Minimal Model Theory for Log Canonical Pairs

by Zhengyu Hu*

Abstract. This is a brief exposition on some of the developments in birational geometry focusing on the minimal model program for log canonical pairs.

1. Introduction

We work over an algebraically closed field k of characteristic zero. Varieties are all quasi-projective.

Given an algebraic variety, several questions arise about its geometry, volume, distribution, curvature and so on. Instead of answering each question for all varieties, mathematicians first find a simpler model for a given variety, which preserves most of the information about the original one. Finding such a model is a huge qualitative task, as preserving geometric properties restricts our manoeuvres significantly.

Two varieties are *birational* if they contain isomorphic open subsets (in Zariski topology). In fact, smooth birational varieties have good common properties such as plurigenera, Kodaira dimension and irregularity. In dimension one, a nice element in a birational class is simply a smooth and projective element. In higher dimension though there are infinitely many such elements in each class. By Hironaka's resolution theorem, every quasi-projective variety is birational to a smooth projective variety.

Question 1.1. *How to classify smooth projective varieties up to birational isomorphism?*

One approach to this has given birth to birational geometry, a theory of more than a century old, with significant progress in the past decades. Several key questions remain unsolved however. A success story is the development of the minimal model program (MMP for short). One fundamental insight that arises from MMP is the necessity of working with *singularities*. We can quantify the type of singularities which occur: the simplest MMP-singularities are called "terminal" and the most complex ones are the so-called "log canonical" (lc for short). There are others in between, for example the "Kawamata log terminal" (klt for short) singularities.

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1.2 Canonical Divisor

When X is smooth, the canonical divisor is the divisor whose associated sheaf $\mathcal{O}_X(K_X)$ is the canonical sheaf $\omega_X := \det \Omega_X$. When X is only normal, K_X is the closure of the canonical divisor of the smooth locus. This plays a central role in birational geometry.

1.3 Classical Theory

Projective curves are one dimensional projective varieties, which are classified by

$$g(X) = \begin{cases} 0 & \text{iff } X \simeq \mathbb{P}^1 \\ 1 & \text{iff } X \text{ is elliptic} \\ \geq 2 & \text{iff } X \text{ is of general type} \end{cases}$$

And in terms of the canonical divisor we have:

$$g(X) = \begin{cases} 0 & \text{iff } \deg K_X < 0 \\ = 1 & \text{iff } \deg K_X = 0 \\ \geq 2 & \text{iff } \deg K_X > 0 \end{cases}$$

We note that, of particular importance is the numerical property of the canonical divisor. It is conjectured that its numerical property fully represents its geometric property, which is known in lower dimensions.

Definition 1.4. For a smooth projective variety X, define the m-th plurigenus as

$$P_m(X) := h^0(X, \omega_X^{\otimes m})$$

Note that $P_1(X) = g(X)$. Define the Kodaira dimension $\kappa(X)$ of X as the largest integer a satisfying

$$0 < \limsup_{m\to\infty} \frac{P_m(X)}{m^a}$$

if $P_m(X) > 0$ for some $m > 0$. Otherwise let $\kappa(X) = -\infty$.

Geometrically, for each m, sections of $H^0(X, \omega_X^{\otimes m})$ define a rational map $X \dashrightarrow Y_m$, and $\dim Y_m = \kappa(X)$ when m is sufficiently divisible.

To get the above classification for surfaces one can use the classical MMP as follows. Pick a smooth projective surface X over k. If there is a -1-curve E (i.e. $E \simeq \mathbb{P}^1$ and $E^2 = -1$) on X, then by Castelnuovo theorem we can contract E by a birational morphism $f: X \to X_1$ where X_1 is also smooth. Now replace X with X_1 and continue the process. In each step, the Picard number $\rho(X)$ drops by one. Therefore, after finitely many steps, we get a smooth projective variety Y with no -1-curves.

$$X \to X_1 \to \ldots \to X_n = Y$$

Such a Y turns out to have strong numerical properties. In fact, it is not hard to show that $Y = \mathbb{P}^2$ or Y is a ruled surface over some curve or that K_Y is nef (i.e. $K_Y \cdot C \geq 0$ for every curve C).

