. It is now generally believed that the classification is settled and that the list we shall display later is complete.

One could say that most effort in the CFSG program was directed towards achieving an upper bound on the possibilities for finite simple groups. Those who sought new groups and tried to construct them worked to achieve a lower bound. These two bounds met eventually. Some people belonged in both camps.

Things I learned from working on CFSG helped me find my way in the world of sporadic groups. This article is focused on the story of sporadic groups and is not an attempt to survey the CFSG. I will give certain anecdotes about discoveries, constructions and uniqueness proofs of sporadic groups. Apologies to the many researchers on CFSG, finite geometries and representations whose work will not be mentioned. The interested reader may consult [216, 88, 9, 197] and other surveys.

See Appendices for list of finite group terminology and list of the finite simple groups.

In Memoriam. Players in the sporadic group story who have passed: John Conway, Bernd Fischer, Marshall Hall, Donald Higman, Graham Higman, Jeffrey Leon, Émile Mathieu, Jack McLaughlin, Simon Norton, Mike O'Nan, Charles Sims, Michio Suzuki, Ernst Witt. I mention the related historic articles about Donald Higman [11] and John Conway [120].

2. Main Themes Which Developed for the Sporadic Groups

Most finite simple groups are of Lie type (analogues of Lie groups over finite fields, and variations) and can be treated uniformly by Lie theory. Definitions for symmetric groups and alternating groups are easy to understand. The sporadic groups are not easily described. Listed below are several themes which are relevant for discovering or describing most of the sporadic groups.

For certain sporadic groups, more than one category applies.

External themes:

Multiply transitive and rank 3 permutation representations (rank 3: transitive permutation representation for which point stabilizer has just three orbits).

Isometries of lattices in Euclidean space (Leech lattice and related lattices).

Automorphisms of commutative nonassociative algebras.

Internal themes:

ω-transposition groups (definition given below).

Pure group theoretic characterizations (such as characterization by centralizer of involution, transitivity on flags of elementary abelian 2-subgroups).

Recall that in any group, two involutions generate a dihedral group.

Definition (Bernd Fischer)**.** Let ω be a subset of $\{3,4,5,6,\dots\}$. Given a finite group G, a subset D of involutions is a class of ω-transpositions in G if D is a conjugacy class in G and whenever $x,y \in D$ and $xy \neq yx$, then the subgroup generated by x,y is dihedral of order $2n$ for some $n \in \omega$ (so n is the order of xy). When this is the case and G is generated by D, we say that *G is an ω-transposition group for the subset D.*

This class of groups will be discussed later. Sometimes, one sees a variation of this condition which allows D to be a union of conjugacy classes.

3. The List of FSG, Changing Over Time

Now, shall present *the lists, at several moments in history, of known or putative finite simple groups,* including counts for sporadic groups known up to those moments. For some groups, existence and uniqueness proofs came much later. To keep exposition in this section efficient, the lists are sketchy. See the Appendix of Notations and FSG orders for more details including group orders and discussions of nonsimplicity and multiple occurrences of isomorphism types.

The date given for sporadic groups is year of discovery, as well as I remember. Publication dates are found in the reference section.

3.1 1910 View of FSG

Cyclic groups of prime order;

Alternating groups of degree at least 5;

Some classical groups over finite fields (general linear, unitary, orthogonal, symplectic), done by Galois and Jordan; the nonabelian simple group associated to a classical group is the quotient of its commutator subgroup by the center (with a few exceptions in small dimensions);

Groups of type G_2, E_6 (and possibly F_4) by L. E. Dickson (around 1901 or so), at least in odd characteristic;

Proposals by Émile Mathieu of the Mathieu groups: $M_{11}, M_{12}, M_{22}, M_{23}, M_{24}$ [174, 175, 176] (1861–1873); Their existence was first made rigorous in 1937 by Ernst Witt [243, 244];

[sporadic count=5].

3.2 1959 View of FSG

Cyclic groups of prime order;

Alternating groups of degree at least 5;

LIE TYPE:

1955 Chevalley groups over finite fields

(types $A_n(q), B_n(q), C_n(q), D_n(q), E_{6,7,8}(q), F_4(q), G_2(q)$, $q =$ prime power; some restrictions on n, q) [28]; includes classical groups: for example, $A_n(q) \cong PSL(n+1, q)$;

LIE TYPE:

1959 Steinberg variations of Chevalley groups, associated to graph times field automorphisms (types ${}^2A_n(q), {}^2D_n(q), {}^3D_4(q), {}^2E_6(q)$, $q =$ prime power; some restrictions on n, q); includes classical groups involving field automorphism: for example, ${}^2A_n(q) \cong PSU(n+1, q)$;

The groups of Galois, Jordan and Dickson are included in the Chevalley-Steinberg series;

Mathieu groups;

[sporadic count=5].

3.3 1962 View of FSG

Same as in 1959 plus these:

LIE TYPE:

1960 The *series of Suzuki groups* $Sz(q), q =$ *odd power of 2*, $q \geq 8$, found by pure group theoretic internal characterization [221]; these *might* have been considered "sporadic" but in 1961 were shown by Takashi Ono to be of Lie type, ${}^2B_2(q)$, associated to an isomorphism of the group $B_2(q)$ but which does not come from a morphism of the Lie algebra [191, 192].

LIE TYPE:

1961 The *two series of groups defined by Ree,* ${}^2G_2(q)$, $q \geq 27$, $q =$ *odd power of 3; and* ${}^2F_4(q)'$, *for* q *an odd power of 2*. The group ${}^2F_4(2)$ is nonsimple; its derived group ${}^2F_4(2)'$, called the Tits group [232], has index 2 and is simple [232];

[sporadic count=5]

3.4 1970 View of FSG: The Deluge, Part 1

Same as in 1962 plus these:

NEW SPORADIC:

1965 Janko group J_1;

1967 Hall-Janko group ($HJ = J_2$); Janko group J_3; Higman-Sims group; McLaughlin group; Suzuki group;

1968 Conway's Co_1, Co_2, Co_3; Held; Fischer's simple groups $Fi_{22}, Fi_{23}, Fi_{24}'$ (the first two are 3-transposition groups; the third is the commutator subgroup of the 3-transposition group Fi_{24}).

1969 Lyons group;

[sporadic count=5+14=19].

3.5 Early 1970s Blues

New genuine and putative sporadic groups were fun to examine. There were no announcements about discoveries during 1970 and 1971. A mild depression spread within the finite group community.

3.6 1973 View of FSG: The Deluge, Part 2

Same view as in 1970 with these additions:

NEW SPORADIC:

Springtime 1972: Ru (Rudvalis rank 3 group) order $2^{14}3^3 5^3 7 \cdot 13 \cdot 29$;

before mid-May 1973: $O'N$ (the O'Nan group, of order $2^9 3^4 5 \cdot 7^3 19 \cdot 31$);

summer 1973: F_2 (Fischer's Baby Monster; a $\{3,4\}$-transposition group);

November 1973: $\mathbb{M} = F_1$ (Monster, discovered independently by Fischer and Griess; a $\{3,4,5,6\}$-transposition group);

F_3 (Thompson);

F_5 (Harada-Norton);

[sporadic group count=25].

3.7 1975 View of FSG

Same as 1973, plus

NEW SPORADIC:

May, 1975: Janko's fourth group J_4, of order $2^{21}3^35{\cdot}11^3{\cdot}23{\cdot}29{\cdot}31{\cdot}37{\cdot}43$, found by centralizer of involution characterization for $2^{1+12}3M_{22}2$.

[sporadic group count=26].

3.8 The Final List of FSG (After Decades of CFSG)

The final list here is given in compact form. See Appendices for more detailed list of notations, group orders and duplicate listings.

(a) cyclic groups of prime order;

(b) the alternating groups (even permutations on a set of n symbols, $n \geq 5$);

(c) groups of Lie type over finite fields (17 families): Chevalley groups $A_n(q) \cong PSL(n+1,q)$, $B_n(q) \cong PSO(2n+1,q),\ldots,E_8(q)$; Steinberg, Suzuki and Ree variations: ${}^2A_n(q) \cong PSU(n+1,q),\ldots{}^2F_4(2^{2m+1})'$;

(d) 26 sporadic groups = 5 groups of Mathieu from 1860s, plus 21 others, discovered during period 1965–1975.

[sporadic count = 26].

Of the 26 sporadic groups, 20 are subquotients of the Monster, the largest sporadic. These twenty groups form *The Happy Family*. The set of six remaining groups are called *The Pariahs*. There is no single, simply stated theme which explains or describes the sporadic groups in a useful or efficient manner. The broadest theme so far is membership in the Happy Family.

4. Fuzzy Boundary Between Sporadic Simple Groups and the Others

The Lie theoretic viewpoint describes most of the finite simple groups. The symmetric groups could be viewed as general linear groups over the "field of one element", an idea introduced by Jacques Tits [230] (for background, see [241]). The sporadic groups are the outsiders. However, other viewpoints for finite groups do not particularly exclude the sporadic groups.

(1) We see an easily checked 3-transposition condition for transpositions in symmetric groups. The classification of 3-transposition groups with trivial solvable normal subgroups includes familiar groups but also three previously unknown sporadic groups, Fi_{22}, Fi_{23} and Fi_{24}. *What resulted from such a simple hypothesis is amazing.*

(2) The finite real and complex reflection groups were classified a long time ago. Their composition factors involve only cyclic groups, alternating groups and certain classical matrix groups over the fields of 2 and 3 elements.

The finite quaternionic reflection groups were classified in the 1970s by Arjeh Cohen [36, 37, 38]. Their composition factors involve only cyclic groups, alternating groups, a few classic matrix groups of small dimension over small fields, *and the sporadic group of Hall-Janko.*

(3) Timmesfeld's classification [229] of $\{4,odd\}^+$ transposition groups with no normal solvable subgroups gave most groups of Lie type in characteristic 2 (the groups ${}^2F_4(q)$ do not occur here), all of the 3-transposition groups of Fischer, *plus the sporadic Hall-Janko group* (which is not a 3-transposition group). The group HJ embeds into the group $G_2(4)$ and its $\{4,odd\}^+$-transpositions are contained in those of $G_2(4)$. So, HJ is close to being a group of Lie type in characteristic 2. There is a beautiful description of subgroups K of $G_2(4)$ which are isomorphic to HJ and which contain the derived group of the natural $G_2(2)$ subgroup ($G_2(2)' \cong PSU(3,3)$) [236]. Such K do not contain the natural $G_2(2)$ subgroup.

(4) Also the $\{3,4,5,6\}$-transposition property of the $2A$-involutions in $\mathbb{M}$ (the Monster) feels like a property of Weyl groups in Lie algebra theory, particularly because of the theory of Miyamoto involutions [182] for vertex operator algebras and Sakuma's theorem [202]. See [45] for discussions of Y-diagrams for sets of $2A$ elements in the Monster. Many finite groups which come up in vertex algebra theory are generated by Miyamoto involutions; see later section *Lattices, vertex algebras and applications* and references therein.

(5) The groups of Lie type have well-known geometries based on their parabolic subgroups. Some actions of sporadic groups on local subgroups (normalizers of p-groups, for a prime p) have features in common with the latter actions, including diagrams which are analogues of Dynkin diagrams. See the article of Mark Ronan and Stephen Smith [198]. The interesting example for the Monster gives an E_8-diagram. These authors call their study of geometries for sporadic groups "2-local geometry" or "parabology". Geoffrey Mason and Stephen Smith [173] extended this analysis to the groups of Held and Rudvalis.

If G is a group of Lie type and P is a parabolic subgroup with unipotent radical U, then P is a split extension $P = UL$, due to a so-called Levi subgroup L. If M is a finite dimensional highest weight module, then the subspace of vectors fixed by U is P-invariant and is irreducible as a module for $P/U \cong L$. Steve Smith noted analogous behavior for some 2-local subgroups in sporadic groups [214] and observed that the analogues of parabolic subgroups are often not split For some constructions of big p-locals in sporadic and other finite groups see the methods of [42, 104, 106, 105, 111, 109].

(6) There are occasional references to "the 27th sporadic group", meaning the Tits group ${}^2F_4(2)'$ [232]. While ${}^2F_4(2)$ is a standard group of Lie type (meaning from the series of Chevalley, Steinberg or Ree), its commutator subgroup ${}^2F_4(2)'$ is not. Compare ${}^2G_2(3)$, whose commutator subgroup has index 3 and is isomorphic to $PSL(2,8)$; this is a genuine group of Lie type, though not from a series of characteristic 3. Compare also $B_2(2) \cong Sym_6$ and its commutator subgroup $B_2(2)' \cong Alt_6 \cong A_1(9) \cong PSL(2,9)$.

5. Beginnings of CFSG and Sporadic Encounters

The CFSG starts in early 1950s and, as a consequence, encourages a search for more finite simple groups.

In any group, two involutions generate a dihedral group. The following theorem [18] extends a thesis result of Kenneth Fowler at The University of Michigan [76], under the direction of Richard Brauer.

Theorem 5.1 (Brauer-Fowler). *There exists a function $f : \mathbb{N} \to \mathbb{R}$ so that if G is a finite simple group (of even order) and $t \in G$ is an involution, then $|G| \le f(|C_G(t)|)$.*

In other words, if H is a finite group then, up to isomorphism, only finitely many finite simple groups have an involution whose centralizer is isomorphic to H. (Usually, that number is zero.)

The function f is extravagant, of no practical value. However, *the psychological impact of limiting isomorphism types of a finite simple group by a centralizer of involution was powerful.*

In Brauer's talk at the International Congress of Mathematicians, in Amsterdam (1954) [18], he gave an early theorem along this line. We first give some notation.

Let q be an odd prime power and let H be the centralizer of an involution $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ (mod scalars) in the simple group $PSL(3,q)$. So, H is the group of all matrices over the field of q elements of the form $\begin{pmatrix} c & \\ & A \end{pmatrix}$ (mod scalars in $SL(3,q)$) where A is an invertible 2×2 matrix and $c \cdot det(A) = 1$. So, for all odd q, H is isomorphic to $GL(2,q)/Z$, where Z is the central subgroup of order $(3, q-1)$.

Theorem 5.2. *Assume that (i) G is a finite simple group with involution u so that $C_G(u) \cong GL(2,q)/Z$;*

(ii) for $x \ne 1$ in $Z(C_G(u))$, $C_G(x) = C_G(u)$;

Then $G \cong PSL(3,q)$ or $q = 3$ and $G \cong M_{11}$, the Mathieu group of order $7920 = 2^4 3^2 5 \cdot 11$.

Hypothesis (ii) was eventually removed. For details, see [19, 245].

Note that we get the expected answers $PSL(3,q)$ but in the proof there is a case which leads to a sporadic group. *If you had never met M_{11} before, you would meet it this way.*

Brauer's strategy applied "only" to simple groups of even order. In 1963, Feit and Thompson proved that all finite groups of odd order are solvable [70]. After their theorem, it was clear that Brauer's viewpoint was more significant.

Theorem 5.3 (Janko-Thompson [161]). *Let $q \ge 5$ be a prime power and G be a finite simple group with abelian Sylow 2-subgroups and an involution t so that $C_G(t) \cong 2 \times PSL(2,q)$. Then q is an odd power of 3 with $q \ge 27$ or $q = 5$.*

The case $q = 5$ had been mistakenly eliminated due to an error in a character table for $PSL(2,11)$. The error was found later by Janko, who obtained the following result.

