Advances in Path Homology Theory of Digraphs

by Alexander Grigor'yan*

Contents

Introduction

The purpose of this paper is to introduce a new emerging area of research – the theory of path homology on digraphs, that is also known as GLMY-homology.

There exists a number of ways to define the notion of homology for graphs and digraphs, for example, clique homology ([6], [33]) or singular homology ([3], [33], [37]). However, the path

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homology has certain advantages as it enjoys adequate functorial properties with respect to graph-theoretical operations, such as morphisms of digraphs, Cartesian products, joins, homotopy etc. The notion of path homology has a rich mathematical content, and I hope that it will become a useful tool in various areas of pure and applied mathematics.

Sections 1 and 3 contain a survey of the results obtained in [18], [20], [22], [26], [29], [30], while the results of Sections 2, 4, 5 and 6 are entirely new.

For further reading on this subject and related topics I recommend [1], [2], [4], [5], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [19], [21], [23], [24], [25], [27], [28], [31], [32], [35], [36].

1. Spaces of ∂-Invariant Paths

The material of this section is based on [20] and [22].

1.1 Paths and the Boundary Operator

Let V be a finite set whose elements will be called vertices. For any $p \geq 0$, an *elementary p-path* is any sequence $i_0, \ldots, i_p$ of $p+1$ vertices of V (allowing repetitions). Fix a field $\mathbb{K}$ and denote by $\Lambda_p = \Lambda_p(V, \mathbb{K})$ the $\mathbb{K}$-linear space that consists of all formal $\mathbb{K}$-linear combinations of elementary p-paths in V. Any element of Λ_p is called a p-path.

An elementary p-path $i_0, \ldots, i_p$ as an element of Λ_p will be denoted by $e_{i_0 \ldots i_p}$. For example, we have

$$\Lambda_0 = \langle e_i : i \in V \rangle, \quad \Lambda_1 = \langle e_{ij} : i, j \in V \rangle, \quad \Lambda_2 = \langle e_{ijk} : i, j, k \in V \rangle$$

Any p-path u can be written in a form $u = \sum_{i_0, i_1, \ldots, i_p \in V} u^{i_0 i_1 \ldots i_p} e_{i_0 i_1 \ldots i_p}$, where $u^{i_0 i_1 \ldots i_p} \in \mathbb{K}$.

Definition. Define for any $p \geq 1$ a linear *boundary operator* $\partial : \Lambda_p \to \Lambda_{p-1}$ by

$$\partial e_{i_0 \ldots i_p} = \sum_{q=0}^{p} (-1)^q e_{i_0 \ldots \widehat{i_q} \ldots i_p}, \tag{1.1}$$

where $\widehat{}$ means omission of the index. Set $\Lambda_{-1} = \{0\}$ and define $\partial : \Lambda_0 \to \Delta_{-1}$ by $\partial = 0$.

For example, $\partial e_i = 0$, $\partial e_{ij} = e_j - e_i$ and $\partial e_{ijk} = e_{jk} - e_{ik} + e_{ij}$.

Lemma 1.1 ([20], [22, Lemma 2.1]). *We have $\partial^2 = 0$.*

Proof. Indeed, for any $p \geq 2$ we have

$$\begin{aligned}
\partial^2 e_{i_0 \ldots i_p} &= \sum_{q=0}^{p} (-1)^q \partial e_{i_0 \ldots \widehat{i_q} \ldots i_p} \\
&= \sum_{q=0}^{p} (-1)^q \left(\sum_{r=0}^{q-1} (-1)^r e_{i_0 \ldots \widehat{i_r} \ldots \widehat{i_q} \ldots i_p} \right. \\
&\qquad \left. + \sum_{r=q+1}^{p} (-1)^{r-1} e_{i_0 \ldots \widehat{i_q} \ldots \widehat{i_r} \ldots i_p} \right) \\
&= \sum_{0 \leq r < q \leq p} (-1)^{q+r} e_{i_0 \ldots \widehat{i_r} \ldots \widehat{i_q} \ldots i_p} \\
&\quad - \sum_{0 \leq q < r \leq p} (-1)^{q+r} e_{i_0 \ldots \widehat{i_q} \ldots \widehat{i_r} \ldots i_p}.
\end{aligned}$$

After switching q and r in the last sum we see that the two sums cancel out, whence $\partial^2 e_{i_0 \ldots i_p} = 0$. This implies $\partial^2 u = 0$ for all $u \in \Lambda_p$. □

Hence, we obtain a chain complex $\Lambda_*(V)$:

$$0 \leftarrow \Lambda_0 \overset{\partial}{\leftarrow} \Lambda_1 \overset{\partial}{\leftarrow} \ldots \overset{\partial}{\leftarrow} \Lambda_{p-1} \overset{\partial}{\leftarrow} \Lambda_p \overset{\partial}{\leftarrow} \ldots$$

Definition. An elementary p-path $e_{i_0 \ldots i_p}$ is called *regular* if $i_k \neq i_{k+1}$ for all $k = 0, \ldots, p-1$, and irregular otherwise.

Let $\mathcal{I}_p$ be the subspace of Λ_p spanned by irregular p-paths $e_{i_0 \ldots i_p}$. We claim that $\partial \mathcal{I}_p \subset \mathcal{I}_{p-1}$. Indeed, if $e_{i_0 \ldots i_p}$ is irregular then $i_k = i_{k+1}$ for some k. We have

$$\begin{aligned}
\partial e_{i_0 \ldots i_p} &= e_{i_1 \ldots i_p} - e_{i_0 i_2 \ldots i_p} + \ldots \\
&\quad + (-1)^k e_{i_0 \ldots i_{k-1} i_{k+1} i_{k+2} \ldots i_p} + (-1)^{k+1} e_{i_0 \ldots i_{k-1} i_k i_{k+2} \ldots i_p} \\
&\quad + \ldots + (-1)^p e_{i_0 \ldots i_{p-1}}.
\end{aligned} \tag{1.2}$$

By $i_k = i_{k+1}$ the two terms in the middle line of (1.2) cancel out, whereas all other terms are non-regular, whence $\partial e_{i_0 \ldots i_p} \in \mathcal{I}_{p-1}$.

Hence, ∂ is well-defined on the quotient spaces $\mathcal{R}_p := \Lambda_p / \mathcal{I}_p$, and we obtain the chain complex $\mathcal{R}_*(V)$:

$$0 \leftarrow \mathcal{R}_0 \overset{\partial}{\leftarrow} \mathcal{R}_1 \overset{\partial}{\leftarrow} \ldots \overset{\partial}{\leftarrow} \mathcal{R}_{p-1} \overset{\partial}{\leftarrow} \mathcal{R}_p \overset{\partial}{\leftarrow} \ldots$$

By setting all irregular p-paths to be equal to 0, we can identify $\mathcal{R}_p$ with the subspace of Λ_p spanned by all regular paths. For example, if $i \neq j$ then $e_{iji} \in \mathcal{R}_2$ and

$$\partial e_{iji} = e_{ji} - e_{ii} + e_{ij} = e_{ji} + e_{ij}$$

because $e_{ii} = 0$ in $\mathcal{R}_2$.

1.2 Chain Complex Ω_*

Definition. A *digraph (directed graph)* is a pair $G = (V, E)$ of a set V of vertices and $E \subset \{V \times V \setminus \text{diag}\}$ is a set of arrows (directed edges). If $(i, j) \in E$ then we write $i \to j$.

Definition. Let $G = (V, E)$ be a digraph. An elementary p-path $i_0 \ldots i_p$ on V is called *allowed* if $i_k \to i_{k+1}$ for any $k = 0, \ldots, p-1$, and *non-allowed* otherwise.

Let $\mathcal{A}_p = \mathcal{A}_p(G)$ be $\mathbb{K}$-linear subspace of Λ_p spanned by allowed elementary p-paths:

$$\mathcal{A}_p = \langle e_{i_0 \ldots i_p} : i_0 \ldots i_p \text{ is allowed} \rangle.$$

The elements of $\mathcal{A}_p$ are called *allowed* p-paths. Since any allowed path is regular, we have $\mathcal{A}_p \subset \mathcal{R}_p$.

We would like to build a chain complex based on subspaces $\mathcal{A}_p$ of $\mathcal{R}_p$. However, the spaces $\mathcal{A}_p$ are in general *not* invariant for ∂. For example, in the digraph

$$\overset{a}{\bullet} \longrightarrow \overset{b}{\bullet} \longrightarrow \overset{c}{\bullet}$$

we have $e_{abc} \in \mathcal{A}_2$ but $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \notin \mathcal{A}_1$ because e_{ac} is non-allowed.

Consider the following subspace of $\mathcal{A}_p$

$$\Omega_p \equiv \Omega_p(G) := \left\{ u \in \mathcal{A}_p : \partial u \in \mathcal{A}_{p-1} \right\}.$$

We claim that $\partial \Omega_p \subset \Omega_{p-1}$. Indeed, $u \in \Omega_p$ implies $\partial u \in \mathcal{A}_{p-1}$ and $\partial(\partial u) = 0 \in \mathcal{A}_{p-2}$, whence $\partial u \in \Omega_{p-1}$.

Definition. The elements of Ω_p are called ∂-*invariant* p-paths.

Thus, we obtain a chain complex $\Omega_* = \Omega_*(G)$:

$$(1.3)\quad 0 \leftarrow \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots$$

By construction we have $\Omega_0 = \mathcal{A}_0$ and $\Omega_1 = \mathcal{A}_1$, while in general $\Omega_p \subset \mathcal{A}_p$.

Proposition 1.2 ([20]). *If* $\dim \Omega_n \leq 1$ *then* $\Omega_p = \{0\}$ *for all* $p \geq n+1$.

We say that a pair a, b forms a *double arrow* if $a \to b$ and $b \to a$.

Proposition 1.3 ([20]). *If G contains no double arrow and* $\dim \Omega_n \leq 2$ *then* $\Omega_n = \{0\}$ *for all* $p \geq n+2$.

1.3 Path Homology

Definition. *Path homologies* of G are defined as the homologies of the chain complex $\Omega_*(G)$:

$$H_p = H_p(G) = \ker \partial|_{\Omega_p} / \operatorname{Im} \partial|_{\Omega_{p+1}}.$$

For a vector space U over $\mathbb{K}$ we write

$$|U| = \dim_{\mathbb{K}} U.$$

Define the Betti numbers of G by

$$\beta_p = \beta_p(G) = |H_p|.$$

For any $N \in \mathbb{N}$ define the Euler characteristic of G of the order N by

$$\chi^{(N)} = \chi^{(N)}(G) = \sum_{p=0}^{N} (-1)^p |\Omega_p|.$$

If the sequence $\{\Omega_p\}$ is finite in the sense that $\Omega_p = \{0\}$ for large enough p, then, for large enough N,

$$\chi^{(N)} = \chi := \sum_{p=0}^{\infty} (-1)^p |\Omega_p| = \sum_{p=0}^{\infty} (-1)^p \beta_p.$$

Proposition 1.4. *If X and Y are two disjoint digraphs then*

$$(1.4)\qquad \beta_p(X \sqcup Y) = \beta_p(X) + \beta_p(Y).$$

Proof. Clearly, any allowed elementary p-path on $X \sqcup Y$ is contained in X or Y. It follows that the same property is true for ∂-invariant paths, so that

$$\Omega_p(X \sqcup Y) = \Omega_p(X) \oplus \Omega_p(Y).$$

Hence, the same identity holds for homology groups, whence (1.4) follows. □

Proposition 1.5. *We have $\beta_0(G) =$ #of connected components of G.*

Proof. It suffices to prove that if G is connected then $\beta_0 = 1$. We have $\beta_0 = |\Omega_0| - |\partial \Omega_1|$. Let the set of vertices of G be $\{1, \dots, n\}$ so that $|\Omega_0| = n$. Since Ω_1 is spanned by all arrows e_{ij}, $i \to j$, the space $\partial \Omega_1$ is spanned by all differences $e_j - e_i$ where $i \to j$. Since there is an edge path between the vertex 1 and any other vertex i, it follows that $\partial \Omega_1$ contains $e_i - e_1$ for any vertex $i > 1$. These $n-1$ elements of $\partial \Omega_1$ are linearly independent while any other difference $e_j - e_i$ is expressed as $(e_j - e_1) - (e_i - e_1)$. Hence, $|\partial \Omega_1| = n-1$ and $\beta_0 = 1$. □

1.4 Digraph Morphisms

Let X and Y be two digraphs. For simplicity of notations, we denote the sets of vertices of X and Y by the same letters X resp. Y.

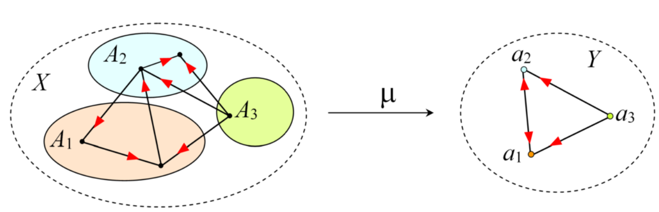
Definition. A *mapping* $f : X \to Y$ between the sets of vertices of X and Y called a *digraph map (or morphism)* if

$$a \to b \text{ on } X \;\Rightarrow\; f(a) \to f(b) \text{ or } f(a) = f(b) \text{ on } Y.$$

In other words, any arrow of X under the mapping f either goes to an arrow of Y or collapses to a vertex of Y.

We say that a digraph Y is a *subgraph* of a digraph X if the sets of vertices and arrows of Y are subset of the sets of vertices and arrows of X, respectively. In this case we have a natural inclusion $i : Y \to X$ that is clearly a digraph morphism. A subgraph Y of X is called *induced* if, for any two vertices a, b of Y such that there is an arrow $a \to b$ in X, there is also an arrow $a \to b$ in Y.

To give another example of a morphism, assume that a vertex set of a digraph X splits into a disjoint union of n subsets $A_1, \dots, A_n$, and construct a digraph Y of n vertices $a_1, \dots, a_n$ that is obtained from X by merging all the vertices from A_i into a single vertex a_i of Y. More precisely, we have an arrow $a_i \to a_j$ in Y if and only if there are $x \in A_i$ and $y \in A_j$ such that $x \to y$ in X.



An example of a merging map μ

We have a natural merging map $\mu : X \to Y$ such that $\mu(x) = a_i$ for any $x \in A_i$. Clearly, a merging map is a digraph morphism that keeps any arrow $x \to y$ if x and y belong to different sets A_i and collapses an arrow $x \to y$ into a vertex if x, y belong to the same A_i.

Any digraph morphism $f: X \to Y$ induces a mapping $f_*: \Lambda_n(X) \to \Lambda_n(Y)$ as follows: first set

$$f_*\left(e_{i_0 \dots i_n}\right) = e_{f(i_0)\dots f(i_n)},$$

and then extend f_* by linearity to all of $\Lambda_n(X)$.

Proposition 1.6. *Let $f: X \to Y$ be a digraph morphism. Then the induced mapping $f_*: \Lambda_n(X) \to \Lambda_n(Y)$ extends to a chain mapping $f_*: \Omega_n(X) \to \Omega_n(Y)$ and, hence, to homomorphism $f_*: H_n(X) \to H_n(Y)$.*

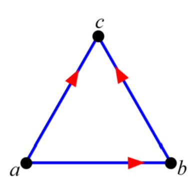
Proof. If $e_{i_0\dots i_n}$ is irregular then $f_*\left(e_{i_0\dots i_n}\right)$ is also irregular. Therefore, f_* maps the space $\mathcal{I}_n(X)$ of irregular paths on X into $\mathcal{I}_n(Y)$. It follows that f_* maps $\mathcal{R}_n(X) = \Lambda_n(X)/\mathcal{I}_n(X)$ into $\mathcal{R}_n(Y)$.

Next, f_* maps the space $\mathcal{A}_n(X)$ of allowed paths into $\mathcal{A}_n(Y)$: if $e_{i_0\dots i_n}$ is allowed then $i_k \to i_{k+1}$ for all k, which implies that either $f(i_k) \to f(i_{k+1})$ for all k and, hence, $f_*\left(e_{i_0\dots i_n}\right)$ is also allowed, or $f(i_k) = f(i_{k+1})$ for some k so that $f_*\left(e_{i_0\dots i_n}\right)$ is irregular, thus $f_*\left(e_{i_0\dots i_n}\right) = 0$.

Clearly, f_* commutes with ∂, which implies that f_* maps $\Omega_n(X)$ into $\Omega_n(Y)$ and f_* is a chain mapping. Consequently, we obtain a homomorphism of homology groups $f_*: H_n(X) \to H_n(Y)$. □

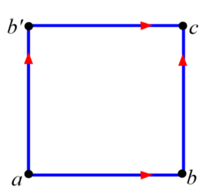
Further examples of digraph morphisms will be given in Sections 1.8 and 2.3.

1.5 Examples of ∂-Invariant Paths



A *triangle* is a sequence of three distinct vertices a, b, c such that $a \to b \to c$, $a \to c$.

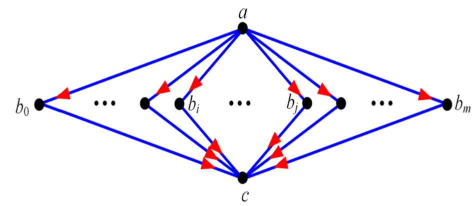
It determines a 2-path $e_{abc} \in \Omega_2$ because $e_{abc} \in \mathcal{A}_2$ and $\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} \in \mathcal{A}_1$.



A *square* is a sequence of four distinct vertices a, b, b', c such that $a \to b \to c$, $a \to b' \to c$ while $a \not\to c$.

It determines a 2-path $u = e_{abc} - e_{ab'c} \in \Omega_2$ because $u \in \mathcal{A}_2$ and

$$\begin{aligned}\partial u &= \left(e_{bc} - \underline{e_{ac}} + e_{ab}\right) - \left(e_{b'c} - \underline{e_{ac}} + e_{ab'}\right) \\ &= e_{ab} + e_{bc} - e_{ab'} - e_{b'c} \in \mathcal{A}_1.\end{aligned}$$



An *m-square* is a sequence of $m+3$ distinct vertices

$$a,\ b_0,\ b_1,\ \dots,\ b_m,\ c$$

such that $a \to b_k \to c\ \forall k = 0, \dots, m$, while $a \not\to c$.

An m-square determines ∂-invariant 2-paths

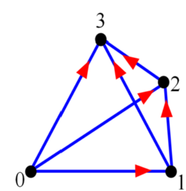
$$u_{ij} = e_{ab_ic} - e_{ab_jc} \in \Omega_2 \quad \text{for all } i, j = 0, \dots, m,$$

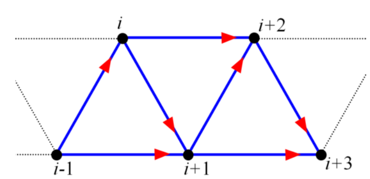
and among them the following m paths are linearly independent:

$$u_{0j} = e_{ab_0c} - e_{ab_jc}, \quad j = 1, \dots, m.$$

Clearly, an 1-square is a square in the above sense. Any m-square with $m \geq 2$ is called a *multisquare*.

A *p-simplex* (or p-clique) is a configuration of $p+1$ distinct vertices, say, $0, 1, \dots, p$, such that $i \to j\ \forall i < j$. It determines a p-path $e_{01\dots p} \in \Omega_p$. Here is a 3-simplex:





A *p-snake* is a configuration of $p+1$ distinct vertices, say $0, 1, \dots, p$, with the following arrows:

$$\begin{aligned}&i \to i+1 \ \text{ for all } i = 0, \dots, p-1, \\ &i \to i+2 \ \text{ for all } i = 0, \dots, p-2.\end{aligned}$$

In particular, any triple $i(i+1)(i+2)$ forms a triangle for $i = 0, \dots, p-2$.

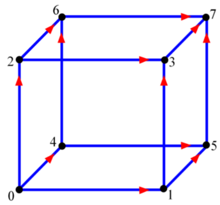
A p-snake determines a ∂-invariant p-path $e_{01\dots p}$. Indeed, this path is obviously allowed, and its boundary

$$\partial e_{01\dots p} = \sum_{q=0}^{p} (-1)^q e_{0\dots(q-1)(q+1)\dots p}$$

is also allowed because $q-1 \to q+1$. Hence, $e_{i_0\dots i_p} \in \Omega_p$.

A toy snake

Clearly, a p-simplex contains a p-snake.

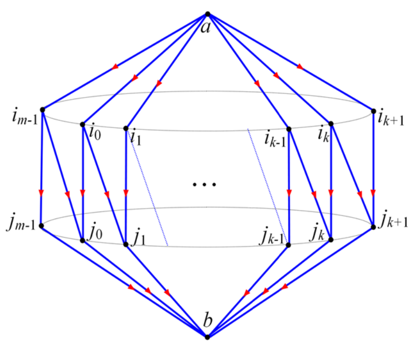


A 3-*cube* is a sequence of 8 vertices $0,1,2,3,4,5,6,7$, connected by arrows as shown here: A 3-cube determines a ∂-invariant 3-path

$$u = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267} \in \Omega_3$$

because $u \in \mathcal{A}_3$ and

$$\partial u = (e_{013} - e_{023}) + (e_{157} - e_{137}) + (e_{237} - e_{267}) - (e_{046} - e_{026}) - (e_{457} - e_{467}) - (e_{015} - e_{045}) \in \mathcal{A}_2.$$



A *trapezohedron* of order $m \geq 2$ is a configuration of $2m+2$ distinct vertices

$$a,\ b,\ i_0,\ \ldots,\ i_{m-1},\ j_0,\ \ldots,\ j_{m-1}$$

with $4m$ arrows:

$$a \to i_k, \quad j_k \to b$$

and

$$i_k \to j_k, \quad i_k \to j_{k+1},$$

for all $k = 0, \ldots, m-1$, where k is understood $\bmod m$.

The trapezohedron gives rise to the following ∂-invariant 3-path:

$$\tau_m = \sum_{k=0}^{m-1} \left(e_{ai_kj_kb} - e_{ai_kj_{k+1}b}\right). \tag{1.5}$$

Indeed, τ_m is clearly allowed, and its boundary is also allowed because

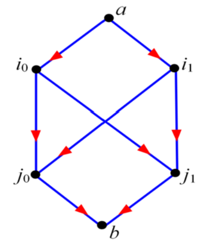
$$\partial \tau_m = \sum_{k=0}^{m-1} \partial \left(e_{ai_kj_kb} - e_{ai_kj_{k+1}b}\right)$$

$$= \sum_{k=0}^{m-1} \left(e_{i_kj_kb} - e_{i_kj_{k+1}b}\right) - \sum_{k=0}^{m-1} \left(e_{ai_kj_k} - e_{ai_kj_{k+1}}\right) \tag{1.6}$$

$$- \sum_{k=0}^{m-1} \left(e_{aj_kb} - e_{aj_{k+1}b}\right) + \sum_{k=0}^{m-1} \left(e_{ai_kb} - e_{ai_kb}\right), \tag{1.7}$$

where the both sums in (1.6) are allowed, while both sums in (1.7) vanish.

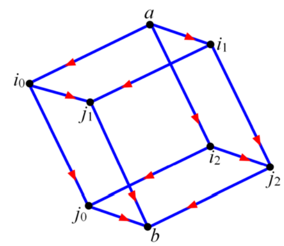
A trapezohedron of order $m = 2$ is shown here:



In this case we have

$$\tau_2 = e_{ai_0j_0b} - e_{ai_0j_1b} + e_{ai_1j_1b} - e_{ai_1j_0b}.$$

A trapezohedron of order $m \geq 3$ can be realized as a convex polyhedron in $\mathbb{R}^3$ with flat faces. For example, a trapezohedron of order $m = 3$ coincides with a 3-cube:

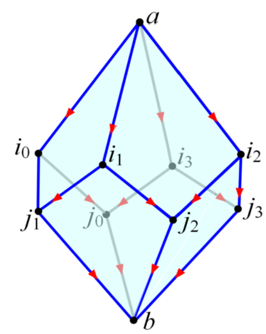


In this case we have

$$\tau_3 = e_{ai_0j_0b} - e_{ai_0j_1b} + e_{ai_1j_1b} - e_{ai_1j_2b} + e_{ai_2j_2b} - e_{ai_2j_0b},$$

and τ_3 coincides (up to a sign) with the aforementioned 3-path determined by a 3-cube.

A trapezohedron of order $m = 4$ is a *tetragonal trapezohedron*:

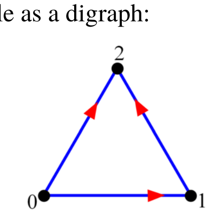


In this case we have

$$\tau_4 = e_{ai_0j_0b} - e_{ai_0j_1b} + e_{ai_1j_1b} - e_{ai_1j_2b} + e_{ai_2j_2b} - e_{ai_2j_3b} + e_{ai_3j_3b} - e_{ai_3j_0b}.$$

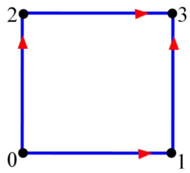
1.6 Examples of Spaces Ω_p and H_p

Here is a triangle as a digraph:



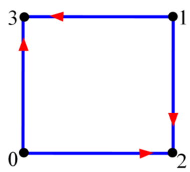
We have $\Omega_1 = \langle e_{01}, e_{02}, e_{12}\rangle$, $\Omega_2 = \langle e_{012}\rangle$. Since $\ker \partial|_{\Omega_1} = \langle e_{01} - e_{02} + e_{12}\rangle$ and $e_{01} - e_{02} + e_{12} = \partial e_{012}$, it follows that $H_1 = \{0\}$, $\Omega_p = \{0\}$ for $p \geq 3$ and $H_p = \{0\}$ for $p \geq 2$.

Here is a square as a digraph:



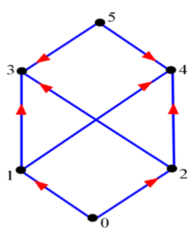
We have $\Omega_1 = \langle e_{01}, e_{02}, e_{13}, e_{23}\rangle$, $\Omega_2 = \langle e_{013} - e_{023}\rangle$. Since $\ker \partial|_{\Omega_1} = \langle e_{01} - e_{02} + e_{13} - e_{23}\rangle$ and $e_{01} - e_{02} + e_{13} - e_{23} = \partial(e_{013} - e_{023})$ it follows that $H_1 = \{0\}$, $\Omega_p = \{0\}$ for $p \geq 3$ and $H_p = \{0\}$ for $p \geq 2$.

Here is a 4-cycle that is called a *diamond*:



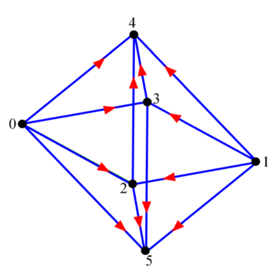
We have $\Omega_1 = \langle e_{02}, e_{03}, e_{12}, e_{13}\rangle$, $H_1 = \ker \partial|_{\Omega_1} = \langle e_{02} - e_{03} - e_{12} + e_{13}\rangle$, $\Omega_p = \{0\}$ and $H_p = \{0\}$ for all $p \geq 2$.

Consider a hexagon with two diagonals:



Here $\Omega_2 = \langle e_{013} - e_{023},\ e_{014} - e_{024}\rangle$, $H_1 = \langle e_{13} - e_{53} + e_{54} - e_{14}\rangle$, $\Omega_p = \{0\}$ for $p \geq 3$ and $H_p = \{0\}$ for $p \geq 2$.

