Modular Forms and Anomaly Cancellations in String Theory and M-Theory

by Fei Han[*] and Kefeng Liu[†‡]

1. Physics Background

String theory is a candidate theory in particle physics for reconciling quantum mechanics and general relativity, i.e. quantum gravity. Moreover, it's also a candidate of TOE ("theory of everything"), describing all the known fundamental forces and matters in our universe in a mathematically complete system. In conventional theories, elementary particles are mathematical points, whereas, the fundamental objects in string theory are 1-dimensional oscillating lines or loops.

[*] Department of Mathematics, National University of Singapore, Block S17, 10 Lower Kent Ridge Road, Singapore 119076
E-mail: mathanf@nus.edu.sg
[†] Department of Mathematics, University of California at Los Angeles, Los Angeles, CA 90095, USA
E-mail: liu@math.ucla.edu
[‡] Mathematical Science Research Center, Chongqing University of Technology, Chongqing, 400054, China

In the 1983's seminal paper of Alvarez-Gaumé and Witten [3], it was shown that in certain parity-violating gravity theory in $4k+2$ dimensions, when Weyl fermions of spin-$\frac{1}{2}$ or spin-$\frac{3}{2}$ or self-dual anti-symmetric tensor field are coupled to gravity, perturbative anomalies $\widehat{I}_{1/2}$, $\widehat{I}_{3/2}$ and $\widehat{I}_A$ occur and moreover the authors showed that there are cancellation formulas for these anomalies in dimensions $2,6,10$. More precisely, in dimensions 2, 6, 10, the following anomaly cancellation formulas hold respectively,

$$-\widehat{I}_{1/2}+\widehat{I}_A=0, \tag{1.1}$$

$$21\widehat{I}_{1/2}-\widehat{I}_{3/2}+8\widehat{I}_A=0, \tag{1.2}$$

and

$$-\widehat{I}_{1/2}+\widehat{I}_{3/2}+\widehat{I}_A=0. \tag{1.3}$$

Type IIB superstring theory is such an anomaly-free ten-dimensional parity-violating theory.

Before 1984, many physicists were still skeptical about string theory, as there was "what they believed was almost certainly the fatal weakness of the theory: it could not explain why left–right symmetry is broken in some radioactive decays, as Lee and Yang had successfully predicted almost three decades before." (Chap. 9 in [7]). The great discovery of Green and Schwarz [8] in 1984 changed this situation and triggered the first revolution of string theory. "As is well known, in the early 1980s, it appeared that superstrings could not describe parity violating theories, because of quantum mechanical inconsistencies due to anomalies. The discovery that in certain cases the anomaly could cancel was important for convincing many theorists that string theory is a promising approach to unification.", Schwarz remarked in [18]. The discovery of Green–Schwarz is that in type I string theory with 10 dimensional spacetime and gauge group $SO(32)$, there is the following factorization formula of anomalies,

$$I_{\mathrm{sugra}} + I_{\mathrm{gau}} = Y_4 \cdot Y_8, \tag{1.4}$$

where I_{sugra} is the anomaly contributed by supergravity and I_{gau} is the anomaly contributed by gauginos; and then one can design a Chern–Simons term to add to the effective action to cancel the Yang–Mills anomaly. In other types of string theories, when the gauge group is $E_8 \times E_8$, there are similar factorization formulas like (1.4). The formulas (1.1)–(1.3) and (1.4) were all obtained by hand after long computation.

2. Mathematical Understanding of the Anomalies

The mathematical understanding of these anomalies is related to the Atiyah–Singer index theory [2, 1], which was one of the greatest achievements in mathematics of the 20th century. This theory establishes connections among analysis, topology and geometry on differentiable manifolds.

In mathematical lingo, the above anomaly cancellation problem is the following. Consider a fiber bundle $Z \to M \to Y$, where a family of spin manifold Z is parametrized by another connected manifold Y and the family as a whole forms a manifold M. In the string theory situation, Z is a $4k+2$ dimensional spin manifold and Y is the quotient space of the space of metrics on Z by the action of certain subgroup of diffeomorphism group of Z. Let D^Z be the family of Atiyah–Singer Dirac operators and D^Z_{sig} be the family of signature operators on the fiber bundle respectively. D^Z and D^Z_{sig} are first order elliptic operators. Let $T^{\mathrm{v}}Z$ be the bundle of vertical tangent spaces on M. Then expressed by mathematical concepts, the anomalies are