Theorem 5.4 (Janko [155]). *Let G be a finite simple group with abelian Sylow 2-subgroups and an involution t so that $C_G(t) \cong 2 \times PSL(2,5)$. Then G has order $175560 = 2^3 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$. Furthermore, such a G exists and is unique up to isomorphism.*

Assuming that such a G exists, Janko gave two matrices A, B in $GL(7,11)$, which would generate such a simple group (and prove uniqueness). In [155] M. A. Ward gave a nice proof that A and B do generate a simple group of the right order and W. A. Coppel gave a nice proof that this subgroup lies in a $G_2(11)$-subgroup of $GL(7,11)$ [155].

The group J_1 is another example of how a sporadic group comes up as a special case in a classification result.

The discovery, existence proof and uniqueness proof for J_1 were done by hand and the articles give full details, assuming background in basic finite group theory and representation theory. This sense of a "self-contained" treatment for a new sporadic group was not to last. As the CFSG expanded, there were dramatically increasing demands for broad specialized knowledge, starting around the mid 1960s.

5.1 Reaction to J_1, the First New Sporadic Simple Group in a Century

There was a lot of discussion about what the appearance of J_1 could mean. The Suzuki series discovered in 1960 turned out to be groups of Lie type. Were there more sporadics waiting to be discovered?

(1) Think about the order of $GL(n,q)$,

$$q^{\binom{n}{2}}(q^n-1)(q^{n-1}-1)\cdots(q-1)$$

There are similar polynomial expressions for orders of groups of Lie type over finite fields of q elements. Maybe the new group J_1, of order [155, 156]

$$175560 = 11{\cdot}12{\cdot}1330 = 11(11+1)(11^3-1)$$

is part of a series of groups of order $q(q+1)(q^3-1)$ where q is a prime power (or maybe a power of 11). This was a nice idea, but it did not lead anywhere.

(2) Consider these amusing factorizations (see [155, 156] and [233], page 188):

$$175560 = 2^3 3{\cdot}5{\cdot}7{\cdot}11{\cdot}19 = 19{\cdot}20{\cdot}21{\cdot}22 = 55{\cdot}56{\cdot}57$$

As far as I know, no one has done anything special with this. Noam Elkies points out that 175560 is the largest integer which is both the product of three consecutive integers and the product of four consecutive integers [69].

(3) J_1 contains a subgroup isomorphic to $PSL(2,11)$ of index $266 = 2\cdot 133$. Since 133 is the dimension of the E_7 Lie algebra, one might wonder if something is going on with the exceptional Lie group E_7. The group J_1 has a 7-dimensional representation over $\mathbb{F}_{11}$ which embeds J_1 in $G_2(11)$! There is a containment of exceptional algebraic groups $G_2 \le F_4 \le E_6 \le E_7 \le E_8$, hence a connection between J_1 and E_7, though it is somewhat distant.

6. Centralizer of Involution Examples

The strategy of characterization by centralizer of involution was pursued, refined and replaced as the years went by. One can not hope to try all finite groups as centralizer of involution candidates to finish CFSG. Still, it is remarkable that many small groups occurred as centralizers of involutions in previously unknown finite simple groups.

6.1 The $C = 2^{1+4}{:}Alt_5$ Dichotomy and HJ and J_3

I am not sure why Janko chose this candidate for the centralizer of involution in a simple group, but it was fortunate. Ronald Solomon points out similarity to involution centralizers in the groups

$$M_{12}, G_2(3), PSp(4,3), PSL(4,3), PSU(4,3).$$

The group $C = 2^{1+4}{:}Alt_5$ has three conjugacy classes of involutions. Take involutions z in the center, w in $O_2(C)\setminus\{z\}$ and $t \in C\setminus O_2(C)$. They represent the three conjugacy classes. The Glauberman Z^*-theorem [83] says that z must be conjugate to one of w,t. The case where z is conjugate to just one of w,t leads to the group HJ (and z is conjugate to w, in fact). The case where z is conjugate to both w and t leads to the group J_3 of order $50232960 = 2^7 3^5{\cdot}5{\cdot}17{\cdot}19$.

6.2 The Double Covers $2{\cdot}Alt_n$ and the Groups of McLaughlin and Lyons

The sporadic group of McLaughlin has one conjugacy class of involutions with centralizer of shape $2{\cdot}Alt_8$, the double cover of Alt_8, described by Schur [203].

Richard Lyons, while a graduate student at The University of Chicago, produced strong evidence that there is a finite simple group with an involution centralizer isomorphic to $2{\cdot}Alt_{11}$ [172]. Such a group has one conjugacy class of involutions and order $2^8 3^7 5^6 7{\cdot}11{\cdot}31{\cdot}37{\cdot}67$. The Lyons group was proven to exist by Charles Sims [210, 139].

Short arguments with the Z^*-theorem [83] show that the groups $2{\cdot}Alt_n$ are not centralizers of involutions in finite simple groups if $n \ge 12$ or $n \le 7$. For $n \le 7$, one may also use the Brauer-Suzuki Theorem [21], which says that a finite simple group does not have a quaternion Sylow 2-group. The cases $n = 9,10$ were eliminated by Janko and Lyons, respectively [158, 171].

6.3 Exceptional Central Extensions of Finite Groups of Lie Type

With finitely many exceptions, finite groups of Lie type over a field of characteristic p have Schur multiplier of order relatively prime to p [218, 219, 97, 98, 100]. Certain associated exceptional central extensions appeared as normal subgroups in normalizers of small p-groups within sporadic groups. Some examples: (a) $2{\cdot}PSU(6,2)$ in Fi_{22}; (b) $3^2{\cdot}PSU(4,3)$ in Co_1; (c) $2{\cdot}F_4(2)$ in the Monster; (d) $2^2{\cdot}PSL(3,4)$ in the Held group; (e) $4{\cdot}PSL(3,4).2$ in the O'Nan group; (f) $2{\cdot}Sp(6,2)$ in Co_3.

I do not recall a case of a sporadic group being *discovered* by centralizer of involution procedure starting from an exceptional central extension of a group of Lie type (or such extended upwards by outer automorphisms).

For connections between exceptionally nonvanishing cohomology and sporadic groups, see [106].

6.4 The $2_+^{1+6}{:}GL(3,2)$ Trichotomy, $PSL(5,2)$, M_{24} and the Held Sporadic Group

Dieter Held [140] studied the group $2_+^{1+6}{:}GL(3,2)$, which is a centralizer of involution in both $GL(5,2)$ and M_{24} and found evidence for a third finite simple group, where it is also a centralizer. Such a latter group would have order $4030387200 = 2^{10}3^3 5^2 7^3 17$ and would become known as the Held group.

From CFSG, we know that, given a particular group H, the number of finite simple groups, up to isomorphism, having H as centralizer of involution is at most 3. Only $H = 2_+^{1+6}{:}GL(3,2)$ achieves the upper bound of 3. Several groups occur as centralizer twice.

The pair of simple groups $PSL(2,7), Alt_6 \cong PSL(2,9)$ each have one conjugacy class of involutions and common involution centralizer Dih_8. The pair of simple groups HJ, J_3 have involutions with common centralizer $2_-^{1+4}{:}Alt_5$.

Dieter Held writes:

> "Dear Professor Griess,
>
> Thank you very much for your email. [In] 1968 I had been considering a variety of problems which included the groups $PSL(5,2)$ and M_{24} (Mathieu-group on 24 letters) as 'small' cases. Thus, I started with a characterization of these two simple groups by the centralizer of a 2-central involution. For a successful start, I assumed that if H is the centralizer of an involution in the center of a Sylow-2 subgroup of the group G to be determined, then H is also the full normalizer in G of each of the two elementary abelian subgroups E and E_1. I knew that this is the case neither in M_{24} nor in $PSL(5,2)$. Thus, I could hope to arrive soon at a contradiction while investigating G locally. But I did not produce a contradiction in this manner. On the contrary, I was able to decide which involutions of H were G-conjugate and which were not. I found that G has precisely two classes of involutions and I determined the structure of the centralizer of an involution not contained in the center of a Sylow-2 subgroup of G. At that point I wanted to determine the order of G. The idea was to make use of the Suzuki-Order-Formula; see Janko's characterization of the smallest group of Ree associated with the simple Lie Algebra of type (G_2), Journal of Algebra Vol. 4, No. 2, September 1966, page 295, Lemma of Suzuki. After I had computed the relevant character tables I found the necessary further calculations as too tiresome. By chance I got an invitation to take part at an Oberwolfach-Meeting where I could talk to John Thompson. I gave him an account of the up to now known structure of G: So far no contradiction, precisely two classes of involutions, and the structures of the centralizers of two non-conjugate involutions. It sounded unbelievably to my ears when Thompson said that precisely for such a situation he could give me a rather simple formula – now known as The Thompson Order Formula – for computing the order of G. A few weeks later I computed 4030387200 as the order of G which was a first hint for the existence of a new sporadic simple group. More information gathered supported this. At this state of affairs I felt that it would be best to publish my results as soon as possible as I wanted to use it for habilitation at a german university. At that time I was research fellow of DFG, Deutsche Forschungsgemeinschaft, and I had not been affiliated to any university.
>
> So far the story of the exciting days in 1968. By the way, it had been an unexpected stroke of luck that with John Thompson I met somebody who understood and was interested in what I had been looking for.
>
> I hope that this report is of some historical value to you.
>
> Best regards
>
> Dieter Held"

Such a simple group was first constructed by Graham Higman and John McKay using computer work (unpublished). Existence of such a group follows from existence of the Monster [103]. See also later section *Computer and manual constructions of sporadic groups*.

6.5 Janko's Long Quest

In the early 1970s, Ulrich Dempwolff told me about a weekly seminar at Ohio State University in which Koichiro Harada and Zvonimir Janko were the main speakers. They gave ongoing reports on their current research. The seminar was a good opportunity for young participants to learn about techniques. Janko spoke about cases he considered for centralizers of an involution in a finite simple group. Dempwolff said "In particular he tried a number of centralizers H with $E = O_2(H)$ extraspecial and an irreducible action of H/E on the E/E'". He noted Janko's repeated use of the phrases "large extraspecial groups" and "large amounts of time" during his lectures.

Janko even studied series of centralizer candidates. Jon Alperin told me that Janko considered $q^{1+8}Sp(6,q)$ for q a power of 2. This series generalizes the case of an involution centralizer $2^{1+8}Sp(6,2)$ in the group Co_2. No simple group occurs for such a centralizer when $q > 2$. At a 1972 meeting in Gainesville, Florida, Janko proposed another infinite family of centralizers for possible new simple groups. They had the approximate form $2^{1+2m}PSL(2,2^m)$. Within a few weeks after the conference, a story circulated that these possibilities were eliminated for all but finitely many m. In the proceedings [159], Janko reported on a family of then still-unresolved centralizer of involution problems including the series which created excitement at the conference.

Janko was the most openly energetic explorer of centralizer of involution problems. He found success four times, starting with centralizer candidates which puzzled observers. They wondered whether he had extremely good insight or just an amazing lucky streak. The most exotic-looking centralizer for his groups was $2^{1+12}.3.M_{22}.2$ in the pariah J_4. The context of his successes surely included a great number of trials which led to no new groups but strengthened his instincts.

7. Don Higman's Rank 3 Theory

A *permutation representation* of a group G on a set Ω is a group homomorphism $G \to Sym(\Omega)$. Its *degree* is the cardinality $|\Omega|$.

The *rank* of a transitive permutation representation of the group G on a set Ω is the number of orbits for the natural action on $\Omega \times \Omega$. Equivalently, it is the number of orbits of a point stabilizer $G_a := \{g \in G \mid ga = a\}$ on Ω.

Notation 7.1. *Let the finite group G act on the set Ω transitively with rank 3. For $a \in \Omega$, let the orbits of the point-stabilizer G_a be $\{a\}, \Delta(a), \Gamma(a)$. Assume that if $g \in G$, then $\Delta(g \cdot a) = g \cdot \Delta(a)$ and $\Gamma(g \cdot a) = g \cdot \Gamma(a)$.*

Define $n := |\Omega|$, the degree;

$k := |\Delta(a)|$,

$\ell := |\Gamma(a)|$

$\lambda := |\Delta(a) \cap \Delta(b)|$ *for* $b \in \Delta(a)$,

$\mu := |\Delta(a) \cap \Delta(b)|$ *for* $b \in \Gamma(a)$.

Call k,ℓ the subdegrees and call k,ℓ,λ,μ the rank 3 parameters or the Higman parameters of the rank 3 representation.

Lemma 7.2 ([141]). $\mu\ell = k(k-\lambda-1)$ *(the Higman condition).*

Call a sequence of nonnegative integers k,ℓ,λ,μ a *Higman quadruple* if they satisfy the Higman condition. A Higman quadruple may arise from a rank 3 group, or not. Don Higman kept a list of such quadruples which might be relevant to finite groups.

There are other numerical conditions in [141]; above is all I need now.

The next example may be verified by counting. No specialized group theory is required.

Example 7.3. *Let G be 4-transitive subgroup of Sym_m for $m \geq 4$, $\Omega =$ the set of unordered pairs of distinct integers from $\{1,2,3,\ldots,m\}$. The action of G is transitive. The stabilizer in G of (i,j) has two nontrivial orbits:*

$\Delta((i,j))$ *:= the pairs which contain just one of i, j (cardinality $k = 2(m-2)$);*

$\Gamma((i,j))$*: = the pairs which avoid i, j (cardinality $\ell = \binom{m-2}{2}$).*

The stabilizer of (i,j) in G is transitive on the sets $\Delta((i,j))$ and $\Gamma((i,j))$. So, we have a rank 3 permutation representation on $n = 1+k+\ell = 1+2(m-2)+\frac{(m-2)(m-3)}{2} = \binom{m}{2}$ points. The remaining parameters for the Higman condition are:

$\lambda = m-3+1 = m-2$*;* $\mu = 4$.

The Higman condition $\mu\ell = k(k-\lambda-1)$ here would say $4\binom{m-2}{2}$ equals $2(m-2)(2(m-2)-(m-2)-1) = 2(m-2)(m-3)$, which is true.

Remark 7.4. *To a rank 3 group, there are two naturally associated graphs. Define a graph by connecting distinct points $a,b \in \Omega$ with an edge if and only if $b \in \Delta(a)$. The second graph is defined using the function Γ.*

8. The Groups of Hall-Janko and Higman-Sims

Now we jump to year 1967 and closely related stories about the sporadic groups HJ and HS.

8.1 The Hall-Janko Group

The Hall-Janko group, HJ, which has order $604800 = 2^7 3^3 5^2 7$, was discovered independently by Zvonimir Janko and Marshall Hall, Jr. As explained earlier, Janko started by using $C := 2_-^{1+4}{:}Alt_5$ (split extension) as candidate for the centralizer of an involution in a simple group.

Marshall Hall, Jr. had been pursuing an account of simple groups of order at most one million [130, 131]. Leonard Eugene Dickson [57] listed 53 known finite simple groups of order less than a million. Most numbers less than a million can be eliminated as possible simple group orders with elementary arguments. The number $604800 = 2^7 3^3 5^2{\cdot}7$ resisted elimination and attracted Marshall Hall, Jr. to work out properties of a simple group of this order. His computational methods, with character theory, subgroups and permutations, were intense. He relied on some then-recent fundamental results in CFSG, listed on page 139 of [130]. In particular, he refers to the announced classification of N-groups by John Thompson, which implies the classification of minimal simple groups. The latter was used to restrict composition factors within subgroups of G. The 1965 announcement of Janko's group J_1, of order 175560, added a group to the Dickson list and encouraged Marshall Hall, Jr. to settle whether 604800 was the order of a simple group.