Consider an octahedron based on a diamond:

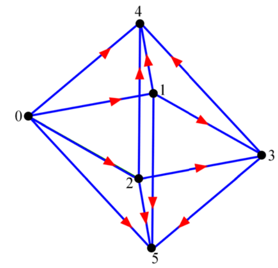


Space Ω_2 is spanned by 8 triangles:

$$\Omega_2 = \langle e_{024}, e_{034}, e_{025}, e_{035}, e_{124}, e_{134}, e_{125}, e_{135}\rangle,$$
$$H_2 = \langle e_{024} - e_{034} - e_{025} + e_{035} - e_{124} + e_{134} + e_{125} - e_{135}\rangle,$$
$$\Omega_p = \{0\} \text{ for } p \geq 3 \text{ and } H_p = \{0\} \text{ for } p = 1 \text{ and } p \geq 3.$$

Consider an octahedron based on a square:

$$\Omega_2 = \langle e_{024}, e_{025}, e_{014}, e_{015}, e_{234}, e_{235}, e_{134}, e_{135}, e_{013} - e_{023}\rangle,$$
$$\Omega_3 = \langle e_{0234} - e_{0134},\ e_{0235} - e_{0135}\rangle,\ \Omega_p = \{0\}\ \forall p \geq 4.$$



We have $\ker \partial|_{\Omega_2} = \langle u, v\rangle$ where

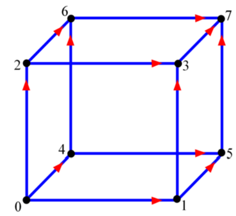
$$u = e_{024} + e_{234} - e_{014} - e_{134} + (e_{013} - e_{023})$$
$$v = e_{025} + e_{235} - e_{015} - e_{135} + (e_{013} - e_{023})$$

but $H_2 = \{0\}$ because

$$u = \partial(e_{0234} - e_{0134}) \text{ and } v = \partial(e_{0235} - e_{0135}).$$

In fact, $H_p = \{0\}$ for all $p \geq 1$.

Consider a 3-cube:



Space Ω_2 is spanned by 6 squares:

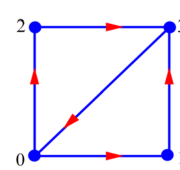
$$\Omega_2 = \langle e_{013} - e_{023},\ e_{015} - e_{045},\ e_{026} - e_{046},\ e_{137} - e_{157},\ e_{237} - e_{267},\ e_{457} - e_{467}\rangle$$

Space Ω_3 is spanned by one 3-cube:

$$\Omega_3 = \langle e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267}\rangle$$
$$\Omega_p = \{0\} \text{ for all } p \geq 4 \text{ and } H_p = \{0\} \text{ for all } p \geq 1.$$

1.7 An Example of Computation of Ω_p and H_p

Consider a square with a diagonal:



We have $\Omega_0 = \mathcal{A}_0 = \langle e_0, e_1, e_2, e_3\rangle$, $|\Omega_0| = 4$, $\Omega_1 = \mathcal{A}_1 = \langle e_{01}, e_{02}, e_{13}, e_{23}, e_{30}\rangle$, $|\Omega_1| = 5$, and $\mathcal{A}_2 = \langle e_{013}, e_{023}, e_{130}, e_{230}, e_{301}, e_{302}\rangle$, $|\mathcal{A}_2| = 6$. To determine Ω_2, let us first compute $\partial|_{\mathcal{A}_2} \bmod \mathcal{A}_1$:

$$\partial e_{013} = e_{13} - e_{03} + e_{01} = -e_{03} \bmod \mathcal{A}_1$$
$$\partial e_{023} = e_{23} - e_{03} + e_{02} = -e_{03} \bmod \mathcal{A}_1$$
$$\partial e_{130} = e_{30} - e_{10} + e_{13} = -e_{10} \bmod \mathcal{A}_1$$
$$\partial e_{230} = e_{30} - e_{20} + e_{23} = -e_{20} \bmod \mathcal{A}_1$$
$$\partial e_{301} = e_{01} - e_{31} + e_{30} = -e_{31} \bmod \mathcal{A}_1$$

$$\partial e_{302} = e_{02} - e_{32} + e_{30} = -e_{32} \bmod \mathcal{A}_1$$

We have

$$D := \text{matrix of } \partial|_{\mathcal{A}_2} \bmod \mathcal{A}_1$$
$$= \begin{pmatrix} & e_{013} & e_{023} & e_{130} & e_{230} & e_{301} & e_{302} \\ e_{03} & -1 & -1 & & & & 0 \\ e_{10} & & & -1 & & & \\ e_{20} & & & & -1 & & \\ e_{31} & & & & & -1 & \\ e_{32} & 0 & & & & & -1 \end{pmatrix}$$

and

$$\Omega_2 = \ker \partial|_{\mathcal{A}_2} \bmod \mathcal{A}_1 = \operatorname{nullspace} D = \langle e_{013} - e_{023} \rangle.$$

One can show that $\{\Omega_p\} = 0$ for all $p \geq 3$ (which also follows from Proposition 1.2) and, hence, $\{H_p\} = 0$ for all $p \geq 3$.

Let us compute H_1 and H_2. We have for the basis in Ω_1:

$$\partial e_{01} = -e_0 + e_1$$
$$\partial e_{02} = -e_0 + e_2$$
$$\partial e_{13} = -e_1 + e_3$$
$$\partial e_{23} = -e_2 + e_3$$
$$\partial e_{30} = e_0 - e_3$$

Therefore,

$$D := \text{matrix of } \partial|_{\Omega_1} = \begin{pmatrix} & e_{01} & e_{02} & e_{13} & e_{23} & e_{30} \\ e_0 & -1 & -1 & 0 & 0 & 1 \\ e_1 & 1 & 0 & -1 & 0 & 0 \\ e_2 & 0 & 1 & 0 & -1 & 0 \\ e_3 & 0 & 0 & 1 & 1 & -1 \end{pmatrix}$$

and

$$\ker \partial|_{\Omega_1} = \operatorname{nullspace} D = \langle e_{01} + e_{13} - e_{02} - e_{23},\ e_{01} + e_{13} + e_{30} \rangle.$$

Similarly, for the basis in Ω_2 we have

$$\partial (e_{013} - e_{023}) = (e_{13} - e_{03} + e_{01}) - (e_{23} - e_{03} + e_{02})$$
$$= e_{01} + e_{13} - e_{02} - e_{23}$$

whence

$$\operatorname{Im} \partial|_{\Omega_2} = \langle e_{01} + e_{13} - e_{02} - e_{23} \rangle \quad \text{and} \quad \ker \partial|_{\Omega_2} = \{0\}.$$

It follows that $H_2 = \{0\}$ and

$$H_1 = \ker \partial|_{\Omega_1} / \operatorname{Im} \partial|_{\Omega_2} = \langle e_{01} + e_{13} + e_{30} \rangle.$$

As we have seen, computation of the spaces $\Omega_p(G)$ and $H_p(G)$ amounts to computing ranks and null-spaces of matrices. We currently use for numerical computation of $H_p(G, \mathbb{F}_2)$ a C++ program written by Chao Chen in 2012.

Problem 1.7. *Devise an efficient algorithm/software for computation of the spaces Ω_p for arbitrary digraphs, possibly avoiding null-spaces of large matrices. Such algorithms exist for Ω_2 and Ω_3. Are there simpler ways of computing directly $\dim \Omega_p$ without computing the bases of Ω_p?*

1.8 Structure of Ω_2

As we know, $\Omega_0 = \langle e_i \rangle$ consists of all vertices and $\Omega_1 = \langle e_{ij} : i \to j \rangle$ consists of all arrows.

Definition. Let us call a *semi-arrow* any pairs (x, y) of distinct vertices x, y such that $x \not\to y$ but $x \to z \to y$ for some vertex z. We write in this case $x \rightharpoonup y$

Theorem 1.8 ([21, Proposition 2.9], [20])**.**

(a) *We have $|\Omega_2| = |\mathcal{A}_2| - s$ where s is the number of semi-arrows.*

(b) *The space Ω_2 is spanned by all triangles e_{abc}, squares $e_{abc} - e_{ab'c}$ and double arrows e_{aba}.*

Proof. (a) Recall that

$$\mathcal{A}_2 = \operatorname{span}\{e_{abc} : abc \text{ is allowed}\}$$

and

$$\Omega_2 = \{v \in \mathcal{A}_2 : \partial v \in \mathcal{A}_1\} = \{v \in \mathcal{A}_2 : \partial v = 0 \bmod \mathcal{A}_1\}.$$

If abc is allowed then ab and bc are arrows, whence

$$\partial e_{abc} = e_{bc} - e_{ac} + e_{ab} = -e_{ac} \bmod \mathcal{A}_1.$$

If $a = c$ or $a \to c$ then $e_{ac} = 0 \bmod \mathcal{A}_1$. Otherwise ac is a semi-arrow, and in this case

$$e_{ac} \neq 0 \bmod \mathcal{A}_1.$$

For any $v \in \mathcal{A}_2$, we have

$$v = \sum_{\{a \to b \to c\}} v^{abc} e_{abc}$$

from which it follows that

$$\partial v = -\sum_{\{a \to b \to c, a \rightharpoonup c\}} v^{abc} e_{ac} \bmod \mathcal{A}_1.$$

The condition $\partial v = 0 \bmod \mathcal{A}_1$ is equivalent to

$$\sum_{\{a \to b \to c,\ a \rightharpoonup c\}} v^{abc} e_{ac} = 0 \bmod \mathcal{A}_1,$$

which in turn is equivalent to

$$\sum_{b \in V} v^{abc} = 0 \quad \text{for any semi-arrow } ac. \tag{1.8}$$

The number of the equations in (1.8) is exactly s, and they all are linearly independent for different semi-arrows, because a triple abc determines at most one semi-arrow. Hence, Ω_2 is obtained from $\mathcal{A}_2$ by imposing s linearly independent conditions, which implies $|\Omega_2| = |\mathcal{A}_2| - s$.

(b) Any allowed 2-path ω can be represented as a sum of elementary 2-paths e_{ijk} with $i \to j \to k$ multiplied with a scalar $c \neq 0$. If $k = i$ then e_{ijk} is a double arrow. If $i \neq k$ and $i \to k$ then e_{ijk} is a triangle. Subtracting from ω all double arrows and triangles, we can assume that ω has no such terms any more.

Then, for any term e_{ijk} in ω we have $i \neq k$ and $i \not\to k$. Fix such a pair i,k and consider any vertex j with $i \to j \to k$. Assume that e_{ijk} enters ω with a coefficient $c_j \neq 0$. Set

$$\omega_{ik} = \sum_j c_j e_{ijk} \tag{1.9}$$

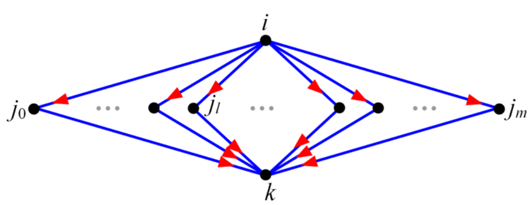
so that $\omega = \sum_{ik} \omega_{ik}$. It suffices to verify that each ω_{ik} is a linear combination of squares. The 1-path $\partial\omega$ is the sum of 1-paths of the form

$$\partial\left(c_j e_{ijk}\right) = c_j e_{ij} - c_j e_{ik} + c_j e_{jk}.$$

Since $\partial\omega$ is allowed but e_{ik} is not allowed, the term $c_j e_{ik}$ should cancel out after we sum up all such terms over all possible j, that is,

$$\sum_j c_j = 0. \tag{1.10}$$

Denote by $\{j_0, j_1, ..., j_m\}$ the sequence of all possible vertices j with $i \to j \to k$ so that we obtain an m-square:



An m-square $\{i, \{j_l\}_{l=0}^m, k\}$

Then we obtain from (1.9)

$$\omega_{ik} = \sum_{l=0}^m c_{j_l} e_{ij_lk} = \sum_{l=1}^m c_{j_l}\left(e_{ij_lk} - e_{ij_0k}\right)$$

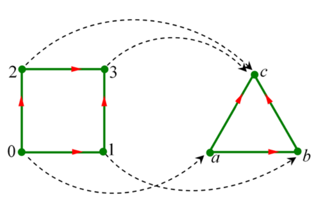
because by (1.10)

$$c_{j_0} = -\sum_{l=1}^l c_{j_l}.$$

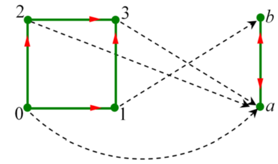
We conclude that ω_{ik} is a linear combination of squares. □

Example 1.9. Let the digraph G be an m-square shown on the above picture. It has one semi-arrow $i \rightharpoonup k$ so that $s = 1$. Since $|\mathcal{A}_2| = m+1$, we conclude that $|\Omega_2| = m$. Indeed, the basis in Ω_2 is given by the sequence of m squares $\left\{e_{ij_0k} - e_{ij_lk}\right\}_{l=1}^m$.

Observe that a triangle e_{abc} and a double arrow e_{aba} are images of a square $e_{013} - e_{023}$ under merging maps (cf. Subsection 1.4) as shown on these pictures:



a merging map from a square onto a triangle
$e_{013} - e_{023} \mapsto e_{abc} - e_{acc} = e_{abc}$



a merging map from a square onto a double arrow
$e_{013} - e_{023} \mapsto e_{aba} - e_{aaa} = e_{aba}$

Hence, we can rephrase Theorem 1.8 as follows: Ω_2 is spanned by squares and their morphism images. Or: squares are *basic shapes* of Ω_2.

1.9 Path Complex

The material of this section is based on [20], [22] and [26]. We discuss here the notion of *path complex* that unifies digraphs and simplicial complexes.

Definition. A path complex on a finite set V is a collection $\mathcal{P}$ of elementary paths on V such that if $i_0 i_1 ... i_{p-1} i_p \in \mathcal{P}$ then also $i_1 ... i_p$ and $i_0 ... i_{p-1}$ belong to $\mathcal{P}$.

For example, each digraph $G = (V,E)$ gives rise to a path complex $\mathcal{P}$ that consists of all allowed elementary paths, that is, of the paths $i_0 \to i_1 \to ... \to i_p$. In general, all paths in a path complex $\mathcal{P}$ are also called allowed.

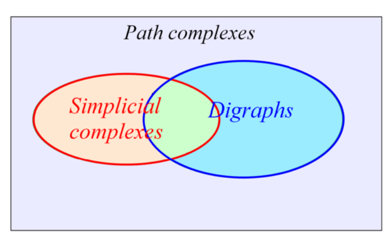
The above definitions of ∂-invariant paths, spaces Ω_p and H_p go through without any change to general path complexes in place of digraphs because they are based on the notion of allowed paths only. In fact, most of the results that are proved for path homology theory for digraphs remain true also for a more general setting of path complexes.

Let us recall the definition of an abstract simplicial complex.

Definition. A simplicial complex with the set of vertices V is a collection $\mathcal{S}$ of subsets of V such that if $\sigma \in \mathcal{S}$ then any subset of σ is also an element of $\mathcal{S}$.

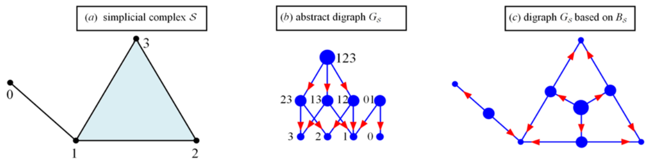
Let us enumerate all elements of V so that any subset σ of V can be regarded as a path $i_0 ... i_p$ with $i_0 < i_1 < .... < i_p$. The above definition means that if $i_0 ... i_p \in \mathcal{S}$ then also any sub-path $i_{k_0} ... i_{k_q}$ with $0 \leq k_0 < k_1 < ... < k_q \leq p$ belongs to $\mathcal{S}$. Hence, a simplicial complex $\mathcal{S}$ is a path complex, and the theory of path homologies applies for $\mathcal{S}$.

In this case, $\mathcal{A}_p$ consists of linear combinations of all p-dimensional simplexes in $\mathcal{S}$ and $\Omega_p = \mathcal{A}_p$ because $\partial e_{i_0...i_p}$ is always allowed if $e_{i_0...i_p}$ is allowed. Hence, the path homology theory of a path complex $\mathcal{S}$ coincides with the simplicial homology theory of $\mathcal{S}$.



Schematic relation between path complexes, digraphs and simplicial complexes

Let $\mathcal{S}$ be a simplicial complex with the vertex set V as above. Define a digraph $G_{\mathcal{S}}$ as follows: the vertex set of $G_{\mathcal{S}}$ is $\mathcal{S}$, and for two simplexes $a,b \in \mathcal{S}$ we have an arrow $a \to b$ provided $a \supset b$ and $|a| = |b|+1$, that is, when b is a face of a of codim $= 1$. The digraph $G_{\mathcal{S}}$ is called the *Hasse diagram* of $\mathcal{S}$.



If $\mathcal{S}$ is realized geometrically as a collection of simplexes in $\mathbb{R}^n$ then $G_{\mathcal{S}}$ can be realized on the set of vertices $B_{\mathcal{S}}$ consisting of barycenters of the simplexes of $\mathcal{S}$ as on the picture. The relation between simplicial homology H^{simpl} with the path homology H is given by the following theorem.

Theorem 1.10 ([26, Corollary 5.4]). *We have*

$$H_*^{simpl}(\mathcal{S}) \cong H_*(G_{\mathcal{S}}).$$

1.10 Triangulation as a Closed Path

Given a closed oriented n-dimensional manifold M, let T be its triangulation, that is, a partition into n-dimensional simplexes. Denote by $V = \{0,1,...\}$ the set of all vertices of the simplexes from T and by E – the set of all edges, so that (V,E) is a graph embedded on M.

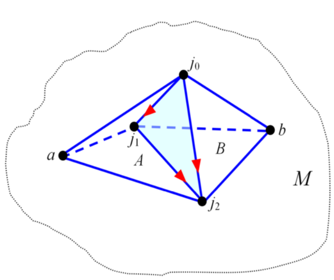
Let us introduce make each edge $(i,j) \in E$ into an arrow $i \to j$ if $i < j$ and into $j \to i$ if $i > j$. Then each simplex from T becomes a digraph-simplex. Denote by $\overrightarrow{T}$ the set of all digraph simplexes constructed in this way. That is, $i_0...i_n \in \overrightarrow{T}$ if $i_0...i_n$ is a monotone increasing sequence that determines a simplex from T. Clearly, any such path $i_0...i_p$ is allowed.

For any simplex from T with the vertices $i_0...i_n$ define the quantity $\sigma^{i_0...i_n}$ to be equal to 1 if the orientation of the simplex $i_0...i_n$ matches the orientation of the manifold M, and -1 otherwise. Then consider the following allowed n-path on the digraph $G = (V,E)$:

$$\sigma = \sum_{i_0...i_n \in \overrightarrow{T}} \sigma^{i_0...i_n} e_{i_0...i_n}. \tag{1.11}$$

Lemma 1.11 ([20]). *The path σ is closed, that is, $\partial\sigma = 0$, which, in particular, implies that σ is ∂-invariant.*

Proof. Observe that $\partial\sigma$ is a linear combination with coefficients ± 1 of the terms $e_{j_0...j_{n-1}}$ where the sequence $j_0,...,j_{n-1}$ is monotone increasing and forms an $(n-1)$-dimensional face of one of the n-simplexes from T.



In fact, every $(n-1)$-face arises from two n-simplexes, say, from

$$A = j_0...j_{k-1}aj_k...j_{n-1} \quad \text{and} \quad B = j_0...j_{l-1}bj_l...j_{n-1}.$$

That is, the n-simplexes A and B have a common $(n-1)$-dimensional face $j_0...j_{n-1}$.

We have

$$\partial e_{j_0...j_{k-1}aj_k...j_{n-1}} = ... + (-1)^k e_{j_0...j_{k-1}j_k...j_{n-1}} + ...$$

Since interchanging the order of two neighboring vertices in an n-simplex changes its orientation, we have

$$\sigma^{j_0...j_{k-1}aj_k...j_{n-1}} = (-1)^k \sigma^{aj_0...j_{k-1}j_k...j_{n-1}}.$$

Multiplying the above lines, we obtain

$$\partial(\sigma^A e_A) = ... + \sigma^{aj_0...j_{n-1}} e_{j_0...j_{n-1}} + ... ,$$

and in the same way

$$\partial(\sigma^B e_B) = ... + \sigma^{bj_0...j_{n-1}} e_{j_0...j_{n-1}} + ...$$

However, the vertices a and b are located on the opposite sides of the face $j_0...j_{n-1}$, which implies that the simplexes $aj_0...j_{n-1}$ and $bj_0...j_{n-1}$ have the opposite orientations relative to that of M. Hence,

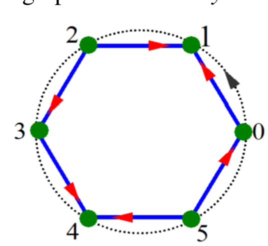
$$\sigma^{aj_0...j_{n-1}} + \sigma^{bj_0...j_{n-1}} = 0,$$

which means that the term $e_{j_0...j_{n-1}}$ cancels out in the sum $\partial(\sigma^A e_A + \sigma^B e_B)$ and, hence, in $\partial\sigma$. This proves that $\partial\sigma = 0$. ☐

The closed path σ defined by (1.11) is called a *surface path* on M.

There is a number of examples when a surface path σ happens to be exact, that is, $\sigma = \partial v$ for some $(n+1)$-path v. In this case v is called a *solid path* on M because v represents a "solid" shape whose boundary is given by a surface path. If σ is not exact then σ determines a non-trivial homology class from $H_n(G)$ and, hence, represents a "cavity" in triangulation T.

Example 1.12. $M = \mathbb{S}^1$. A triangulation of $\mathbb{S}^1$ is a polygon, and the corresponding digraph G is called *cyclic*.



On each edge (i,j) of a polygon we choose an arrow $i \to j$ arbitrary (not necessarily if $i < j$). We have

$$\sigma = \sum_{i \to j} \sigma^{ij} e_{ij}$$

where we have $\sigma^{ij} = 1$ if the arrow $i \to j$ goes counterclockwise, and $\sigma^{ij} = -1$ otherwise.

For the digraph on the picture we have

$$\sigma = e_{01} - e_{21} + e_{23} + e_{34} - e_{54} + e_{50}.$$

If a polygon G is a triangle or a square then $\Omega_p = \{0\}$ for $p \geq 3$ and $H_p = \{0\}$ for all $p \geq 1$. Otherwise we have the following statement.

Proposition 1.13 ([20]). *If a polygon G is neither a triangle nor a square then $\Omega_p = \{0\}$ and $H_p = \{0\}$ for all $p \geq 2$ while $H_1 = \langle \sigma \rangle$.*

Proof. We have $\Omega_p = \{0\}$ for all $p \geq 2$ by Theorem 1.8. Hence, $\dim H_p = 0$ for $p \geq 2$. For the Euler characteristic, we have

$$\chi = \dim \Omega_0 - \dim \Omega_1 = 0.$$

Since also

$$\chi = \dim H_0 - \dim H_1$$

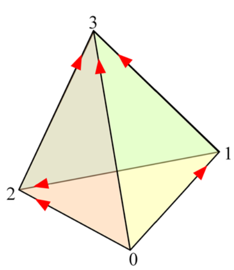
and $\dim H_0 = 1$, we obtain $\dim H_1 = 1$.

It remains to see that σ is a non-zero element of H_1. The path σ is closed by Lemma 1.11. In this case this can also be seen directly because by construction we have $\sigma^{i(i+1)} - \sigma^{(i+1)i} \equiv 1$ whence, for any vertex i,

$$\begin{aligned}(\partial\sigma)^i &= \sum_{j \in V} \left(\sigma^{ji} - \sigma^{ij}\right) \\ &= \sigma^{(i-1)i} + \sigma^{(i+1)i} - \sigma^{i(i-1)} - \sigma^{i(i+1)} = 1 - 1 = 0.\end{aligned}$$

Finally, $\sigma \neq 0$ in H_1 because $\operatorname{Im} \partial|_{\Omega_2} = \{0\}$. □

Example 1.14. Let $M = \mathbb{S}^n$ and let a triangulation of the n-sphere $\mathbb{S}^n$ be given by the surface of an $(n+1)$-simplex.



Then G is a $(n+1)$-simplex digraph. On this picture $n = 2$ and

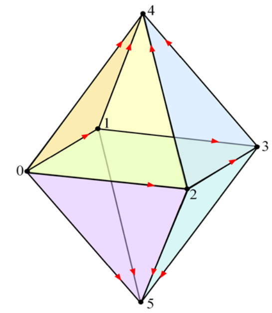
$$\sigma = e_{123} - e_{023} + e_{013} - e_{012} = \partial e_{0123}$$

so that e_{0123} is a solid path representing a tetrahedron.

For an arbitrary n we also have $\sigma = \partial e_{0\ldots n+1}$ so that $e_{0\ldots n+1}$ is a solid path representing an $(n+1)$-simplex.

Example 1.15. Let $M = \mathbb{S}^2$ and let a triangulation of $\mathbb{S}^2$ be given by an octahedron (see also Subsection 1.6). Consider two cases of numbering of vertices and, respectively, orientation of arrows.

An octahedron based on a square:

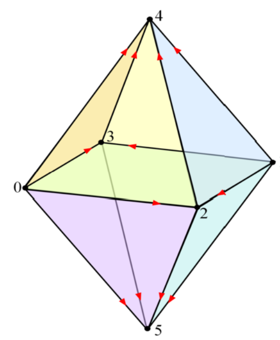


We have $H_2 = \{0\}$; it is easy to see that

$$\begin{aligned}\sigma &= e_{024} - e_{025} - e_{014} + e_{015} - e_{234} + e_{235} + e_{134} - e_{135} \\ &= \partial\left(e_{0134} - e_{0234} + e_{0135} - e_{0235}\right)\end{aligned}$$

Hence, $v = e_{0134} - e_{0234} + e_{0135} - e_{0235}$ is a solid path and the octahedron represents a solid shape.

An octahedron based on a diamond:

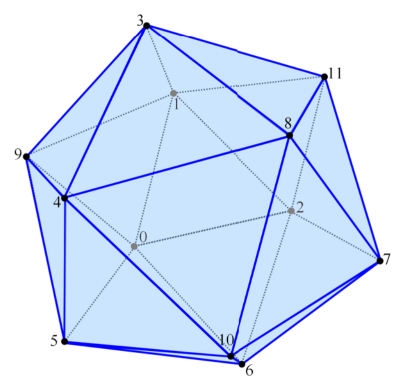


We have $H_2 = \langle \sigma \rangle$ where

$$\sigma = e_{024} - e_{034} - e_{025} + e_{035} - e_{124} + e_{134} + e_{125} - e_{135}$$

so that this octahedron represents a cavity.

Example 1.16. Let $M = \mathbb{S}^2$ and let a triangulation of $\mathbb{S}^2$ be given by an *icosahedron*:



Chose a numbering of vertices as shown here and arrows $i \to j$ if $i \sim j$ and $i < j$.

We have $|V| = 12$, $|E| = 30$, $H_1 = \{0\}$, and $H_2 = \langle \sigma \rangle$ where

$$\begin{aligned}\sigma = &-e_{019} + e_{012} - e_{1211} + e_{026} + e_{059} - e_{056} + e_{5610} \\ &- e_{139} + e_{1311} - e_{267} + e_{6710} - e_{2711} - e_{349} + e_{348} \\ &- e_{4810} + e_{3811} - e_{459} + e_{4510} + e_{7810} - e_{7811}.\end{aligned}$$

Hence, the icosahedron represents a cavity.

Conjecture 1.17. *For icosahedron* $\dim H_2(G) = 1$ *for any numbering of the vertices.*

Conjecture 1.18. *For a general triangulation of* $\mathbb{S}^n$*, the homology group* $H_n(G)$ *is either trivial or is generated by* σ*. All other homology groups* $H_p(G)$ *are trivial.*

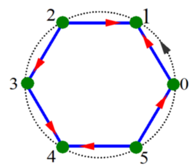
1.11 Homological Dimension

In this section $\mathbb{K} = \mathbb{F}_2$.

Definition. Define the *homological dimension* of a digraph G by

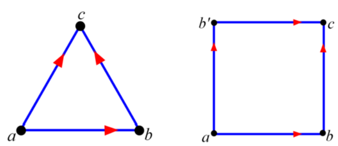
$$\dim_h G = \sup\{k : |H_k(G)| > 0\}.$$

Let G be a polygon (a cyclic digraph).



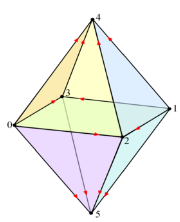
If G is neither triangle nor square, then $|H_1| = 1$ and $|H_p| = 0$ for $p \geq 2$ whence $\dim_h G = 1$.