$$\begin{aligned}
\widehat{I}_{1/2} &= R^{\mathcal{L}_{D^Z}} = \left\{ \int^Z \widehat{A}(T^{\mathrm{v}}Z) \right\}^{(2)},\\
\widehat{I}_{3/2} &= R^{\mathcal{L}_{D^Z \otimes T_{\mathbf{C}}Z}} - R^{\mathcal{L}_{D^Z}}\\
&= \left\{ \int^Z \widehat{A}(T^{\mathrm{v}}Z)\mathrm{ch}(T^{\mathrm{v}}_{\mathbf{C}}Z) - \int^Z \widehat{A}(T^{\mathrm{v}}Z) \right\}^{(2)},\\
\widehat{I}_A &= -\frac{1}{8} R^{\mathcal{L}_{D^Z_{sig}}} = -\frac{1}{8} \left\{ \int^Z \widehat{L}(T^{\mathrm{v}}Z) \right\}^{(2)},
\end{aligned}$$

where $\mathcal{L}$ represents certain line bundle, called the determinant line bundle, on the base Y associated to the elliptic operators, R stands for the curvature of the determinant line bundle, $\widehat{A}(T^{\mathrm{v}}Z)$ and $\widehat{L}(T^{\mathrm{v}}Z)$ are the Hirzebruch $\widehat{A}$-class and $\widehat{L}$-class of $T^{\mathrm{v}}Z$. Therefore (1.1)–(1.3) become

$$R^{\mathcal{L}_{D^Z_{sig}}} + 8R^{\mathcal{L}_{D^Z}} = 0, \tag{2.1}$$

$$R^{\mathcal{L}_{D^Z_{sig}}} + R^{\mathcal{L}_{D^Z \otimes T_{\mathbf{C}}Z}} - 22R^{\mathcal{L}_{D^Z}} = 0, \tag{2.2}$$

and

$$R^{\mathcal{L}_{D^Z_{sig}}} - 8R^{\mathcal{L}_{D^Z \otimes T_{\mathbf{C}}Z}} + 16R^{\mathcal{L}_{D^Z}} = 0. \tag{2.3}$$

We see that the above anomaly cancellations are just cancellations of the curvatures of determinant line bundles of certain elliptic operators.

Now let Z be 10 dimensional and F a vector bundle on M with structure group $SO(32)$. The Green–Schwarz factorization formula (1.4) can be written in mathematical lingo as

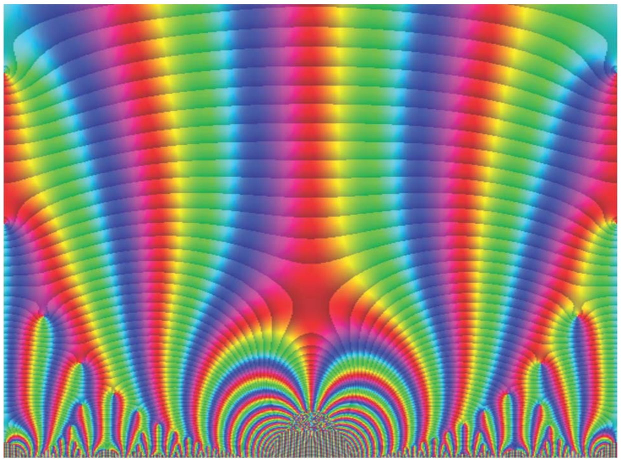
$$\begin{aligned}
&\{\widehat{A}(T^{\mathrm{v}}Z)\mathrm{ch}(\wedge^2 F_{\mathbf{C}})\}^{(12)} + \{\widehat{A}(T^{\mathrm{v}}Z)\mathrm{ch}(T^{\mathrm{v}}_{\mathbf{C}}Z)\}^{(12)}\\
&\quad - 2\{\widehat{A}(T^{\mathrm{v}}Z)\}^{(12)}\\
&= (p_1(T^{\mathrm{v}}Z) - p_1(F))\\
&\quad \cdot \frac{1}{24}\Bigg(\frac{-3p_1(T^{\mathrm{v}}Z)^2 + 4p_2(T^{\mathrm{v}}Z)}{8}\\
&\quad - 2p_1(F)^2 + 4p_2(F) + \frac{1}{2}p_1(T^{\mathrm{v}}Z)p_1(F) \Bigg),
\end{aligned} \tag{2.4}$$

where $p_i(T^{\mathrm{v}}Z)$ and $p_i(F), 1 \le i \le 2$ are the Pontryagin forms of $(T^{\mathrm{v}}Z, \nabla^{T^{\mathrm{v}}Z})$ and (F, ∇^F) respectively. Here for simplicity we omit the process of integration along the fiber to express the terms in the left hand side as curvatures of determinant line bundles.