Hall proposed three irreducible characters of degrees 1, 36, 63 such that their sum χ could be a permutation character for G. If so, in an associated permutation representation, a one point stabilizer would have order 6048 and it is easy (quoting hard theorems) to show that it must be isomorphic to $PSU(3,3)$. The form of the permutation character indicates that the associated permutation representation of G would have rank 3 in the sense of Donald Higman [141]. The relevant Higman quadruple would be $k = 36, \ell = 63, \lambda = 14, \mu = 12$ and the 2-point stabilizers have shape $PSL(2,7)$ and $4^2{:}Dih_6$.

After Janko announced discovery of a simple group of order 604800 in early 1967, Hall acted quickly to conclude his studies in summer 1967. Hall proposed explicit permutations on 100 letters representing elements of the possible group G and used them in a long series of calculations. Finally, Peter Swinnerton-Dyer used the Titan computer at Cambridge University to verify that the permutations generated a group of order 604800, thus proving existence.

David Wales and Marshall Hall, Jr. later proved uniqueness with computer [132, 133]. The review of [132] adds later information about other constructions of HJ, both by computer and by hand.

The rank 3 group HJ was not *discovered* starting from the rank 3 theory, as were the sporadic groups of Higman-Sims, McLaughlin, Rudvalis and Suzuki.

The announcements of Zvonimir Janko and Marshall Hall, Jr. referenced each other's work [132, 155, 156]. It is nice to read about such courtesy. The original character table for HJ circulated by Janko had some errors. Hall introduced computational methods to the study of and search for new simple groups, which aided the searches by other researchers. See the full article [130] which, for example, proves existence and computes the character table of $Sz(8)$, the simple Suzuki group of order $29160 = 2^6 5 \cdot 7 \cdot 13$. The existence proof of HJ has been incorrectly attributed to both Hall and Wales. For a more common misattribution, see item (3) in the Final Remarks section.

8.2 The Higman-Sims Group

The story I tell of the Higman-Sims group discovery and existence proof is taken from testimony of Charles Sims (see [11, 145]).

It took place at a conference "Computational problems in abstract algebra" in Oxford in 1967.

Marshall Hall, Jr. lectured on his construction of his simple group HJ, order $2^7 3^3 5^2 7$. It acted in a rank 3 fashion on a graph on 100 points and valency 36.

On the last day of the conference, *2 September, 1967*, Higman and Sims thought about 100 and wondered if that number could come up in other ways for rank 3 groups. They may not have been so curious but for the fact that our number system is written in base 10 and $100 = 10^2$. Right away, they thought of the wreath product $Sym_{10} \wr 2$ acting on the Cartesian product of two 10-sets. This is rank 3 with subdegrees 1, 18, 81. The Higman parameters are $(18, 81, 8, 2)$.

Higman had a table of Higman quadruples. One quadruple was $(22, 77, 0, 6)$. The number 22 suggested that the Mathieu group M_{22} could be a point stabilizer in a rank 3 group with these parameters. The symmetric and alternating groups on 22 points will not work here since they do not act transitively on a set of 77 points, while it is well known that M_{22} acts on a Steiner system $\mathfrak{S}(3, 6, 22)$, which has 77 blocks ($77 = \binom{22}{3} / \binom{6}{3}$).

So, they defined a graph. For nodes of the graph, they used a set Ω of 100 points: *, with the 22 points Δ affording M_{22}, and with the 77 blocks Γ. It is clear that $Aut(M_{22}) = M_{22}{:}2$ acts on this set of 100 points. Work of Ernst Witt on existence and uniqueness of Steiner systems associated to Mathieu groups [243, 244] was very helpful to Higman and Sims.

The edges in the graph are defined as follows: * is connected to just the 22 points of Δ. A point p in Δ is connected to * and the 21 blocks containing it. A block is connected to the 6 points in the block and the 16 blocks disjoint from it. (So, the Higman parameters have values $\lambda = 0$, $\mu = 6$.)

They needed to prove existence of some permutation π on Ω which preserved the graph and moved *. Existence would prove that the group G generated by π and the action of $Aut(M_{22})$ on the set Ω is a rank 3 group with parameters $(22, 77, 0, 6)$.

Higman and Sims talked all night and got such a π. By the morning of *Sunday, 3 September, 1967*, it was clear that their group G or a subgroup of index 2 was a new simple group. (It turns out that the commutator subgroup G' has index 2 and is simple of order $2^9 3^2 5^3 7 \cdot 11$.) Time from conception to existence proof for this sporadic group was about a day. Their performance was unique. For other sporadic groups, gap between discovery and construction ranged from weeks to years.

I learned in 2007 that Dale Mesner had constructed this Higman Sims graph in his 1956 doctoral thesis at the Department of Statistics, Michigan State University [180]. This 291 page thesis explored several topics, including integrality conditions for strongly regular graphs (association schemes with two classes) related to Latin squares. Mesner's thesis does not mention concerns about the graph's automorphism group or acknowledge connections with Mathieu groups and Steiner systems. Jon Hall gives an account of this in [11]. Connections between rank 3 groups and connected strongly regular graphs are discussed in [122].

9. Discoveries of More Rank 3 Groups by Search for Higman Quadruples

Don Higman's original motivation for his rank 3 theory was probably to study parameter sets. Marshall Hall's multiple innovative techniques for HJ and Higman and Sims' elegant analysis for HS dramatically suggested the wealth of opportunities for investigating possible new simple groups. The group theory community took great interest.

Higman maintained a list of parameter sets which met his conditions. Some corresponded to actual rank 3 groups. *There are relevant group theoretical conditions besides arithmetic ones.* If G is a rank 3 group with point stabilizer H, the group H must have subgroups of indices k and ℓ. See [94], p.125 for Higman's table of parameters which apply to actual rank 3 groups.

In the next few subsections, we shall discuss other simple groups discovered from this viewpoint. Fischer's 3-transposition groups are indeed rank 3 groups by virtue of their action on a class of 3-transpositions, but their discovery came about by Fischer's theory of ω-transposition groups. They are discussed elsewhere in this article.

9.1 Discovery of the McLaughlin Group

When I was in graduate school at University of Chicago, Jack McLaughlin was in residence there during a sabbatical year (1968–1969, as I recall) from the University of Michigan. He was thinking about the Higman-Sims group and Don Higman's rank 3 theory. McLaughlin considered the group $H = PSU(4, 3)$, order $2^7 3^6 5 \cdot 7$ and its maximal parabolic subgroup of index 112. He next studied Higman quadruples k, ℓ, λ, μ with

$k = 112$. Since ℓ must be the index of a subgroup of H, he reviewed ones he knew about and thought of a subgroup isomorphic to $PSL(3,4)$, order $2^6 3^2 5{\cdot}7 = 20160$, described by H. H. Mitchell around 1918 [181]. This gives $\ell = 162$ and the Higman condition forces $\lambda = 30$ and $\mu = 56$. McLaughlin defined a graph on 275 nodes and valency 112 at each node. Using the strategy of Higman and Sims, he constructed an automorphism of the graph and thereby exhibited a new sporadic group of order $2^7 3^6 5^3 7{\cdot}11$ [179]. I omit details.

That year, Janko came to the University of Chicago to give a colloquium. I joined the dinner party, which included Jack and Doris McLaughlin and possibly George Glauberman. I remember that Janko and McLaughlin were in good moods.

9.2 The Suzuki Rank 3 Sporadic and the Suzuki Chain

Suzuki describes a new sporadic simple group which acts as a rank 3 permutation group on 1782 points [224]. It has order $2^{13} 3^7 5^2 7{\cdot}11{\cdot}13$. It is part of a chain $Sym_4 < PSL(2,7) < PSU(3,3) < HJ < G_2(4) < Suz$, where each step represents a rank 3 group and point stabilizer. The rank 3 representation of Suz has Higman parameters $k = 416, \ell = 1365, \lambda = 100, \mu = 96$. At the 1972 Gainesville conference, Suzuki said that there was no upwards extension of this chain. That is, there does not exist a finite group with a rank 3 representation whose one point stabilizer is Suz. This chain may be observed acting on the Leech lattice modulo 2 [40]. In [224], Suzuki credits recent work of Marshall Hall, Jr, Don Higman and Charles Sims for inspiration and mentions the HS construction as a model.

9.3 Discovery of the Rudvalis Group

Arunas Rudvalis announced evidence for his sporadic group around spring 1972 [200]. He was very excited when he called me at home in Ann Arbor to break the news. For months while on the Michigan State University faculty, he had been searching for new sporadic groups by creating lists of candidates for a point stabilizer and a 2-point stabilizer in some unknown rank 3 group, then determining possible associated Higman quadruples using computer searches and group theory methods. Finally, he struck gold: the group ${}^2F_4(2)$ was a good candidate for the point stabilizer with 2-point stabilizers $PSL(2,25).2$ and a parabolic subgroup $2^{1+4+4+1}.Frob(20)$. The associated quadruple $(2304, 1755, 1280, 1328, 1280)$ satisfies the Higman condition (Table 10A1, page 125 in [94] erroneously lists these four parameters as $(2304, 1255, 1280, 1328, 1208)$; my apologies, and thanks to Rob Wilson).

10. Looking for More Sporadics – Other Methods

Lots of group theorists looked for new sporadics. Some probably did so in secret. While in graduate school, I played with the Higman criterion and "rediscovered" the parameters which McLaughlin used, as well as finding quadruples which led nowhere.

Some sporadic groups are multiply transitive permutation groups: all Mathieu groups; Higman-Sims group; Co_3 (on cosets of a subgroup isomorphic to $McL{:}2$).

A putative simple group which has a doubly transitive representation with point stabilizer isomorphic to $PSU(3,5)$ was investigated by Graham Higman, but he did not complete the work before Donald Higman and Charles Sims discovered and constructed their group. The group studied by Graham Higman and the Higman-Sims group are isomorphic. See [142, 211].

The degree 276 doubly transitive representation of Co_3 turned up as a consequence of Conway's analysis of the Leech lattice and isometry group. A *transitive extension* of a transitive representation of the group H on a set Γ is a group G with transitive action on a set $\Omega = \Gamma \cup \{\alpha\}, \alpha \notin \Gamma$, so that the stabilizer G_α of α in G is isomorphic to H and G_α acts on Γ as H acts on Γ. The *Graph Extension Theorem* of Ernest Shult [209] gives a general sufficient condition for construction of doubly transitive groups. It can produce all known doubly transitive groups except triply transitive groups and certain $PSL(2,q)$. With this criterion, Shult *could* have discovered and constructed the sporadic group Co_3 before Conway did via his study of the Leech lattice. In above notation, H would be isomorphic to $McL{:}2$ and Γ would have cardinality 275. The relevant action of H on Γ is the rank 3 representation discussed in Section 9.2.

Computer work was significant for the sporadic groups, from the mid-1960s. John McKay was involved with Graham Higman on the first construction of the Held group (unpublished) and with Conway on studies including trine groups (mentioned in [120]). David Wales collaborated with Marshall Hall on uniqueness of HJ [129].

While in graduate school in Chicago, I attended a two-week meeting on finite and infinite group theory in Ann Arbor, around July 1968, hosted by University of Michigan faculty Donald Higman, Donald Livingstone and Roger Lyndon and Jack McLaughlin. Graham Higman was visiting that summer, giving a course and writing his notes Odd Characterisations of Finite Simple Groups [144]. I observed copies of newly minted character table(s) of the Higman-Sims group being circulated. J. Sutherland Frame was an author of one of these tables [77]. Graham Higman was in contact (by phone, I think) with John McKay during the meeting. They were trying to find new finite simple groups by presentations and computer work. I do not recall the presentations, but at least one relation involved the 19th power of a word in a free group. Audience members speculated that Graham Higman and John McKay noted the prime 19 in the orders of the Janko groups J_1 ($175560 = 2^3 3{\cdot}5{\cdot}7{\cdot}11{\cdot}19$) and J_3 ($50232960 = 2^7 3^5{\cdot}5{\cdot}17{\cdot}19$) and studied presentations inspired by these Janko groups. No new simple group resulted. In fall 1968, after ten years at the University of Michigan, Donald Livingstone left for the University of Birmingham University in the UK. In the late 1970s, he would collaborate with Bernd Fischer and Michael Thorne on the character table of the Monster.

My impression was that the search for sporadic groups was more systematic in the world of centralizer of involution studies than in that of permutation groups. Centralizer of involution results directly engaged the ongoing CFSG program. Sometimes, a new sporadic group was a surprise conclusion of a standard centralizer of involution characterization, such as the Held group [140] and the Harada-Norton group F_5 [138].

10.1 O'Nan's Simple Group

I learned about the O'Nan sporadic group from Jon Alperin's lecture in Warwick, May, 1973. Mike O'Nan was interested in classifying finite groups G with the following property: given E, F, a pair of elementary abelian 2-group contained in G of maximal rank, and two maximal flags $1 = E_0 < E_1 < \cdots < E_r = E$ and $1 = F_0 < F_1 < \cdots < F_r = F$ (meaning, each E_i and F_i has order 2^i), then there exists an element $g \in G$ so that $gE_ig^{-1} = F_i$ for $i = 0, 1, \ldots r$. This property could be called *transitivity on maximal flags of elementary abelian 2-subgroups.* O'Nan thereby found a new sporadic group, now called *the O'Nan group.* The order is $2^9 3^4 5{\cdot}7^3 11{\cdot}19{\cdot}31$. See [189, 190].

The group J_1 has this property because the Sylow 2-normalizer is a semidirect product of an elementary abelian group of order 8 by a nonabelian group of order 21, acting faithfully. See [190] for a list.

The O'Nan group is the only sporadic group found by this strategy.

11. The Leech Lattice

This single object, the Leech lattice, is a rich mathematical world with some remarkable number theory, combinatorics and group theory. It was discovered by John Leech in the mid-1960s, as a dense lattice packing in 24-dimensional Euclidean space [165, 166]. My understanding is that he had been looking for someone to analyze the isometry group. At the International Congress of Mathematicians in 1966, John McKay (then a graduate student) suggested this to John Conway, who took up the challenge.

First, I give a few definitions. A *lattice* L in Euclidean n-space is a $\mathbb{Z}$-linear combination of a basis. It is *integral* if all inner products $\langle x \mid y\rangle$ are integers and is *even* if all inner products are integers and $\langle x \mid x\rangle \in 2\mathbb{Z}$ for all $x \in L$. A *Gram matrix* for L with respect to the $\mathbb{Z}$-basis $v_1, \ldots, v_n$ of L is the $n \times n$ matrix whose i, j entry is $\langle v_i \mid v_j\rangle$. The *determinant* of L is the determinant of any Gram matrix. If a lattice is even and unimodular, n is divisible by 8. If $n = 24$, there are, up to isometry, just 24 even unimodular lattices of determinant 1. The *Leech lattice* is the only one without vectors of norm 2; its minimum norm is 4. A common notation for the Leech lattice is Λ.

Conway's story, reported in [197], is that he figured it all out in a single session of 12.5 hours. Some people told me that, briefly, there was more than one candidate for the group order of the isometry group. The final result is that the order of the isometry group is $2^{22}3^9 5^4 7^2 11{\cdot}13{\cdot}23$. Common notation for this isometry group is Co_0 or $O(\Lambda)$.