Let G be either a triangle or a square:



Then $|H_p| = 0$ for $p \geq 1$ and $\dim_h G = 0$.

Let G be an octahedron based on a diamond:



Then $|H_2| = 1$, $|H_p| = 0$ for $p \geq 3$, whence $\dim_h G = 2$.

Let us give an example of a digraph with ∞ homological dimension that is due to Gabor Lippner and Paul Horn [34]. Fix some $n \geq 5$. We construct a digraph $LH(n)$ of $2n$ vertices that are denoted by

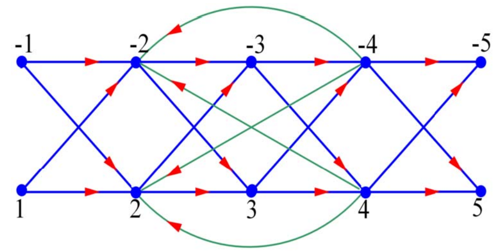
$$1, 2, \ldots, n \quad \text{and} \quad -1, -2, \ldots, -n,$$

and the arrows between vertices x, y in $LH(n)$ are defined as follows:

$$x \to y \quad \text{if } |y| = |x| + 1 \quad \text{or} \quad \text{if } |x| = n-1 \text{ and } |y| = 2, \tag{1.12}$$

so that $LH(n)$ has $4n$ edges. In fact, $LH(n)$ is obtained from the complete multipartite digraph $\overrightarrow{K}_{\underbrace{2,2,\ldots,2}_{n}}$ by adding the last 4 arrows.

Example 1.19. Here is the digraph $LH(5)$.



It is obtained from $\overrightarrow{K}_{2,2,2,2,2}$ by adding four arrows.

For this digraph $\beta_p > 0$ provided

$$p = 1 \bmod 3.$$

Proposition 1.20 ([34])**.** *If* $p = 1 \bmod (n-2)$ *and* $p \geq n-1$ *then the homology group* $H_p(LH(n))$ *is non-trivial.*

Hence, for the digraph $LH(n)$, non-trivial homology groups H_p occur for arbitrarily large p. Consequently, we have

$$\dim_h LH(n) = \infty.$$

There are digraphs with non-trivial homology group H_p for *all* value of p – see below Example 3.27.

Proof. Let $p = (n-2)k + 1$ for some $k \geq 1$. Let us construct a family of allowed paths in $LH(n)$ as follows. First, consider a numerical sequence of $p + 1 = (n-2)k + 2$ numbers:

$$1, \underbrace{2, 3, \ldots, n-1}, \underbrace{2, 3, \ldots, n-1}, \ldots, \underbrace{2, 3, \ldots, n-1}, n, \tag{1.13}$$

where the group $2, 3, \ldots, n-1$ is repeated k times, and then give arbitrarily the signs $+$ and $-$ to each number in this sequence. Clearly, we obtain in this way an allowed elementary p-path in $LH(n)$. For any such a path u, denote by $\sigma(u)$ the number of '$-$' in u, and consider the path

$$\omega = \sum_u (-1)^{\sigma(u)} u, \tag{1.14}$$

where the summation is taken over all paths u obtained in this way from the sequence (1.13).

Let us verify that $\partial\omega = 0$ (and, hence, $\omega \in \Omega_p$). Indeed, let $u = u_0 \ldots u_p$ be one of the elementary paths in the sum (1.14). The boundary ∂u is the sum of the terms

$$(-1)^i u_0 \ldots u_{i-1} u_{i+1} \ldots u_p \tag{1.15}$$

that are obtained from u by omitting u_i. Fix some i and consider a path

$$\widetilde{u} = u_0 \ldots u_{i-1} (-u_i) u_{i+1} \ldots u_p,$$

where only the sign of u_i is changed. Then $\partial\widetilde{u}$ contains also the term (1.15). However, u and $\widetilde{u}$ enter ω with opposite signs so that the term (1.15) cancels out in the sum (1.14). Hence, we obtain $\partial\omega = 0$.

Let us verify that $\omega \neq \partial v$ for any allowed path v, which will imply that ω determines a non-trivial element in H_p. Assume from the contrary that $\omega = \partial v$ for some $v \in \mathcal{A}_{p+1}$. For that, v has to contain an allowed elementary $p+1$-path that contains both a vertex 1 and a vertex n (otherwise, 1 and n cannot appear in the same path (1.13)). Let

$$u = u_0 \dots u_{p+1}$$

be such an allowed elementary $p+1$-path, where

$$|u_0| = 1 \quad \text{and} \quad |u_{p+1}| = n.$$

We have $u_i \to u_{i+1}$ and, hence, as it follows from the definition of arrows in (1.12),

$$|u_{i+1}| = |u_i| + 1 \operatorname{mod} (n-2).$$

Therefore,

$$|u_{p+1}| = |u_0| + p + 1 \operatorname{mod} (n-2),$$

from which it follows that

$$n = p + 2 \operatorname{mod} (n-2)$$

and

$$p = 0 \operatorname{mod} (n-2),$$

which contradicts the hypotheses. □

2. Trapezohedra and Structure of Ω_3

2.1 Spaces Ω_p and H_p for Trapezohedron

For any integer $m \geq 2$, define a *trapezohedron* T_m of order m as follows:

T_m is a digraph of $2m+2$ vertices

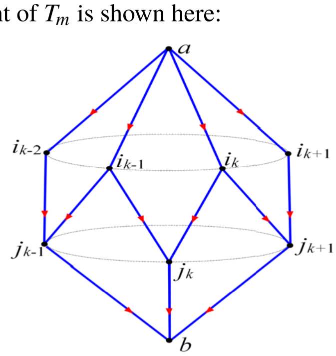
$$a, b, i_0, \dots, i_{m-1}, j_0, j_1, \dots, j_{m-1}$$

and $4m$ arrows

$$a \to i_k \to j_k \to b, \quad i_k \to j_{k+1}$$

for all $k = 0, \dots, m-1 \operatorname{mod} m$.

A fragment of T_m is shown here:



It is clear that all allowed paths in T_m have the length ≤ 3, whence $\Omega_p(T_m) = \{0\}$ for all $p > 3$.

Proposition 2.1. *For the trapezohedron T_m we have*

$$|\Omega_2| = 2m, \quad |\Omega_3| = 1,$$

and $H_p = \{0\}$ for all $p \geq 1$.

Proof. It is easy to detect all squares in T_m:

(2.16) $$e_{ai_{k-1}j_k} - e_{ai_kj_k} \quad \text{and} \quad e_{i_kj_kb} - e_{i_kj_{k+1}b},$$

where $k = 0, \dots, m-1$. Hence, T_m contains $2m$ squares, and they are linearly independent. Since there are no triangles in T_m, we conclude by Theorem 1.8 that $|\Omega_2| = 2m$.

All allowed 3-paths in T_m are as follows:

$$e_{ai_kj_kb} \quad \text{and} \quad e_{ai_kj_{k+1}b},$$

also for all $k = 0, \dots, m-1$. Let us find all linear combinations of these paths that are ∂-invariant. Consider such a linear combination

$$\omega = \sum_{k=0}^{m-1} \left(\alpha_k e_{ai_kj_kb} + \beta_k e_{ai_kj_{k+1}b}\right)$$

with coefficients α_k, β_k, and assume that ω is ∂-invariant. We have

$$\partial\omega = \sum_{k=0}^{m-1} \partial\left(\alpha_k e_{ai_kj_kb} + \beta_k e_{ai_kj_{k+1}b}\right)$$

(2.17) $$= \sum_{k=0}^{m-1} \left(\alpha_k e_{i_kj_kb} + \beta_k e_{i_kj_{k+1}b}\right) - \sum_{k=0}^{m-1} \left(\alpha_k e_{ai_kj_k} + \beta_k e_{ai_kj_{k+1}}\right)$$

(2.18) $$- \sum_{k=0}^{m-1} \left(\alpha_k e_{aj_kb} + \beta_k e_{aj_{k+1}b}\right) + \sum_{k=0}^{m-1} \left(\alpha_k e_{ai_kb} + \beta_k e_{ai_kb}\right).$$

Both sums in (2.17) consist of allowed paths. In the rightmost sum in (2.18) the path e_{ai_kb} is not allowed and, hence, must cancel out, which yields

$$\alpha_k = -\beta_k.$$

The leftmost sum in (2.18) is then equal to

$$\sum_{k=0}^{m-1} \left(\alpha_k e_{aj_kb} - \alpha_k e_{aj_{k+1}b}\right) = \sum_{k=0}^{m-1} \left(\alpha_k - \alpha_{k-1}\right) e_{aj_kb},$$

and it must vanish as e_{aj_kb} is not allowed, whence

$$\alpha_k = \alpha_{k-1}.$$

Setting $\alpha_k \equiv \alpha$ and, hence, $\beta_k = -\alpha$, we obtain that

$$\omega = \alpha \sum_{k=0}^{m-1} \left(e_{ai_kj_kb} - e_{ai_kj_{k+1}b}\right) = \alpha\tau_m$$

so that $\Omega_3 = \langle\tau_m\rangle$ and $|\Omega_3| = 1$.

It follows from (2.17)–(2.18) that

$$\partial\tau_m = \sum_{k=0}^{m-1} \left(e_{i_kj_kb} - e_{i_kj_{k+1}b}\right) - \sum_{k=0}^{m-1} \left(e_{ai_kj_k} - e_{ai_kj_{k+1}}\right) \neq 0.$$

This implies $\ker \partial|_{\Omega_3} = 0$, whence $H_3 = \{0\}$.

Let us show that $H_2 = \{0\}$. Since $\dim \operatorname{Im} \partial|_{\Omega_3} = 1$, it suffices to show that

(2.19) $$\dim \ker \partial|_{\Omega_2} = 1.$$

Consider the following general element of Ω_2:

$$u=\sum_{k=0}^{m-1}\alpha_k\left(e_{ai_{k-1}j_k}-e_{ai_kj_k}\right)+\beta_k\left(e_{i_kj_kb}-e_{i_kj_{k+1}b}\right)$$

with arbitrary coefficients α_k,β_k. We have

$$\begin{aligned}\partial u=&\sum_{k=0}^{m-1}\alpha_k\left(e_{ai_{k-1}}+e_{i_{k-1}j_k}-e_{ai_k}-e_{i_kj_k}\right)\\&+\beta_k\left(e_{j_kb}+e_{i_kj_k}-e_{j_{k+1}b}-e_{i_kj_{k+1}}\right)\\=&\sum_{k=0}^{m-1}\left(\alpha_{k+1}-\alpha_k\right)e_{ai_k}+\sum_{k=0}^{m-1}\left(\beta_k-\beta_{k-1}\right)e_{j_kb}\\&+\sum_{k=0}^{m-1}\left(\beta_k-\alpha_k\right)e_{i_kj_k}+\sum_{k=0}^{m-1}\left(\alpha_{k+1}-\beta_k\right)e_{i_kj_{k+1}}.\end{aligned}$$

The condition $\partial u=0$ is equivalent to

$$\alpha_{k+1}=\alpha_k=\beta_k=\beta_{k-1}\text{ for all }k=0,\ldots,m-1$$

which implies (2.19).

Finally, we determine $|H_1|$ by means of the Euler characteristic

$$\chi=|\Omega_0|-|\Omega_1|+|\Omega_2|-|\Omega_3|=(2m+2)-4m+2m-1=1.$$

Hence, we obtain

$$|H_0|-|H_1|+|H_2|-|H_3|=1,$$

which yields $|H_1|=0$. □

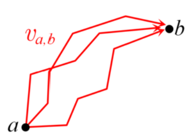
2.2 A Cluster Basis in Ω_p

We start with the following definition.

Definition. A p-path $v=\sum v^{i_0\ldots i_p}e_{i_0\ldots i_p}$ is called an (a,b)-*cluster* if all the elementary paths $e_{i_0\ldots i_p}$ with non-zero values of $v^{i_0\ldots i_p}$ have $i_0=a$ and $i_p=b$. A path v is called a *cluster* if it is an (a,b)-cluster for some a,b.

Lemma 2.2. *Any ∂-invariant p-path is a sum of ∂-invariant clusters.*

Proof. Let $v\in\Omega_p$. For any points $a,b\in V$, denote by $v_{a,b}$ the sum of all terms $v^{i_0\ldots i_p}e_{i_0\ldots i_p}$ with $i_0=a$ and $i_p=b$.



Then $v_{a,b}$ is a cluster and $v=\sum_{a,b\in V}v_{a,b}$, that is, v is a sum of clusters. Let us prove that each non-zero cluster $v_{a,b}$ is ∂-invariant.

Since v is allowed, also all non-zero terms $v^{i_0\ldots i_p}e_{i_0\ldots i_p}$ are allowed, whence $v_{a,b}$ is also allowed. Let us prove that $\partial v_{a.b}$ is allowed, which will yield the ∂-invariance of $v_{a.b}$. The path $v_{a,b}$ is a linear combination of allowed paths of the form $e_{ai_1\ldots i_{p-1}b}$. We have

$$\partial e_{ai_1\ldots i_{p-1}b}=e_{i_1\ldots i_{p-1}b}+(-1)^p e_{ai_1\ldots i_{p-1}}+\sum_{k=1}^{p-1}(-1)^k e_{ai_1..\widehat{i_k}\ldots i_{p-1}b}.$$

The terms $e_{i_1\ldots i_{p-1}b}$ and $e_{ai_1\ldots i_{p-1}}$ are clearly allowed, while among the terms $e_{ai_1..\widehat{i_k}\ldots i_{p-1}b}$ there may be non-allowed. In the full expansion of

$$\partial v=\sum_{a,b\in V}\partial v_{a,b}$$

all non-allowed terms must cancel out. Since all the terms $e_{ai_1..\widehat{i_k}\ldots i_{p-1}b}$ form a (a,b)-cluster, they cannot cancel with terms containing different values of a or b. Therefore, they have to cancel already within $\partial v_{a,b}$, which implies that $\partial v_{a,b}$ is allowed. □

Definition. For any p-path $v=\sum v^{i_0\cdots i_p}e_{i_0\ldots i_p}$ define its *width* $\|v\|$ as the number of non-zero coefficients $v^{i_0\cdots i_p}$.

Definition. A ∂-invariant path ω is called *minimal* if ω cannot be represented as a sum of other ∂-invariant paths with smaller widths.

Example 2.3. A square $\omega=e_{abc}-e_{ab'c}$ has width 2 and is minimal because e_{abc} and $e_{ab'c}$ having width 1 are not ∂-invariant.

Let $a,\{b_0,b_1,b_2\},c$ be a 2-square. The following path

$$\omega=e_{ab_0c}+e_{ab_1c}-2e_{ab_2c}$$

is ∂-invariant, has width 3 but is not minimal because it can be represented as a sum of two squares:

$$\omega=\left(e_{ab_0c}-e_{ab_2c}\right)+\left(e_{ab_1c}-e_{ab_2c}\right),$$

where each square has width 2.

Lemma 2.4. *Every ∂-invariant cluster is a sum of minimal ∂-invariant clusters.*

Proof. Let ω be a ∂-invariant cluster that is not minimal. Then we have

$$\omega=\sum_{k=1}^{n}\omega^{(k)},\tag{2.20}$$

where each $\omega^{(k)}$ is a ∂-invariant path with $\|\omega^{(k)}\|<\|\omega\|$. By Lemma 2.2, each $\omega^{(k)}$ is a sum of clusters $\omega_{a,b}^{(k)}$, and it is clear from the definition of $\omega_{a,b}^{(k)}$ that

$$\|\omega_{a,b}^{(k)}\|\leq\|\omega^{(k)}\|.$$

Hence, we can replace in (2.20) each $\omega^{(k)}$ by $\sum_{a,b}\omega_{a,b}^{(k)}$ and, hence, assume without loss of generality that all terms $\omega^{(k)}$ in (2.20) are ∂-invariant clusters.

If some $\omega^{(k)}$ in this sum is not minimal then we replace it further with a sum of ∂-invariant clusters with smaller widths. Continuing this procedure we obtain in the end a representation ω as a sum of minimal ∂-invariant clusters. □

Proposition 2.5. *The space Ω_p has a basis that consists of minimal ∂-invariant clusters.*

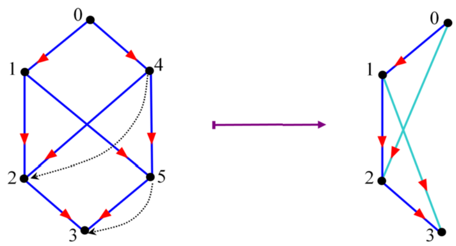
Proof. Indeed, let $\mathcal{M}$ denote the set of all minimal ∂-invariant clusters in Ω_p. By Lemma 2.4, every element of Ω_p is a sum of elements of $\mathcal{M}$. Choosing in $\mathcal{M}$ a maximal linearly independent subset, we obtain a basis in Ω_p. $\square$

2.3 Structure of Ω_3

We use here the trapezohedra T_m and associated trapezohedral paths τ_m defined in Sections 1.5 and 2.1 (see (1.5)), that are ∂-invariant 3-paths for all $m \geq 2$. We prove here in Theorem 2.10 that if G contains no multisquare (see Subsection 1.5) then $\Omega_3(G)$ has a basis that consists of trapezohedral paths and their morphism images.

We start with some examples.

Example 2.6. Here is a merging map from T_2 onto a 3-snake:



The trapezohedral path τ_2 is given by

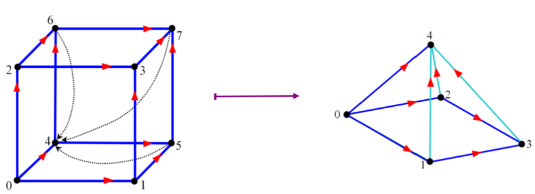
$$\tau_2 = e_{0123} - e_{0153} + e_{0453} - e_{0423},$$

and its merging image is the 3-path

$$v = e_{0123} - e_{0133} + e_{0233} - e_{0223} = e_{0123},$$

that is, the 3-path e_{0123} associated with a 3-snake.

Example 2.7. Here is a merging morphism of T_3 (= a 3-cube) onto a pyramid:



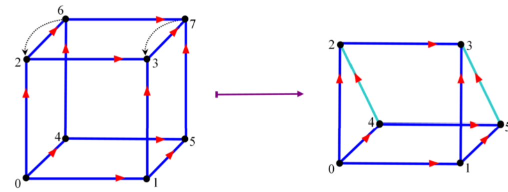
The cubical 3-path is given by

$$\tau_3 = e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267}$$

and its merging image of τ_3 is the following ∂-invariant 3-path in a pyramid:

$$v = e_{0234} - e_{0134} + e_{0144} - e_{0444} + e_{0444} - e_{0244} = e_{0234} - e_{0134}.$$

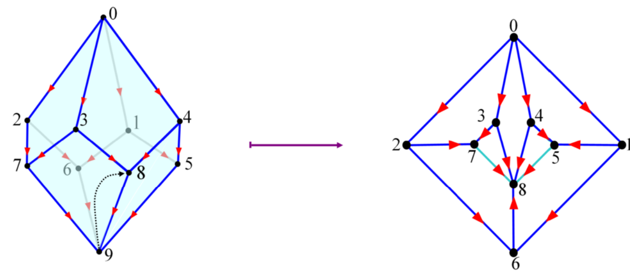
Example 2.8. Consider another merging morphism of T_3 onto a prism:



The merging image of τ_3 is the following ∂-invariant 3-path in the prism:

$$\begin{aligned} u &= e_{0233} - e_{0133} + e_{0153} - e_{0453} + e_{0423} - e_{0223} \\ &= e_{0153} - e_{0453} + e_{0423}. \end{aligned}$$

Example 2.9. Here is a merging morphism $\mu : T_4 \to G$ where the digraph G is a *broken cube* that is shown in the right panel:



The path τ_4 in the present notation is given by

$$\tau_4 = e_{0159} - e_{0169} + e_{0269} - e_{0279} + e_{0379} - e_{0389} + e_{0489} - e_{0459},$$

and the merging image of τ_4 is the following ∂-invariant 3-path on the broken cube:

(2.21)
$$\begin{aligned} v &= e_{0158} - e_{0168} + e_{0268} - e_{0278} + e_{0378} - e_{0388} + e_{0488} - e_{0458} \\ &= e_{0158} - e_{0168} + e_{0268} - e_{0278} + e_{0378} - e_{0458}. \end{aligned}$$

One can show that $\Omega_3(G) = \langle v \rangle$.

The next theorem describes the structure of $\Omega_3(G)$ for a general digraph G but under the following hypothesis:

(2.22) *G contains neither multisquares nor double arrows.*

Under the hypothesis (2.22), $\Omega_2(G)$ has a basis that consists of triangles and squares. The condition (2.22) implies that if $a \to b \to c$ and $a \not\to c$ then there is at most one $b' \neq b$ such that $a \to b' \to c$.

Theorem 2.10. *Under the hypothesis* (2.22)*, there is a basis in $\Omega_3(G)$ that consists of trapezohedral paths τ_m with $m \geq 2$ and their merging images.*

Hence, trapezohedra are *basic shapes* for Ω_3.

Proof. By Proposition 2.5, Ω_3 has a basis that consists of minimal ∂-invariant clusters. Let a path $\omega \in \Omega_3$ be a minimal ∂-invariant (a,b)-cluster. It suffices to prove that ω is a merging image of one of the trapezohedral paths τ_m up to a constant factor.

Denote by P the set of all elementary terms e_{aijb} of ω. Clearly, the number $|P|$ of elements in P is equal to $\|\omega\|$. We claim that, for any $e_{aijb} \in P$,

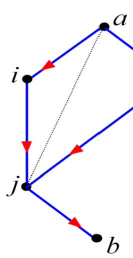
$$\text{either} \quad a \to j \quad \text{or} \quad a \nearrow j$$

where the notation $a \nearrow j$ means that a and j form a diagonal of a square.

Indeed, if $a \not\to j$ then the term e_{ajb} appearing in ∂e_{aijb} is non-allowed and should be cancelled in $\partial\omega$ by the boundary of another elementary 3-path from P that can only be of the form $e_{ai'jb}$ with

$$a \to i' \to j$$

Hence, a and j form diagonal of a square a, i, i', j.



By hypothesis (2.22), the vertex i' with these properties is unique. Hence, in this case we have

$$\omega = ce_{aijb} - ce_{ai'jb} + \ldots \tag{2.23}$$

for some scalar $c \neq 0$. In the same way, we have

$$\text{either} \quad i \to b \quad \text{or} \quad i \nearrow b.$$

and, for some $e_{aij'b} \in P$ and $c \neq 0$,

$$\omega = ce_{aijb} - ce_{aij'b} + \ldots . \tag{2.24}$$

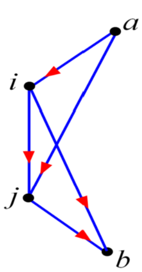
If for some path $e_{aijb} \in P$ we have both conditions

$$a \to j \quad \text{and} \quad i \to b$$

then e_{aijb} is ∂-invariant and, by the minimality of ω,

$$\omega = \text{const}\, e_{aijb}.$$

Since e_{aijb} is in this case a 3-snake, the path ω is a merging image of τ_2.



Next, we can assume that, for any path $e_{aijb} \in P$, we have

$$a \not\to j \quad \text{or} \quad i \not\to b$$

which is equivalent to

$$a \nearrow j \quad \text{or} \quad i \nearrow b. \tag{2.25}$$

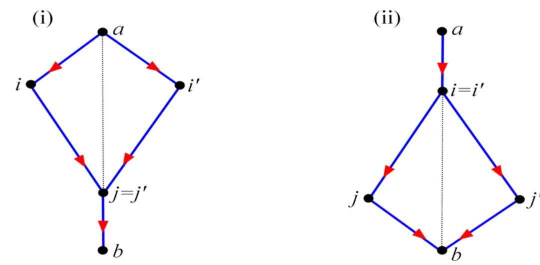
Define a graph structure on P with edges of two types (i) and (ii) as follows: for two distinct elements e_{aijb} and $e_{ai'j'b}$ of P we write

$$e_{aijb} \overset{\text{(i)}}{\sim} e_{ai'j'b} \text{ if } a \nearrow j \text{ and } j = j'.$$

and

$$e_{aijb} \overset{\text{(ii)}}{\sim} e_{ai'j'b} \text{ if } i \nearrow b \text{ and } i' = i.$$

Clearly, both relations $\overset{\text{(i)}}{\sim}$ and $\overset{\text{(ii)}}{\sim}$ are symmetric. We refer to the relations $\overset{\text{(i)}}{\sim}$ and $\overset{\text{(ii)}}{\sim}$ as the edges in P of the first and, respectively, second type.



Cases $e_{aijb} \overset{\text{(i)}}{\sim} e_{ai'j'b}$ and $e_{aijb} \overset{\text{(ii)}}{\sim} e_{ai'j'b}$

By the hypothesis (2.22), for any $e_{aijb} \in P$ there is at most one edge of the first type and at most one edge of the second type. In particular, the degree of any vertex of the graph $(P, \sim)$ is at most 2.

Fix a path $e_{aijb} \in P$. By the above argument, if $a \nearrow j$ then there exists $e_{ai'jb} \in P$ such that $e_{aijb} \overset{\text{(i)}}{\sim} e_{ai'jb}$ and ω satisfies (2.23). Similarly, if $i \nearrow b$ then there exists $e_{aij'b} \in P$ such that $e_{aijb} \overset{\text{(ii)}}{\sim} e_{aij'b}$ and ω satisfies (2.24). In particular, the degree of any vertex of the graph P is at least 1.

Let us prove that the graph $(P, \sim)$ is connected. If P not connected then P is a disjoint union of its connected components $\{P_k\}_{k=1}^{n}$ where $n > 1$. Denote by $\omega^{(k)}$ the sum of all elementary terms of ω lying in P_k, with the same coefficients as in ω, so that

$$\omega = \sum_{k=1}^{n} \omega^{(k)}. \tag{2.26}$$

Let us verify that each $\omega^{(k)}$ is ∂-invariant. Clearly, $\omega^{(k)}$ is allowed, and let us prove that $\partial\omega^{(k)}$ is allowed. Indeed, let $\partial\omega^{(k)}$ contain a non-allowed term. The latter comes from the boundary ∂e_{aijb} of some elementary term e_{aijb} of $\omega^{(k)}$ and, hence, is either e_{aib} or e_{ajb}, let it be e_{aib}, which means $i \not\to b$. The term e_{aib} cancels out in $\partial\omega$, which can only happen when ω contains another term of the form $e_{aij'b}$. However, then

$$e_{aijb} \sim e_{aij'b}$$

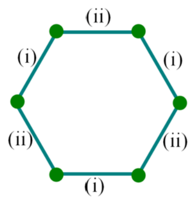
so that $e_{aij'b}$ belongs to the same connected component P_k and, hence, must be an elementary term of $\omega^{(k)}$. This proves that $\partial\omega^{(k)}$ is allowed and, hence, $\omega^{(k)}$ is ∂-invariant.

If the number n of the terms in (2.26) is greater than 1 then the number of vertices in each P_k is strictly less that in P, which implies $\|\omega_k\| < \|\omega\|$. However, in this case the representation (2.26) is not possible because ω is minimal. Hence, $n = 1$ and P is connected.

Since each vertex of P has at most two adjacent edges, there are only two possibilities:

(A) P is a simple closed polygon;
(B) P is a linear graph.

Consider first the case (A). In this case every vertex of P has two edges: exactly one edge of each type (i), (ii).