We have:

- If $\kappa(Y) = -\infty \implies Y = \mathbb{P}^2$ or Y is a ruled surface over some curve.
- If $\kappa(Y) = 0 \implies Y$ is a K3 surface, an Enriques surface or an étale quotient of an abelian surface.
- If $\kappa(Y) = 1 \implies Y$ is a minimal elliptic surface, that is, it is fibred over a curve with the general fibre being an elliptic curve.
- If $\kappa(Y) = 2 \implies Y$ is of general type.

Moreover, the last three cases correspond to the situation when K_Y is nef. Here the linear system $|mK_Y|$ is base point free for some $m > 0$.

1.5 Minimal Model Program in Arbitrary Dimension

The skeleton of birational classification is designed by MMP, which aims to find a geometrically simple model for a given algebraic variety, by means of geometric surgeries that eliminate "negative" components at each step. As we presented before, classification of algebraic varieties in dimensions 1 and 2 is classical. The theory of MMP in dimension 3 was settled in the eighties by the work of Mori, Kawamata, Kollár, Reid, Shokurov and others.

One of the major obstacles was that it was not clear how to generalise -1-curves and their contractions to higher dimension. This problem was essentially solved by Mori who replaced -1-curves by the so-called K_X*-negative extremal rays*. Another conceptual progress due to extremal rays was the fact that one could define analogues of ruled surfaces in higher dimension called Mori fibre spaces. A *Mori fibre space* is defined as a fibre type contraction $Y \to Z$ which is a K_Y-negative extremal fibration with connected fibres. And a *minimal* variety is defined as Y having K_Y nef.

Conjecture 1.6 (Minimal model conjecture). *Let X be a smooth projective variety. Then,*

- *If $\kappa(X) = -\infty$, then X is birational to a Mori fibre space $Y \to Z$.*
- *If $\kappa(X) \geq 0$, then X is birational to a minimal variety Y.*

Conjecture 1.7 (Abundance conjecture). *Let Y be a minimal variety. Then, there is a fibration $\phi: Y \to S$ with connected fibres and an ample divisor H on S such that*

$$mK_Y = \phi^* H$$

for some $m > 0$. In particular,

- $\phi(C) = \text{pt.} \iff K_Y \cdot C = 0$ *for any curve C on Y.*
- $\dim S = \kappa(Y) \geq 0$.

The minimal model conjecture holds in dimension ≤ 4 [Mo88, Sho03, Sho93, Kaw92b], etc. in full generality, and in any dimension for varieties of general type [BCHM10] while the abundance conjecture is proved in dimension ≤ 3 [Mi88, Kaw92a].

Definition 1.8 (Flip). Let X be a projective variety with "mild singularities" and $f: X \to Y$ the contraction of a K_X-negative extremal ray of small type (i.e. contracting no divisors). The flip of this flipping contraction is a diagram such that

- X^+ is a normal projective variety,
- f^+ is a small projective birational contraction,
- $-K_X$ is ample over Y (by assumption), and K_{X^+} is ample over Y.

Definition 1.9 (MMP). Let X be a projective variety with "mild singularities". The following process is called MMP if every step exists: If K_X is not nef, then there is an K_X-extremal ray R and its contraction $f: X \to Y$. If $\dim Y < \dim X$, then we get a Mori fibre space and we stop. If f is a divisorial contraction, we replace X with Y and continue. If f is a flipping contraction, we replace X with the right hand side X^+ of the flip and continue.

One guarantees the existence of above contractions [Sho85, Kaw84] and flips [BCHM10].

Conjecture 1.10. *After finitely many steps, we get a minimal model or a Mori fibre space.*

So, the termination of MMP implies the minimal model conjecture.