Here is the *standard description of the Leech lattice.* Details may be found in [94]. The Leech lattice is built from sublattices starting with a sublattice J which had 24×24 Gram matrix $diagonal(4, 4, \ldots, 4, 4)$. Take an orthogonal basis for J, say $v_i, i \in \Omega$, where Ω is an index set of size 24. Then $\langle v_i \mid v_j\rangle = 4\delta_{ij}$. For a subset S of Ω, we use the notation $v_S := \sum_{i \in S} v_i$. Next, we take a *binary Golay code* $\mathcal{G}$, which means a 12-dimensional linear subspace of $\mathbb{F}_2^\Omega$ so that the minimum weight of a vector is 8 (weight means the number of nonzero coordinates). There is a natural identification of $\mathbb{F}_2^\Omega$ with the family of subsets of Ω obtained by considering coordinates of a vector. This sublattice J is then enlarged to a lattice K by taking the $\mathbb{Z}$-span of J and all sums $\frac{1}{2}v_A$, where A is the subset of Ω corresponding to a word in $\mathcal{G}$. Finally, we get the Leech lattice $\Lambda := K + \mathbb{Z}(-v_i + \frac{1}{4}v_\Omega)$ for any i (this definition of Λ is independent of choice of $i \in \Omega$). See [165, 166, 39, 40, 107].

The simple group M_{24}, the group of the Golay code $\mathcal{G}$, acts on 24-dimensional space by permuting the basis $v_i, i \in \Omega$, as it permutes the index set. For this action, M_{24} preserves each of the lattices J, K, Λ. Using the same basis, there is for every subset S of Ω, a linear transformation defined by $\varepsilon_S : v_i \mapsto \begin{cases} -v_i & \text{if } i \in S \\ v_i & \text{if } i \notin S \end{cases}$. Self duality of the Golay code shows that ε_S takes Λ to itself if and only if S corresponds to a Golay codeword. All these transformations give a monomial group H of shape $2^{12}{:}M_{24}$.

The full isometry group of Λ is larger than H. The order of the isometry group follows from the mass formula [204] involving all rank 24 even unimodular lattices. Conway gave an explicit formula for an isometry u not in H, then showed that the full isometry group of Λ is generated by u and H and has order $2^{22}3^9 5^4 7^2 11{\cdot}13{\cdot}23$. His isometry was useful in computations.

A different style analysis of the Leech lattice, its properties and its isometry group was given by me in [95]. It emphasizes configurations of $\sqrt{2}E_8$-sublattices. It is relatively free of calculations with matrices, special counting arguments, etc. Its use of uniqueness results yields easy proofs of transitivity of the isometry group $O(\Lambda)$ on sets of vectors of norms 2, 3, 4 and triangles of type 222 *before deducing the order of* $O(\Lambda)$.

11.1 Consequences of Leech Lattice Theory for Finite Groups

The quotient $Co_1 := Co_0/\{\pm 1\}$ is simple of order $2^{21}3^9 5^4 7^2 11{\cdot}13{\cdot}23$.

Stabilizers of sublattices gave then-new sporadic groups

Co_2 of order $2^{18}3^6 5^3 7{\cdot}11{\cdot}23$;

Co_3 of order $2^{10}3^7 5^3 7{\cdot}11{\cdot}23$;

and some familiar ones

HS re-discovered, order $2^9 3^2 5^3 7{\cdot}11$;

McL re-discovered, order $2^7 3^6 5^3 7{\cdot}11$;

Suz re-discovered, order $2^{13}3^7 5^2 7{\cdot}11{\cdot}13$ (see paragraphs following).

Centralizers of certain isometries gave the groups HJ of order $2^7 3^3 5^2 7$ and Suz of order $2^{13}3^7 5^2 7{\cdot}11{\cdot}13$; both re-discovered. (More precisely, perfect groups $2{\cdot}HJ$ and $6{\cdot}Suz$ occur as subgroups of centralizers within Co_0.) I remark that if g is an element of order 3 in $O(\Lambda)$ with minimum polynomial x^2+x+1 in $End(\Lambda)$, then $(g-1)^2 = -3g$, whence $(g-1)\Lambda$ is a lattice between 3Λ an Λ, of index 3^{12} in Λ, which is stabilized by $N_{O(\Lambda)}(\langle g\rangle) \cong 6{\cdot}Suz{:}2$. The appearance of Suz within $O(\Lambda)$ is typically described as a composition factor of a centralizer, but it could be described as a composition factor of a sublattice stabilizer. A similar observation can be made for HJ, that it occurs as a composition factor in lattice stabilizers. If we take $g \in O(\Lambda)$, g of order 5 with minimum polynomial $x^4+x^3+x^2+x+1$, then the commutator subgroup $C_{O(\Lambda)}(\langle g\rangle)' \cong 2{\cdot}HJ$ is a normal subgroup of the stabilizers of the sublattices $(g-1)^k$ for $k=1,2,3$ which are between Λ and 5Λ and have respective indices 5^{6k}.

Studies of linear groups of degree 6 by John H. Lindsey led him to construct the double cover of the Hall-Janko group [168] then, by extending the process, to the six fold covering of the Suzuki group [167, 169, 170]. He gave invariant lattices over rings of integers for these groups which, when considered as integral lattices, led to the Leech lattice. These came after the initial discoveries of these sporadic groups, but can be considered independent approaches to the Leech lattice and certain subgroups of its isometry group.

12. Fischer's ω-Transposition Group Theory

We recall a definition given earlier. Let ω be a nonempty subset of $\{3,4,5,\dots\}$. An *ω-transposition group* is a finite group G generated by D, a conjugacy class of involutions, so that for $x \neq y$ in D, then either x,y commute or xy has order $|xy| \in \omega$. So, $\langle x,y\rangle$ is a dihedral group of order $2|xy|$.

Examples for the case $\omega = \{3\}$:

(a) D = the set of transpositions (i,j) in a symmetric group $G = Sym_n$. If (i,j) and (k,ℓ) do not commute, their product is a 3-cycle. So the group generated by (i,j) and (k,ℓ) is a copy of $Sym_3 \cong Dih_6$.

(b) Transvections in orthogonal groups $O^\varepsilon(2m,2)$, ($\varepsilon = \pm$), symplectic groups $Sp(2m,2)$ and unitary groups $PSU(m,2)$. Each of the two conjugacy classes of reflections in orthogonal groups over fields of 3 elements.

Fischer described and classified (with some qualifications) such groups provided that each solvable normal subgroup is in the center. The list of conclusions includes symmetric groups, some classical groups over small fields *and* three previously unknown almost-simple groups $Fi_{22}, Fi_{23}, Fi_{24}$. *To me, finding these sporadic groups from the simple-looking 3-transposition property is one of the most surprising events in finite simple group theory.* Fischer's published existence proofs for Fi_{23} and Fi_{24} are not accepted as complete. See [8] and its detailed review. The book [8] proves existence by deducing their existence from existence of the Monster (done in [103, 7]).

Examples for the case $\omega = \{3,4\}$:

(a) $GL(n,2)$, for D the conjugacy class of *transvections* = identity + rank 1 nilpotent, e.g. $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It is easy to check that two different transvections generate a dihedral group of order at most 8.

(b) Some sporadic groups: Co_2, Baby Monster F_2. The Baby Monster was Fischer's fourth sporadic group.

(c) The Monster provides an example for $\omega = \{3,4,5,6\}$.

Examples of "disconnected" ω-transposition theories: A finite group generated by a conjugacy class of involutions is an ω-transposition group for some ω. Such a set may be difficult to work with compared to $\omega = \{3\}$. The Suzuki group $Sz(8)$, order $29120 = 2^6 5{\cdot}7{\cdot}13$ has one class of involutions. Two distinct involutions commute or their product has order 5, 7 or 13.

For $PSL(2,2^n)$, these involutions are the transvections (with Jordan canonical form $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$) and ω must contain all non-1 divisors of 2^n+1 and 2^n-1.

For most odd transposition groups G generated by a conjugacy class D of odd transpositions, one can connect two involutions $x, y \in D$ by a sequence $x = d_0, d_1, \ldots, d_r = y$ of elements of D so that d_i and d_{i+1} commute for $i = 0, 1, \ldots, r-1$. When D is connected in this sense, one can begin to understand the group G by how intersections $C(d_i) \cap C(d_{i+1})$ embed in each of $C(d_i)$ and $C(d_{i+1})$.

The study of disconnected odd transposition groups follows a different strategy. Works of Suzuki, Bender [12, 223], Fischer and Aschbacher [72, 73, 5, 6] are used. The groups $PSL(2,2^n), Sz(2^n)$ and $PSU(3,2^n)$ occur this way.

Infinite nonsolvable ω-transformation groups have been studied by H. Cuypers, J. Hall and L. Soicher [123, 124, 125, 126, 127, 128].

13. Computer and Manual Constructions of Sporadic Groups

The Higman-Sims group, the Mathieu groups and several rank 3 groups were was envisioned and constructed by hand. Several sporadic groups were first constructed by computer. The first such constructions were for *HJ*, J_3, *Held*, *Lyons*, *O'Nan*, *Rudvalis*, *Baby Monster*, J_4 by McKay, Sims, Leon, Norton, Benson, Conway, Wales, et al. [184]. Computational challenges generally got tougher with larger groups. In some cases there were several proofs of existence, even cases of groups with a computer constructions and a manual constructions. I discuss a few examples.

The Rudvalis group has a subgroup of index "only" $4060 = 2304 + 1755 + 1$. Conway and Wales constructed it with machine work by exhibiting a 28-dimensional representation of the double cover of the Rudvalis group over the field $\mathbb{Q}[\sqrt{-1}]$. The group so constructed permutes a set of $4 \cdot 4060$ vectors in $\mathbb{Q}[\sqrt{-1}]^{28}$. Other proofs (one by computer and one by hand) are in [118, 163]. Some sporadic groups required computations with permutation representations on cosets of subgroups with large indices. For example, the O'Nan group has a permutation representation on 122760 symbols and the Lyons group has a permutation representation on 8835156 symbols. For each of the latter two groups, Sims took two years or so for construction with computer work [139, 212]. Ryba gave a shorter existence proof for the O'Nan group [201].

An interesting story is that of J_3. The first construction was done by Graham Higman and John McKay [143], using computer. A construction of J_3 as automorphisms on a nonassociative commutative algebra of dimension 170 was done by Daniel Frohardt [82], by hand. His proof also gives uniqueness.

There is an embedding of the triple cover $3{\cdot}J_3$ in $GL(9, \overline{\mathbb{F}_4})$. Alex Ryba writes:

> "The 9-dimensional rep[resentation] of $3J_3$ dates from before I was a graduate student. I remember an undergraduate talk by Conway where he showed the tiny matrices (size 9×9) and then said that it should be possible to find an invariant to prove that they generate a proper subgroup of the unitary group. By the time I was a graduate student he knew such an invariant. I think it was never published and I don't remember what it was. The 9-dimensional representation had originally been discovered (and constructed) by Richard Parker in the course of his first meataxe experiments. It's an interesting case because the matrices are so small, but [were] initially only known through computer work.
>
> I think $O'N$ is another interesting case too, and I know more about that since I was involved in one of the constructions that appears in the ATLAS. In both the $O'N$ and J_3 constructions the classification as hand or machine is just a matter of taste. A compact and human readable definition of the group is written down (and appears in the ATLAS). There is one computation to do: that some quantity is preserved. It could be done in a fairly ugly way by hand or invisibly by machine. My view is that the machine is more reliable (and convincing) but I know other people who would prefer the long hand calculation be done. There may still remain some cases that I would view as purely machine constructions. Here a human readable definition of generators can't be put on paper. Any cases like that really ought to be replaced by something a human could read. The original constructions of several sporadics were of this nature."

Dan Frohardt comments:

> "Alex's quote reminded me of a brief conversation I had with Charlie Sims in 1985 when I had given a talk about my work on J_3. He said that I would not learn any more about it with my approach than he had with his programming. I realized at the time that he was probably right."

14. The Simple Group of Fischer and Griess

During summer 1973, I received a letter from Ulrich Dempwolff who reported that Fischer had evidence for a new group, a $\{3,4\}$-transposition group, of order $2^{41}3^{13}5^6 7^2 11{\cdot}13{\cdot}17{\cdot}19{\cdot}23{\cdot}31{\cdot}47$, about 4×10^{33}; it would become Fischer's fourth sporadic group. This group was eventually called the Baby Monster. For the moment, I will denote it H. Certain properties of this putative group led me to imagine that there could be a larger group.

For example, H contains a subgroup $2^{1+22}Co_2$. To me this suggested a larger group of shape $2^{1+24}Co_1$. Also H contains a subgroup $3^{1+10}PSU(5,2)$. To me, this suggested a larger group $3^{1+12}2Suz$. These larger groups, plus other information about H came together to suggest a simple group G with such larger groups as subgroups.

After initial studies, I felt that there were no obvious reasons to reject the possibility that such a G

may exist. I add that H was not a subgroup of G but instead G would have a subgroup $2{\cdot}H$ which is a non-split central extension of H by $\mathbb{Z}_2$. The first weekend in November 1973, in Ann Arbor, was when I felt there was a serious chance of a new sporadic group. At a meeting in Bielefeld the same weekend, Fischer spoke about his ideas for the same group. We had no direct communications about our respective discoveries until weeks or months after that.

Fischer's thinking about enlarging local subgroups may have overlapped with mine. He certainly was thinking about his class of $\{3,4\}$-transpositions in H and what they would correspond to in a larger simple group which contained $2{\cdot}H$ as a centralizer of involution. This group would eventually be called the Monster and become Fischer's fifth sporadic group, and my first (and only).

In late 1973, I began working out some internal properties of the Monster and a version of the Thompson group order formula (such a formula may have first appeared in [140]). My announcement [101] included a proof that the smallest degree of a nontrivial complex irreducible of the Monster is at least $196883 = 47{\cdot}59{\cdot}71$ and conjectured that 196883 was in fact an irreducible degree. My initial group order work was not published, but a proof of the group order was given later by Ulrich Meierfrankenfeld, Yoav Segev and myself [108].

14.1 What Happened at Bielefeld and Cambridge in 1973

As I worked alone in Ann Arbor in fall 1973, much was happening elsewhere. I pass on some accounts of the Bielefeld meeting and afterwards.

The Monster contains certain elements of respective orders $1,2,3,4,5$ so that their respective centralizers in the Monster have form F_1, $2{\cdot}F_2$, $3\times F_3$, $4{\cdot}F_4(2)$ and $5\times F_5$; the group $4{\cdot}F_4(2)$ contains the covering group $2{\cdot}F_4(2)$ of $F_4(2)$; one could also write $4{\cdot}F_4(2)$ as $4\circ(2{\cdot}F_4(2))$.