Thus, the number of edges is even, let $2m$, and P has necessarily the following form:

(2.27)
$$e_{ai_0j_0b} \overset{\text{(ii)}}{\sim} e_{ai_0j_1b} \overset{\text{(i)}}{\sim} e_{ai_1j_1b} \overset{\text{(ii)}}{\sim} \dots \overset{\text{(i)}}{\sim} e_{ai_{m-1}j_{m-1}b} \overset{\text{(ii)}}{\sim} e_{ai_{m-1}j_0b} \overset{\text{(i)}}{\sim} e_{ai_0j_0b}$$

for some vertices $\{i_k\}_{k=0}^{m-1}$ and $\{j_k\}_{k=0}^{m-1}$ of G. Note that necessarily $m \geq 2$ because if $m = 1$ then (2.27) becomes

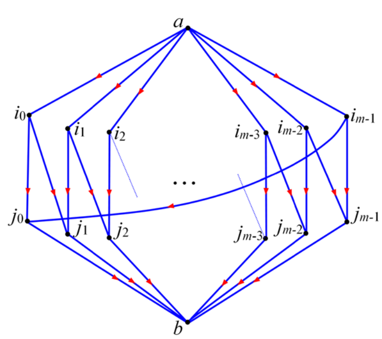
$$e_{ai_0j_0b} \overset{\text{(ii)}}{\sim} e_{ai_0j_1b} \overset{\text{(i)}}{\sim} e_{ai_0j_0b},$$

which is impossible because edges of different types between the same vertices of P do not exist.

Since all the terms in (2.27) enter ω with the same coefficients $\pm c$ (cf. (2.23) and (2.24)), we see that

$$\begin{aligned}\omega = c(&e_{ai_0j_0b} - e_{ai_0j_1b} + e_{ai_1j_1b} - e_{ai_1j_2b} + \dots \\ &+ e_{ai_{m-1}j_{m-1}b} - e_{ai_{m-1}j_0b}).\end{aligned} \tag{2.28}$$

If all vertices a, $\{i_k\}_{k=0}^{m-1}$, $\{j_k\}_{k=0}^{m-1}$, b are distinct then they form a trapezohedron T_m:



In this case we have by (1.5) and (2.28)

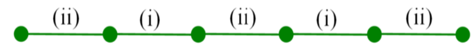
$$\omega = c\tau_m.$$

If some of these vertices coincide then the configuration (2.27) is a merging image of T_m, and ω is a merging image of $c\tau_m$.

Consider now the case (B). In this case the linear graph P has two end vertices of degree 1, while all other vertices have degree 2. Depending on the type of edges at the end vertices of P, we have two essentially different subcases:

case (B_1): the end vertices of P have edges of different types.

case (B_2): the end vertices of P both have edges of type (ii) (the case of type (i) is similar).



Consider first the case (B_1) when the graph P must have the form

(2.29)
$$e_{ai_0j_0b} \overset{\text{(ii)}}{\sim} e_{ai_0j_1b} \overset{\text{(i)}}{\sim} e_{ai_1j_1b} \overset{\text{(ii)}}{\sim} e_{ai_1j_2b} \overset{\text{(i)}}{\sim} \dots \overset{\text{(ii)}}{\sim} e_{ai_{m-1}j_mb} \overset{\text{(i)}}{\sim} e_{ai_mj_mb}.$$

Consequently, we have

$$\begin{aligned}\omega = c\big(&e_{ai_0j_0b} - e_{ai_0j_1b} + e_{ai_1j_1b} - e_{ai_1j_2b} + \dots \\ &- e_{ai_{m-1}j_mb} + e_{ai_mj_mb}\big).\end{aligned} \tag{2.30}$$

Since

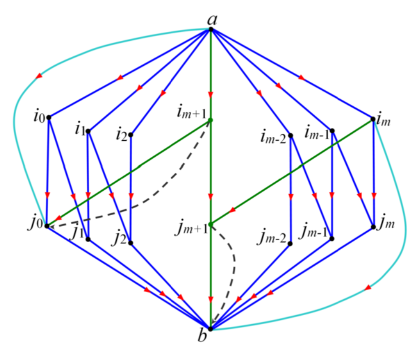
$$\partial\omega = c\left(-e_{aj_0b} + e_{ai_mb}\right) \bmod \mathcal{A}_2 \tag{2.31}$$

and $\partial\omega \in \mathcal{A}_2$, we must have either $e_{aj_0b} = e_{ai_mb}$ or both e_{aj_0b} and e_{ai_mb} are allowed, that is,

$$a \to j_0 \quad \text{and} \quad i_m \to b. \tag{2.32}$$

In the former case we have $j_0 = i_m$ whence (2.32) follows again so that (2.32) is satisfied in both cases.

We claim that in the case (B_1) the configuration (2.29) is a merging image of T_{m+2}.



Indeed, denote the vertices of T_{m+2} also by $a, \{i_k\}_{k=0}^{m+1}, \{j_k\}_{k=0}^{m+1}, b$, and map all the vertices of T_{m+2}, except for i_{m+1}, j_{m+1}, to the vertices of G with the same names; then merge

$$i_{m+1} \mapsto j_0 \quad \text{and} \quad j_{m+1} \mapsto b.$$

The arrows $a \to i_{m+1}, i_m \to j_{m+1}, i_{m+1} \to j_{m+1}$ in T_{m+2} are mapped to the arrows

$$a \to j_0,\ i_m \to b,\ j_0 \to b$$

in G (cf. (2.32)), while the arrows $i_{m+1} \to j_0$ and $j_{m+1} \to b$ go to vertices. It follows that this mapping of T_{m+2} into G is a digraph morphism. Since by (1.5)

$$\tau_{m+2} = (e_{ai_0j_0b} - e_{ai_0j_1b}) + (e_{ai_1j_1b} - e_{ai_1j_2b}) + \dots$$

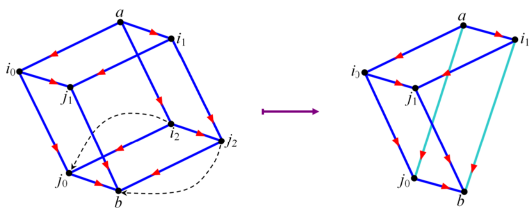
$$+(e_{ai_m j_m b} - e_{ai_m j_{m+1} b}) + (e_{ai_{m+1} j_{m+1} b} - e_{ai_{m+1} j_0 b}),$$

the image of τ_{m+2} is the following path, where we replace i_{m+1} by j_0 and j_{m+1} by b:

$$\begin{aligned} u &= (e_{ai_0 j_0 b} - e_{ai_0 j_1 b}) + (e_{ai_1 j_1 b} - e_{ai_1 j_2 b}) + \dots \\ &\quad + (e_{ai_m j_m b} - \underline{e_{ai_m bb}}) + (\underline{e_{aj_0 bb}} - \underline{e_{aj_0 j_0 b}}) \\ &= e_{ai_0 j_0 b} - e_{ai_0 j_1 b} + e_{ai_1 j_1 b} - e_{ai_1 j_2 b} + \dots - e_{ai_{m-1} j_m b} + e_{ai_m j_m b}. \end{aligned}$$

Comparison with (2.30) shows that $\omega = cu$, that is, ω is a merging image of $c\tau_{m+2}$.

For example, in the case $m = 1$, this merging morphism of T_3 is shown here:



Clearly, it coincides with the merging morphism of Example 2.8 mapping a 3-cube onto a prism.

Consider now the case (B_2) when the graph P has the form

$$(2.33)\qquad \begin{aligned} &e_{ai_0 j_0 b} \overset{(ii)}{\sim} e_{ai_0 j_1 b} \overset{(i)}{\sim} e_{ai_1 j_1 b} \overset{(ii)}{\sim} e_{ai_1 j_2 b} \overset{(i)}{\sim} \dots \overset{(i)}{\sim} e_{ai_{m-1} j_{m-1} b} \\ &\overset{(ii)}{\sim} e_{ai_{m-1} j_m b}, \end{aligned}$$

so that

$$(2.34)\qquad \begin{aligned} \omega = c(&e_{ai_0 j_0 b} - e_{ai_0 j_1 b} + e_{ai_1 j_1 b} - e_{ai_1 j_2 b} + \dots \\ &+ e_{ai_{m-1} j_{m-1} b} - e_{ai_{m-1} j_m b}). \end{aligned}$$

Since

$$\partial\omega = c\left(-e_{aj_0 b} + e_{aj_m b}\right) \bmod \mathcal{A}_2,$$

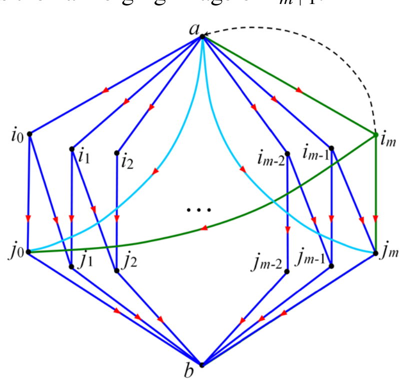
it follows that either $j_0 = j_m$ or

$$(2.35)\qquad a \to j_0 \quad \text{and} \quad a \to j_m.$$

However, $j_0 = j_m$ is not possible because it would imply that

$$e_{ai_0 j_0 b} \overset{(i)}{\sim} e_{ai_{m-1} j_0 b}$$

and the line graph P would close into a polygon, which gives the case (A). Hence, (2.35) is satisfied. We claim that the configuration (2.33) is then a merging image of T_{m+1}.



Indeed, we denote the vertices of T_{m+1} also by $a, \{i_k\}_{k=0}^m, \{j_k\}_{k=0}^m, b$, and then map all the vertices of T_{m+1}, except for i_m, to the vertices of G with the same names; then map i_m to a.

Clearly, the following arrows

$$i_m \to j_0 \quad \text{and} \quad i_m \to j_m$$

in T_{m+1} are mapped to the arrows

$$a \to j_0 \quad \text{and} \quad a \to j_m$$

in G as in (2.35), and the arrow $a \to i_m$ goes to a vertex. Hence, we obtain a merging morphism of T_{m+1} into G. Since by (1.5)

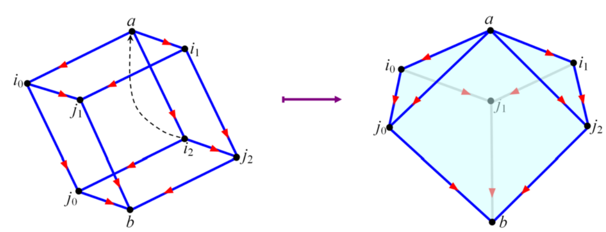
$$\begin{aligned} \tau_{m+1} = (&e_{ai_0 j_0 b} - e_{ai_0 j_1 b}) + (e_{ai_1 j_1 b} - e_{ai_1 j_2 b}) + \dots \\ &+ (e_{ai_{m-1} j_{m-1} b} - e_{ai_{m-1} j_m b}) + (e_{ai_m j_m b} - e_{ai_m j_0 b}), \end{aligned}$$

the image of τ_{m+1} is the following path, where we replace i_m by a:

$$\begin{aligned} v &= (e_{ai_0 j_0 b} - e_{ai_0 j_1 b}) + (e_{ai_1 j_1 b} - e_{ai_1 j_2 b}) + \dots \\ &\quad + (e_{ai_{m-1} j_{m-1} b} - e_{ai_{m-1} j_m b}) + (\underline{e_{aaj_m b}} - \underline{e_{aaj_0 b}}) \\ &= e_{ai_0 j_0 b} - e_{ai_0 j_1 b} + e_{ai_1 j_1 b} - e_{ai_1 j_2 b} + \dots \\ &\quad + e_{ai_{m-1} j_{m-1} b} - e_{ai_{m-1} j_m b}. \end{aligned}$$

Comparison with (2.34) shows that $\omega = cv$ so that ω is a merging image of $c\tau_{m+1}$. □

For example, in the case $m = 3$, the above morphism is equivalent to the merging morphism of Example 2.9 mapping T_4 onto a broken cube. In the case $m = 2$ we obtain the following merging image of a 3-cube:



Problem 2.11. *Prove Theorem 2.10 in the general case without the hypothesis (2.22).*

Problem 2.12. *Devise an algorithm for computing a basis in Ω_3 based on Theorem 2.10.*

Problem 2.13. *State and prove similar results for Ω_4. Are the basic shapes in Ω_4 given by polyhedra in $\mathbb{R}^4$? Devise an algorithm for computing a basis in Ω_4. The same questions for Ω_p with $p > 4$.*

3. Künneth Formulas

The material in this section is based on [22] and [29].

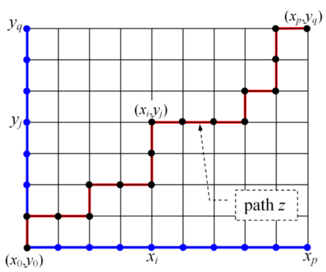
3.1 Cross Product of Paths

Given two finite sets X,Y, consider their product

$$Z = X \times Y = \{(a,b) : a \in X \text{ and } b \in Y\}.$$

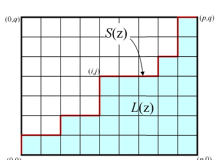
Let $z = z_0 z_1 \dots z_r$ be a regular elementary r-path on Z, where $z_k = (a_k, b_k)$ with $a_k \in X$ and $b_k \in Y$. We say that z is *stair-like* if, for any $k = 1,\dots,r$, either $a_{k-1} = a_k$ or $b_{k-1} = b_k$ is satisfied. That is, any couple $z_{k-1}z_k$ of consecutive vertices is either vertical (when $a_{k-1} = a_k$) or horizontal (when $b_{k-1} = b_k$).

Given a stair-like path z on Z, define its projection onto X as an elementary path x on X obtained from z by removing Y-components in all the vertices of z and then by collapsing in the resulting sequence any subsequence of repeated vertices to one vertex.



In the same way define projection of z onto Y and denote it by y.

The projections $x = x_0 \dots x_p$ and $y = y_0 \dots y_q$ are regular elementary paths, and $p + q = r$.



Every vertex (x_i, y_j) of the path z can be represented as a point (i,j) of $\mathbb{Z}^2$ so that the path z is represented by a *staircase* $S(z)$ in $\mathbb{Z}^2$ connecting $(0,0)$ and (p,q).

Define the *elevation* $L(z)$ of z as the number of cells in $\mathbb{Z}^2_+$ below the staircase $S(z)$.

For given elementary regular paths x on X and y on Y, denote by $\Sigma_{x,y}$ the set of all stair-like paths z on Z whose projections on X and Y are respectively x and y.

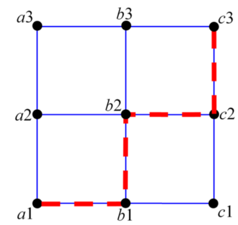
Definition. Define the *cross product* of the paths e_x and e_y as a path $e_x \times e_y$ on Z as follows:

$$e_x \times e_y = \sum_{z \in \Sigma_{x,y}} (-1)^{L(z)} e_z \tag{3.36}$$

and extend it by linearity to all $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ so that $u \times v \in \mathcal{R}_{p+q}(Z)$.

Example 3.1. Let us denote the vertices on X by letters a,b,c etc and the vertices on Y by integers $1,2,3,$ etc so that the vertices on Z can be denoted as $a1, b2$ etc as the fields on a chessboard. Then we have

$$\begin{aligned}
e_a \times e_{12} &= e_{a1\,a2}, \quad e_{ab} \times e_1 = e_{a1\,b1}\\
e_{ab} \times e_{12} &= e_{a1\,b1\,b2} - e_{a1\,a2\,b2}\\
e_{ab} \times e_{123} &= e_{a1\,b1\,b2\,b3} - e_{a1\,a2\,b2\,b3} + e_{a1\,a2\,a3\,b3}\\
e_{abc} \times e_{123} &= e_{a1\,b1\,c1\,c2\,c3} - e_{a1\,b1\,b2\,c2\,c3} + e_{a1\,b1\,b2\,b3\,c3}\\
&\quad + e_{a1\,a2\,b2\,c2\,c3} - e_{a1\,a2\,b2\,b3\,c3} + e_{a1\,a2\,a3\,b3\,c3}
\end{aligned}$$



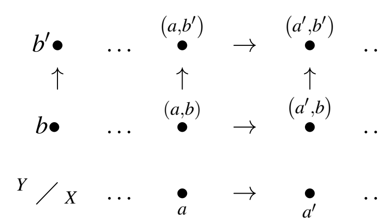
Lemma 3.2 ([29, Proposition 4.4]). *If $u \in \mathcal{R}_p(X)$ and $v \in \mathcal{R}_q(Y)$ where $p,q \geq 0$, then*

$$\partial(u \times v) = (\partial u) \times v + (-1)^p u \times (\partial v). \tag{3.37}$$

3.2 Cartesian Product of Digraphs

Denote a digraph and its set of vertices by the same letters to simplify notation. Given two digraphs X and Y, define their Cartesian product as a digraph $Z = X \Box Y$ as follows:

- the set of vertices of Z is $X \times Y$, that is, the vertices of Z are the couples (a,b) where $a \in X$ and $b \in Y$;
- the edges in Z are of two types: $(a,b) \to (a',b)$ where $a \to a'$ (a *horizontal* edge) and $(a,b) \to (a,b')$ where $b \to b'$ (a *vertical* edge):



It follows that any allowed elementary path in Z is stair-like.

Moreover, any regular elementary path on Z is allowed if and only if it is stair-like and its projections onto X and Y are allowed.

It follows from definition (3.36) of the cross product that

$$u \in \mathcal{A}_p(X) \text{ and } v \in \mathcal{A}_q(Y) \;\Rightarrow\; u \times v \in \mathcal{A}_{p+q}(Z). \tag{3.38}$$

Furthermore, the following is true.

Lemma 3.3 ([29, Proposition 4.6]). *If $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then*

$$u \times v \in \Omega_{p+q}(Z).$$

Proof. $u \times v$ is allowed by (3.38). Since ∂u and ∂v are allowed, by (3.38) also $\partial u \times v$ and $u \times \partial v$ are allowed. By (3.37), $\partial(u \times v)$ is also allowed. Hence, $u \times v \in \Omega_{p+q}(Z)$. □

Theorem 3.4 ([29, Theorem 5.1]). *Any ∂-invariant path w on $Z = X\Box Y$ admits a representation of the form*

$$w = \sum_{i=1}^{m} u_i \times v_i$$

for some finite m, where u_i and v_i are ∂-invariant paths on X and Y, respectively.

3.3 Künneth Formula for Product

Here is the main result of this section.

Theorem 3.5 (Künneth formula for product [29, Theorem 4.7]). *Let X, Y be two finite digraphs. Then, for any $r \geq 0$,*

$$\Omega_r(X\Box Y) \cong \bigoplus_{\{p,q\geq 0: p+q=r\}} \Omega_p(X)\otimes\Omega_q(Y), \tag{3.39}$$

where the isomorphism is given by

$$u\otimes v \mapsto u\times v$$

for $u \in \Omega_p(X)$ and $v\in\Omega_q(Y)$.

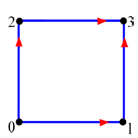
Consequently, we have

$$H_r(X\Box Y) \cong \bigoplus_{\{p,q\geq 0: p+q=r\}} H_p(X)\otimes H_q(Y) \tag{3.40}$$

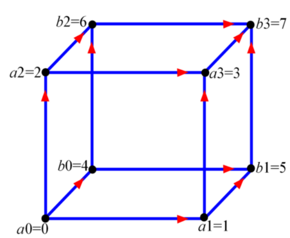
and

$$\beta_r(X\Box Y) = \sum_{\{p,q\geq 0: p+q=r\}} \beta_p(X)\,\beta_q(Y).$$

Example 3.6. Let X be an interval and Y be a square:

$X = {}^a\bullet \to \bullet^b$ and $Y =$ 

Then $Z = X\Box Y$ is a 3-cube:

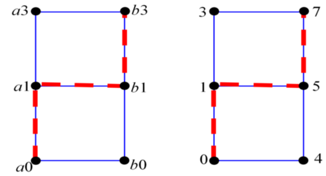


We have:

$$\Omega_1(X) = \langle e_{ab}\rangle,\ \Omega_p(X) = 0 \text{ for } p\geq 2,$$
$$\Omega_1(Y) = \langle e_{01}, e_{13}, e_{23}, e_{02}\rangle,$$
$$\Omega_2(Y) = \langle e_{013} - e_{023}\rangle,\ \Omega_q(Y) = 0 \text{ for } q\geq 3.$$

By (3.39) we obtain

$$\Omega_3(Z) \cong \Omega_1(X)\otimes\Omega_2(Y) = \langle e_{ab}\times(e_{013}-e_{023})\rangle.$$



Let us compute the cross-products:

$$\begin{aligned} e_{ab}\times e_{013} &= e_{a0b0b1b3} - e_{a0a1b1b3} + e_{a0a1a3b3} \\ &= e_{0457} - e_{0157} + e_{0137} \end{aligned}$$

and

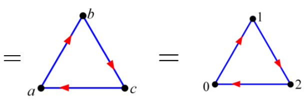
$$e_{ab}\times e_{023} = e_{0467} - e_{0267} + e_{0237}.$$

Hence, we obtain

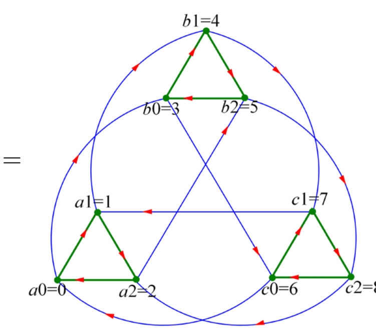
$$\Omega_3(Z) = \langle e_{0457} - e_{0157} + e_{0137} - e_{0467} + e_{0267} - e_{0237}\rangle.$$

That is, Ω_3 is generated by a single ∂-invariant 3-path that is associated with the 3-cube.

Example 3.7. Denote by T the following 3-cycle (= 1-torus):

$T =$ 

Consider the 2-torus $G = T\Box T$ that is shown here:

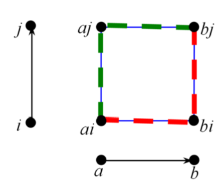
$G =$ 

Let us compute $\Omega_r(G)$ and $H_r(G)$. We have

$$\Omega_0(T) = \langle e_0, e_1, e_2\rangle,$$
$$\Omega_1(T) = \langle e_{01}, e_{12}, e_{20}\rangle,$$
$$\Omega_p(T) = \{0\} \text{ for } p\geq 2.$$

By (3.39) we obtain $\Omega_r = \{0\}$ for $r\geq 3$ and

$$\begin{aligned} \Omega_2(G) &= \Omega_1(T)\otimes\Omega_1(T) \\ &= \langle e_{ab}\times e_{01}, e_{ab}\times e_{12}, e_{ab}\times e_{20}, e_{bc}\times e_{01}, e_{bc}\times e_{12}, \\ &\quad e_{bc}\times e_{20}, e_{ca}\times e_{01}, e_{ca}\times e_{12}, e_{ca}\times e_{20}\rangle. \end{aligned}$$



Using

$$e_{ab}\times e_{ij}=e_{aibibj}-e_{aiajbj}$$

we obtain that

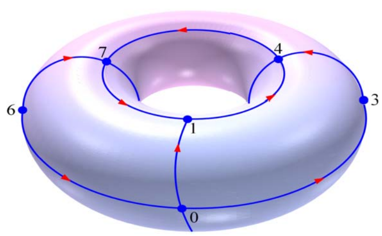
$$\begin{aligned}\Omega_2(G)=\langle & e_{a0b0b1}-e_{a0a1b1},\ e_{a1b1b2}-e_{a1a2b2},\\ & e_{a2b2b0}-e_{a2a0b0},\ e_{b0c0c1}-e_{b0b1c1},\\ & e_{b1c1c2}-e_{b1b2c2},\ e_{b2c2c0}-e_{b2b0c0},\\ & e_{c0a0a1}-e_{c0c1a1},\ e_{c1a1a2}-e_{c1c2a2},\\ & e_{c2a2a0}-e_{c2c0a0}\rangle.\end{aligned}$$

That is,

$$\begin{aligned}\Omega_2(G)=\langle & e_{034}-e_{014},\ e_{145}-e_{125},\ e_{253}-e_{203},\\ & e_{367}-e_{347},\ e_{478}-e_{458},\ e_{586}-e_{536}\\ & e_{601}-e_{671},\ e_{712}-e_{782},\ e_{820}-e_{860}\rangle \end{aligned}\tag{3.41}$$

so that $\Omega_2(G)$ is generated by 9 squares.

This can be visualized using the following embedding of G onto a topological torus:



Let us compute the homology groups of G. We know that

$$\begin{aligned}&H_0(T)=\langle e_0\rangle,\quad H_1(T)=\langle e_{01}+e_{12}+e_{20}\rangle,\\ &H_p(T)=\{0\}\text{ for }p\geq 2.\end{aligned}$$

By (3.40) we obtain

$$H_1(G)=H_0(T)\otimes H_1(T)+H_1(T)\otimes H_0(T)=\langle v_1,v_2\rangle$$

where

$$\begin{aligned}v_1&=e_a\times(e_{01}+e_{12}+e_{20})=e_{a0a1}+e_{a1a2}+e_{a2a0}\\ &=e_{01}+e_{12}+e_{20}\\ v_2&=(e_{ab}+e_{bc}+e_{ca})\times e_0=e_{a0b0}+e_{b0c0}+e_{c0a0}\\ &=e_{03}+e_{36}+e_{60}.\end{aligned}$$

Again by (3.40) we get

$$H_2(G)=H_1(T)\otimes H_1(T)=\langle u\rangle,$$

where

$$u=(e_{ab}+e_{bc}+e_{ca})\times(e_{01}+e_{12}+e_{20}),$$

Hence,

$$\begin{aligned}u=\ & e_{a0b0b1}-e_{a0a1b1}+e_{a1b1b2}-e_{a1a2b2}+e_{a2b2b0}-e_{a2a0b0}\\ &+e_{b0c0c1}-e_{b0b1c1}+e_{b1c1c2}-e_{b1b2c2}+e_{b2c2c0}-e_{b2b0c0}\\ &+e_{c0a0a1}-e_{c0c1a1}+e_{c1a1a2}-e_{c1c2a2}+e_{c2a2a0}-e_{c2c0a0},\end{aligned}$$

that is,

$$\begin{aligned}u=\ &(e_{034}-e_{014})+(e_{145}-e_{125})+(e_{253}-e_{203})\\ &+(e_{367}-e_{347})+(e_{478}-e_{458})+(e_{586}-e_{536})\\ &+(e_{601}-e_{671})+(e_{712}-e_{782})+(e_{820}-e_{860}).\end{aligned}\tag{3.42}$$

Finally, $H_r(G)=0$ for all $r\geq 3$.

3.4 An Example: n-Cube

Define the n-cube as follows:

$$n\text{-cube}=I^{\Box n}=\underbrace{I\Box I\Box\ldots\Box I}_{n},$$

where $I=\{0\to 1\}$ and $n\in\mathbb{N}$. Hence, each vertex a of the n-cube can be identified with a binary sequence $(a_1,\ldots,a_n)$. For example, $\mathbf{0}=(0,\ldots,0)$ and $\mathbf{1}=(1,\ldots,1)$ are the corners of the n-cube.

For two vertices a, b of the n-cube, there is an arrow $a\to b$ if $b_k=a_k+1$ for exactly one value of k and $b_k=a_k$ for all other values of k. Denote

$$|a|=a_1+\ldots+a_n.$$

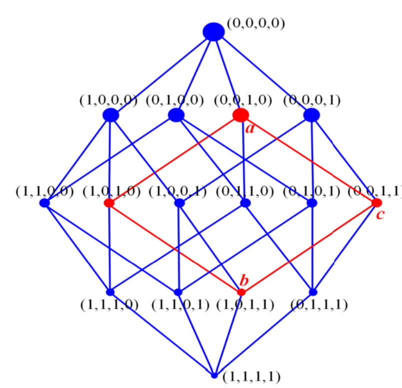
We write $a\preceq b$ if there is an allowed path from a to b, that is

$$a\preceq b\ \Leftrightarrow\ a_k\leq b_k\text{ for all }k=1,\ldots,n.$$

For any pair $a\preceq b$ consider an induced subgraph $D_{a,b}$ of the n-cube as follows: the vertices of $D_{a,b}$ are all vertices c of $I^{\Box n}$ such that

$$a\preceq c\preceq b$$

and an arrow $c_1\to c_2$ exists in $D_{a,b}$ exactly when this arrow exists in $I^{\Box n}$. Here is a 4-cube and its subgraph $D_{a,b}$ (the arrows go from top to bottom):



The mapping $c\mapsto c-a$ provides an isomorphism of $D_{a,b}$ onto a p-cube with

$$p=|b|-|a|.$$

Assuming that $a\preceq b$, denote by $P_{a,b}$ the set of all elementary allowed paths going from a to b. All paths of $P_{a,b}$ lie in $D_{a,b}$, each path in $P_{a,b}$ has the length $p=|b|-|a|$, and the total number of the paths in $P_{a,b}$ is $p!$.