3. Unify the Anomaly Cancellations via Modular Forms

It was discovered in [16] that formulas (2.1)–(2.3) can all be derived from modular forms in number theory, and by using modular forms one can generalize the above cancellation formulas to general $4k+2$ dimension.

A modular form is a holomorphic function on the upper half complex plane, which possesses certain symmetry with respect to the Möbius transformation $\tau \to \frac{a\tau+b}{c\tau+d}$. Modular forms are very important in arithmetics and the generalizations of modular forms are automorphic forms. Let $q = e^{2\pi\sqrt{-1}\tau}$. One can actually show that $R^{\mathcal{L}_{D^Z_{sig}}}$ is a piece in the q-series of a modular form $P_1(\tau)$ while $R^{\mathcal{L}_{D^Z \otimes T_{\mathbf{C}}Z}}$ as well as $R^{\mathcal{L}_{D^Z}}$ are pieces in the q-series of another modular form $P_2(\tau)$ and moreover $P_1(-\frac{1}{\tau}) = (2\tau)^{2k+2}P_2(\tau)$. Then $R^{\mathcal{L}_{D^Z_{sig}}}$ and $R^{\mathcal{L}_{D^Z \otimes T_{\mathbf{C}}Z}}, R^{\mathcal{L}_{D^Z}}$ are related because $P_1(\tau)$ and $P_2(\tau)$ are modularly related, i.e. there is some hidden symmetry. This gives us the generalizations of the anomaly



cancellation formulas (1.4)–(2.2): if the fiber is $8m+2$ dimensional,

$$R^{\mathcal{L}_{D^Z_{sig}}} - 8\sum_{r=0}^{m} 2^{6m-6r} R^{\mathcal{L}_{D^Z\otimes b_r(T^V_{\mathbf{C}}Z)}} = 0; \tag{3.1}$$

if the fiber be $8m-2$ dimensional,

$$R^{\mathcal{L}_{D^Z_{sig}}} - \sum_{r=0}^{m} 2^{6m-6r} R^{\mathcal{L}_{D^Z\otimes z_r(T^V_{\mathbf{C}}Z)}} = 0. \tag{3.2}$$

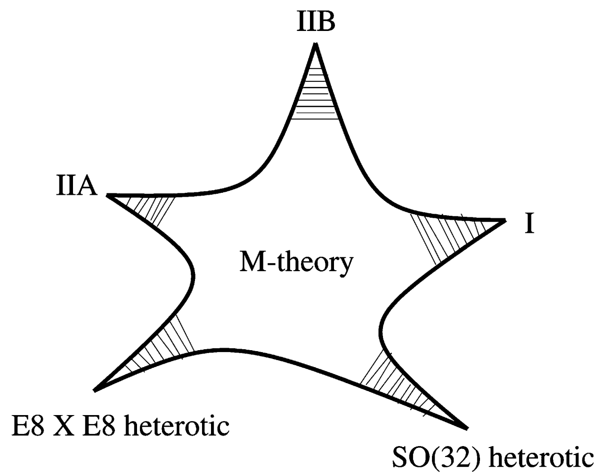
The b_r and z_r are certain operations of the vertical tangent bundle determined by the q-series:

$$-\frac{1}{8} - 3\sum_{n=1}^{\infty}\sum_{\substack{d|n\\ d\ odd}} dq^{n/2},\ \sum_{n=1}^{\infty}\sum_{\substack{d|n\\ n/d\ odd}} d^3q^{n/2}.$$

Along this clue, in [10, 11, 12, 13, 9], we systematically use modular forms to understand anomaly cancellation and factorization formulas.

In [12, 13], we find that the factorization formulas of Green–Schwarz for $SO(32)$ and $E_8\times E_8$ can both be derived from modular forms. In [12], we construct a modular form of weight 6 over some index 2 subgroup of $SL(2,\mathbf{Z})$ by twisting the Witten class (a q-deformed version of the $\widehat{A}$-class, which gives the famous Witten genus) with the Chern character of some q-series of bundle coefficients manufactured out from the positive energy representation of $LSpin(32)$ and a correction factor of E_2, the second Eisenstein series. Then the Green–Schwarz factorization formula for $SO(32)$ turns out to be the consequence of the modularity of this modular form. For the $E_8\times E_8$ case, the corresponding modular form is constructed by twisting the Witten class with the Chern character of some q-series of bundle coefficients manufactured out from the affine representation of LE_8 as well as a correction factor of E_2. This modular form is weight 14 over $SL(2,\mathbf{Z})$. The general pattern for the derivation of the above mentioned 3 fundamental anomaly cancellation formulas is the following: **twist the Witten class by the Chern character of certain q-series of bundle coefficients manufactured out from the auxiliary bundles and a correction factor of the second Eisenstein series. In the case of Alvarez-Gaumé–Witten, the auxiliary bundle is the spinor bundle; in the case of Green–Schwarz, the auxiliary bundles are the vector bundles with structure group $SO(32)$ and $E_8\times E_8$.**

There are five types superstring theories: I, IIA, IIB, HE and HO. In the second revolution of string theory, physicists realized that the five superstring theories are special cases a more fundamental theory: M-theory.