Bernd Stellmacher provided these memories of the 1973 Bielefeld meeting, but cautions that recollection may not be accurate:

> "Thursday, Aug 20, 2020
> Dear Bob,
> I was at this conference in Bielefeld. But I was a young postdoc at that time and not included in the discussion, only a Zaungast as one would say in German. And of course I do not need to say that after nearly 50 years I would not take an oath on what I remember.
> So what I remember is an episode between two talks. Fischer was standing [at] the blackboard with Thompson, and Fischer was writing his usual diagrams of (local sub)groups: extra special groups as upside down triangles with the acting group as a vertical bar on top of it, and double stroke vertical bars for quasisimple groups. All of it decorated with the names or orders of the groups in question.
> In this way I saw (what is now known to be) the centralizer of (the) two involutions and the centralizer of a 3-element (with Suz as factor group). The discussion was then how these subgroups emerge, fit together and could live in a larger group. But it was more the visual memory that stuck, I do not remember any details of the discussion.
> And I guess this was not the only discussion the two had at this conference, but the only one I was present. Anyway about a week after the conference Fischer had the group order of the Monster. I think he got it by mail from Thompson, but I am not sure at all and I do not know who else contributed. Shortly after Fischer also talked of two other new sporadic groups found "inside" centralizers of 3-elements and 5-elements.
> Here is now the sad end of the story: Fischer died last week. I will be at the funeral on Monday. He had a marvelous memory and surely could have given me all the information you are interested in – and corrected what I may have gotten wrong or not in the correct order. I will miss him a lot.
> Best wishes,
> Bernd"

My impression is that John Thompson found the simple group F_3 (known as the Thompson group) at the 1973 Bielefeld meeting or shortly thereafter. The story of Harada-Norton group, F_5, is a bit more complicated. Before the meeting, Koichiro Harada had been in the process of investigating a possible simple group having an involution whose centralizer has a normal subgroup $2{\cdot}HS$. It turns out that F_5 met his conditions. Simon Norton began working on this group F_5 after news of Fischer's lecture reached him. Koichiro Harada tells me that Helmut Bender, John Thompson, Franz Timmesfeld and John Walter were at the Bielefeld meeting but that Simon Norton was not.

Koichiro Harada writes:

> "Wednesday, August 19, 2020.
> …Energetic and mathematical activities on the Monster (later named) began after we all came back [to Cambridge] from Germany, I recall. Thompson was teaching something and at the end of each day's teaching, he often talked about the progress on the Monster. …Gainesville, Bielefeld, Cambridge!! Super good days for group theory!! You and I, to mention only a few, had a very good time.
> Koichiro".

15. Moonshine

In the late 1970s, the term *moonshine* became established in finite group theory. It meant an unexpected connection between sporadic groups and an area of mathematics different from finite group theory. We mention a few examples.

15.1 Monstrous Moonshine

Before existence of possible Monster-like simple group was resolved, we heard a surprising story about modular forms in the late 1970s.

The starting point for *Monstrous Moonshine* of Conway and Norton was a pair of surprising ideas.

First, John McKay's observed around 1977 that 196884, the first nontrivial coefficient of the elliptic modular function $j(z)$, equals 1+196883 (z varies over the upper half complex plane). The number $196883 = 47{\cdot}59{\cdot}71$ was expected to be the smallest degree of a nontrivial irreducible representation of the Monster [101]. It is easy to show that the degree of a faithful matrix representation of the Monster is at least 196883.

Second, John Thompson [228] looked at a few of the higher coefficients of

$$j(z) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots,$$

where $q = e^{2\pi iz}$ and noticed that they were nonnegative integer linear combinations of degrees of irreducible representations of the Monster,

$$1, 196883, 21296876, 842609326, \ldots.$$

He then asked whether there could be a graded space $V = \bigoplus_{n\geq -1} V_n$, where V_n is a finite dimensional module for the Monster, so that the formal series $\sum_{n\geq -1} dim(V_n)q^n$ equals $j(z) - 744$ and that the series $\sum_{n\geq -1} tr(g|_{V_n})q^n$ for all g in the Monster are interesting.

These series were indeed interesting. It was the basis of the Monstrous Moonshine theory of Conway and Norton [43], a near-bijective correspondence between conjugacy classes of the Monster and a family of genus 0 function fields on the upper half plane. This was an unexpected connection between deep parts of finite group theory and number theory. The impact on mathematics would be great.

15.2 Other Moonshine

(a) John McKay noted a connection between the extended E_8-diagram and pairs of $2A$-involutions in the Monster. See the later section *Moonshine involving the E_8-diagram*.

(b) A second connection between he extended E_8-diagram and pairs of $2A$-involutions in the Monster is that the orders of xy for a pair of $2A$ elements (x, y) of the Monster, one pair from each of the 9 orbits, occur as the coefficients of the standard null vector in the affine root system. For just the nodes occurring within the E_8 subdiagram, the coefficients are those of the highest root. This observation may be due to McKay. I am not aware of any explanation.

(c) Andrew Ogg observed that the fifteen prime divisors of the Monster order were exactly those primes p for which the discrete groups $\Gamma_0(p)$ have genus 0. He offered a bottle of Jack Daniels for an explanation. The bottle was claimed but not awarded.

16. Considering an Existence Proof for the Monster

16.1 Impressions of Difficulty

The order of the Monster was about 8×10^{53}, so construction was expected to be difficult. Computer constructions of certain sporadic groups took years. The problem with trying a computer construction of the Monster was that there were no small representations. By comparison, the symmetric group of degree 44 has order about 10^{54}, yet can be represented by 44 symbols written on a piece of paper. Also, the group $GL(9,5)$ has order about 10^{56} and the group $GL(14,2)$ has order about 10^{58}. Both are represented by small matrices.

The smallest index of a subgroup in the Monster was believed to be about 10^{20}, so a permutation representation would involve that many symbols. The smallest degree of a faithful matrix representation in characteristic 0 had been known since 1973 to be at least 196883 [101]; in fact the smallest degree of a faithful matrix representation over a field of characteristic not 2 or 3 turns out to be at least 196883, and the smallest degree in characteristics 2 and 3 is at least 196882 by a result of Steve Smith and myself [119].

For comparison, consider the group J_4 of order

$$2^{21}3^35{\cdot}11^3{\cdot}23{\cdot}29{\cdot}31{\cdot}37{\cdot}43 = 86,775,571,046,077,562,880 \sim 8.6 \times 10^{19}.$$

Its lowest degree nontrivial complex character is 1333. In characteristic 2, J_4 has a representation of degree 112, which was considered "small" [184]. The smallest degree representation for the Baby Monster, F_2, in characteristic 0 is 4371 and in positive characteristic, the smallest degree I know of is 4370 in characteristic 2.

16.2 Worth the Effort?

In the mid 1970s, there was increasing concern about encountering larger and larger sporadic groups. The Monster was really large compared to predecessors. The difficulty of constructing it seemed orders of magnitude beyond past experiences. Not only that, but the challenges in such a project might even be small relative to ones posed by sporadic groups yet to be discovered. The sense of what was important to CFSG could change.

I recall a group theorist telling me in the mid-1970s that he/she could have envisioned the same expansion of ideas from Baby Monster to Monster as I did, and even declared them "obvious". This person did not want to investigate seriously because they foresaw only a thankless great labor with little likelihood of payoff. Looking away from Monster issues at that time, a group theorist could see no lack of challenging problems to take up in the ongoing CFSG.

16.3 Shifting Winds in Late 1970s

In the late 1970s, there was a sense that the classification of finite groups might close in the near future since Daniel Gorenstein and Richard Lyons had outlined an end game [87]. No sporadics had been discovered since Janko's J_4 in May 1975. Also Monstrous Moonshine had arrived and suddenly made resolving existence of the Monster more important. I began to think more about how a construction would go.

17. The Attempt, Late 1979

In fall 1979, about six years after Fischer and I discovered evidence for the Monster, I decided to make a serious try at a construction. I was at the Institute for Advanced Study, on a one year sabbatical from the University of Michigan.

To me, the most reasonable setting seemed to be a degree 196883 complex representation, which was expected to be writeable over the rationals. Work of Simon Norton *suggested* that if B is an irreducible 196883-dimensional representation of the Monster, B is self-dual and has a degree 3 invariant symmetric tensor. This means that B would have the structure of a commutative algebra with an associative bilinear form, for which the Monster acts as algebra automorphisms.

Let us say a finite simple group has *Monster type* if it has an involution whose centralizer has the form $2^{1+24}Co_1$. My goal was to create a finite simple group which has Monster type.

Now, I summarize the program I carried out. I started by constructing a dimension 196883 representation B for a *suitable* group C of shape $2^{1+24}Co_1$ and consider the family of C-invariant algebra structures. Then I had to (1) make a choice of C-invariant algebra structure which *might* be invariant under a finite group larger than C; (2) define an invertible linear transformation σ on B, then prove that σ preserves the algebra structure; (3) Show that the group $\langle C,\sigma\rangle$ generated by C and σ is a finite simple group in which C is the centralizer of an involution. Then $\langle C,\sigma\rangle$ would be a group of Monster type.

We can describe C with a fiber product, $\widehat{C}$:

$$\begin{array}{ccc} \widehat{C} & \dashrightarrow & C_\infty \\ \vdots & & \downarrow \\ Co_0 & \rightarrow & Co_1 \end{array}$$

where C_∞ is a subgroup of $GL(2^{12},\mathbb{Q})$ of the form $2^{1+24}.Co_1$. We can think of $\widehat{C}$ as the subgroup of the direct product $Co_0\times C_\infty$ consisting of all pairs (u,v) so that the images in Co_1 of $u\in Co_0$ and $v\in C_\infty$ are equal. Then we take $C:=\widehat{C}/\langle(-1,-1)\rangle$ where the first component of $(-1,-1)$ means the scalar -1 on the rational span of the Leech lattice and where the second component means -1 in $GL(2^{12},\mathbb{Q})$.

The smallest faithful representation of C has dimension $98304=24{\cdot}2^{12}$.

We use notation similar to that in [103]. Let z be the involution which generates the center of C and let $R:=O_2(C)$.

Define $B:=U\oplus V\oplus W$, a direct sum of irreducible C-modules, where U has dimension 299, $dim(V)=98280$ and W has dimension $98304=24{\cdot}2^{12}$.

Think of U as 24×24 symmetric matrices of trace 0; V has a basis of all unordered pairs $\{\lambda,-\lambda\}$ where λ is a minimum norm vector in the Leech lattice; and W can be thought of as a tensor product of a degree 24 representation of C_0 and a degree 2^{12} representation of C_∞.

The spaces $Hom_C(X\otimes Y,Z)$ were described, where $X,Y,Z\in\{U,V,W\}$. This information enables a description of the multi-parameter space of C-invariant algebra structures $B\times B\to B$. A choice of algebra product and automorphism $\sigma\notin C$ of the chosen algebra were sought.

If the Monster were to exist, there would be a subgroup K of the form $2^{2+11+22}[M_{24}\times Sym_3]$ for which $C\cap K$ would look like $2^{2+11+22}[M_{24}\times 2]$. The right hand factor in $[M_{24}\times 2]$ can be thought of as representing a subgroup generated by the transposition $(1,2)$ inside the symmetric group on $\{1,2,3\}$. My choice of σ would be an element of K which, in the quotient $[M_{24}\times Sym_3]$ of K, represents the transposition $(2,3)$ in the right hand factor.

Such a σ would not leave the subspaces U,V,W invariant. I found that certain direct sum decompositions

$$U=U_1\oplus U_2\oplus\ldots,\quad V=V_1\oplus V_2\oplus\ldots,\quad W=W_1\oplus W_2\oplus\ldots$$

were helpful to imagine an approximation of a good σ of order 2. For example, one could see how such a σ would permute these smaller summands, for example, fixing certain ones while switching some U_i and V_j and some V_k and W_ℓ, etc.

Getting signs right in a matrix for σ was a big problem, solved by trial and error. Without knowing signs exactly, the procedure of the previous paragraph enabled me to quickly determine a nonzero product, unique up to scalar multiple. Call the product *.

When I chose an invertible linear transformation σ, I had to check whether it preserved the algebra product. This involved taking a convenient basis b_i of B, then asking whether $\sigma(b_i * b_j)$ equals $(\sigma b_i) * (\sigma b_j)$, for all i, j.

Checking the above set of equalities typically took about a week of calculations by hand. Failures of equality were studied and new formulas for σ were proposed. I tested a long series of candidates before finding one which worked. Sometimes, I ran a second test for a candidate using a different basis to understand failures better.

This construction took a few months, roughly October 1979 to early January, 1980. I worked around the clock, sleeping as needed and taking little time off. The atmosphere at IAS was great for work and discussions. Enrico Bombieri, an IAS faculty member, encouraged me a lot during this intense time. I was and still am very grateful for his understanding of my commitment and giving such ardent support. He really wanted me to succeed. During this otherwise serious time, we socialized a bit. An extraordinary event was his visit to my office at IAS (23 ECP building) at 4 in the morning. We walked to his home for conversation over spaghetti and beer until the sun came up, a fun event! One of Enrico's memorable stories was about encouragement he received from Harold Davenport.

The ECP office building was quiet overnight, excellent for concentration. Other IAS members were infrequently in their offices during the wee hours. The building and offices within seemed to be always unlocked. I noticed a man occasionally entering the building mid-evenings to make long phone calls from the pay phone in the lobby. An individual whom I did not recognize came a few times with a sleeping bag, opened an office, and went to sleep on the floor. Most days between late evening and dawn, a service person entered the building. While his station wagon motor was left running, he made an inspection then left promptly, except a few times when we had a chat.

17.1 Finale

I announced the construction of the Friendly Giant on 14 January, 1980, by mailing copies of a typed announcement to many group theorists. I gave a formal lecture about it at IAS May, 1980. The audience included Enrico Bombieri, Armand Borel, Freeman Dyson, Howard Garland, Daniel Gorenstein, James Lepowsky, Richard Lyons, Jill Mesirov, Nick Patterson, Mike O'Nan, Charles Sims, Karen Uhlenbeck, Shing-Tung Yau, et al. An article about the talk and its significance to finite simple group theory appeared in the New York Times; the author, Jonathan Friendly attended my talk. I left the next day to give a talk at The University of Chicago.

Publicity about my research was unusually high for mathematics at that time. A partial list:

National Science Foundation News, by Ralph Kazarian; March 14, 1980; NSF PR80-17; Scientist takes big step in solving major problem in mathematics;

Scientific American, Science and the Citizen, May, 1980, p. 82.

Scientific American, Martin Gardner Mathematical Games column in June, 1980.

Science News, 27 September, 1980, A monstrous piece of research, by Lynn Arthur Steen, p. 204–206.

Encyclopedia Britannica Book of the Year, 1981 (events of 1980), p. 529 (the article is flawed).

Mosaic, A Friendly Giant, by Henry T. Simmons, September-October 1981, 23–36.

Jonathan Friendly, Ideas and Trends, "School of theorists works itself out of a job", 22 June, 1980, New York Times.

Clive Cookson, "Mathematics: 'The Monster' unveiled", London Times, 31 May, 1980.

London Times Higher Education Supplement, "Michigan doctor claims monster maths breakthrough", about May, 1980.

Frances Buekenhout, Les Groupes Sporadiques, La Recherche, 131, Mars 1982. 348–355.

Freeman Dyson, "Unfashionable Pursuits", Mathematical Intelligencer, vol. 5, no. 3, 1983, 47–54.

Freeman Dyson, "Unfashionable Pursuits", article in book "From Eros to Gaia."

The next mathematical result to get wide publicity was the 1983 work of Gert Faltings on the Mordell conjecture. Mathematics coverage in the popular media became more frequent thereafter.

Later in 1980, I finished writing up consequences of the construction, including short existence proofs for certain sporadic groups and table of involvements of sporadic groups in one another. The article was submitted to Inventiones and appeared in 1982 [103].

18. Reactions to the Original Proof

On 13 November, 2020, Richard Borcherds wrote:

> "Thanks for the history.
>
> John Conway once told me how astonished the group theorists at Cambridge were when your construction came out. He said that although they had known that in theory one could try to construct the monster using an algebra structure on 196883, everyone had felt that the calculations would simply be too complicated for anyone to carry out. I attended a series of seminars by Conway and Norton on your construction while I was an undergraduate, but must admit I was rather lost after the first couple of weeks.
>
> Best regards, Richard Borcherds."