Lemma 3.8. *There is a function* $\sigma : P_{a,b} \to \{0,1\}$ *such that the following p-path*

$$\omega_{a,b} = \sum_{x \in P_{a,b}} (-1)^{\sigma(x)} e_x \tag{3.43}$$

is ∂*-invariant.*

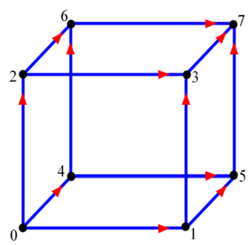
For example, in a 3-cube as shown here, we have

$$\omega_{0,1} = e_{01},$$
$$\omega_{0,3} = e_{013} - e_{023},$$

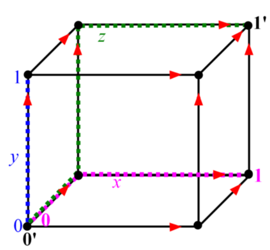
and

$$\omega_{0,7} = e_{0137} - e_{0237} - e_{0157} + e_{0457} + e_{0267} - e_{0467}$$

(cf. Example 3.6).



Proof. Without loss of generality, we can assume that $a = \mathbf{0}$, $b = \mathbf{1}$, and prove the claim by induction in $n = p$. The induction basis for $n = 1$ is obvious. For the induction step from n to $n+1$ we use Lemma 3.3 that says that the cross product of ∂-invariant paths is ∂-invariant. Denote by $\mathbf{0}' = (\mathbf{0}, 0)$ and $\mathbf{1}' = (\mathbf{1}, 1)$ the corners of the $(n+1)$-cube.



A path $x \in P_{\mathbf{0},\mathbf{1}}$ and $z \in \Sigma_{x,y}$

Taking the cross product of the n-path $\omega_{\mathbf{0},\mathbf{1}}$ on $I^{\square n}$ and the 1-path $y = e_{01}$ on I, and using (3.36), we obtain the following ∂-invariant $(n+1)$-path on $I^{\square(n+1)}$:

$$\begin{aligned}\omega_{\mathbf{0},\mathbf{1}} \times e_{01} &= \sum_{x \in P_{\mathbf{0},\mathbf{1}}} (-1)^{\sigma(x)} e_x \times e_y \\ &= \sum_{x \in P_{\mathbf{0},\mathbf{1}}} \sum_{z \in \Sigma_{x,y}} (-1)^{\sigma(x)} (-1)^{L(z)} e_z,\end{aligned}$$

where z is any stair-like path on $(n+1)$-cube that projects onto x and y, respectively.

Clearly, z runs over all paths $P_{\mathbf{0}',\mathbf{1}'}$. Setting

$$\sigma(z) = \sigma(x) + L(z) \bmod 2$$

and

$$\omega_{\mathbf{0}',\mathbf{1}'} = \omega_{\mathbf{0},\mathbf{1}} \times e_{01},$$

we obtain

$$\omega_{\mathbf{0}',\mathbf{1}'} = \sum_{z \in P_{\mathbf{0}',\mathbf{1}'}} (-1)^{\sigma(z)} e_z,$$

which concludes the proof. □

Proposition 3.9. *For any* $p \geq 0$*, we have*

$$\Omega_p(n\text{-cube}) = \left\langle \omega_{a,b} : a \preceq b \text{ and } |b| - |a| = p \right\rangle.$$

Moreover, $\{\omega_{a,b}\}$ *is a basis of* $\Omega_p(n\text{-cube})$*.*

Proof. The proof is again by induction in n. The induction basis for $n = 1$ is obvious. For the induction step from n to $n+1$ we use the Künneth formula (3.39). By this formula and by the induction hypothesis, we obtain that the basis in $\Omega_p((n+1)\text{-cube})$ consists of the following p-paths:

$$\begin{aligned}&\left\{\omega_{a,b} \times e_{01} : \omega_{a,b} \in \Omega_{p-1}(n\text{-cube})\right\} \\ &\quad \cup \left\{\omega_{a,b} \times e_i : \omega_{a,b} \in \Omega_p(n\text{-cube}), i = 0,1\right\}\end{aligned}$$

As above, the products $\omega_{a,b} \times e_{01}$ give us all the p-paths $\omega_{(a,0),(b,1)}$, while $\omega_{a,b} \times e_i$ give us all the p-paths $\omega_{(a,0),(b,0)}$ and $\omega_{(a,1),(b,1)}$. Clearly, we obtain in this way all p-paths $\omega_{a',b'}$ with $a', b' \in (n+1)$-cube, which concludes the proof. □

3.5 Augmented Chain Complex

In this section we use the augmented chain complexes

$$0 \leftarrow \mathbb{K} \overset{\partial}{\leftarrow} \Lambda_0 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Lambda_{p-1} \overset{\partial}{\leftarrow} \Lambda_p \overset{\partial}{\leftarrow} \dots \tag{3.44}$$

$$0 \leftarrow \mathbb{K} \overset{\partial}{\leftarrow} \mathcal{R}_0 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \mathcal{R}_{p-1} \overset{\partial}{\leftarrow} \mathcal{R}_p \overset{\partial}{\leftarrow} \dots \tag{3.45}$$

and

$$0 \leftarrow \mathbb{K} \overset{\partial}{\leftarrow} \Omega_0 \overset{\partial}{\leftarrow} \dots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \dots, \tag{3.46}$$

with the added space $\Lambda_{-1} = \mathcal{R}_{-1} = \Omega_{-1} = \mathbb{K}$. The operator $\partial : \Lambda_0 \to \Lambda_{-1}$ is define by

$$\partial e_i = e = \text{ the unity of } \mathbb{K}$$

which matches the definition (1.1) for $p = 0$.

The homology groups of (3.46) are called the *reduced* homology groups of G and are denoted by $\widetilde{H}_p(G)$. We have

$$\widetilde{H}_p(G) = H_p(G) \text{ for } p \geq 1 \quad \text{and} \quad \widetilde{H}_0(G) = H_0(G)/\mathbb{K}.$$

Define the reduced Betti numbers: $\widetilde{\beta}_p(G) = \dim \widetilde{H}_p(G)$. We have

$$\widetilde{\beta}_p(G) = \beta_p(G) \text{ for } p \geq 1 \quad \text{and} \quad \widetilde{\beta}_0(G) = \beta_0(G) - 1.$$

For a disjoint union $X \sqcup Y$ of two digraphs we have by (1.4)

$$\widetilde{\beta}_r(X \sqcup Y) = \widetilde{\beta}_r(X) + \widetilde{\beta}_r(Y) + \mathbf{1}_{\{r=0\}}. \tag{3.47}$$

The augmented chain complex (3.46) as opposed to (1.3) will also be used in Subsection 6.9. In all other places we continue using the chain complex (1.3).

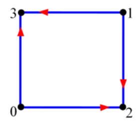
3.6 A Join of Two Digraphs

Let X, Y be two digraphs.

Definition. The *join* $X * Y$ of the digraphs X, Y is a digraph whose set of vertices is a disjoint union of the sets of vertices of X and Y, and the set of arrows consists of all arrows of X and Y as well as from all arrows $x \to y$ where $x \in X$ and $y \in Y$.

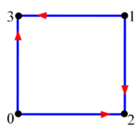
Example 3.10. For example, for the digraphs $\{\cdot,\cdot\}$ of two vertices and no arrows, we have

$$\{0,1\} * \{2,3\} =$$

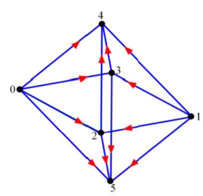


a diamond

and



$$* \{4,5\} =$$

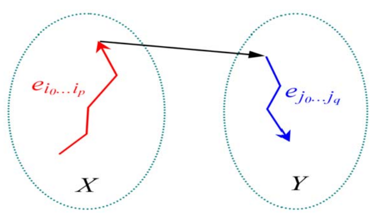


an octahedron

Definition. Let $p, q \geq -1$. For a p-path u on X and a q-path v on Y, define the *join* uv as a $(p+q+1)$-path on $X * Y$ as follows: first define it for elementary paths by

$$e_{i_0 \dots i_p} e_{j_0 \dots j_q} = e_{i_0 \dots i_p j_0 \dots j_q},$$

and then extend this definition by linearity to all u and v.



A join path $e_{i_0 \dots i_p} e_{j_0 \dots j_q}$ on $X * Y$

If u and v are allowed on X, resp. Y, then uv is clearly allowed on $Z = X * Y$.

Lemma 3.11 (Product rule for join [20], [29, Lemma 2.4])**.** *For all $p, q \geq -1$ and $u \in \Lambda_p$, $v \in \Lambda_q$ we have*

$$\partial (uv) = (\partial u) v + (-1)^{p+1} u \partial v. \tag{3.48}$$

If $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ then ∂u and ∂v are allowed, which implies by (3.48) that $\partial (uv)$ is also allowed, that is, $uv \in \Omega_{p+q+1}(Z)$. The product rule implies also that the join uv is well defined for the reduced homology classes: if $u \in \widetilde{H}_p(X)$ and $v \in \widetilde{H}_q(Y)$ then $uv \in \widetilde{H}_{p+q+1}(Z)$.

3.7 Künneth Formula for Join

Let X, Y be two digraphs.

Theorem 3.12 (Künneth formula for join [29, Theorem 3.3])**.** *We have the following isomorphism: for any $r \geq -1$,*

$$\Omega_r(X * Y) \cong \bigoplus_{\{p,q \geq -1: p+q=r-1\}} (\Omega_p(X) \otimes \Omega_q(Y)) \tag{3.49}$$

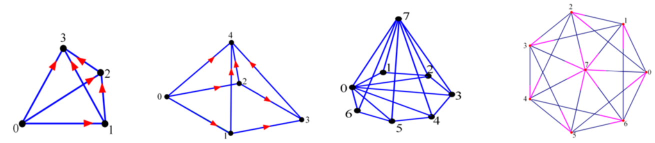
that is given by the map $u \otimes v \mapsto uv$ with $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$, and, for any $r \geq 0$,

$$\widetilde{H}_r(X * Y) \cong \bigoplus_{\{p,q \geq 0: p+q=r-1\}} \widetilde{H}_p(X) \otimes \widetilde{H}_q(Y) \tag{3.50}$$

$$\widetilde{\beta}_r(X * Y) \cong \sum_{\{p,q \geq 0: p+q=r-1\}} \widetilde{\beta}_p(X) \widetilde{\beta}_q(Y). \tag{3.51}$$

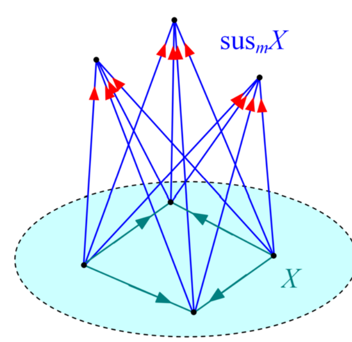
The identity (3.49) means that any path in $\Omega_r(Z)$ can be obtained as linear combination of joins uv where $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ with $p+q+1 = r$, and (3.50) means the same for homology classes.

Example 3.13. Let Y consist of a single vertex. In this case the join $X * Y$ is called a *cone* over X. Since all homology groups $\widetilde{H}_*(Y)$ are trivial, the cone $X * Y$ is also homologically trivial by (3.50). For example, the following digraphs are cones and, hence, they are homologically trivial.



Example 3.14. Let Y consist of m vertices without arrows. Then the join $X * Y$ is called the *m-suspension* of X and is denoted by $\mathsf{sus}_m X$.

Here is an example of $\mathsf{sus}_m X$ with $m = 3$:



Since $\widetilde{\beta}_0(Y) = m - 1$ and $\widetilde{\beta}_p(Y) = 0$ for $p \geq 1$, we obtain from (3.51) that

$$\widetilde{\beta}_r(\mathsf{sus}_m X) = (m-1)\widetilde{\beta}_{r-1}(X).$$

For example, on this picture $X = \mathsf{sus}_2\{\cdot,\cdot\}$ whence $\widetilde{\beta}_1(X) = 1$ and $\widetilde{\beta}_p(X) = 0$ for $p \neq 1$.

For $G = \mathsf{sus}_3 X$ we have $\widetilde{\beta}_2(G) = 2$ and $\widetilde{\beta}_r(G) = 0$ for $r \neq 2$.

Observe that the operation $*$ of digraphs is associative. For a sequence $X_1, \dots, X_l$ of l digraphs we obtain by induction from

(3.49), (3.50) and (3.51) that

(3.52)
$$\Omega_r(X_1 * X_2 * \ldots * X_l) \cong \bigoplus_{\{p_i \geq -1:\ p_1+p_2+\ldots+p_l = r-l+1\}} \Omega_{p_1}(X_1) \otimes \ldots \otimes \Omega_{p_l}(X_l)$$

(3.53)
$$\widetilde{H}_r(X_1 * X_2 * \ldots * X_l) \cong \bigoplus_{\{p_i \geq 0:\ p_1+p_2+\ldots+p_l = r-l+1\}} \widetilde{H}_{p_1}(X_1) \otimes \ldots \otimes \widetilde{H}_{p_l}(X_l)$$

(3.54)
$$\widetilde{\beta}_r(X_1 * X_2 * \ldots * X_l) = \sum_{\{p_i \geq 0:\ p_1+p_2+\ldots+p_l = r-l+1\}} \widetilde{\beta}_{p_1}(X_1) \ldots \widetilde{\beta}_{p_l}(X_l).$$

Example 3.15. Consider an octahedron $Z = X_1 * X_2 * X_3$ where

$$X_1 = \{0,1\}, \ X_2 = \{2,3\}, \ X_3 = \{4,5\}.$$

(see Example 3.10). Then we have

$$\begin{aligned} \Omega_2(Z) &= \bigoplus_{\{p_i \geq -1:\ p_1+p_2+p_3 = 2-3+1\}} \Omega_{p_1}(X_1) \otimes \Omega_{p_2}(X_2) \otimes \Omega_{p_3}(X_3) \\ &= \Omega_0(X_1) \otimes \Omega_0(X_2) \otimes \Omega_0(X_3) \\ &= \langle e_0, e_1 \rangle \otimes \langle e_2, e_3 \rangle \otimes \langle e_4, e_5 \rangle \\ &= \langle e_{024}, e_{025}, e_{034}, e_{035}, e_{124}, e_{125}, e_{134}, e_{135} \rangle \end{aligned}$$

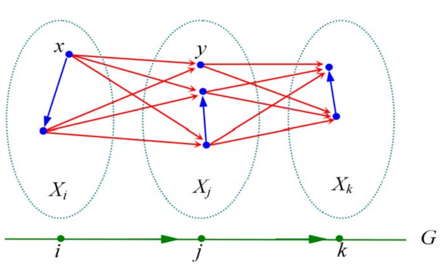
and

$$\begin{aligned} H_2(Z) = \widetilde{H}_2(Z) &= \bigoplus_{\{p_i \geq 0:\ p_1+p_2+p_3 = 2-3+1\}} \widetilde{H}_{p_1}(X_1) \otimes \widetilde{H}_{p_2}(X_2) \otimes \widetilde{H}_{p_3}(X_3) \\ &= \widetilde{H}_0(X_1) \otimes \widetilde{H}_0(X_2) \otimes \widetilde{H}_0(X_3) \\ &= \langle e_0 - e_1 \rangle \otimes \langle e_2 - e_3 \rangle \otimes \langle e_4 - e_5 \rangle \\ &= \langle e_{024} - e_{025} - e_{034} + e_{035} - e_{124} + e_{125} + e_{134} - e_{135} \rangle. \end{aligned}$$

3.8 Linear Join

The material in this section is based on [30]. Given a digraph G of l vertices $\{1, 2, \ldots, l\}$ and a sequence $X_1, \ldots, X_l$ of l digraphs, define their *generalized join* $(X_1 \ldots X_l)_G = X_G$ as follows: X_G is obtained from the disjoint union $\bigsqcup_i X_i$ of digraphs X_i by keeping all the arrows in each X_i and by adding arrows $x \to y$ whenever $x \in X_i$, $y \in X_j$ and $i \to j$ in G.

The digraph X_G is also referred to as a *G-join* of $X_1, \ldots, X_l$, and G is called the *base* of X_G.



The main problem to be discussed here is

how to compute the homology groups and Betti numbers of X_G.

Denote by K_l a complete digraph with vertices $\{1, \ldots, l\}$ and arrows

$$i \to j \Leftrightarrow i < j,$$

that is, K_l is an $(l-1)$-simplex. For example, $K_2 = \{1 \to 2\}$ and $K_3 = \{1 \to 2 \to 3, 1 \to 3\}$ is a triangle.

The digraph X_{K_l} is called a *complete* join of $X_1, \ldots, X_l$. It is easy to see that

$$X_{K_l} = X_1 * X_2 * \ldots * X_l$$

It follows from (3.54) that, for any $r \geq 0$,

(3.55)
$$\widetilde{\beta}_r(X_{K_l}) = \sum_{\{p_i \geq 0:\ p_1+p_2+\ldots+p_l = r-l+1\}} \widetilde{\beta}_{p_1}(X_1) \ldots \widetilde{\beta}_{p_l}(X_l).$$

Denote by I_l the *monotone linear digraph* with the vertices $\{1, \ldots, l\}$ and arrows $i \to i+1$:

(3.56)
$$I_l = \{1 \to 2 \to \ldots \to l\}.$$

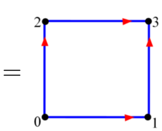
If $G = I_l$ then we use the following simplified notation:

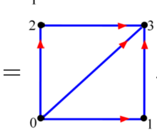
$$(X_1 X_2 \ldots X_l)_{I_l} = X_1 X_2 \ldots X_l$$

and refer to this digraph as a *monotone linear join* of $X_1, \ldots, X_l$.

Clearly, $X_1 X_2 \ldots X_n$ can be constructed as follows: first take a disjoint union $\bigsqcup_{i=1}^{l} X_i$ and then add arrows from any vertex of X_i to any vertex of X_{i+1} (see Example 4.13).

In the case $l = 2$ we obviously have $X_1 X_2 = X_1 * X_2$ but in general $X_1 X_2 \ldots X_l$ is a subgraph of $X_1 * X_2 * \ldots * X_l$. For example, we have

$\{0\}\{1,2\}\{3\} =$ 

while $\{0\} * \{1,2\} * \{3\} =$  .

Theorem 3.16 ([30]). *We have*

(3.57)
$$\widetilde{H}_r(X_1 X_2 \ldots X_l) \cong \bigoplus_{\{p_i \geq 0:\ p_1+p_2+\ldots+p_l = r-l+1\}} \widetilde{H}_{p_1}(X_1) \otimes \ldots \otimes \widetilde{H}_{p_l}(X_l)$$

and

(3.58)
$$\widetilde{\beta}_r(X_1 X_2 \ldots X_l) = \sum_{\{p_i \geq 0:\ p_1+p_2+\ldots+p_l = r-l+1\}} \widetilde{\beta}_{p_1}(X_1) \ldots \widetilde{\beta}_{p_l}(X_l).$$

Moreover, if $\dim_p X_i < \infty$ *for all i, then also* $\dim_p(X_1 \ldots X_l) < \infty$.

It follows from comparison of (3.53) and (3.57), that the linear join $X_1 X_2 \ldots X_l$ and the complete join $X_1 * X_2 * \ldots * X_l$ are homologically equivalent.

Example 3.17. Assume that *one* of the digraphs X_i is homologically trivial, that is, $\widetilde{\beta}_p(X_i) = 0$ for all p and some i. Then by (3.58) the digraph $X_1X_2...X_l$ is also homologically trivial.

Example 3.18. Assume that all digraphs X_i have no arrows. In this case the only non-trivial Betti numbers are $\widetilde{\beta}_0(X_i)$, and we obtain from (3.58) that the only non-trivial Betti number of $X_1X_2...X_l$ is

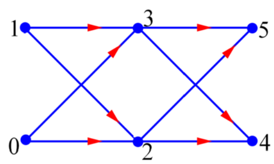
$$\widetilde{\beta}_{l-1}(X_1X_2...X_l) = \widetilde{\beta}_0(X_1)...\widetilde{\beta}_0(X_l). \tag{3.59}$$

This particular case of Theorem 3.16 was proved in [7].

Here is an example of a monotone linear join:

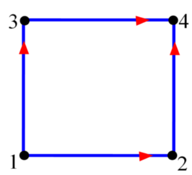
$$X = X_1X_2X_3$$

where each $X_i = \{\cdot,\cdot\}$.



Since $\widetilde{\beta}_0(X_i) = 1$, it follows from (3.59) that the only non-trivial Betti number of X is $\beta_2(X) = 1$.

Example 3.19. Let the base G be a square:



We have

$$G = \{1\}\{2,3\}\{4\}$$

which implies that

$$X_G = X_1(X_2 \sqcup X_3)X_4.$$

By Theorem 3.16 and (3.47) we obtain that

$$\begin{aligned}\widetilde{\beta}_r(X_G) &= \sum_{\{p_i\geq 0:\ p_1+p_2+p_3=r-2\}} \widetilde{\beta}_{p_1}(X_1)\widetilde{\beta}_{p_2}(X_2\sqcup X_3)\widetilde{\beta}_{p_3}(X_4)\\ &= \sum_{\{p_i\geq 0:\ p_1+p_2+p_3=r-2\}} \widetilde{\beta}_{p_1}(X_1)\big(\widetilde{\beta}_{p_2}(X_2)+\widetilde{\beta}_{p_2}(X_3)\\ &\qquad + \mathbf{1}_{\{p_2=0\}}\big)\widetilde{\beta}_{p_3}(X_4)\\ &= \widetilde{\beta}_r(X_1X_2X_4) + \widetilde{\beta}_r(X_1X_3X_4) + \widetilde{\beta}_{r-1}(X_1X_4). \end{aligned} \tag{3.60}$$

For a general base G, if $i_1...i_k$ is an arbitrary sequence of vertices in G then denote

$$X_{i_1...i_k} = X_{i_1}X_{i_2}...X_{i_k}.$$

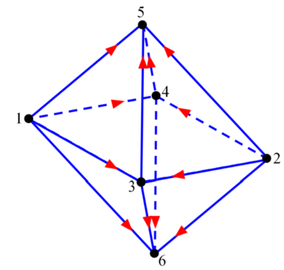
Note that by (3.58)

$$\widetilde{\beta}_r(X_{i_1...i_k}) = \sum_{\substack{p_1+...+p_k=r-(k-1)\\ p_1,...,p_k\geq 0}} \widetilde{\beta}_{p_1}(X_{i_1})...\widetilde{\beta}_{p_k}(X_{i_k}).$$

Using this notation, we can rewrite (3.60) as follows: if G is a square then

$$\widetilde{\beta}_r(X_G) = \widetilde{\beta}_r(X_{124}) + \widetilde{\beta}_r(X_{134}) + \widetilde{\beta}_{r-1}(X_{14}).$$

Example 3.20. Let G be an octahedron based on the diamond:



We have

$$G = \{1,2\} * \{3,4\} * \{5,6\}$$

whence

$$X_G = (X_1 \sqcup X_2) * (X_3 \sqcup X_4) * (X_5 \sqcup X_6).$$

By (3.55) we obtain

$$\begin{aligned}\widetilde{\beta}_r(X_G) & \sum_{\{p_i\geq 0:\ p_1+p_2+p_3=r-2\}} \widetilde{\beta}_{p_1}(X_1\sqcup X_2)\widetilde{\beta}_{p_2}(X_3\sqcup X_4)\\ &\qquad \times \widetilde{\beta}_{p_3}(X_5\sqcup X_6)\\ = & \sum_{\{p_i\geq 0:\ p_1+p_2+p_3=r-2\}} (\widetilde{\beta}_{p_1}(X_1)+\widetilde{\beta}_{p_1}(X_2)+\mathbf{1}_{\{p_1=0\}})\\ &\qquad \times (\widetilde{\beta}_{p_2}(X_3)+\widetilde{\beta}_{p_2}(X_4)+\mathbf{1}_{\{p_2=0\}})\\ &\qquad \times (\widetilde{\beta}_{p_3}(X_5)\sqcup\widetilde{\beta}_{p_3}(X_6)+\mathbf{1}_{\{p_3=0\}})\\ = & \widetilde{\beta}_r(X_{135}) + \widetilde{\beta}_r(X_{145}) + \widetilde{\beta}_r(X_{235}) + \widetilde{\beta}_r(X_{245}) + \widetilde{\beta}_r(X_{136})\\ & + \widetilde{\beta}_r(X_{146}) + \widetilde{\beta}_r(X_{236}) + \widetilde{\beta}_r(X_{246}) + \widetilde{\beta}_{r-1}(X_{13}) + \widetilde{\beta}_{r-1}(X_{23})\\ & + \widetilde{\beta}_{r-1}(X_{14}) + \widetilde{\beta}_{r-1}(X_{24}) + \widetilde{\beta}_{r-1}(X_{15}) + \widetilde{\beta}_{r-1}(X_{25})\\ & + \widetilde{\beta}_{r-1}(X_{35}) + \widetilde{\beta}_{r-1}(X_{45}) + \widetilde{\beta}_{r-1}(X_{16}) + \widetilde{\beta}_{r-1}(X_{26})\\ & + \widetilde{\beta}_{r-1}(X_{36}) + \widetilde{\beta}_{r-1}(X_{46}) + \widetilde{\beta}_{r-2}(X_1) + \widetilde{\beta}_{r-2}(X_2) + \widetilde{\beta}_{r-2}(X_3)\\ & + \widetilde{\beta}_{r-2}(X_4) + \widetilde{\beta}_{r-2}(X_5) + \widetilde{\beta}_{r-2}(X_6) + \mathbf{1}_{\{r=2\}}.\end{aligned}$$

3.9 Subgraphs and Mayer-Vietoris Exact Sequence

The material of this section is based on [18].

A digraph Y is called a *subgraph* of a digraph X if both sets of vertices and arrows of Y are subsets of those sets of X. Any allowed path in Y is therefore also allowed in X. Since the natural inclusion map $i : Y \to X$ commutes with ∂, we obtain that every ∂-invariant path in Y is also ∂-invariant in X.

A converse is not always true: even if $e_{a_0...a_p}$ is an allowed path in X and all the vertices $a_0,\ldots,a_p$ lie in Y, this path is not necessarily allowed in Y because some of its arrows may not be in Y.

A subgraph Y is called *induced* if together with two vertices $a,b \in Y$ it contains also the arrow $a \to b$ if this arrow is present in X. For an induced subgraph Y, if $e_{a_0...a_p}$ is an allowed path in X

and all the vertices $a_0,\dots,a_p$ lie in Y then $e_{a_0\dots a_p}$ is also allowed in Y. Consequently, if ω is a ∂-invariant path in X and if all the vertices of ω are contained in Y then ω is also ∂-invariant in Y.

If Y_1 and Y_2 are two subgraphs of X then their union $Y_1\cup Y_2$ is a subgraph of X whose sets of vertices and arrows are unions of those of Y_1 and Y_2, respectively. In the same way one defines the intersection $Y_1\cap Y_2$. If Y_1 and Y_2 are induced then $Y_1\cap Y_2$ is also induced.

Assume that a digraph X is a union of two subgraphs Y_1 and Y_2, that is,

$$X=Y_1\cup Y_2.$$

In particular, every arrow of X lies in Y_1 or Y_2. Denote

$$Z=Y_1\cap Y_2.$$

Then we have the following commutative diagram of the natural inclusions of the digraphs:

$$\begin{array}{ccc} Z & \xrightarrow{i^1} & Y_1 \\ {\scriptstyle i^2}\downarrow & & \downarrow{\scriptstyle j^1} \\ Y_2 & \xrightarrow{j^2} & X. \end{array} \tag{3.61}$$

For any $p\ge -1$ the commutative diagram (3.61) induces a commutative diagram

$$\begin{array}{ccc} \mathcal{R}_p(Z) & \xrightarrow{i^1_*} & \mathcal{R}_p(Y_1) \\ \downarrow{\scriptstyle i^2_*} & & \downarrow{\scriptstyle j^1_*} \\ \mathcal{R}_p(Y_2) & \xrightarrow{j^2_*} & \mathcal{R}_p(X), \end{array} \tag{3.62}$$

where all homomorphisms are injective. Observe that all homomorphisms i_* and j_* commute with the boundary operator ∂ and map allowed paths to the allowed ones.