2. Log Minimal Model Theory for klt Pairs

One may notice that, in Definition 1.9, the variety Y or X^+ appeared in the process of MMP is not necessarily smooth, even if the initial object X is assumed to be smooth. To guarantee the process, we need to expand our category from "smooth varieties" to "varieties with mild singularities".

Moreover, we usually consider *pairs* instead of varieties because they are more flexible under the definition of "mild singularities" and behave nicely under divisorial adjunctions.

Definition 2.1. A *pair* (X,B) consists of a normal variety X, and an $\mathbb{R}$-divisor B on X with coefficients in $[0,1]$ such that K_X+B is $\mathbb{R}$-Cartier.

Why pairs? The main reason for considering pairs is the various kinds of adjunction, that is, relating the canonical divisor of two varieties which are closely related. We have already seen the adjunction formula $(K_X+S)|_S = K_S$ where X,S are smooth and S is a prime divisor on X. It is natural to consider (X,S) rather than just X. From this observation we are able to arrange an inductive argument on dimension, which is one of the most powerful methodology in higher dimensional geometry.

Now let $f\colon X\to Z$ be a finite morphism. It often happens that $K_X = f^*(K_Z+B)$ for some boundary B. For example, if $f\colon X=\mathbb{P}^1\to Z=\mathbb{P}^1$ is a finite map of degree 2 ramified at two points P and P', then $K_X\sim f^*(K_Z+\frac{1}{2}P+\frac{1}{2}P')$. Similarly, when f is a contraction and $K_X\sim_{\mathbb{R}} 0/Z$, then under good conditions $K_X\sim_{\mathbb{R}} f^*(K_Z+B)$ for some boundary B on Z. Kodaira's canonical bundle formula for an elliptic fibration of a surface is a clear example.

Singularities play a crucial role in birational geometry and MMP. They are often measured by numbers called *log discrepancies*, and the "worst" type of singularities for which MMP works are called *lc*. To be precise, lc singularities require the minimal log discrepancies to be non-negative. In the hierarchy of singularities, the next type of MMP singularities are called *klt*, and requires that the log discrepancies are strictly positive. For precise definitions and basic properties the reader may consult [KM98]. There is an inclusion of categories:

$$\{\text{klt pairs}\}\subset\{\text{lc pairs}\}.$$

Definition 2.2. An MMP on a log canonical divisor K_X+B is called a log minimal model program (log MMP).

Remark 2.3. The log minimal model theory in the category of $\mathbb{Q}$-factorial klt (and some special lc, such as *"dlt"*) pairs is basically the same as starting from a smooth variety. However, the log minimal model theory in the category of lc pairs are much more complicated [Fuj-book17]. The existence of lc flips is proved by [Bir12, HMX14].

Thanks to the negativity lemma, we can blow-up an lc pair to a dlt pair.

Though the minimal model conjecture is still widely open but some very important cases have already been established, in particular, the famous theorem of Birkar-Cascini-Hacon-McKernan.

Theorem 2.4 ([BCHM10])**.** *Let (X,B) be a projective klt pair where $B\geq A$ for some ample divisor A. Then:*

(1) *if K_X+B is pseudo-effective, then (X,B) has a log minimal model (Y,B_Y). Moreover, the abundance holds, that is, K_Y+B_Y is semi-ample;*
(2) *if K_X+B is not pseudo-effective, then (X,B) has a Mori fibre space.*

As a corollary one deduces finite generation of a log canonical ring.

Corollary 2.5 (Finite generation)**.** *Let (X,B) be a projective klt pair such that K_X+B is a $\mathbb{Q}$-divisor. Then, the log canonical algebra*

$$\mathcal{R}(X,B) := \bigoplus_{m\geq 0} H^0(X,\lfloor m(K_X+B)\rfloor)$$

is a finitely generated $\mathbb{C}$-algebra.

In [Hu20] we establish a generalisation of the above corollary with real coefficients.