In 1981, I heard similar accounts from Jan Saxl about the opinions around Cambridge.

Jacques Tits made many improvements and studies, both on the 196883-dimensional algebra and determination of the automorphism group [234, 235].

In 1984, Conway gave a more efficient construction of the Monster and an associated 196884-dimensional algebra [42]. He used the Parker loop, which is a Moufang loop (a kind of nonassociative group) built from the binary Golay code of dimension 12 over the field of integers mod 2. It has 2^{13} elements. The loop enabled Conway to write down compact formulas for the algebra multiplication and show existence of an "extra automorphism", a point which was particularly difficult in [103]. His formulas were compatible with my original formulas. Such a loop was not known to exist until the early 1980s, several years after the original construction of the Monster [102, 103]. For background on related loops, see [104, 106].

In 2012, Ching Hung Lam and I made a much easier existence proof by vertex algebra theory. For a summary, look ahead in the section *Short existence proof of the Monster and MVOA*.

19. Uniqueness of the Monster and the Algebra B

Uniqueness of the Monster was proved by Griess, Meierfrankenfeld and Segev in 1989 [108], using hypotheses on centralizer of involutions. The method was to study the graph of $2A$ involutions and use results from the CFSG to determine stabilizers of pairs, then eventually prove that the graph is uniquely determined and that the Monster was the full automorphism group of the graph. See the Final Remarks section for more detail.

There is still no uniqueness result for $(B,*)$ as an algebra (though it is essentially unique, *assuming* that a group of Monster type acts on it as algebra automorphisms). One could consider ring-theoretic hypotheses such as an invariant bilinear form (i.e. $(x*y,z)=(z,y*z)$ for all $x,y,z\in B$) and a polynomial identity. A low degree homogeneous polynomial identity (besides the commutative law) satisfied by $(B,*)$ is not known explicitly. There is an identity of degree $1+2{\cdot}dim(B)$, made by adapting the standard degree $2n$ identity for $Mat_{n\times n}(\mathbb{Q})$ [2]. Since the algebra is commutative, there is a nontrivial identity of degree $2+dim(B)$ [199]. The degree of any polynomial identity (independent of $xy-yx$) for $(B,*)$ is at least 6 [93]. Basic theories for Lie algebras and Jordan algebras start with their defining identities, of degrees 3 and 4 respectively. It could be hard to start a classical-style theory for algebras depending mainly on a high degree polynomial identity.

20. Commutative, Nonassociative Algebras

The Norton proposal in the 1970s that there could be a good 196883-dimensional commutative algebra for studying the Monster created a general interest in finite groups as automorphisms of commutative nonassociative algebras. Such algebras did not generally have a unit. Several examples were found as follows. If G is a finite group and M is a finite dimensional $\mathbb{C}G$ module with character χ, then a G-invariant commutative algebra structure on M corresponds to G-homomorphisms of modules $S^2M\to M$, where S^n denotes symmetric tensors of degree n. The space of such maps has dimension equal to the inner product of characters χ and $S^2\chi$ which is defined by $S^2\chi(g)=\frac{1}{2}(\chi(g)^2+\chi(g^2))$. The space of invariant symmetric trilinear forms on M has dimension equal to the inner product of the principal character with $S^3\chi$, which is defined by $S^3(\chi)(g)=\frac{1}{6}(\chi(g)^3+3\chi(g^2)\chi(g)+2\chi(g^3))$. In general, one could try any finite group and any module. In practice, more interesting examples came up by taking a group G and a permutation representation, then taking an irreducible constituent of a permutation module. See articles of Smith [213] and Frohardt [82].

Later in this article, I describe how finite dimensional commutative nonassociative algebras and finite groups come up in vertex operator algebra theory. An example of dimension 156 is analyzed in [110].

21. Graded Spaces and VOAs

A graded space for the Monster was announced by Igor Frenkel, James Lepowsky and Arne Meurman in 1983, a response to the Thompson suggestion. They used a blend of theory for highest weight modules for affine Lie algebras and the techniques from the Monster construction [103]. Their book [81] came a few years later after redoing their work to include the then-new vertex operator algebra (VOA) theory [15]. Their main achievement was construction of the *Moonshine VOA*, whose automorphism group is the Monster [81]. They use the symbol $V^\natural$ for this VOA. The graded dimension for the Moonshine VOA is $q\cdot(j(z)-744)$, representing removal of constant term from the elliptic modular function, then a shift of degree.

Sometimes one refers to *a Moonshine VOA (MVOA)*, meaning a VOA whose graded dimension is $q(j(z)-744)$ for which the degree 2 component is essentially the algebra created to prove existence of the Monster [92]. Frenkel, Lepowsky and Meurman conjectured that their MVOA, denoted $V^\natural$ [81], should be characterized by the properties of being a VOA (a) whose only irreducible module is itself; (b) whose central charge 24; and (c) whose degree 1 summand is zero. So far, there is no such uniqueness result. For some partial results, see [66, 164, 1]. As mentioned before, the 196884-dimensional algebra $(MVOA_2, 1^{st})$ has no characterization so far without assuming an action of the Monster.

Frenkel, Lepowsky and Meurman proved graded traces were right for many but not all group elements of the Monster. Later, Borcherds proved that the traces were right for all group elements [17].

22. Some Consequences of VOA Axioms

A VOA is a graded space over a field of characteristic 0 with countably many products. The definition of a VOA is too long to present here; see [81]. I mention a few points about the case of VOAs graded over the nonnegative integers.

Given a VOA $V = \oplus_{i\geq 0} V_i$, the k-th product gives a bilinear map $V_i \times V_j \longrightarrow V_{i+j-k-1}$. So, V_n under the $(n-1)^{th}$ product is a finite dimensional algebra, denoted $(V_n, (n-1)^{th})$.

In addition, (a) if $dim(V_0) = 1$, $(V_1, 0^{th})$ is a Lie algebra; (b) if $dim(V_0) = 1$ and $dim(V_1) = 0$, then $(V_2, 1^{st})$ is a commutative algebra with a symmetric, associative form $(ab, c) = (a, bc)$.

Algebras $(V_2, 1^{st})$ as in (b) are sometimes called *Griess algebras* [182]. A VOA which meets condition (b) is sometimes called an *OZ algebra*, which refers to one-zero for the dimensions of V_0 and V_1.

A *vertex algebra (VA)* over a commutative ring K is a graded K-module with a set of axioms similar to the VOA axioms. There is an analogue of a vacuum element but there is not necessarily an analogue of a Virasoro element [116, 117].

In the Frenkel-Lepowsky-Meurman VOA $V^\natural$, $(V_2^\natural, 1^{st})$ is a commutative nonassociative algebra of dimension 196884, essentially the algebra I defined to construct the Monster.

The automorphism group of a finitely generated VOA is an algebraic group, by a theorem of Chongying Dong and myself [59]. Our paper has some results about derivation algebras of VOAs.

23. Graded Complex Representations for Other Groups?

The authors John Duncan, Michael Mertens and Ken Ono [68] have constructed graded spaces for the O'Nan sporadic group, order $2^9 3^4 5 \cdot 7^3 11 \cdot 19 \cdot 31$ with number theoretic properties. The graded traces in one version are weight $\frac{3}{2}$ modular forms. This is a very interesting advance. These graded spaces do not (yet?) have a wealth of algebraic properties like a VOA does. Sophisticated graded algebraic structures for other pariahs have been sought for decades. This "O'Nanshine" seems to be the best contribution so far.

There is an automorphism of order 2 of the O'Nan group whose fixed point subgroup is isomorphic to J_1. So the O'Nanshine space gives a kind of Moonshine for J_1.

See [52] in which authors Samuel DeHority, Samuel Xavier Gonzalez, Neekon Vafa, and Roger Van Peski give general constructions of graded spaces for arbitrary finite groups which have specific number theoretic properties. It shows that some finite groups which do not embed in the Monster can have genus 0 kind of properties. There seems to be nothing new about particular sporadic groups. The use of "moonshine" in the title of [52] seems inappropriate because the traditional meaning of moonshine is a surprising connection between sporadic groups and other areas of mathematics.

24. Lattices, Vertex Algebras and Applications

The articles [63, 64] by Chongying Dong and myself establish a beginning to a theory of (group-invariant) integral forms in VOAs, and give an integral form in $V^\natural$ which is invariant under the Monster. Scott Carnahan [29] shows that there is even one which is self-dual. None of these Monster-invariant forms is given an explicit description, unfortunately.

There is a standard integral form in the lattice VOA V_L, for any even lattice L, given with explicit generators [63]. Moreover, when L is a root lattice of type ADE, Ching Hung Lam and I showed [116, 117] that those natural generators which lie in the degree 1 term form a standard Chevalley basis of the finite dimensional simple complex Lie algebra $((V_L)_1, 0^{th})$ as well as a basis for the intersection of the standard integral form in V_L with $(V_L)_1$. *So, we have a natural generalization of usual Chevalley basis and root lattice to an integral form in the lattice VOA V_L.*

24.1 Is Every Finite Group the Automorphism Group of a VOA?

There is a natural question for VOAs, in the spirit of Noether's inverse Galois problem (given a finite group, G, is there a Galois field extension K of the rationals $\mathbb{Q}$ so that $Gal(K/\mathbb{Q}) \cong G$?). The Noether problem is not settled.

Given a finite group G, is there a VOA whose automorphism group is G? The answer is unknown in general but is yes for $G = \mathbb{M}$ and a variety other finite groups. Automorphism groups of VOAs are studied in articles of Dong, Griess, Nagatomo and Ryba [67, 60, 61, 62, 206]. One of the interesting cases of large rank is the occurrence of $2^{27}.E_6(2)$; see Shimakura [208].

For VAs over finite fields, there is a fairly natural affirmative answer for a finite group which is an adjoint form Chevalley or Steinberg group extended upwards by diagonal and graph automorphisms, proved by Ching Hung Lam and myself [116, 117]. This result makes use of the standard integral form in the lattice VOA V_L.

25. Short Existence Proof of the Monster and MVOA

A new existence proof for a Moonshine VOA was given by Miyamoto [183]. Shimakura [207] gave a relatively short existence proof of a Moonshine VOA using a theory of Miyamoto about simple current modules for a VOA [182]. Ching Hung Lam and I used this construction of an MVOA to give a relatively short existence proof for the Monster [115]. The very long calculations in earlier proofs [103, 42] are now avoidable, a sign of progress.

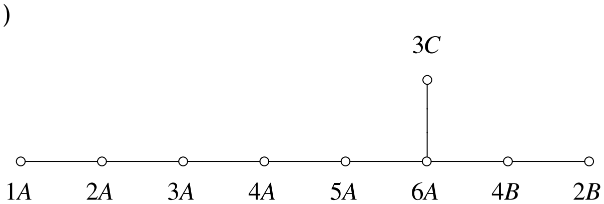
The relationship between VOA theory and finite simple groups has become stronger.

26. Moonshine Involving the E_8-Diagram

This section is essentially the introduction to the article [114] by Ching Hung Lam and myself.

In 1979, John McKay [178] noticed a remarkable correspondence between $\tilde{E}_8$, the extended E_8-diagram, and pairs of $2A$-involutions in $\mathbb{M}$, the Monster (the largest sporadic finite simple group).

(1)



There are 9 conjugacy classes of such pairs (x,y), and the orders of the 8 products $|xy|$, for $x \neq y$, are the coefficients of the highest root in the E_8-root system. Thus, the 9 nodes are labeled with 9 conjugacy classes of $\mathbb{M}$. There is no obvious reason why there should be such a correspondence involving high-level theories from different parts of the mathematical universe.

In 2001, George Glauberman and Simon Norton [85] enriched this theory by adding details about the centralizers in the Monster of such pairs of involutions and relations involving the associated modular forms. Let (x,y) be such a pair and let $n(x,y)$ be its associated node. Let $n'(x,y)$ be the subgraph of $\tilde{E}_8$ which is supported at the set of nodes complementary to $\{n(x,y)\}$. If (x,y) is a pair of $2A$ involutions and z is a $2B$ involution which commutes with $\langle x,y\rangle$, Glauberman and Norton give a lot of detail about $C_{\mathbb{M}}(x,y,z)$. In particular, they proposed how $C_{\mathbb{M}}(x,y,z)$ gives a "new" relation to the extended E_8-diagram, namely that $C_{\mathbb{M}}(x,y,z)/O_2(C_{\mathbb{M}}(x,y,z))$ looks roughly like "half" of the Weyl group corresponding to the subdiagram $n'(x,y)$.

The important and provocative McKay-Glauberman-Norton observations seemed like looking across a great foggy space, from one high mountain top to another. We want to realize their connections in a manner which is more down-to-earth, like walking along a path, making natural steps with familiar mathematical objects. These objects are lattices, vertex operator algebras, Lie algebras, Lie groups and finite groups.

In [114], we propose a specific moonshine path for the $3C$-case (i.e., $n'(x,y)$ is an A_8-diagram). The $3C$-case seems to be especially rich. Several Niemeier lattices are involved. They include E_8^3 and the Leech lattice Λ. Triality for D_4 plays a role. An explanation for occurrence of just "half" the Weyl group (of type A_8) arises naturally.

We subsequently developed similar moonshine paths for some other nodes [112, 113]. The suggested "half Weyl group" property turns out to be true in several cases though not for $6A$.

27. Final Remarks

27.1 Significance of Uniqueness and the Nine-Orbit Theorem for the Monster

Uniqueness of the Monster was proved by Ulrich Meierfrankenfeld, Yoav Segev and myself [108]. This article also contains a first proof of the group order

$$2^{46}3^{20}5^97^611^213^317{\cdot}19{\cdot}23{\cdot}29{\cdot}31{\cdot}41{\cdot}47{\cdot}59{\cdot}71$$

and the first proof that the action of the Monster on ordered pairs of $2A$-involutions has nine orbits (the *nine orbit theorem*).

Progress on a uniqueness proof for the Monster was claimed by Simon Norton in [185] but was not followed by a published proof. The article [185] refers to claims of exactly nine orbits and the group order, attributing these to unpublished work of Bernd Fischer and John Thompson. I find no publications to support this attribution. My recollection is that, in the early 1980s, these assertions had been considered very likely to be true, but unproved. I could find no proof or solid reference to a proof for the nine orbit theorem in Norton's publications [185, 186, 187, 188]. Nine cases of the so-called dihedral algebras (generated by a pair of idempotents associated to $2A$-involutions) are listed in [45], page 230, but without a reference to a proof. My impression is that Norton deserves credit for describing those nine dihedral algebras, but not for proving completeness of the list, which would appear to depend on the nine orbit theorem. Sakuma [202] proved that those nine dihedral algebras described by Norton are the only isomorphism types possible for generation by a pair of such idempotents. Sakuma's result does not seem to imply the nine orbit theorem.

27.2 About the Atlas of Finite Groups

The Atlas of Finite Groups is a remarkable book full of well-organized material about many of the finite simple groups, including character tables, classifications of maximal subgroups and lists of many special properties. I consult it frequently and appreciate its great convenience. The tables could be called *enriched character tables*, meaning that they include power maps on classes, indicators and fusion under outer automorphisms. However, the user should be cautious due to lack of verifications, unstated assumptions and problems with attributions. See a review in MathSciNet [45]. When I make reference to the Atlas material in a paper, I may treat it as a hypothesis.