Consider the following homomorphisms:

$$0\longrightarrow\mathcal{R}_p(Z)\xrightarrow{\delta}\mathcal{R}_p(Y_1)\oplus\mathcal{R}_p(Y_2)\xrightarrow{\gamma}\mathcal{R}_p(X)\longrightarrow 0, \tag{3.63}$$

where

$$\delta(z)=(i^1_*(z),i^2_*(z))\quad\text{and}\quad\gamma(y_1,y_2)=j^1_*(y_1)-j^2_*(y_2) \tag{3.64}$$

for all $z\in Z$ and $y_i\in Y_i$. The map δ is evidently injective.

Lemma 3.21 ([18, Lemma 3.23])**.** *In the sequence (3.63) we have* $\operatorname{Im}\delta=\ker\gamma$.

Proof. For any $z\in Z$ we have

$$\gamma(\delta(z))=j^1_*\circ i^1_*(z)-j^2_*\circ i^2_*(z)=0,$$

so that $\gamma\circ\delta=0$ and, hence, $\operatorname{Im}\delta\subset\ker\gamma$. To prove the opposite inclusion, observe that

$$\ker\gamma=\left\{(y_1,y_2)\in\mathcal{R}_p(Y_1)\oplus\mathcal{R}_p(Y_2): j^1_*(y_1)=j^2_*(y_2)\right\},$$

that is, y_1 and y_2 coincide as paths in X. Since y_1 is a path in Y_1 and y_2 is a path in Y_2, it follows that y_1 and y_2 can be identified with the same path z in $Z=Y_1\cap Y_2$. It follows that $\delta(z)=(y_1,y_2)$ and, hence, $(y_1,y_2)\in\operatorname{Im}\delta$, which finishes the proof of $\operatorname{Im}\delta=\ker\gamma$. □

For all $(y_1,y_2)\in\mathcal{R}_p(Y_1)\oplus\mathcal{R}_p(Y_2)$ set

$$\partial(y_1,y_2):=(\partial y_1,\partial y_2)\in\mathcal{R}_{p-1}(Y_1)\oplus\mathcal{R}_{p-1}(Y_2).$$

Also, we say that (y_1,y_2) is allowed if both y_1,y_2 are allowed.

Since i_* and j_* commute with the boundary operator ∂, it follows that δ and γ also commute with ∂, that is, the following diagram is commutative:

$$\begin{array}{ccccccccc} & & & 0 & & 0 & & \\ & & & \downarrow & & \downarrow & & \\ 0 & \dots & \leftarrow & \mathcal{R}_{n-1}(Z) & \overset{\partial}{\leftarrow} & \mathcal{R}_n(Z) & \overset{\partial}{\leftarrow} & \dots \\ & & & \downarrow^{\delta} & & \downarrow^{\delta} & & \\ 0 & \dots & \leftarrow & \mathcal{R}_{n-1}(Y_1)\oplus\mathcal{R}_{n-1}(Y_2) & \overset{\partial}{\leftarrow} & \mathcal{R}_n(Y_1)\oplus\mathcal{R}_n(Y_2) & \overset{\partial}{\leftarrow} & \dots \\ & & & \downarrow^{\gamma} & & \downarrow^{\gamma} & & \\ 0 & \dots & \leftarrow & \mathcal{R}_{n-1}(X) & \overset{\partial}{\leftarrow} & \mathcal{R}_n(X) & \overset{\partial}{\leftarrow} & \dots \\ & & & \downarrow & & \downarrow & & \\ & & & 0 & & 0 & & \end{array}$$

Indeed, for $z\in\mathcal{R}_n(Z)$ we have

$$\delta\circ\partial(z)=\left(i^1_*(\partial z),i^2_*(\partial z)\right)=\left(\partial i^1_*(z),\partial i^2_*(z)\right)=\partial\circ\delta(z)$$

and for $(y_1,y_2)\in\mathcal{R}_n(Y_1)\oplus\mathcal{R}_n(Y_2)$ we have

$$\begin{aligned}\gamma\circ\partial(y_1,y_2)&=j^1_*(\partial y_1)-j^*_2(\partial y_2)=\partial j^1_*(y_1)-\partial j^*_2(y_2)\\&=\partial\circ\gamma(y_1,y_2).\end{aligned}$$

Observe also that δ and γ map allowed paths to allowed ones, which follows from the same properties of i_* and j_*. Since δ and γ commute with ∂, it follows that δ and γ map ∂-invariant paths to ∂-invariant ones. Hence, we obtain the following sequence of homomorphisms

$$0\longrightarrow\Omega_p(Z)\xrightarrow{\delta}\Omega_p(Y_1)\oplus\Omega_p(Y_2)\xrightarrow{\gamma}\Omega_p(X)\longrightarrow 0, \tag{3.65}$$

where δ is injective as above.

Lemma 3.22 ([18, Lemma 3.24])**.** *In (3.65) we have* $\operatorname{Im}\delta=\ker\gamma$. *If in addition*

$$\begin{aligned}&\forall x\in\Omega_p(X)\ \textit{we have } x=y_1+y_2\\&\quad\textit{for some } y_1\in\Omega_p(Y_1)\ \textit{and } y_2\in\Omega_p(Y_2),\end{aligned} \tag{3.66}$$

then γ in (3.65) is surjective and (3.65) is a short exact sequence.

Proof. Since $\gamma\circ\delta=0$, we have $\operatorname{Im}\delta\subset\ker\gamma$. Let us prove the opposite inclusion. Let $y_1\in\Omega_p(Y_1)$ and $y_2\in\Omega_p(Y_2)$ be such that $(y_1,y_2)\in\ker\gamma$, that is, $j^1_*(y_1)=j^2_*(y_2)$. By Lemma 3.21, y_1 and y_2 can be identified with a path $z\in\mathcal{A}_p(Z)$. Then $\partial z=\partial y_1\in\mathcal{A}_{p-1}(Y_1)$ and $\partial z=\partial y_2\in\mathcal{A}_{p-1}(Y_2)$, that is $\partial z\in\mathcal{A}_{p-1}(Z)$ and, hence, $z\in\Omega_p(Z)$. Therefore, $(y_1,y_2)=\delta(z)$, which was to be proved.

Let us prove that the map γ in (3.65) is surjective. For any $x\in\Omega_p(X)$ we have by hypothesis that $x=y_1+y_2$ where $y_1\in\Omega_p(Y_1)$ and $y_2\in\Omega_p(Y_2)$. Then we have $\gamma(y_1,-y_2)=x$ so that γ is surjective. □

The condition (3.66) can be equivalently stated as follows: there is a basis in $\Omega_p(X)$ such that any element of this basis is a sum of elements of $\Omega_p(Y_1)$ and $\Omega_p(Y_2)$.

Theorem 3.23 (Mayer-Vietoris exact sequence [18, Theorem 3.25])**.** *Let*

$$X = Y_1 \cup Y_2,\ Z = Y_1 \cap Y_2$$

and assume that the hypothesis (3.66) is satisfied for any $p \geq 2$. Then we have a long exact sequence of homology groups:

$$\cdots \to \widetilde{H}_n(Z) \overset{\delta}{\to} \widetilde{H}_n(Y_1) \oplus \widetilde{H}_n(Y_2) \overset{\gamma}{\to} \widetilde{H}_n(X) \overset{\beta}{\to} \widetilde{H}_{n-1}(Z)$$

$$\overset{\delta}{\to} \widetilde{H}_{n-1}(Y_1) \oplus \widetilde{H}_{n-1}(Y_2) \to \cdots \tag{3.67}$$

where $\delta = (i_*^1, i_*^2)$, $\gamma(y_1, y_2) = j_*^1(y_1) - j_*^2(y_2)$, and β is a connecting homomorphism.

Proof. Note that (3.66) is trivially satisfied for $p \leq 1$. Hence, this condition is satisfied for all p. By the above construction, we have the following commutative diagram

(3.68)

$$\begin{array}{ccccccccc}
 & & & 0 & & 0 & & \\
 & & & \downarrow & & \downarrow & & \\
0 & \dots & \leftarrow & \Omega_{n-1}(Z) & \overset{\partial}{\leftarrow} & \Omega_n(Z) & \overset{\partial}{\leftarrow} & \dots \\
 & & & \downarrow^{\delta} & & \downarrow^{\delta} & & \\
0 & \dots & \leftarrow & \Omega_{n-1}(Y_1) \oplus \Omega_{n-1}(Y_2) & \overset{\partial}{\leftarrow} & \Omega_n(Y_1) \oplus \Omega_n(Y_2) & \overset{\partial}{\leftarrow} & \dots \\
 & & & \downarrow^{\gamma} & & \downarrow^{\gamma} & & \\
0 & \dots & \leftarrow & \Omega_{n-1}(X) & \overset{\partial}{\leftarrow} & \Omega_n(X) & \overset{\partial}{\leftarrow} & \dots \\
 & & & \downarrow & & \downarrow & & \\
 & & & 0 & & 0 & &
\end{array}$$

where each column is a short exact sequence by Lemma 3.22. The claim follows from the zig-zag lemma and from

$$\widetilde{H}_* \left(\Omega_*(Y_1) \oplus \Omega_*(Y_2)\right) \cong \widetilde{H}_*(Y_1) \oplus \widetilde{H}_*(Y_2).$$

□

Any p-path $u \in \mathcal{R}_p(X)$ has the form

$$u = \sum_{i_0 \dots i_p} u^{i_0 \dots i_p} e_{i_0 \dots i_p}$$

with the coefficients $u^{i_0 \dots i_p} \in \mathbb{K}$. We say that $e_{i_0 \dots i_p}$ (or $u^{i_0 \dots i_p} e_{i_0 \dots i_p}$) is an *elementary term* of u if $u^{i_0 \dots i_p} \neq 0$.

The next lemma provides sufficient conditions for the hypothesis (3.66).

Lemma 3.24. *Assume that the following two conditions are satisfied:*

(i) *For any $p \geq 2$ and for any $x \in \Omega_p(X)$, any elementary term of x lies in one of the subgraphs Y_1, Y_2 and is allowed in this subgraph.*

(ii) *For any square $e_{abc} - e_{ab'c}$ in X, if $a, b, c \in Y_k$ for some $k = 1, 2$ then also $b' \in Y_k$.*

Then the condition (3.66) is satisfied.

Proof. Fix $x \in \Omega_p$ for some $p \geq 2$. Denote by y_1 the sum of all elementary terms of x that lie in Y_1 and are allowed in Y_1. Set $y_2 = x - y_1$. By (i), y_2 is a sum of some elementary terms of x that lie in Y_2 and are allowed in Y_2. Since $x = y_1 + y_2$, it suffices to verify that both y_1 and y_2 are ∂-invariant, that is, ∂y_1 and ∂y_2 are allowed. Assume that ∂y_1 is not allowed. Then ∂y_1 contains a non-allowed elementary term, say

$$\text{const}\, e_{i_0 \dots \widehat{i_q} \dots i_p} \tag{3.69}$$

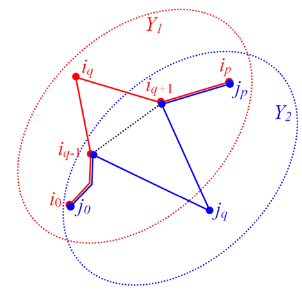
(where $1 \leq q \leq p-1$) that comes from the boundary of a term $e_{i_0 \dots i_p}$ of y_1. This term must cancel out in ∂x, which means that x must contain another elementary term $e_{j_0 \dots j_p}$ with

$$i_0 \dots i_{q-1}\, \widehat{i_q}\, i_{q+1} \dots i_p = j_0 \dots j_{q-1}\, \widehat{j_q}\, j_{q+1} \dots j_p.$$

Consequently, $i_k = j_k$ for all $k \neq q$. Hence, we obtain the following square in X:

$$e_{i_{q-1} i_q i_{q+1}} - e_{i_{q-1} j_q i_{q+1}}. \tag{3.70}$$

Since i_{q-1}, i_q and i_{q+1} belong to Y_1 then by (ii) also $j_q \in Y_1$. Hence, $e_{j_0 \dots j_p}$ lies in Y_1 and the non-allowed term (3.69) cancels also in ∂y_1. Therefore, ∂y_1 is allowed and y_1 is ∂-invariant. In the same way also y_2 is ∂-invariant. □



In this picture we show a situation when each of the paths $i_0 \dots i_p$, $j_0 \dots j_p$ belongs to one of the digraphs Y_1, Y_2, while the condition (ii) is not satisfied: the square (3.70) has the vertices i_{q-1}, i_q, i_{q+1} in Y_1 while $j_q \notin Y_1$.

Corollary 3.25. *Assume that the hypothesis (3.66) is satisfied.*

(*a*) *If, for some n, the homology groups $\widetilde{H}_n(Z)$ and $\widetilde{H}_{n-1}(Z)$ are trivial, then*

$$\widetilde{H}_n(X) \cong \widetilde{H}_n(Y_1) \oplus \widetilde{H}_n(Y_2). \tag{3.71}$$

(*b*) *If, for some n, the homology groups $\widetilde{H}_n(Y_1)$, $\widetilde{H}_n(Y_2)$, $\widetilde{H}_{n-1}(Y_1)$, $\widetilde{H}_{n-1}(Y_2)$ are trivial, then*

$$\widetilde{H}_n(X) \cong \widetilde{H}_{n-1}(Z). \tag{3.72}$$

(*c*) *If, for some n, the homology groups $\widetilde{H}_{n-1}(Y_1)$, $\widetilde{H}_{n-1}(Y_2)$ and $\widetilde{H}_n(Z)$ are trivial, then*

$$\dim \widetilde{H}_n(X) = \dim \widetilde{H}_n(Y_1) + \dim \widetilde{H}_n(Y_2) + \dim \widetilde{H}_{n-1}(Z). \tag{3.73}$$

Proof. (*a*) We have the following fragment of (3.67):

$$0 = \widetilde{H}_n(Z) \to \widetilde{H}_n(Y_1) \oplus \widetilde{H}_n(Y_2) \to \widetilde{H}_n(X) \to \widetilde{H}_{n-1}(Z) = 0,$$

whence (3.71) follows.

(*b*) We have the following fragment of (3.67):

$$0 = \widetilde{H}_n(Y_1) \oplus \widetilde{H}_n(Y_2) \to \widetilde{H}_n(X) \to \widetilde{H}_{n-1}(Z) \to \widetilde{H}_{n-1}(Y_1) \oplus \widetilde{H}_{n-1}(Y_2) = 0,$$

whence (3.72) follows.

(c) We have the following fragment of (3.67):

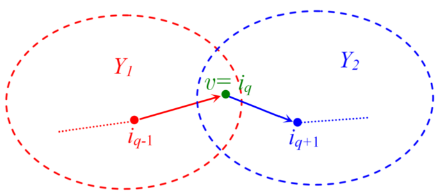
$$0 = \widetilde{H}_n(Z) \to \widetilde{H}_n(Y_1) \oplus \widetilde{H}_n(Y_2) \stackrel{\gamma}{\to} \widetilde{H}_n(X) \stackrel{\beta}{\to} \widetilde{H}_{n-1}(Z) \to \widetilde{H}_{n-1}(Y_1) \oplus \widetilde{H}_{n-1}(Y_2) = 0.$$

Hence, γ is injective and β is surjective, and $\operatorname{Im}\gamma = \ker\beta$. By the rank-nullity theorem we have

$$\begin{aligned} \dim \widetilde{H}_n(X) &= \dim\ker\beta + \dim\operatorname{Im}\beta \\ &= \dim\operatorname{Im}\gamma + \dim\operatorname{Im}\beta \\ &= \dim\widetilde{H}_n(Y_1) + \dim\widetilde{H}_n(Y_2) + \dim\widetilde{H}_{n-1}(Z), \end{aligned}$$

which was to be proved. □

Example 3.26. Assume that Z consists of a single vertex v. In this case Y_1 and Y_2 are necessarily induced subgraphs. Alternatively, one can say that X is obtained by merging digraphs Y_1 and Y_2 at one vertex v. Let us verify that the hypotheses (i) and (ii) of Lemma 3.24 are satisfied. For any $x \in \Omega_p(X)$ with $p \geq 2$ consider an elementary term $ce_{i_0\ldots i_p}$ of x and show that $e_{i_0\ldots i_p}$ lies in Y_1 or in Y_2. Assume that this is not the case, that is, one of the vertices $i_1, \ldots, i_{p-1}$ is v, say $v = i_q$, while i_{q-1} and i_{q+1} belong to different Y_1, Y_2.



The path $\partial e_{i_0\ldots i_p}$ contains the term

$$e_{i_0\ldots i_{q-1} i_{q+1}\ldots i_p}$$

that is not allowed because $i_{q-1} \not\to i_{q+1}$. This term must be cancelled in ∂x using another elementary term of x.

However if another elementary term $e_{j_0\ldots j_p}$ of x contains $e_{i_0\ldots i_{q-1} i_{q+1}\ldots i_p}$ in its boundary then

$$i_0\ldots i_{q-1} i_{q+1}\ldots i_p = j_0\ldots j_{q-1} j_{q+1}\ldots j_p$$

which implies $j_q = v$ because this is the only choice of j_q to make $j_0\ldots j_p$ allowed. Hence, $e_{i_0\ldots i_p} = e_{j_0\ldots j_p}$ and the above cancellation is not possible, which proves (i).

The condition (ii) is obvious: if $e_{abc} - e_{ab'c}$ is a square in X and $a, b, c \in Y_1$ while $b' \notin Y_1$ then both a and c must coincide with v, which is not possible.

Since $\widetilde{H}_*(Z) = \{0\}$, Corollary 3.25 (a) applies in this case and yields (3.71) for all n. Consequently, we have

$$\widetilde{\beta}_n(X) = \widetilde{\beta}_n(Y_1) + \widetilde{\beta}_n(Y_2). \tag{3.74}$$

Example 3.27. Denote by Y_1 the digraph $LH(5)$ from Example 1.19. For this digraph

$$\beta_p(Y_1) > 0 \quad \text{for all } p = 1 \bmod 3.$$

More precisely, $\beta_1(Y_1) = 1$ and $\beta_p(Y_1) = 4$ if $p = 1 \bmod 3$ and $p > 1$. Set

$$Y_2 = \mathsf{sus}_2 Y_1 \quad \text{and} \quad Y_3 = \mathsf{sus}_2 Y_2.$$

Using the formula $\widetilde{\beta}_r(\mathsf{sus}_2 G) = \widetilde{\beta}_{r-1}(G)$ from Example 3.14, we obtain that

$$\beta_p(Y_2) > 0 \text{ for all } p = 2 \bmod 3$$

and

$$\beta_p(Y_3) > 0 \text{ for all } p = 0 \bmod 3.$$

Let X be a digraph that is obtained from disjoint digraphs Y_1, Y_2 and Y_3 by merging them at one vertex. By (3.74) we obtain for all $p \geq 1$

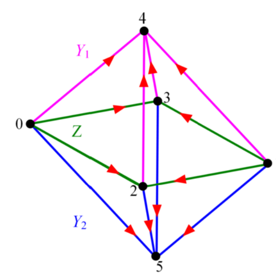
$$\beta_p(X) = \beta_p(Y_1) + \beta_p(Y_2) + \beta_p(Y_3).$$

Since $\beta_p(Y_i) > 0$ for $p = i \bmod 3$, it follows that

$$\beta_p(X) > 0 \text{ for all } p.$$

Hence, we obtain an example of a digraph with non-trivial homology groups H_p for *all* p.

Example 3.28. Let X be an octahedron as here:



Let Y_1 and Y_2 be induced subgraphs consisting of the upper and lower pyramids. Then Z is the diamond in the middle section of X.

The space $\Omega_2(X)$ is spanned by 8 triangles:

$$e_{024}, e_{034}, e_{025}, e_{035}, e_{124}, e_{134}, e_{125}, e_{135},$$

each of them lying in Y_1 or Y_2, and $\Omega_p(X) = \{0\}$ for all $p \geq 3$.

Hence, the hypothesis of Theorem 3.23 is satisfied.

Note that all $\widetilde{H}_*(Y_1)$ and $\widetilde{H}_*(Y_2)$ are trivial, while the only nontrivial group $\widetilde{H}_p(Z)$ is

$$H_1(Z) = \langle e_{02} - e_{12} + e_{13} - e_{03} \rangle.$$

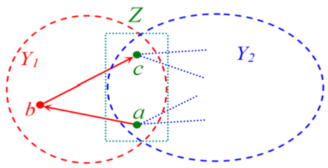
By Corollary 3.25 (b) we conclude that $H_2(X) \cong H_1(Z)$. Indeed, we have seen in Example 3.15 that $H_2(X)$ is one-dimensional.

Example 3.29. Let Y_2 be an induced connected subgraph of X such that $X \setminus Y_2$ has a single vertex b and two arrows $a \to b$ and $b \to c$ where a, c are distinct vertices of Y_2. We assume further that $a \not\to c$ in Y_2 (while in X we have either $a \to c$ or $a \rightharpoonup c$). Let us related $H_p(X)$ to $H_p(Y_2)$.

Denote by Y_1 an induced subgraph of X with the vertices a, b, c, and set $Z = Y_1 \cap Y_2$.

Then Z is an induced subgraph with two vertices a and c.

Here is an example of this configuration:



Let us verify that the conditions (i), (ii) of Lemma 3.24 are satisfied.

Let $\alpha e_{i_0...i_p}$ be an elementary term of $x \in \Omega_p(X)$ where $p \geq 2$. Let us show that the path $i_0...i_p$ lies in Y_1 or Y_2. If $i_0...i_p$ does not contain b then it lies in Y_2. Let b be one of the vertices $i_0...i_p$, say $b = i_k$.

If

$$p = 2 \quad \text{and} \quad k = 1, \tag{3.75}$$

then $e_{i_0...i_p} = e_{abc}$ and the path abc is contained in Y_1.

Assume that (3.75) is not satisfied, so that either $k \geq 2$ or $k \leq p-2$.

If $k \geq 2$ then $e_{i_0...i_p} = e_{...i_{k-2}ab...}$ and $\partial e_{i_0...i_p}$ contains the term $e_{...i_{k-2}b...}$ that is non-allowed and cannot be cancelled by other terms of x.

Similarly, if $k \leq p-2$ then $e_{i_0...i_p} = e_{...bci_{k+2}...}$ and $\partial e_{i_0...i_p}$ contains a non-allowed term $e_{...bi_{k+2}...}$ that cannot be cancelled by other terms of x. Hence, the condition (i) is satisfied.

The condition (ii) is obvious: if s is a square in X that does not lie in Y_2 then s must contain the vertex b and, hence,

$$s = e_{abc} - e_{ab'c}$$

where $b' \in Y_2$. However, since ac is not a semi-arrow in Y_2, the path $ab'c$ cannot be allowed.

Since

$$H_n(Z) = \{0\} \ \forall n \geq 1 \quad \text{and} \quad H_n(Y_1) = \{0\} \quad \forall n \geq 2,$$

we obtain by Corollary 3.25 (a) that

$$H_n(X) \cong H_n(Y_2) \text{ for all } n \geq 2.$$

In order to determine $H_1(X)$, observe that $\widetilde{H}_0(Y_1)$, $\widetilde{H}_0(Y_2)$ and $\widetilde{H}_1(Z)$ are trivial, and we conclude by Corollary 3.25 (c) that

$$\dim H_1(X) = \dim H_1(Y_1) + \dim H_1(Y_2) + \dim \widetilde{H}_0(Z).$$

Next, consider three cases.

Case 1. Let $a \to c$. Then $H_1(Y_1) = \{0\}$ and $\widetilde{H}_0(Z) = \{0\}$ whence

$$\dim H_1(X) = \dim H_1(Y_2).$$

Case 2. Let $a \not\to c$ and $c \to a$. Then $\widetilde{H}_0(Z) = \{0\}$ and

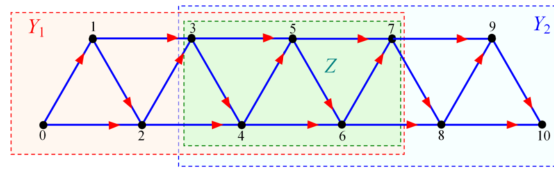
$$H_1(Y_1) = \langle e_{ab} + e_{bc} + e_{ca} \rangle,$$

whence

$$\dim H_1(X) = \dim H_1(Y_2) + 1. \tag{3.76}$$

Case 3. Let $a \not\to c$ and $c \not\to a$. Then $H_1(Y_1) = \{0\}$, $\dim \widetilde{H}_0(Z) = 1$, and we obtain again (3.76).

Example 3.30. Let Y_1, Y_2 be induced subgraphs of X as shown here:

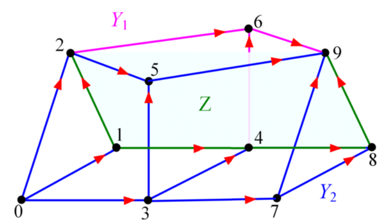


The digraph X contains a ∂-invariant snake $e_{012...10}$ that does not lie in any of the subgraphs Y_1, Y_2. Hence, the hypothesis (3.66) of Theorem 3.23 is not satisfied, and the condition (i) of Lemma 3.24 fails as well.

Example 3.31. Consider the following digraph X of 10 vertices and induced subgraphs Y_1 and Y_2 as follows:

- Y_1 contains the vertices $\{1,2,4,6,8,9\}$,
- Y_2 contains all the vertices except for 6.

Hence, Z contains the vertices $\{1,2,4,8,9\}$. Digraphs Y_1, Y_2, Z are homologically trivial, while $\dim H_2(X) = 1$.



In fact, we have

$$\begin{aligned} H_2(X) = \langle e_{012} &- (e_{014} - e_{034}) + (e_{025} - e_{035}) - (e_{126} - e_{146}) \\ &- (e_{259} - e_{269}) - (e_{348} - e_{378}) + (e_{359} - e_{379}) \\ &- (e_{469} - e_{489}) - e_{789} \rangle. \end{aligned} \tag{3.77}$$

Therefore, (3.71) fails for $n = 2$. The condition (3.66) fails as well because the square

$$e_{259} - e_{269} \tag{3.78}$$

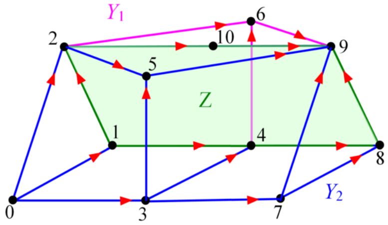
is ∂-invariant on X but it not a sum of ∂-invariant paths on Y_1 and Y_2.

For the same reason also the hypothesis (ii) of Lemma 3.24 fails: in the square (3.78) the vertices $2,6,9$ belong to Y_1 while 5 does not. Note that the hypothesis (i) of Lemma 3.24 is satisfied in this case. Indeed, one can show that

$$\begin{aligned} \Omega_2 = \langle & e_{012}, e_{789}, e_{014} - e_{034}, e_{025} - e_{035}, e_{126} - e_{146}, \\ & e_{259} - e_{269}, e_{348} - e_{378}, e_{359} - e_{379}, e_{469} - e_{489} \rangle, \end{aligned} \tag{3.79}$$

and $\Omega_p = \{0\}$ for $p > 2$ so that (i) follows from the observation that every elementary term in (3.79) lies in Y_1 or Y_2.

Example 3.32. Consider the following modification of the previous example with an added vertex 10 and arrows $2 \to 10 \to 9$.



The digraphs Y_1, Y_2 are still homologically trivial, while Z is a polygon so that $\dim H_1(Z) = 1$, $H_p(Z) = \{0\}$ for $p \geq 2$.