Corollary 2.6 ([Hu20])**.** *Let (X,B) be a projective klt pair. Then, (X,B) has the canonical model.*

Regarding the log MMP for klt pairs, [BCHM10] proves the next.

Corollary 2.7. *Let (X,B) be a projective klt pair such that $B\geq A$ where $A\geq 0$ is an ample divisor. Then, any log MMP on K_X+B with scaling terminates.*

Definition 2.8. Let (X,B) be a projective pair. We say (X,B) is *log Fano* if it is lc and $-(K_X+B)$ is ample; if $B=0$ we just say X is Fano. We say X is *of Fano type* if (X,B) is klt log Fano for some choice of B. We say X is *of log Fano type* if (X,B) is log Fano for some choice of B.

A variety X is a *Mori dream space* if its cox ring is finitely generated. Geometrically, if X is $\mathbb{Q}$-factorial, then it means we can run MMP on every divisor on X which terminates after finitely many steps.

Corollary 2.9. *A variety X of Fano type is a Mori dream space.*

3. Log Minimal Model Theory for lc Pairs

Some of the crucial breakthroughs in log MMP hold in the klt category but are unknown for the lc category. Extending results from the klt category to the lc category is important, both from the theoretical viewpoint and in applications.

Birkar and Hu [BH14] generalised the main results of [BCHM10] from the klt category to the lc category under an extra assumption on the base loci.

Theorem 3.1. *Let (X,B) be a projective lc pair such that B is a $\mathbb{Q}$-divisor, and that there is a surjective morphism $f: X \to Z$ onto a normal projective variety Z satisfying:*

- *$K_X + B \sim_{\mathbb{Q}} f^* M_Z$ for some big $\mathbb{Q}$-divisor M_Z,*
- *$\mathbf{B}_+(M_Z)$ does not contain the image of any lc centre of (X,B).*

Then, (X,B) has a good log minimal model. In particular, the log canonical algebra $\mathcal{R}(K_X+B)$ is finitely generated.

Later on, a generalisation without extra assumptions is achieved [Hu17]. The argument is improved and simplified in [HH20].

Theorem 3.2 ([HH20, Hu17])**.** *Let (X,B) be a projective lc pair where $B \geq A$ for some ample divisor $A \geq 0$. Then:*

(1) *if $K_X + B$ is pseudo-effective, then (X,B) has a log minimal model (Y,B_Y). Moreover, the abundance holds, that is, $K_Y + B_Y$ is semi-ample;*
(2) *if $K_X + B$ is not pseudo-effective, then (X,B) has a Mori fibre space.*

We immediately obtain the following corollaries.

Corollary 3.3 (Finite generation)**.** *Let (X,B) be a projective lc pair such that $B \geq A$ where $A \geq 0$ is an ample $\mathbb{R}$-divisor. Then, (X,B) has the canonical model. In particular, if K_X+B is $\mathbb{Q}$-Cartier, then the divisorial ring $\mathcal{R}(X,B)$ is finitely generated.*

Corollary 3.4 ([Hu17])**.** *Let X be a $\mathbb{Q}$-factorial projective variety of log Fano type. Then, X is a Mori dream space.*

Remark 3.5. The previous corollary is false if we remove the assumption "$\mathbb{Q}$-factorial".

Regarding the log MMP on $\mathbb{Q}$-factorial lc pairs, Hu [Hu17] proves the next corollary.

Corollary 3.6 ([Hu17])**.** *Let (X,B) be a projective $\mathbb{Q}$-factorial lc pair such that $B \geq A$ where $A \geq 0$ is an ample divisor. Then, any log MMP on K_X+B with scaling terminates.*

One the other hand, Hashizume and Hu [HH20] prove the existence of a log MMP which terminates.

Corollary 3.7 ([HH20])**.** *Let (X,B) be a projective lc pair such that $B \geq A$ where $A \geq 0$ is an ample divisor. Then, there exists a log MMP on K_X+B with scaling which terminates.*

It is also expected that the termination above holds for all such log MMP.

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