For revisions to the original version of the Atlas, see [46, 13]. Concerns about accepting the Atlas material as proved or at least reproducible were presented in a lecture of Jean-Pierre Serre ([205] is available on youtube.com) and in a letter to Donna Testerman [13]. Subsequently, electronic re-computations of the Atlas tables were redone in [22, 23, 25], resulting in confirmations plus some revisions. Only a few cases currently remain.

Thomas Breuer, and Robert A. Wilson [24] are completing a check of the Monster character table in the Atlas (coauthor Kay Magaard recently passed away). Author Rob Wilson writes:

> [19 March, 2021] "...we rely on your reference [108] for most of the relevant facts about the Monster. I don't think we need anything else, although the paper is not completely written so there may be some details still to consider. We also use character tables of various subgroups of the Monster, either previously verified or re-computed as necessary from verifiable and/or published information. In particular we do not need any classification of elementary subgroups or anything like this – nor did we use this kind of information in [22]. The bulk of the work is in calculating the class list and the power maps. Strictly speaking, we did not re-calculate the character table, but verified (proved) that the published table is correct."
>
> [26 March, 2021] "Yes, we need existence of the Monster, and existence of the 196883 character. For that matter, we assume CFSG and any other reasonable background material. We are not trying to prove anything from first principles.
>
> It is only in the case of facts about the Monster itself that it is imperative we do not use any results that might need the character table – for example, we do not use the isomorphism type of $3^{1+12}.2Suz$, since there are two non-isomorphic non-split extensions of this type, only one of which is a subgroup of the Monster, and ultimately we use the character table to determine which one it is, not the other way round.
>
> It is hard to articulate a 'general strategy' for obtaining the conjugacy classes. We collect a whole load of classes from involution centralizers first, then centralizers of elements of order 3 and 5, and then use various ad hoc arguments to deal with all the primes one at a time, not necessarily in the order you might expect – ending in fact with 7. We use the known values of the permutation character to give us (bounds on) some centralizer orders, but sometimes have to fight quite hard to get the fusion or lack of it."

More comments on the Atlas (several refer to my work):

(a) The uniqueness result [108] is needed for justification of the character table of the Monster, found in [45]. *That table was based on the assumption that a group of Monster type has an irreducible complex representation of dimension 196883.* Dependence on this assumption was not stated in the Atlas. Existence of an irreducible of degree 196883 was shown in [92] for a *particular group of Monster type*, but not for an arbitrary group of Monster type. I am not aware of any publications about the table by the authors Bernd Fischer, Donald Livingstone and Michael Thorne.

(b) The table of involvement of one sporadic group in another, [45], page 238, is an update of a table taken from [103], but credit is not given.

(c) The reference [104] was not included; it gave the first published rigorous treatment to the Parker loop theory and furthermore showed that such loops could be viewed as based on binary codes. Some further applications of loops to finite simple groups may be found in [106, 195, 196].

(d) Page 228 of [45] is subtitled: Sporadic Fischer-Griess-Monster or "Friendly Giant" group $M \cong FG \cong F_1$. The latter pair of isomorphisms is ambiguous since uniqueness of the Monster was not resolved until the 1989 article [108].

(e) The Atlas system of notations for simple groups is uncomfortable to many finite group theorists. Look ahead to the section *Notations for the finite simple groups.*

27.3 Fischer's Early Work on Y-Diagrams

In early 1975, during the Rutgers special year in finite groups, Bernd Fischer visited for a few weeks and lectured on aspects of his ω-transposition groups, especially his $\{3,4\}$-transposition groups.

In conversations, he showed me work on subgroups of the Monster generated by various sets of $2A$-involutions. There were examples where such involutions form diagrams which look like the letter Y, with arms of various lengths. Two nodes are disconnected if the pair of involutions commute and are connected if the two involutions generate a dihedral group of order 6.

Interesting results on Y-diagrams are discussed in the Atlas of Finite Groups [45], though sometimes without attributions or indications of proofs.

27.4 Early Hint About the Monster?

I have mentioned the 1972 Gainesville group theory meeting a few times, where Suzuki and Janko discussed possibilities for finding new simple groups. My talk [89, 90] was about extraspecial groups 2_ε^{1+2n} and their upwards extensions by subgroups of their outer automorphism group $O^\varepsilon(2n,2)$. I gave an easy sufficient condition for such an extension to be nonsplit. One application was that an upwards extension $2_+^{1+24}.Co_1$ is nonsplit. During my lecture, I wondered aloud about the possibility of a simple group with a subgroup of shape $2_+^{1+24}.Co_1$ as centralizer of an involution. The audience was attentive but reserved. Conway, in the front row, half-smiled for a half-second. This possibility actually came to life in late 1973.

27.5 The Dempwolff Group

Because sporadic group world is full of exceptional phenomena, there was special interest in group extensions. Unusual extensions were regular things to be observed in sporadic groups, e.g. centralizers and other local subgroups. For general background, see [106].

Around 1972, Ulrich Dempwolff worked on $H^2(GL(n,2),\mathbb{F}_2^n)$. It is zero except at $n = 3,4,5$ where it is 1-dimensional. The cases $n = 3,4$ had been well known. The case $n = 5$ was Dempwolff's result [53]. It turns out that the nonsplit extension E with $O_2(E) \cong 2^5$ and $E/O_2(E) \cong GL(5,2)$ appears as a subgroup of the Thompson group F_3 and in the exceptional Lie group $E_8(\mathbb{C})$. The group F_3 does not embed in $E_8(\mathbb{C})$ but it does embed in $E_8(3)$. For more on the Dempwolff group and E_8, see [99]. The Thompson group arises in a point about modular moonshine and vertex algebras [16]. The articles [116, 117] resolve the Borcherds-Ryba concern and give a vertex algebra style proof that F_3 embeds in $E_8(3)$.

27.6 Naming Sporadic Groups

Policies for naming sporadic groups have generally included the discoverers' names but sometimes also the constructors'. As time passed, the constructors' names were gradually dropped. The groups once named Higman-Janko-McKay and Higman-Held-McKay (the "Higman" in each of these two names refers to Graham Higman), Lyons-Sims and O'Nan-Sims became J_3, $Held$, Ly, $O'N$, respectively. If timing had been different for Graham Higman's work or that of Donald Higman and Charles Sims, the Higman-Sims group *could* have been named the Higman-Higman-Sims group.

One unusual case is the Hall-Janko group, HJ, found independently by Marshall Hall, Jr. (as part of his studies in the mid 1960s of groups of order less than a million) and by Zvonimir Janko. When one is about to speak the sentence "The groups Janko discovered are J_1, HJ, J_3 and J_4", there is understandable temptation to substitute J_2 for HJ. Yielding would be an injustice to Marshall Hall, Jr. I encourage everyone to use the notation HJ, not J_2.

Perhaps the most unusual case is the Monster, which, following tradition, might have been called the Fischer-Griess group if Conway had not promoted the name Monster so aggressively. Conway's explanation was that the group was so special that it should belong to no individuals. He added that this policy was an analogue of his introducing the names $\cdot 0, \cdot 1, \cdot 2, \cdot 3$ in [39, 40] around 1968 for his newly found groups, though he publicly admitted wishing that someone would change those "ridiculous names". He got his wish in [45]: they are now called Co_0, Co_1, Co_2, Co_3. This represents improvement: in the Baby Monster, the centralizer of a 2-central involution is written $2^{1+22}.\cdot 2$ or $2^{1+22}.(\cdot 2)$ in the old system but $2^{1+22}.Co_2$ in the new system.

A consequence of the popular use of Monster terminology was that I was not widely credited for my 1973 discovery until the late 1970s, even though I told group theorists about it regularly. Fischer did get general credit for his discovery from the beginning.

At an algebra conference at Ohio State University in 1993, the twentieth year of Monster-awareness, I introduced the notation $\mathbb{M}$ for the Monster. Conference participants, including Conway, immediately liked the idea. It is now standard. Actually, I prefer to double stroke the first vertical (similar to IM, as when I write by hand) rather than the last vertical, but such a style is not available in Latex.

27.7 Commutative Nonassociative Algebras

The study of commutative nonassociative algebras associated to finite groups has continued. Finite dimensional, commutative nonassociative algebras occurred as homogeneous summands of certain VOAs, as described in a previous section. Their properties [182] inspired several sets of axioms for classes of finite dimensional algebras. I am aware of axioms for the theories *Majorana representations* [151] and *axial algebras* [136, 137]. The special involutions in both of these theories ought to be called *Miyamoto involutions* since they are analogues of the important original idea of Miyamoto [182]. The book [151] attributes the axioms for the algebra properties and involutions to the mathematical physicist Majorana, but this claim is very doubtful. I followed the references given in [151] and looked through Majorana's writings and came up with no corroborating evidence. Literature on these algebras includes [33, 78, 79, 80, 151, 152, 150, 149, 153, 154, 148, 147] and [134, 135, 136, 137, 33, 34, 50, 51, 162, 177].

The book [151] claims an existence and uniqueness proof of the Monster by finite geometries and group amalgam methods, but does not acknowledge the first uniqueness proof and nine orbit theorem proof in [108] (although it does refer to [108] for some information).

27.8 About the Suzuki Series

The series of simple Suzuki groups [221] is now denoted $Sz(q)$ or ${}^2B_2(q)$, $q = 2^{1+2k}$; it is simple for $k \geq 1$. The group has order $q^2(q-1)(q^2+1)$ and acts doubly transitively on the set of q^2+1 Sylow 2-groups. In 1971, Jacques Tits told me that when news of the Suzuki series of groups and their orders reached him, he gave his own construction without seeing the proof. See [231]. Marshall Hall, Jr. made an original construction of $Sz(8)$ [130].

Each group in the Suzuki series has order relatively prime to 3. About 1970, John Thompson proved that they are the only nonabelian simple groups with this property. The first published proof of this result was by George Glauberman [84]. A shorter proof was given by Bernd Stellmacher [220].

The group $B_2(2) \cong Sym_6$ and $Sz(2) = {}^2B_s(2) \cong Frob(20)$, the unique group of order 20 with trivial center. Here is a way to understand $Sz(2)$. For Sym_n, every automorphism is inner except for $n = 6$ in which case $Aut(Sym_6)/Inn(Sym_6) \cong \mathbb{Z}_2$ and there is a non-inner automorphism of order 2 whose fixed point subgroup is isomorphic to $Frob(20)$. It interchanges the conjugacy classes of (12) and $(12)(34)(56)$. One way to see existence of such an automorphism is to use the Frattini argument for the normalizer of a Sylow 5-group in the normal subgroup $Inn(Sym_6)$ of $Aut(Sym_6)$ and the fact that every automorphism of $Frob(20)$ is inner.

27.9 The Groups of Ree Type

Definition 27.1. *A finite group of Ree type is a finite simple group which contains an involution whose centralizer has the form $2 \times PSL(2,q)$, for q an odd power of 3, $q \geq 27$, and for which the Sylow 2-subgroups are elementary abelian of order 8. We denote such a group by $R(q)$.*

Harold Ward [237] considered G, a finite simple group of with elementary abelian Sylow-2 subgroup and an involution centralizer of shape $2 \times PSL(2,q)$, for $q > 5, q \equiv 3,5 (mod\,8)$. Under some mild additional hypotheses, he showed that q is an odd power of 3 and that G and the Ree group ${}^2G_2(q)$ both share the group order $q^3(q-1)(q^3+1)$ and both possess doubly transitive representations of degree q^3+1.

Uniqueness of the groups of Ree type was an especially difficult problem in the classification of finite simple groups. The desired result was that $R(q) \cong {}^2G_2(q)$ for all q. This was considered an analogue of characterization problems for the doubly transitive groups $PSL(2,q), PSU(3,q)$ for prime powers q (say $q = p^e$ for a prime $p \geq 2$) and $Sz(q) = {}^2B_2(q)$ for $q = 2^{1+2n}, n \geq 1$.

Let G be one of the above groups, $\mathbb{F}_q$ its associated finite field, q a power of the prime p, B a 1-point stabilizer and H a 2-point stabilizer. The subgroup H is cyclic. There is an element $w \in G$ of order 2 so that w normalizes H and $G = B \cup BwB$. An element of G has the form uh or $uwhv$, where $h \in H$ and $u, v \in U := O_p(B)$. The multiplication table of G depends on knowing wxw for $x \in K$. Since $wxw \in BwB$ if $x \neq 1$, we have an expression $wxw = y_1 w y_2 z$ for unique $y_1, y_2 \in U \setminus \{1\}$ and $z \in H$. The functions which take x to y_1, y_2, z, respectively, can be described by using the field $\mathbb{F}_q$. Uniqueness of G would follow from a proof that these functions are unique. The difficulty of proving the latter increases with the nilpotence class of U, which is 1 (i.e., U is abelian) for the $PSL(2,q)$ case, class 2 for the cases $PSU(3,q)$ and $Sz(q)$ and finally class 3 for the groups of Ree type.

John Thompson [225] studied the above functions for the groups of Ree type to initiate a uniqueness program. A field automorphism on $\mathbb{F}_q$ which appeared in his very technical formulas resisted determination. If its square were the Frobenius map $x \mapsto x^3$, uniqueness of Ree type groups would follow.

By the late 1960s, the uniqueness challenge for the groups of Ree type was viewed as a possible serious obstruction to completing the CFSG. If some groups of Ree type other than ${}^2G_2(q)$ were to exist, should they have been considered sporadic groups, or groups of Lie type, or would their existence establish a new category within the simple groups? Recall that the Suzuki series of groups was discovered and proved to exist, but not initially recognized as belonging to the world of Lie theory.

Enrico Bombieri used elimination theory to deduce a contradiction from the Thompson formulas when the field automorphism is not the right one, provided q is large enough. David Hunt and Andrew Odlyzko, independently, treated small cases which Bombieri's methods did not. See [14] and its review in MathSciNet which has a detailed summary of this remarkable algebraic achievement. Uniqueness of the groups of Ree type were thus established.

When I was looking for a thesis topic in late 1969, I naïvely thought of trying to prove uniqueness of the Ree type groups, which I had seen on a list of open problems near the end of Gorenstein's book [86]. My advisor, John Thompson was skeptical but showed me preprints to some of his articles [225]. About a month later, I gave up. Thompson commented "Good decision" and suggested other ideas.

27.10 Unpublished and Unresolved

As the momentum built towards the CFSG, starting in the 1970s, there was a noticeable pattern of claims announced but left unpublished for a long time. Informal notes with news about finite simple groups were circulated regularly. Citations of such unpublished items appeared in reference lists in published articles. I think of some computer proofs and explorations, announcements about properties of several new sporadic groups (3-transposition groups, Rudvalis group, the Monster), characterizations by certain centralizers of involutions, and uniqueness results ($\mathbb{M}$, J_4). In the early 1980s, when Daniel Gorenstein announced that the CFSG was complete, he was expecting CFSG researchers to resolve ongoing programs in the near future. Unfortunately, some were not completed for years to come. A major work [10] was needed to resolve the quasithin group case.

27.11 Mein Greisenalterstraum

Mein Greisenalterstraum consists of wishes that we would have a useful set of axioms for the sporadic groups. Ideally, axioms should enable efficient existence and uniqueness proofs, fairly uniform procedures for determining representations and conjugacy classes. They should affirm community with the rest of the finite simple groups, demystify the moonshine connections found so far and enable discovery of new ones.

Work on Monstrous Moonshine and finite groups in vertex algebra theory may be good steps in this direction. At this time, however, not every sporadic group is involved.

The terminology Greisenalterstraum makes references to Kronecker's Jugendtraum, the challenging pronunciation of my family name Griess and the fact that I am no longer a young man. About a century ago, my ancestors in the USA gave up the traditional German pronunciation of my name (similar to "Greece") in favor of "grys", which sounds like the German word Greis, meaning a very old man.