Condition (3.66) is satisfied, in particular, because the square (3.78) is a sum of two squares

$$(e_{2\,109} - e_{269}) + (e_{259} - e_{2\,109})$$

lying in Y_1 and Y_2, respectively,

By Corollary 3.25 (b) we conclude that $\dim H_2(X) = \dim H_1(Z) = 1$. Indeed, in this case $H_2(X)$ is also given by (3.77).

Note that the condition (ii) of Lemma 3.24 fails in this case for the same reason as in the previous example.

4. Fixed Point Theorems for Digraph Maps

4.1 Lefschetz Number and a Fixed Point Theorem

Everywhere here $\mathbb{K} = \mathbb{R}$ (or $\mathbb{K} = \mathbb{Q}$). Let $f_n : \Omega_n \to \Omega_n$ be a sequence of linear mappings that commutes with ∂, that is,

$$\partial \circ f_{n+1} = f_n \circ \partial \tag{4.80}$$

for any $n \geq 0$. In other words, the following diagram is commutative:

$$\begin{array}{ccccc} \Omega_{n-1} & \stackrel{\partial}{\longleftarrow} & \Omega_n & \stackrel{\partial}{\longleftarrow} & \Omega_{n+1} \\ \downarrow^{f_{n-1}} & & \downarrow^{f_n} & & \downarrow^{f_{n+1}} \\ \Omega_{n-1} & \stackrel{\partial}{\longleftarrow} & \Omega_n & \stackrel{\partial}{\longleftarrow} & \Omega_{n+1} \end{array} . \tag{4.81}$$

Denote

$$Z_n = \ker \partial|_{\Omega_n}, \;\; B_n = \operatorname{Im} \partial|_{\Omega_{n+1}},$$

so that

$$H_n = Z_n / B_n.$$

It follows from (4.80) that f_n acts on Z_n, B_n and H_n.

Definition. Denote shortly by f the sequence $\{f_n\}$ of the mappings as above. For any non-negative integer N, define the *Lefschetz number* of f of order N by

$$L^{(N)}(f) = \sum_{n=0}^{N} (-1)^n \operatorname{trace} f_n|_{\Omega_n}. \tag{4.82}$$

For example, if each $f_n = \mathrm{id}$ then

$$L^{(N)}(f) = \sum_{n=0}^{N} (-1)^n \dim \Omega_n = \chi^{(N)}.$$

Proposition 4.1. *The following identity holds:*

$$L^{(N)}(f) := \sum_{n=0}^{N} (-1)^n \operatorname{trace} f_n|_{H_n} + (-1)^N \operatorname{trace} f_N|_{B_N} . \tag{4.83}$$

Proof. Using the following identity (that will be proved in Subsection 4.2)

$$\operatorname{trace} f_n|_{H_n} = \operatorname{trace} f_n|_{\Omega_n} - \operatorname{trace} f_{n-1}|_{B_{n-1}} - \operatorname{trace} f_n|_{B_n} , \tag{4.84}$$

we obtain

$$\begin{aligned} &\sum_{n=0}^{N} (-1)^n \operatorname{trace} f_n|_{H_n} \\ &= \sum_{n=0}^{N} (-1)^n \operatorname{trace} f_n|_{\Omega_n} - \sum_{n=1}^{N} (-1)^n \operatorname{trace} f_{n-1}|_{B_{n-1}} \\ &\quad - \sum_{n=0}^{N} (-1)^n \operatorname{trace} f_n|_{B_n} \\ &= \sum_{n=0}^{N} (-1)^n \operatorname{trace} f_n|_{\Omega_n} + \sum_{k=0}^{N-1} (-1)^k \operatorname{trace} f_k|_{B_k} \\ &\quad - \sum_{n=0}^{N} (-1)^n \operatorname{trace} f_n|_{B_n} \\ &= \sum_{n=0}^{N} (-1)^n \operatorname{trace} f_n|_{\Omega_n} - (-1)^N \operatorname{trace} f_N|_{B_N} \\ &= L^{(N)}(f) - (-1)^N \operatorname{trace} f_N|_{B_N}, \end{aligned}$$

whence (4.82) follows. □

Let now $f : G \to G$ be a digraph map, that is,

$$i \to j \Rightarrow f(i) \to f(j) \text{ or } f(i) = f(j).$$

In Subsection 1.4 we have defined an induced mapping $f_* : \Lambda_n \to \Lambda_n$ as follows: first set

$$f_*(e_{i_0 \ldots i_n}) = e_{f(i_0)\ldots f(i_n)},$$

and then extend f to Λ_n by linearity. By Proposition 1.6, f_* extends to linear mappings $\Omega_n \to \Omega_n$ and $H_n \to H_n$.

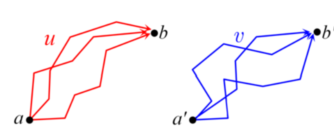
In this section we denote f_* for simplicity also by f. Hence, we obtain the diagram (4.81) where all $f_n = f$. In particular, $L^{(N)}(f)$ is defined.

Theorem 4.2. *Let $f : G \to G$ be a digraph map. If, for some $N \geq 0$, we have $L^{(N)}(f) \neq 0$ then f has a fixed point, that is, a vertex a such that $f(a) = a$.*

We use the definition of a cluster from Subsection 2.2. For example, $e_{abc} - e_{ab'c}$ is an (a,c)-cluster whereas $e_{abc} + e_{acb}$ is not a cluster.

Lemma 4.3. *In each Ω_p there is an orthogonal basis (with respect to the natural inner product $\langle \cdot, \cdot \rangle$) that consists of clusters.*

Proof. Let $\mathcal{C}$ be the set of all ∂-invariant clusters in Ω_p. By Lemma 2.2, Ω_p is spanned by $\mathcal{C}$. Choosing in $\mathcal{C}$ a maximal linearly independent subset, we obtain a basis $\mathcal{B}$ in Ω_p that consists of clusters. Let us show how to make an orthogonal basis of clusters. Let u, v be two elements from $\mathcal{B}$.



Let u be an (a,b)-cluster and v be an (a',b')-cluster. If $(a,b) \neq (a',b')$ then clearly $u \perp v$.

If $\mathcal{B}$ has more than one (a,b)-cluster, then among all (a,b)-clusters in $\mathcal{B}$, we run a Gram-Schmidt orthogonalization process and obtain an orthogonal set of (a,b)-clusters in $\mathcal{B}$. Note that during this process all newly arising elements are again (a,b)-clusters. Doing that for all pairs (a,b), we obtain an orthogonal basis in Ω_p that consists of clusters. □

Proof of Theorem 4.2. Assume that f has no fixed point. We will prove that

$$\text{trace } f|_{\Omega_n} = 0 \quad \text{for any } n \geq 0, \tag{4.85}$$

which gives by (4.82) that $L^{(N)}(f) = 0$ thus contradicting the hypothesis that $L^{(N)}(f) \neq 0$.

By Lemma 4.3, there is an orthogonal basis $u_1, \ldots, u_m$ in Ω_n, where all u_k are clusters. Denote by (c_{ij}) the matrix of the operator $f : \Omega_n \to \Omega_n$ in this basis, that is,

$$f(u_j) = \sum_{i=1}^{m} c_{ij} u_i, \qquad \text{whence} \quad c_{ij} = \frac{\langle f(u_j), u_i \rangle}{\|u_i\|^2}.$$

Consequently, we have

$$\text{trace } f|_{\Omega_n} = \sum_{k=1}^{m} c_{kk} = \sum_{k=1}^{m} \frac{\langle f(u_k), u_k \rangle}{\|u_k\|^2}.$$

It remains to show that $f(u_k) \perp u_k$, which will imply (4.85). Indeed, let u_k be an (a,b)-cluster, that is, u_k is a linear combination of elementary n-paths of the form

$$e_{ai_1 \ldots i_{n-1} b}, \tag{4.86}$$

where a, b are fixed while $i_1, \ldots, i_{n-1}$ are variable. Then $f(u_k)$ is a linear combination of the n-paths

$$e_{f(a) f(j_1) \ldots f(j_{n-1}) f(b)}, \tag{4.87}$$

where $j_1, \ldots, j_{n-1}$ are variable. Since $a \neq f(a)$, we see that the paths (4.86) and (4.87) are orthogonal, which implies that $f(u_k)$ and u_k are orthogonal, too, which was to be proved. □

4.2 Rank-Nullity Formulas for Trace

The purpose of this section is to prove the identity (4.84) – see Lemma 4.6 below. Recall that we have a commutative diagram

$$\begin{array}{ccccc} \Omega_{n-1} & \stackrel{\partial}{\longleftarrow} & \Omega_n & \stackrel{\partial}{\longleftarrow} & \Omega_{n+1} \\ \downarrow^{f_{n-1}} & & \downarrow^{f_n} & & \downarrow^{f_{n+1}} \\ \Omega_{n-1} & \stackrel{\partial}{\longleftarrow} & \Omega_n & \stackrel{\partial}{\longleftarrow} & \Omega_{n+1} \end{array}$$

and

$$Z_n = \ker \partial|_{\Omega_n}, \quad B_n = \operatorname{Im} \partial|_{\Omega_{n+1}}, \quad H_n = Z_n / B_n.$$

Lemma 4.4. *We have*

$$\text{trace } f_n|_{H_n} = \text{trace } f_n|_{Z_n} - \text{trace } f_n|_{B_n}. \tag{4.88}$$

Proof. Let $u_1, \ldots, u_l$ be a basis of B_n. Choose in Z_n elements $v_1, \ldots, v_k$ so that the sequence $u_1, \ldots, u_l, v_1, \ldots, v_k$ is a basis of Z_n. Then

$$f_n(u_i) = \sum_{j=1}^{l} a_{ij} u_j$$

and

$$f_n(v_i) = \sum_{j=1}^{k} b_{ij} v_j + \text{terms with } u_j.$$

For the homology classes we have

$$f_n([v_i]) = \sum_{j=1}^{k} b_{ij} [v_j].$$

It follows that

$$\text{trace } f_n|_{Z_n} = \sum_{i=1}^{l} a_{ii} + \sum_{i=1}^{k} b_{ii} = \text{trace } f_k|_{B_n} + \text{trace } f_n|_{H_n},$$

which is equivalent to (4.88). □

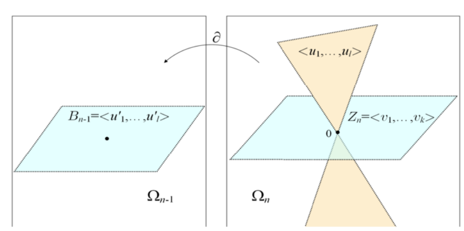
Lemma 4.5. *We have the identity*

$$\text{trace } f_n|_{Z_n} + \text{trace } f_{n-1}|_{B_{n-1}} = \text{trace } f_n|_{\Omega_n}.$$

For example, if f_n and f_{n-1} are the identity operators then this becomes the rank-nullity theorem for the operator ∂:

$$\dim Z_n + \dim B_{n-1} = \dim \Omega_n. \tag{4.89}$$

Proof. Let $v_1, \ldots v_k$ be a basis in Z_n and $u'_1, \ldots, u'_l$ be a basis in B_{n-1}. Choose any vector $u_i \in \partial^{-1}(u'_i)$, that is, $\partial u_i = u'_i$. Let us show that the sequence $v_1, \ldots, v_k, u_1, \ldots, u_l$ is linearly independent in Ω_n.



Indeed, if there is a vanishing linear combination

$$\sum_{i=1}^{l} \alpha_i u_i + \sum_{j=1}^{k} \beta_j v_j = 0,$$

then it follows that

$$0 = \partial \sum_{i=1}^{l} \alpha_i u_i + \partial \sum_{j=1}^{k} \beta_j v_j = \sum_{i=1}^{l} \alpha_i u'_i + 0,$$

whence it follows that all $\alpha_i = 0$. Consequently, $\sum_{j=1}^{k} \beta_j v_j = 0$ and, hence, also all $\beta_j = 0$.

Since by (4.89) $k + l = \dim \Omega_n$, it follows that the sequence $v_1, \ldots, v_k, u_1, \ldots, u_l$ is a basis in Ω_n.

Hence, for some coefficients a_{ij} and b_{ij},

$$f_n(u_i) = \sum_{j=1}^{l} a_{ij} u_j + \text{terms with } v_j \tag{4.90}$$

and

$$f_n(v_i) = \sum_{j=1}^{k} b_{ij} v_j.$$

The latter expansion contains no u_j because $f_n(Z_n) \subset Z_n$. Hence,

$$\operatorname{trace} f_n|_{\Omega_n} = \sum_{i=1}^{l} a_{ii} + \sum_{i=1}^{k} b_{ii}.$$

On the other hand, we have

$$\operatorname{trace} f_n|_{Z_n} = \sum_{i=1}^{k} b_{ii}.$$

It remains to prove that

$$\operatorname{trace} f_{n-1}|_{B_{n-1}} = \sum_{i=1}^{l} a_{ii}.$$

Since f_{n-1} maps B_{n-1} into itself, there are coefficients a'_{ij} such that

$$f_{n-1}(u'_i) = \sum_{j=1}^{l} a'_{ij} u'_j. \tag{4.91}$$

It follows from (4.90) that

$$\partial f_n(u_i) = \sum_{j=1}^{l} a_{ij} \partial u_j + 0 = \sum_{j=1}^{l} a_{ij} u'_j. \tag{4.92}$$

On the other hand, using (4.80) and (4.91), we obtain that

$$\partial f_n(u_i) = f_{n-1}(\partial u_i) = f_{n-1}(u'_i) = \sum_{j=1}^{l} a'_{ij} u'_j.$$

Comparison with (4.92) shows that $a'_{ij} = a_{ij}$ and, hence,

$$\operatorname{trace} f_{n-1}|_{B_{n-1}} = \sum_{i=1}^{l} a'_{ii} = \sum_{i=1}^{l} a_{ii},$$

which finishes the proof. $\square$

Finally, we can prove (4.84).

Lemma 4.6. *The following identity holds*

$$\operatorname{trace} f_n|_{H_n} = \operatorname{trace} f_n|_{\Omega_n} - \operatorname{trace} f_{n-1}|_{B_{n-1}} - \operatorname{trace} f_n|_{B_n}. \tag{4.93}$$

Proof. By Lemma 4.4 we have

$$\operatorname{trace} f_n|_{H_n} = \operatorname{trace} f_n|_{Z_n} - \operatorname{trace} f_n|_{B_n},$$

and by Lemma 4.5

$$\operatorname{trace} f_n|_{Z_n} = \operatorname{trace} f_n|_{\Omega_n} - \operatorname{trace} f_{n-1}|_{B_{n-1}},$$

which yields (4.93). $\square$

4.3 A Fixed Point Theorem in Terms of Homology

Definition. Define the *path dimension* of a digraph G by

$$\dim_p G = \sup\{n : |\Omega_n| > 0\}.$$

Assume that $\dim_p G < \infty$. Then for any $N > \dim_p G$ we have by (4.83)

$$L^{(N)}(f) = \sum_{n=0}^{N} (-1)^n \operatorname{trace} f|_{\Omega_n} = \sum_{n=0}^{N} (-1)^n \operatorname{trace} f|_{H_n}. \tag{4.94}$$

Recall the definition of the homological dimension:

$$\dim_h G = \sup\{n : |H_n| > 0\}.$$

Theorem 4.7. *Let G be a connected digraph. Let $\dim_p G < \infty$ and $\dim_h G = 0$. Then any digraph map $f : G \to G$ has a fixed point.*

Proof. The condition $\dim_h G = 0$ means that $H_n = \{0\}$ for all $n \geq 1$, and the connectedness means that $|H_0| = 1$. The space H_0 is spanned by a single homology class $[e_a]$ where a is one of the vertices. Then $f(e_a) = e_{f(a)} \sim e_a$ so that $f([e_a]) = [e_a]$. It follows that $\operatorname{trace} f|_{H_0} = 1$ while $\operatorname{trace} f|_{H_n} = 0$ for all $n \geq 1$. By (4.94) we obtain $L^{(N)}(f) = 1 \neq 0$, and by Theorem 4.2 we conclude that f has a fixed point. $\square$

The condition that a mapping $f : G \to G$ is a digraph map can be reformulated as follows. Define a *directed distance* between vertices a, b of G by

$$\vec{d}(a,b) = \inf\{n : \exists \text{ a path } \underbrace{a \to i_1 \to \ldots \to i_{n-1} \to b}_{n \text{ arrows}}\}.$$

Then f is a digraph map if and only if

$$\vec{d}(f(a), f(b)) \leq \vec{d}(a,b) \qquad \text{for all } a, b \in V.$$

Let us relax this condition.

Problem 4.8. *Devise a fixed point theorem for maps $f : G \to G$ with*

$$\vec{d}(f(a), f(b)) \leq C \vec{d}(a,b) \qquad \textit{for all } a, b \in V,$$

where $C > 1$ is a constant.

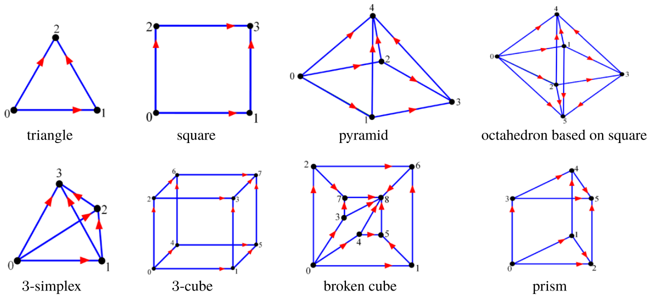
Alternatively, one can strengthen conditions on f, assuming that f is a *digraph isomorphism*, which is equivalent to

$$\vec{d}(f(a), f(b)) = \vec{d}(a,b) \qquad \text{for all } a, b \in V.$$

Problem 4.9. *Devise a fixed point theorem for a digraph isomorphism $f : G \to G$.*

4.4 Examples

Example 4.10. First consider some simple examples of digraphs satisfying the hypotheses of Theorem 4.7.



The triviality of H_* (that is, $\dim_h G = 0$) for each of these digraphs was mentioned in the previous sections. The finiteness of the path dimension follows from the fact that all arrows go in the direction of increase of numbering of the vertices so that the length of allowed paths is bounded.

Note that in all digraphs of Example 4.10, a fixed point theorem can be obtained much simpler from the following elementary result.

Proposition 4.11. *Assume that a digraph $G = (V, E)$ satisfies the following two conditions:*

(i) *there is no closed elementary allowed p-path with $p \geq 2$, that is, for any allowed p-path $e_{i_0 \dots i_p}$, we have $i_0 \neq i_p$;*

(ii) *there exists a vertex a such that there is an elementary allowed path from a to any other vertex x.*

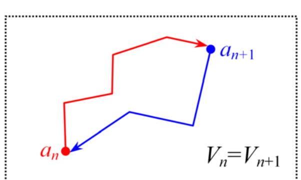
Then any digraph map $f : G \to G$ has a fixed point.

Proof. Consider the sequence of sets $V_n \subset V$ defined by

$$V_0 = V, \quad V_{n+1} = f(V_n) \text{ for } n \geq 0.$$

By induction we have $V_{n+1} \subset V_n$. Since all sets V_n are finite, we obtain that $V_{n+1} = V_n$ for large enough n. Fix such n so that we have $V_{n+1} = V_n$.

For each $x \in V$ set $x_k = f^k(x)$. Then there is an elementary allowed path from a_k to x_k for any $k \geq 0$.



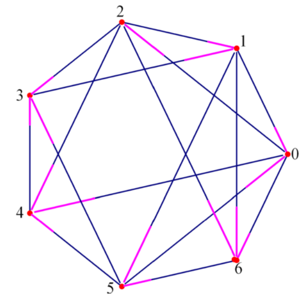
In particular, there is an allowed path from a_n to any other vertex of V_n, and that from a_{n+1} to any other vertex of $V_{n+1} = V_n$.

Hence, if $a_n \neq a_{n+1}$ then there are allowed paths from a_n to a_{n+1} and from a_{n+1} to a_n.

Therefore, there is a closed allowed path starting and ending at a_n, which is not possible. Hence, $a_n = a_{n+1}$, that is, a_n is a fixed point of f. □

Next, we give an example of a digraph that satisfies the hypotheses of Theorem 4.7 but not those of Proposition 4.11.

Example 4.12. Consider the following digraph G with 7 vertices and 16 arrows.



There are closed allowed paths

$$0 \to 2 \to 1 \to 0, \ 5 \to 0 \to 6 \to 5$$

etc. Hence, there are arbitrarily long allowed paths. Nevertheless, one can show that

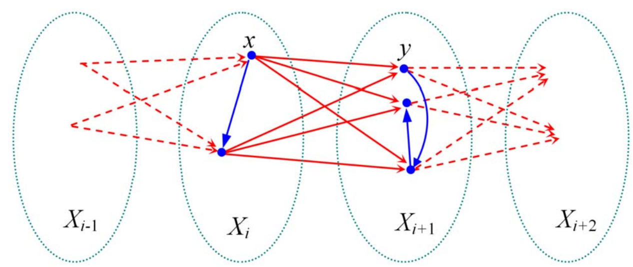
$$\dim_p G < 6,$$

and that G is homologically trivial.

Hence, G satisfies the hypotheses of Theorem 4.7, and we conclude that any digraph map $f : G \to G$ has a fixed point.

The next example provides a large family of digraphs satisfying the hypotheses of Theorem 4.7.

Example 4.13. Given n digraphs $X_1, \dots, X_n$, define *their monotone linear join* $X_1 X_2 \dots X_n$ as follows: take first a disjoint union $\bigsqcup_{i=1}^n X_i$ and then add arrows from any vertex x of X_i to any vertex y of X_{i+1}.



A monotone linear join $X_1 X_2 \dots X_n$

Proposition 4.14. *Assume that the following two conditions are satisfied:*

(i) *for all i, $\dim_p X_i < \infty$;*

(ii) *there exists i such that X_i is connected and $\dim_h X_i = 0$.*

Then any digraph map f in $X = X_1 \dots X_n$ has a fixed point.

Proof. It follows from Theorem 3.16 that the digraph X is homologically trivial and $\dim_p X < \infty$ (see also Example 3.17). Hence, the claim follows from Theorem 4.7. □

Let us now consider some examples when the hypotheses of Theorem 4.7 are not satisfied.

Example 4.15. Assume that G contains a double arrow $\{a \rightleftarrows b\}$. Then

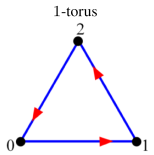
$$\dim_p G = \infty$$

because each Ω_p contains p-paths $e_{ababab\dots}$ and $e_{bababa\dots}$. Define a map $f : G \to G$ by

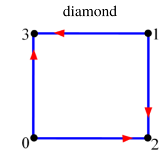
$$f(a) = b \text{ and } f(x) = a \quad \text{for all } x \neq a.$$

Clearly, f is a digraph map without fixed points. Hence, the hypotheses $\dim_p G < \infty$ is essential for Theorem 4.7.

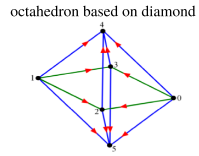
Example 4.16. Here are some examples of digraphs that admit digraph maps f without fixed points. All they have $\dim_p G < \infty$ but $\dim_h G > 0$.



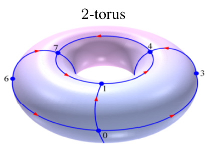
f =rotation
$|H_1| = 1$



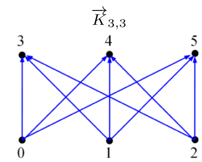
f =central symmetry
$|H_1| = 1$



f =central symmetry
$|H_2| = 1$



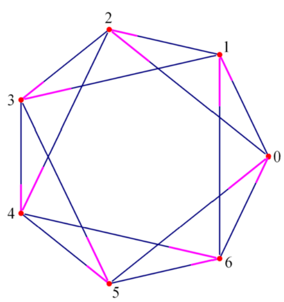
f = rotation
$|H_1| = 2,\ |H_2| = 1$



$f : 0 \mapsto 1 \mapsto 2 \mapsto 0,\ 3 \mapsto 4 \mapsto 5 \mapsto 3$
$|H_1| = 2$

Problem 4.17. *Suppose that $H_1(G)$ contains a non-trivial class $e_{01} + e_{12} + e_{20}$ (like for 1-torus). Is it true that there exists a digraph map $f : G \to G$ without a fixed point?*

Example 4.18. Consider the following digraph G with 7 vertices and 14 arrows:



G has the following arrows:

$$i \to i+1 \text{ and } i \to i+2$$

where addition is considered mod 7.

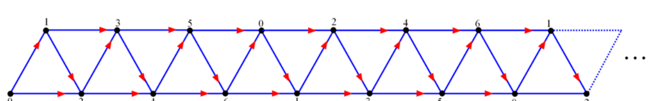
Let us first show that

$$|\Omega_p| = 14 \quad \text{for all } p \geq 1$$

and

$$|H_p| = 0 \text{ for all } p \geq 2.$$

This digraph can also be shown as a *periodic snake*:



where the vertices with the same numbers are merged (like a Möbius band).

Each elementary p-path

$$\omega_i = e_{i(i+1)(i+2)\dots(i+p)} \tag{4.95}$$

is snake-like and, hence, is ∂-invariant. Let us refer to any path (4.95) as a p-snake. Hence, we obtain in Ω_p already 7 linearly independent p-snakes $\{\omega_i\}_{i=0}^6$. Another group of 7 linearly independent p-paths in Ω_p is given by the boundaries $\partial\varpi_i$ of $(p+1)$-snakes

$$\varpi_i = e_{i(i+1)(i+2)\dots(i+p)(i+p+1)}.$$

Hence, we obtain that

$$\Omega_p = \langle \omega_i, \partial\varpi_i \rangle_{i=0}^6$$

and $\dim \Omega_p = 14$. Since $\partial(\partial\varpi_i) = 0$, while $\partial\omega_i$ are linearly independent for $p \geq 2$, we obtain that

$$\dim \ker \partial|_{\Omega_p} = 7.$$

By the rank-nullity theorem we have

$$\dim \operatorname{Im} \partial|_{\Omega_{p+1}} = 14 - 7 = 7,$$

whence $H_p = \{0\}$ for all $p \geq 2$. For the case $p = 1$ we have, in fact,

$$H_1 = \langle e_{01} + e_{12} + e_{23} + e_{34} + e_{45} + e_{56} + e_{60} \rangle.$$

Hence, we have $\dim_p G = \infty$ and $\dim_h G = 0$. The hypothesis $\dim_p G < \infty$ of Theorem 4.7 is not satisfied, and the conclusion of Theorem 4.7 fails as well because the digraph map $f(i) = i+1$ has no fixed point.

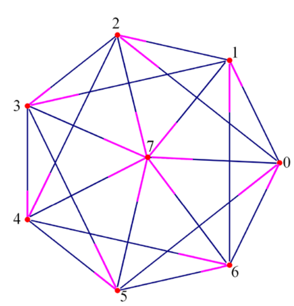
Problem 4.19. *Devise a fixed point theorem that would work with digraphs containing double arrows. For that we need to impose additional restriction on $f : G \to G$, for example, let us assume that f is a digraph isomorphism, that is,*

$$a \to b \Rightarrow \ f(a) \to f(b).$$

Problem 4.20. *Assume that G is connected,* $\dim_h G = 0$ *and that G* ***has no double arrow****. Prove or disprove the claim that any digraph map $f : G \to G$ has a fixed point. Of course, the main interest here lies in the case when*

$$\dim_p G = \infty.$$

Example 4.21. Here is a candidate for a positive example with $\dim_p G = \infty$.



This is the above snake with an additional vertex 7 such that

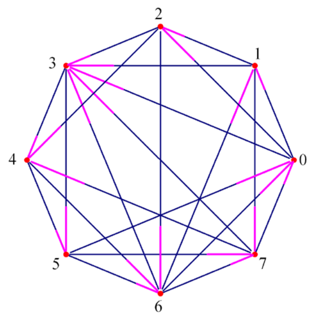
$$7 \to i \text{ for all } i \in \{0, \dots, 6\}.$$

For this digraph

$$\dim_h G = 0 \quad \text{and} \quad \dim_p G = \infty.$$

Problem 4.22. *Prove that any digraph map $f : G \to G$ for the above digraph has a fixed point.*

Example 4.23. Here is a candidate for a counterexample.