The Witten–Freed–Hopkins anomaly cancellation formula [19, 6] in M theory can be expressed mathematically as the following. Let Z be a 12 dimensional smooth closed spin manifold. Let $x\in H^4(Z;\mathbf{Z})$. Following Witten [19], x determines an isomorphism class of principal E_8 bundles on Z. Let $V(x)$ denote the real adjoint vector bundle associated to the principal E_8 bundle determined by the class x. Denote by $V_{\mathbf{C}}(x)$ the complexification of $V(x)$. Let $\lambda\in H^4(Z,\mathbf{Z}), p\in H^8(Z,\mathbf{Z})$ be the spin classes of Z respectively. Let

$$C(x) = \lambda + 2x \in H^4(Z;\mathbf{Z}).$$

Then one has

(3.3)
$$\begin{aligned}\frac{C(x)[p-C(x)^2]}{48} &= \left\{\frac{1}{2}\widehat{A}(TZ)\mathrm{ch}(V_{\mathbf{C}}(x))\right\}^{(12)} + \frac{1}{4}\{\widehat{A}(TZ)\mathrm{ch}(T_{\mathbf{C}}Z)\}^{(12)}\\ &\quad - \{\widehat{A}(TZ)\}^{(12)}.\end{aligned}$$

This deep formula was still discovered by hand after a long computation. The integrality of the right hand side after integration over Z is a consequence of the Atiyah–Singer index theory. This motivated Freed–Hopkins to develop the algebraic theory of cubic forms and showed that the characteristic classes

λ and p solve the following algebraic equation

$$\widehat{p}\cdot\widehat{x} = 4\widehat{x}^3 + 6\widehat{\lambda}\widehat{x}^2 + 3\widehat{\lambda}^2\widehat{x} \pmod{24} \tag{3.4}$$

for all $\widehat{x} \in H^4(Z;\mathbf{Z})\otimes\mathbf{Z}/24\mathbf{Z}$.

In the recent paper [9], still using the general pattern stated above, we can also derive the Witten–Freed–Hopkins anomaly cancellation formula by constructing a modular form. And this pattern allowed us to generalize the formula to spin^c manifolds and more generally, to orientable manifolds. We find that on spin^c 12-manifolds, there exist spin^c classes $\lambda_c \in H^4(Z,\mathbf{Z}), p_c \in H^8(Z,\mathbf{Z})$ such that if $C_c(x) = \lambda_c + 2x \in H^4(Z;\mathbf{Z})$ and ξ is the complex line bundle for the spin^c structure, then one has

$$\begin{aligned}(3.5)\quad & \\ & \frac{C_c(x)[p_c - C_c(x)^2]}{24} \\ &= \left\{\widehat{A}(TZ)e^{c/2}\mathrm{ch}(V_{\mathbf{C}}(x))\right\}^{(12)} + \left\{\frac{1}{2}\widehat{A}(TZ)e^{c/2}\mathrm{ch}(T_{\mathbf{C}}Z)\right\}^{(12)} \\ &\quad - \frac{1}{2}\left\{\widehat{A}(TZ)e^{c/2}\mathrm{ch}[\xi_{\mathbf{C}}\otimes\xi_{\mathbf{C}} - \xi_{\mathbf{C}} + 2]\right\}^{(12)}.\end{aligned}$$

Consequently, we showed that the characteristic classes λ_c and p_c solve the following algebraic equation

$$\widehat{p}_c\cdot\widehat{x} = 4\widehat{x}^3 + 6\widehat{\lambda}\widehat{x}^2 + 3\widehat{\lambda}_c^2\widehat{x} \pmod{12} \tag{3.6}$$

for all $\widehat{x} \in H^4(Z;\mathbf{Z})\otimes\mathbf{Z}/12\mathbf{Z}$.

In the orientable case, we found similar formulas with the $\widehat{L}$-class by constructing a modular form via the $\widehat{L}$-Witten class that first appeared in Liu's thesis [15], and above mod 24 and mod 12 equations are further weakened to mod 3.

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