27.12 Humor

A few examples come to mind.

(a) [94], page. 123. Shortly after [129] was published, Marshall Hall, as an editor of the Journal of Algebra, received a short submission titled "The simple group of order 604801". David Wales confirms this story.

(b) Around 1965–1975, it seemed that one's chances to discover a sporadic group were greater for those employed at state universities.

(c) Dan Frohardt told me that Mike O'Nan jokingly asked "Who wants one now?" after he found his simple group, as if sightings of new sporadic groups could go on indefinitely. He may have been trying to lower feelings of disappointment. There had just been an interval without any discoveries (early 1970 to spring 1972), ended by Rudvalis's announcement.

(d) Serge Lang noted the Monstrous Moonshine excitement, which got started with the number 196884 (=1+196883). Serge, who was very conscious about politics, told me around 1984 that the way he remembers 196884 is to recall 1968 (year of street and societal conflict in Paris and Chicago) and 1984 (the title of George Orwell's famous novel).

(e) In [235], Tits referred to J_1 as "le méchant nain" (the wicked dwarf), due to the difficulty of determining whether it embeds in the Friendly Giant = the Monster [92]. Eventually, Robert Wilson proved that J_1 does not embed [242].

(f) The order of the Hall-Janko group is $604800 = 2^7 3^3 5^2 7 = 4{\cdot}5{\cdot}6{\cdot}7{\cdot}8{\cdot}9{\cdot}10 = 10!/3!$, possibly a new observation. Also, the order of HJ is the number of seconds in a week. Awareness of this amusing fact circulated decades ago. I do not know its origin.

(g) The 1980 film It's My Turn, starring Jill Clayburgh, Michael Douglas, and Charles Grodin, depicted an imaginary early career female mathematician, Kate Gunzinger, played by JC. While a young faculty member at Princeton, Dick Gross was hired as a consultant for that film. Jill Clayburgh attended a math department tea time at Princeton to observe. Early in the film, Kate Gunzinger gave a decent presentation of the Snake Lemma. We saw her consider fusion and other local behaviors which might occur within a putative finite simple group. This young researcher, probably hoping to discover a new finite simple group, was too open about her thoughts in front of an aggressive graduate student. This unrealistic scene struck me as funny.

28. Appendix of Notations and FSG Orders

28.1 Notations and Terminology for Finite Group Theory

Our usage is generally consistent with that of [86, 146]. Later, we discuss notations for the finite simple groups

In a group, $\langle S \rangle$ means the subgroup generated by the subset S.

The order of a group is its cardinality;

The order of a group element is the order of the cyclic group it generates;

An involution is an element of order 2 in a group;

G' or $[G,G]$ denotes the commutator subgroup of the group G;

A perfect group is a group which equals its commutator subgroup;

A quasisimple group is a perfect group so that the quotient by its center is a nonabelian simple group;

$O_p(G)$ denotes the largest normal p-subgroup of the finite group G (p is a prime);

$Z(G)$ denotes the center of G;

$Frob(n)$ refers to some Frobenius group of order n; it has the form KH, where K is a normal subgroup and H is a complement with the property that if $x \in H, x \neq 1$ then $C_K(x) = 1$. One example is $Frob(10) \cong Dih_{10}$ and another is $Frob(20)$, which has a normal subgroup of order 5 and a cyclic complement of order 4.

The finite classical groups associated to sesquilinear forms are $GL(n,q)$, $SL(n,q)$, $O^\varepsilon(n,q)$, $\cdots$ (n refers to the size of the square matrices, q to the cardinality of the finite field $\mathbb{F}_q$, and ε indicates type of quadratic form). For unitary groups and others which involve a degree 2 field extension, q denotes cardinality of the ground field;

Alt_n refers to the alternating group of degree n;

Dih_{2m} refers to the dihedral group of order $2m$;

$Quat_{4m}$ refers to the quaternion group of order $4m$;

FSG means finite simple group(s);

CFSG means the classification of FSG;

$C_G(S)$, $N_G(S)$ means the centralizer, normalizer (resp.) of the subset S in the group G;

When indicating a group, p^n means an elementary abelian group of order p^n;

2^{1+2r}_ε, an extraspecial 2-group of type $\varepsilon = \pm$;

p^{1+2r} an extraspecial p-group;

$p^{a+b+c+\cdots}$ means a group with an increasing chain of normal subgroups of orders $p^a, p^{a+b}, p^{a+b+c}, \ldots$ so that the successive quotients are elementary abelian of order $p^a, p^b, p^c, \ldots$.

Direct product notation: p^n means $p \times \cdots \times p$ (n factors); example: $2 \times 2 \times Alt_5$ means $\mathbb{Z}_2 \times \mathbb{Z}_2 \times Alt_5$;

Central product notation: $A \circ B$ means a group which is a product of subgroups A, B so that $[A,B] = 1$, i.e., $xy = yx$ for all $x \in A, y \in B$. It is isomorphic to a quotient of $A \times B$ by a central subgroup which meets each of A and B trivially.

Group extensions: in general $A.B$ means a group with normal subgroup A and quotient B; $A{:}B$ means split extension, $A{\cdot}B$ means nonsplit extension; example $2{\cdot}Alt_5 \cong SL(2,5)$; a group $2.(2 \times 2)$ could be $2^3, 2 \times 4, Dih_8$ or $Quat_8$. We are not consistent about use of dots, so that $2Alt_5$ would mean some extension $2.Alt_5$; in context, the meaning ought to be clear.

Compound group extensions: $A.B.C\ldots$ indicates a group with normal subgroups isomorphic to A, some $A.B$, some $(A.B).C$, etc.

28.2 Notations for the Finite Simple Groups

Since the 1950s, notations have been fairly stable, influenced by the articles of [28, 32, 4, 3, 217]. The E-notation of Emil Artin for all exceptional groups of Lie type has gone out of fashion, replaced by G_2, F_4, E_6, E_7, E_8. For use of E_2 instead of G_2, see [225, 224].

The authors of the Atlas [45], John Conway, Robert Curtis, Simon Norton, Richard Parker and Robert Wilson, introduced some notations inspired by Monstrous Moonshine [43], like F_{2+} which augments the F_n notation I introduced in the mid-1970s for a few sporadics associated to the Fischer-Griess Monster.

Unfortunately, those authors also promoted notation changes for the classical groups which created discomfort and confusion. Here are some comparisons. Let Q be a quadratic form on an n dimensional vector space over the field K. The orthogonal group is widely denoted $O(Q)$, or by $O^\varepsilon(n,q)$ or $O^\varepsilon_n(q)$ if the vector space has dimension n over the finite field $K = \mathbb{F}_q$ and where ε indicates the Witt index. In the Atlas, $O^\varepsilon(n,q)$ has a different meaning, which is the kernel of the spinor norm modulo scalars if q is odd and is $\Omega^\varepsilon(n,q)$, the kernel of the Dickson homomorphism [58], if q is even. Also, given a quadratic field extension L/K and a Hermitian form Q on an n-dimensional vector space over L, one can define a unitary group $U(Q)$ and when $K = \mathbb{F}_q$, $L = \mathbb{F}_{q^2}$ and the vector space is L^n; this unitary group is widely denoted $U(n,q)$ or $U_n(q)$. The Atlas notation for the latter is $GU(n,q)$ and the Atlas notation for $PSU(n,q)$ is $U(n,q)$ or $U_n(q)$. The group denoted $O^+(8,3)$ in standard notation would, in Atlas notation, be expressed as $2{\cdot}O^+(8,3).2^2$ or $2{\cdot}O_8^+(3).2^2$. I see no real advantage to their classical group notations and strongly discourage their use. See [45], pages x–xiii.

The orders of the Finite Nonabelian Simple Groups

	Alternating groups	
$A_n,\ n \geq 5$	$\frac{1}{2}(n!)$	
Group G of Lie type	$d \cdot \lvert G \rvert$	d
$A_n(q)$	$q^{n(n+1)/2} \prod_{i=1}^{n}(q^{i+1}-1)$	$(n+1, q-1)$
$B_n(q),\ n \geq 1$	$q^{n^2} \prod_{i=1}^{n}(q^{2i}-1)$	$(2, q-1)$
$C_n(q),\ n > 2$	$q^{n^2} \prod_{i=1}^{n}(q^{2i}-1)$	$(2, q-1)$
$D_n(q),\ n > 3$	$q^{n(n-1)}(q^n-1) \prod_{i=1}^{n-1}(q^{2i}-1)$	$(4, q^n-1)$
$G_2(q)$	$q^6(q^6-1)(q^2-1)$	1
$F_4(q)$	$q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$	1
$E_6(q)$	$q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)$	$(3, q-1)$
$E_7(q)$	$q^{63}(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^{10}-1)(q^8-1)(q^6-1)(q^2-1)$	$(2, q-1)$
$E_8(q)$	$q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$	1
${}^2A_n(q),\ n > 1$	$q^{n(n+1)/2} \prod_{i=1}^{n}(q^{i+1}-(-1)^{i+1})$	$(n+1, q+1)$
${}^2B_2(q),\ q = 2^{2m+1}$	$q^2(q^2+1)(q-1)$	1
${}^2D_n(q),\ n > 3$	$q^{n(n-1)}(q^n+1) \prod_{i=1}^{n-1}(q^{2i}-1)$	$(4, q^n+1)$
${}^3D_4(q)$	$q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$	1
${}^2G_2(q),\ q = 3^{2m+1}$	$q^3(q^3+1)(q-1)$	1
${}^2F_4(q),\ q = 2^{2m+1}$	$q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$	1
${}^2E_6(q)$	$q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)$	$(3, q+1)$

Sporadic Groups

M_{11}	$7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$	He	$2^{10}3^3 5^2 \cdot 7^3 \cdot 17$
M_{12}	$95040 = 2^6 \cdot 3^3 \cdot 5 \cdot 11$	Ly	$2^8 3^7 5^6 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$
M_{22}	$443520 = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	$O'N$	$2^9 3^4 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
M_{23}	$10200960 = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	Co_1	$2^{21}3^9 5^4 7^2 11 \cdot 13 \cdot 23$
M_{24}	$244823040 = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	Co_2	$2^{18}3^6 5^3 7 \cdot 11 \cdot 23$
J_1	$175560 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	Co_3	$2^{10}3^7 5^3 7 \cdot 11 \cdot 23$
$HJ = J_2$	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	Fi_{22}	$2^{17}3^9 5^2 7 \cdot 11 \cdot 13$
J_3	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$	Fi_{23}	$2^{18}3^{13}5^2 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
J_4	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$	Fi'_{24}	$2^{21}3^{16}5^2 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
HS	$2^9 3^2 5^3 \cdot 7 \cdot 11$	F_5	$2^{14}3^6 5^6 \cdot 7 \cdot 11 \cdot 19$
McL	$2^7 3^6 5^3 \cdot 7 \cdot 11$	F_3	$2^{15}3^{10}5^3 7^2 13 \cdot 19 \cdot 31$
Suz	$2^{13}3^7 5^2 7 \cdot 11 \cdot 13$	F_2	$2^{41}3^{13}5^6 7^2 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$
Ru	$2^{14}3^3 5^3 7 \cdot 13 \cdot 29$	$F_1 = \mathbb{M}$	$2^{46}3^{20}5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

29. Table of the Finite Nonabelian Simple Groups and Their Orders

29.1 Organization of the Table

According to the CFSG, *a finite nonabelian simple group is a normal subgroup of a group listed in the table above.* The meaning of notations for the alternating groups and sporadic groups should be clear. I use Alt_n for alternating group and Dih_n for dihedral group to avoid possible confusion with Lie theoretic notations A_n and D_n.

For a groups G of Lie type, we explain notations in the next paragraph. In the table, we display $d \cdot |G|$ in column 2 and d is in column 3. Some of the groups of Lie type are not simple, and there are cases of isomorphisms between composition factors of members of different families. The latter situations are discussed in the comment section.

The Chevalley groups, which are listed here as $A_n(q), \cdots, E_8(q)$, are subgroups of the automorphism group of a Lie algebra $L(q)$ obtained from a Chevalley basis, using coefficients a finite field $\mathbb{F}_q$ of cardinality the prime power q. In the usual notation [32], these groups are generated by the automorphisms $x_r(t)$. These groups have trivial center.

The Steinberg groups, notated here by some Chevalley group symbol $X_n(q)$ of type ADE preceded by a superscript ^{2} or ^{3}, are the fixed point subgroups in the Chevalley groups $X_n(q)$ of an automorphism of the Lie algebra $L(q)$.

The Suzuki and Ree groups, notated here by some Chevalley group symbol $X_n(q)$ of type $B_2(q), F_4(q), G_2(q)$ preceded by a superscript ^{2}, are the fixed point subgroups of an automorphism of the groups $X_n(q)$. For ${}^2B_2(q)$ and for ${}^2F_4(q)$, q is an odd power of 2. For ${}^2G_2(q)$, q is an odd power of 3.

29.2 Comments

(a) For each q and for all $n \geq 2$, the groups $B_n(q)$ and $C_n(q)$ have the same order. For all q, $B_2(q) \cong C_2(q)$. Also, for all $n \geq 3$, $B_n(q) \cong C_n(q)$ if and only if q is even.

Note that the well known isomorphisms $PSU(2,q) \cong PSL(2,q) \cong PSO(3,q)$ are not represented by groups listed here due to our restrictions $n \geq 2$ for the families $B_n(q)$ and ${}^2A_n(q)$. Similarly, $P\Omega^-(6,q) \cong PSU(4,q)$ and $P\Omega^+(6,q) \cong PSL(4,q)$ for all q, but these are not represented here since we require $n \geq 4$ for the families $D_n(q)$ and ${}^2D_n(q)$.

(b) For all but finitely many cases, the indicated group of Lie type is simple.

The solvable groups among the finite groups of Lie type are just $A_1(2)$, $A_1(3)$, ${}^2A_2(2)$ and ${}^2B_2(2) \cong Frob(20)$.

There are four nonsolvable and nonsimple cases here. Three of these groups have a normal, nonabelian simple subgroup of index 2: $B_2(2) \cong Sym_6$; $G_2(2)$, whose derived group is isomorphic to $PSU(3,3) \cong {}^2A_2(3)$; and ${}^2F_4(2)$, whose derived group is the Tits simple group. The fourth group in this category is ${}^2G_2(3) \cong PSL(2,8){:}3$, which has a normal subgroup of index 3.

(c) A given finite simple group occurs just once in the above table with the exceptions mentioned in (a) and those isomorphisms listed below, which involve finite fields of multiple characteristics or alternating groups (or a few cases from (a)).

$PSL(2,7) \cong GL(3,2)$, $(A_1(7) \cong A_2(2))$;
$Alt_5 \cong PSL(2,4) \cong PSL(2,5)$ $(A_1(4) \cong A_1(5))$;
$Alt_6 \cong A_1(9) \cong B_2(2)'$;
$Alt_8 \cong GL(4,2) \cong \Omega^+(6,2)$ $(A_3(2) \cong D_3(2))$;
$PSU(3,3) \cong G_2(2)'$;
$PSO(5,3) \cong PSp(4,3) \cong PSU(4,2)$ $(B_2(3) \cong C_2(3) \cong {}^2A_3(2))$.

(d) This table is modeled after the table on p. 169 of [94], though with corrections of notations or orders for the entries $G_2(q)$, $O'N$, Suz, Fi_{24}', F_3, F_5.

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