For this digraph we have

$$\dim_h G = 0 \quad \text{and} \quad \dim_p G = \infty.$$

All spaces Ω_p are non-trivial because G contains a periodic snake

$$e_{\underline{01234560}123456\ldots}$$

Problem 4.24. *Construct for this digraph a digraph map f without fixed points (or prove a fixed point theorem for this digraph). Simple rotations $f(i) = i + a \bmod 8$ are not digraph maps here. For example, for $f(i) = i+4$ the arrow $0 \to 3$ goes to $4 \not\to 7$, for $f(i) = i+5$ the arrow $5 \to 0$ goes to $2 \not\to 5$.*

Problem 4.25. *Devise convenient sufficient conditions for* $\dim_p G < \infty$.

5. Combinatorial Curvature of Digraphs

5.1 Motivation

Let Γ be a finite planar graph. There is the following old notion of a *combinatorial curvature* K_x at any vertex x of Γ:

$$K_x = 1 - \frac{\deg(x)}{2} + \sum_{f \ni x} \frac{1}{\deg(f)}, \tag{5.96}$$

where the sum is taken over all faces f containing x and $\deg(f)$ denotes the number of vertices of f. For example, if all faces are triangles then we obtain

$$K_x = 1 - \frac{\deg(x)}{2} + \frac{\deg_\Delta(x)}{3}, \tag{5.97}$$

where $\deg_\Delta(x)$ is the number of triangles having x as a vertex.

In general, denoting by V, E and F the number of vertices, edges and faces of Γ and observing that

$$\sum_x \deg(x) = 2E \quad \text{and} \quad \sum_x \sum_{f \ni x} \frac{1}{\deg(f)} = \sum_f \sum_{x \in f} \frac{1}{\deg(f)} = F,$$

we obtain

$$\sum_x K_x = V - E + F = \chi.$$

We try to realize this idea on digraph: to "distribute" the Euler characteristic over all vertices and, hence, to obtain an analog of the Gauss curvature that satisfies the Gauss-Bonnet theorem.

5.2 Curvature Operator

Let $G = (V, E)$ be a finite digraph and $\mathbb{K} = \mathbb{R}$. We would like to generalize (5.96) to arbitrary digraphs, so that the faces in (5.96) should be replaced by the elements of a basis in Ω_p. However, the result should be independent of the choice of a basis.

Fix $p \geq 0$. Any function $f : V \to \mathbb{R}$ on the vertices induces an linear operator

$$T_f : \mathcal{R}_p \to \mathcal{R}_p$$

by

$$T_f e_{i_0 \ldots i_p} = (f(i_0) + \ldots + f(i_p))\, e_{i_0 \ldots i_p}.$$

For example, for a constant function $f = \mathbf{1}$ on V, we have $T_{\mathbf{1}} e_{i_0 \ldots i_p} = (p+1)\, e_{i_0 \ldots i_p}$ and, hence,

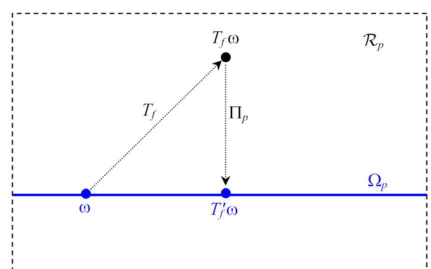
$$T_{\mathbf{1}} \omega = (p+1)\, \omega \quad \text{for any } \omega \in \mathcal{R}_p. \tag{5.98}$$

If $f = \mathbf{1}_x$ where $x \in V$, then

$$T_{\mathbf{1}_x} e_{i_0 \ldots i_p} = m e_{i_0 \ldots i_p}, \tag{5.99}$$

where m is the number of occurrences of x in $i_0, \ldots, i_p$.

Fix in $\mathcal{R}_p$ an inner product $\langle \cdot, \cdot \rangle$. For example, this can be a *natural inner product* when all regular elementary paths $e_{i_0 \ldots i_p}$ form an orthonormal basis in $\mathcal{R}_p$.



Let $\Pi_p : \mathcal{R}_p \to \Omega_p$ be the orthogonal projection onto Ω_p.

Considering T_f as an operator from Ω_p to $\mathcal{R}_p$, we obtain the following operator in Ω_p:

$$T'_f := \Pi_p \circ T_f : \Omega_p \to \Omega_p.$$

Definition. Define the *incidence* of f and Ω_p by

$$[f, \Omega_p] := \operatorname{trace} T'_f.$$

Definition. For any $\omega = \sum \omega^{i_0 \ldots i_p} e_{i_0 \ldots i_p} \in \Omega_p$ define the *incidence* of f and ω by

$$[f, \omega] := \langle T_f \omega, \omega \rangle$$

Lemma 5.1. *For any orthogonal basis* $\{\omega_k\}$ *in* Ω_p *we have*

$$[f,\Omega_p]=\sum_k \frac{[f,\omega_k]}{\|\omega_k\|^2}. \tag{5.100}$$

Proof. It suffices to prove (5.100) for orthonormal basis when $\|\omega_k\|=1$ for all k. By the definition of the trace, we have

$$\operatorname{trace} T'_f=\sum_k \langle T'_f\omega_k,\omega_k\rangle.$$

Moreover, for every $\omega\in\Omega_p$ we have

$$\langle T'_f\omega,\omega\rangle=\langle \Pi_p T_f\omega,\omega\rangle=\langle T_f\omega,\Pi_p\omega\rangle=\langle T_f\omega,\omega\rangle=[f,\omega]$$

from which (5.100) follows. □

Definition. For any $N\in\mathbb{N}$ define the *curvature operator* $K^{(N)}:\mathbb{R}^V\to\mathbb{R}$ of order N by

$$K^{(N)}f=\sum_{p=0}^{N}\frac{(-1)^p}{p+1}[f,\Omega_p].$$

If $\Omega_p=\{0\}$ for all $p>N$, then write $K_f^{(N)}=K_f$.

5.3 The Gauss-Bonnet Formula

For $f=\mathbf{1}_x$ with $x\in V$, we write

$$[x,\Omega_p]:=[\mathbf{1}_x,\Omega_p] \quad \text{and} \quad [x,\omega]:=[\mathbf{1}_x,\omega],$$

If $\{\omega_k\}$ is an orthogonal basis of Ω_p, then by (5.100)

$$[x,\Omega_p]=\sum_k\frac{[x,\omega_k]}{\|\omega_k\|^2}. \tag{5.101}$$

If the inner product is natural so that $\{e_{i_0\dots i_p}\}$ is orthonormal then by (5.99)

$$[x,e_{i_0\dots i_p}]=m,$$

where m is the number of occurrences of x in $i_0,\dots,i_p$. For example,

$$[a,e_{abca}]=2,\qquad [b,e_{abca}]=1,\qquad [d,e_{abca}]=0.$$

In this case, for $\omega=\sum\omega^{i_0\dots i_p}e_{i_0\dots i_p}$ we have

$$[x,\omega]=\sum_{i_0\dots i_p\in V}\left(\omega^{i_0\dots i_p}\right)^2[x,e_{i_0\dots i_p}].$$

Definition. For any $N\in\mathbb{N}$ define the *curvature of order N* at a vertex x by

$$K_x^{(N)}:=K^{(N)}\mathbf{1}_x=\sum_{p=0}^{N}\frac{(-1)^p}{p+1}[x,\Omega_p].$$

Set also

$$K_{total}^{(N)}=\sum_{x\in V}K_x^{(N)}.$$

Recall that the Euler characteristic is given by

$$\chi^{(N)}:=\sum_{p=0}^{N}(-1)^p\dim\Omega_p.$$

Proposition 5.2 (Gauss-Bonnet)**.** *For any choice of the inner product in* $\mathcal{R}_p$ *and for any* N *we have*

$$K_{total}^{(N)}=\chi^{(N)}.$$

Proof. Since $\sum_{x\in V}\mathbf{1}_x=\mathbf{1}$, we obtain that

$$K_{total}^{(N)}=\sum_{x\in V}K_x^{(N)}=\sum_{x\in V}K^{(N)}\mathbf{1}_x=K^{(N)}\mathbf{1}=\sum_{p=0}^{N}(-1)^p\frac{[\mathbf{1},\Omega_p]}{p+1}.$$

On the other hand, by (5.98)

$$[\mathbf{1},\omega]=\langle T_{\mathbf{1}}\omega,\omega\rangle=(p+1)\|\omega\|^2.$$

If $\{\omega_k\}$ is an orthogonal basis in Ω_p then by (5.100)

$$[\mathbf{1},\Omega_p]=\sum_k\frac{[\mathbf{1},\omega_k]}{\|\omega_k\|^2}=(p+1)\dim\Omega_p,$$

which implies

$$K_{total}^{(N)}=\sum_{p=0}^{N}(-1)^p\dim\Omega_p=\chi^{(N)}.$$

□

Remark 5.3. If $\Omega_p=\{0\}$ for all $p>N$ then

$$\chi:=\sum_{p=0}^{N}(-1)^p\dim\Omega_p=\sum_{p=0}^{N}(-1)^p\dim H_p.$$

Remark 5.4. It can happen that $\Omega_p\neq\{0\}$ for all p. An example of such a digraph is given in Example 1.19. A simpler example is $G=\{a\rightleftarrows b\}$. For this digraph we have

$$\Omega_0=\langle e_a,e_b\rangle,\qquad \Omega_1=\langle e_{ab},e_{ba}\rangle,\qquad \Omega_3=\langle e_{aba},e_{bab}\rangle,$$
$$\Omega_4=\langle e_{abab},e_{baba}\rangle,\quad \text{etc},$$

so that $|\Omega_p|=2$ for all $p\geq 0$. Indeed, $e_{aba}\in\mathcal{A}_2$ and

$$\partial e_{aba}=e_{ba}-e_{aa}+e_{ab}=e_{ba}+e_{ab}\in\mathcal{A}_1$$

so that $e_{aba}\in\Omega_2$. Similarly, $e_{abab}\in\mathcal{A}_3$ and

$$\partial e_{abab}=e_{bab}-e_{aab}+e_{abb}-e_{aba}=e_{bab}-e_{aba}\in\mathcal{A}_2$$

so that $e_{abab}\in\Omega_3$, etc.

If $\Omega_p\neq\{0\}$ for all p, then one can always truncate the chain complex to make it finite by setting $\Omega_{N+1}=\{0\}$ for some N:

$$0\ \leftarrow\ \Omega_0\ \overset{\partial}{\leftarrow}\ \Omega_1\ \overset{\partial}{\leftarrow}\ \dots\ \overset{\partial}{\leftarrow}\ \Omega_{N-1}\ \overset{\partial}{\leftarrow}\ \Omega_N\ \leftarrow 0$$

and work with homology groups of this complex. This corresponds to declaring all paths of length $>N$ non-allowed.

5.4 Examples of Computation of Curvature

Let us fix in $\mathcal{R}_p$ the natural inner product. Using the orthonormal basis $\{e_i\}$ in Ω_0 we obtain

$$[x,\Omega_0] = \sum_i [x,e_i] = 1$$

and, using the orthonormal basis $\{e_{ij}\}$ with $i \to j$ in Ω_1, we obtain

$$[x,\Omega_1] = \sum_{i\to j} [x,e_{ij}] = \deg(x).$$

Therefore,

$$K_x^{(1)} = 1 - \frac{\deg(x)}{2}$$

and, for any $N \geq 1$,

$$K_x^{(N)} = 1 - \frac{\deg(x)}{2} + \sum_{p=2}^{N} \frac{(-1)^p}{p+1}[x,\Omega_p]. \tag{5.102}$$

By Theorem 1.8, in the absence of double arrows the space Ω_2 has always a basis of triangles and squares (but this basis is not necessarily orthogonal).

For a triangle $e_{abc} \in \Omega_2$ we have

$$[x,e_{abc}] = \begin{cases} 1, & x \in \{a,b,c\} \\ 0, & \text{otherwise} \end{cases} \tag{5.103}$$

and for a square $e_{abc} - e_{ab'c} \in \Omega_2$

$$[x,e_{abc} - e_{ab'c}] = \begin{cases} 2, & x \in \{a,c\} \\ 1, & x \in \{b,b'\} \\ 0, & \text{otherwise} \end{cases} \tag{5.104}$$

In particular, if G has no square then Ω_2 has a basis $\{\omega_k\}$ that consists of all triangles in G. This basis is orthonormal and

$$[x,\Omega_2] = \sum_k [x,\omega_k] = \deg_\Delta(x) := \#\text{triangles containing } x.$$

It follows that

$$K_x^{(2)} = 1 - \frac{\deg(x)}{2} + \frac{\deg_\Delta(x)}{3},$$

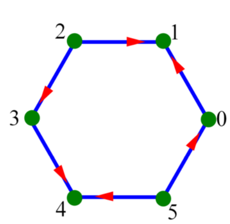
which matches with (5.97).

Example 5.5. Let G be a linear digraph, for example,

$$\cdots \bullet \to \bullet \leftarrow \bullet \to \bullet \ldots$$

Then by (5.102) we have $K_x = \frac{1}{2}$ for the endpoints, and $K_x = 0$ for the interior points.

Example 5.6. Let G be a cyclic digraph (polygon) different from triangle or square:

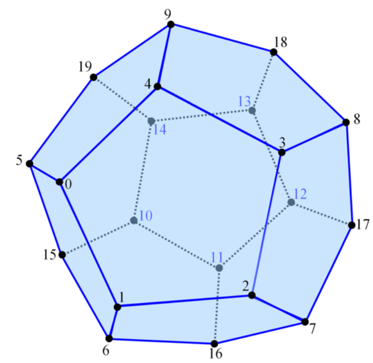


Then we have $\Omega_p = \{0\}$ for $p > 1$.
Hence by (5.102), for any vertex x,

$$K_x = 1 - \frac{\deg(x)}{2} = 0.$$

and $K_{total} = 0$. Note also that $\chi = |\Omega_0| - |\Omega_1| = 6 - 6 = 0$.

Example 5.7. Consider a dodecahedron (with any orientation of edges):



We have $|\Omega_0| = 20$, $|\Omega_1| = 30$, $|\Omega_2| = 0$, and $|H_1| = 11$, $|H_p| = 0$ for $p > 1$.

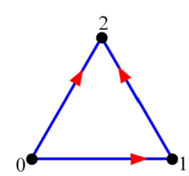
Then, for any vertex x,

$$K_x = 1 - \frac{\deg(x)}{2} = -\frac{1}{2}$$

and $K_{total} = -10$.

For comparison, note that $\chi = 1 - 11 = 20 - 30 = -10$.

Example 5.8. Let G be a triangle. We have $\Omega_2 = \langle e_{012} \rangle$ and $\Omega_p = \{0\}$ for $p > 2$.

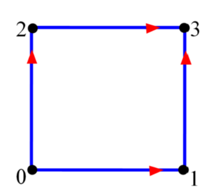


Hence, for each vertex x,

$$K_x = 1 - \frac{\deg(x)}{2} + \frac{\deg_\Delta(x)}{3} = \frac{1}{3}.$$

and $K_{total} = 1$. For comparison, $\chi = |\Omega_0| - |\Omega_1| + |\Omega_2| = 3 - 3 + 1 = 1$.

Example 5.9. Let G be a square. Then $\Omega_2 = \langle e_{013} - e_{023} \rangle$ and $\Omega_p = \{0\}$ for $p > 2$.



Since $\|e_{013} - e_{023}\|^2 = 2$, we obtain

$$[0,\Omega_2] = \frac{1}{2}[0,e_{013} - e_{023}] = 1, \qquad [3,\Omega_2] = 1$$

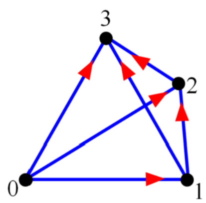
$$[1,\Omega_2] = \frac{1}{2}[1, e_{013} - e_{023}] = \frac{1}{2}, \qquad [2,\Omega_2] = \frac{1}{2}.$$

It follows that

$$K_3 = K_0 = 1 - \frac{\deg(0)}{2} + \frac{1}{3} = \frac{1}{3},$$
$$K_2 = K_1 = 1 - \frac{\deg(1)}{2} + \frac{1}{6} = \frac{1}{6},$$

and $K_{total} = 1 = \chi$.

Example 5.10. Let G be a 3-simplex:



We have

$$\Omega_2 = \langle e_{012}, e_{013}, e_{023}, e_{123}\rangle, \ \Omega_3 = \langle e_{0123}\rangle$$
$$\Omega_p = \{0\} \text{ for } p > 3.$$

It follows that, for any vertex x,

$$[x,\Omega_2] = \deg_\Delta(x) = 3 \quad \text{and} \quad [x,\Omega_3] = 1$$

whence

$$K_x = 1 - \frac{\deg(x)}{2} + \frac{[x,\Omega_2]}{3} - \frac{[x,\Omega_3]}{4} = \frac{1}{4}$$

and $K_{total} = 1 = \chi$.

Example 5.11. Let G be an *n-simplex*, that is, a digraph with a set of vertices $\{0,1,\ldots,n\}$ and edges $i \to j$ whenever $i < j$. Then, for any $p = 0,1,\ldots,n$

$$\Omega_p = \mathcal{A}_p = \langle e_{i_0\ldots i_p} : i_0 < i_1 < \ldots < i_p\rangle$$

so that $\dim\Omega_p = \binom{n+1}{p+1}$. It follows that, for any vertex x,

$$[x,\Omega_p] = \#\left\{e_{i_0\ldots i_p} \text{ such that } x \in \{i_0,\ldots,i_p\}\right\} = \binom{n}{p},$$

and

$$K_x = \sum_{p=0}^{n} (-1)^p \frac{\binom{n}{p}}{p+1}.$$

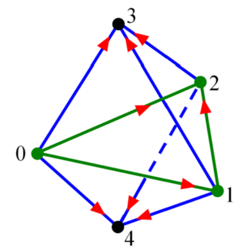
Change $j = p+1$ gives

$$(n+1)K_x = \sum_{j=1}^{n+1} (-1)^{j-1} \frac{(n+1)\binom{n}{j-1}}{j} = \sum_{j=1}^{n+1} (-1)^{j-1}\binom{n+1}{j} = 1,$$

whence

$$K_x = \frac{1}{n+1} \quad \text{and} \quad K_{total} = 1.$$

Example 5.12. Let G be a bipyramid:



We have $|\Omega_0| = 5, \ |\Omega_1| = 9,$

$$\Omega_2 = \langle e_{013}, e_{123}, e_{023}, e_{014}, e_{124}, e_{024}, e_{012}\rangle$$
$$\Omega_3 = \langle e_{0123}, e_{0124}\rangle$$

and $|\Omega_p| = 0$ for $p \geq 4$.

Hence,

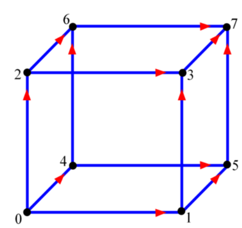
$$\chi = |\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3| = 5 - 9 + 7 - 2 = 1.$$

Let us compute the curvature:

x	$[x,\Omega_2]$	$[x,\Omega_3]$	$1 - \frac{\deg(x)}{2} + \frac{[x,\Omega_2]}{3} - \frac{[x,\Omega_3]}{4}$	$= K_x$
$3,4$	3	1	$1 - \frac{3}{2} + \frac{3}{3} - \frac{1}{4}$	$= \frac{1}{4}$
$0,1,2$	5	2	$1 - \frac{4}{2} + \frac{5}{3} - \frac{2}{4}$	$= \frac{1}{6}$

Consequently, $K_{total} = \frac{2}{4} + \frac{3}{6} = 1$.

Example 5.13. Let G be a 3-cube.



We have

$$\Omega_2 = \langle e_{013} - e_{023}, \ e_{015} - e_{045}, \ e_{026} - e_{046},$$
$$e_{137} - e_{157}, \ e_{237} - e_{267}, \ e_{457} - e_{467}\rangle$$

(note that this basis in Ω_2 is orthogonal),

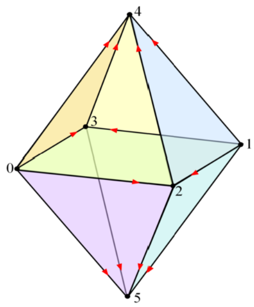
$$\Omega_3 = \langle e_{0237} - e_{0137} + e_{0157} - e_{0457} + e_{0467} - e_{0267}\rangle,$$
$$\chi = |\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3| = 8 - 12 + 6 - 1 = 1.$$

Let us compute the curvature:

x	$[x,\Omega_2]$	$[x,\Omega_3]$	$1 - \frac{\deg(x)}{2} + \frac{[x,\Omega_2]}{3} - \frac{[x,\Omega_3]}{4}$	$= K_x$
$0,7$	$\frac{6}{2} = 3$	$\frac{6}{6} = 1$	$1 - \frac{3}{2} + \frac{3}{3} - \frac{1}{4}$	$= \frac{1}{4}$
$1,2,3,4,5,6$	$\frac{4}{2} = 2$	$\frac{2}{6} = \frac{1}{3}$	$1 - \frac{3}{2} + \frac{2}{3} - \frac{1}{12} = \frac{1}{12}$	$= \frac{1}{12}$

Consequently, $K_{total} = \frac{2}{4} + \frac{6}{12} = 1 = \chi$.

Example 5.14. Consider on octahedron based on a diamond:



We have

$$\Omega_2 = \langle e_{024}, e_{034}, e_{025}, e_{035}, e_{124}, e_{134}, e_{125}, e_{135} \rangle$$

and $\Omega_p = \{0\}$ for all $p \geq 3$.

For any vertex x we obtain

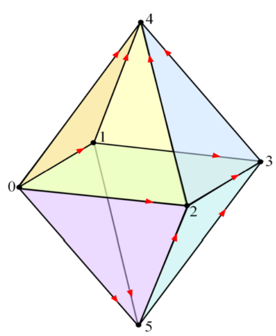
$$[x, \Omega_2] = \deg_\Delta(x) = 4$$

whence

$$K_x = 1 - \frac{\deg(x)}{2} + \frac{\deg_\Delta(x)}{3} = 1 - \frac{4}{2} + \frac{4}{3} = \frac{1}{3}$$

and $K_{total} = \frac{6}{3} = 2 = \chi$.

Example 5.15. Here is yet another octahedron, based on a square, but with the opposite orientation of the edges $2 \sim 5$ and $3 \sim 5$.



In this case we have to orthogonalize the bases:

$$\begin{aligned}\Omega_2 = \langle & e_{014},\ e_{015},\ e_{024},\ e_{052},\ e_{134},\ e_{153},\ e_{234},\ e_{523}, \\ & e_{013} - e_{023},\ e_{013} - e_{053},\ e_{524} - e_{534} \rangle \\ = \langle & e_{014},\ e_{015},\ e_{024},\ e_{052},\ e_{134},\ e_{153},\ e_{234},\ e_{523}, \\ & e_{013} - e_{023},\ e_{013} + e_{023} - 2e_{053},\ e_{524} - e_{534} \rangle \\ \Omega_3 = \langle & e_{0153},\ e_{0523},\ e_{5234},\ e_{0134} - e_{0234}, \\ & e_{0534} - e_{0134} - e_{0524} \rangle \\ = \langle & e_{0153},\ e_{0523},\ e_{5234},\ e_{0134} - e_{0234}, \\ & e_{0134} + e_{0234} - 2e_{0534} + 2e_{0524} \rangle \\ \Omega_4 = \langle & e_{05234} \rangle,\ \Omega_p = \{0\} \text{ for } p \geq 5.\end{aligned}$$

In fact, Ω_4 is generated by a 4-snake 05234.

Here is computation of the curvature:

x	$[x,\Omega_2]$	$[x,\Omega_3]$	$[x,\Omega_4]$	$1-\frac{\deg(x)}{2}+\frac{[x,\Omega_2]}{3}-\frac{[x,\Omega_3]}{4}+\frac{[x,\Omega_4]}{5}$	$=K_x$
0	$4+\frac{2}{2}+\frac{6}{6}=6$	$2+\frac{2}{2}+\frac{10}{10}=4$	1	$1-\frac{4}{2}+\frac{6}{3}-\frac{4}{4}+\frac{1}{5}$	$=\frac{1}{5}$
1	$4+\frac{1}{2}+\frac{1}{6}=\frac{14}{3}$	$1+\frac{1}{2}+\frac{1}{10}=\frac{8}{5}$	0	$1-\frac{4}{2}+\frac{14/3}{3}-\frac{8/5}{4}$	$=\frac{7}{45}$
2	$4+\frac{1}{2}+\frac{1}{6}+\frac{1}{2}=\frac{31}{6}$	$2+\frac{1}{2}+\frac{5}{10}=3$	1	$1-\frac{4}{2}+\frac{31/6}{3}-\frac{3}{4}+\frac{1}{5}$	$=\frac{31}{180}$
3	$4+\frac{2}{2}+\frac{6}{6}+\frac{1}{2}=\frac{13}{2}$	$3+\frac{2}{2}+\frac{6}{10}=\frac{23}{5}$	1	$1-\frac{4}{2}+\frac{13/2}{3}-\frac{23/5}{4}+\frac{1}{5}=\frac{13}{60}$	$=\frac{13}{60}$
4	$4+\frac{2}{2}=5$	$1+\frac{2}{2}+\frac{10}{10}=3$	1	$1-\frac{4}{2}+\frac{5}{3}-\frac{3}{4}+\frac{1}{5}$	$=\frac{7}{60}$
5	$4+\frac{4}{6}+\frac{2}{2}=\frac{17}{3}$	$3+\frac{8}{10}=\frac{19}{5}$	1	$1-\frac{4}{2}+\frac{17/3}{3}-\frac{19/5}{4}+\frac{1}{5}$	$=\frac{5}{36}$

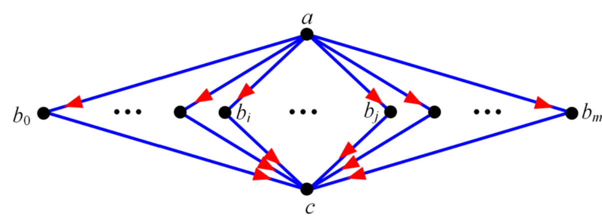
We have

$$\chi = |\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3| + |\Omega_4| = 6 - 12 + 11 - 5 + 1 = 1$$

and

$$K_{total} = \frac{1}{5} + \frac{7}{45} + \frac{31}{180} + \frac{13}{60} + \frac{7}{60} + \frac{5}{36} = 1 = \chi.$$

Example 5.16. Consider the following digraph G that is given by an m-square:



The space Ω_2 consists of squares $e_{ab_ic} - e_{ab_jc}$ and their linear combinations, while $\Omega_p = \{0\}$ for all $p > 2$. It is easy to see that Ω_2 has the following basis:

$$\Omega_2 = \langle e_{ab_0c} - e_{ab_jc} \rangle_{j=1}^m \tag{5.105}$$

so that $|\Omega_2| = m$ and

$$K_{total} = \chi = |\Omega_0| - |\Omega_1| + |\Omega_2| = (m+3) - 2(m+1) + m = 1.$$

Orthogonalization of (5.105) gives the following orthogonal basis for Ω_2:

$$\begin{aligned}\omega_1 &= e_{ab_0c} - e_{ab_1c} \\ \omega_2 &= e_{ab_0c} + e_{ab_1c} - 2e_{ab_2c} \\ &\dots \\ \omega_i &= e_{ab_0c} + \dots + e_{ab_{i-1}c} - ie_{ab_ic} \\ &\dots \\ \omega_m &= e_{ab_0c} + \dots + e_{ab_{m-1}c} - me_{ab_mc}\end{aligned}$$

We have

$$[a, \omega_i] = [c, \omega_i] = \|\omega_i\|^2 = i(i+1)$$

while

$$[b_j, \omega_i] = \begin{cases} 0, & j > i \\ 1, & j < i \\ j^2, & j = i \end{cases},$$

which implies

$$[a, \Omega_2] = \sum_{i=1}^m \frac{[a, \omega_i]}{\|\omega_i\|^2} = m \tag{5.106}$$

and

$$[b_j,\Omega_2]=\sum_{i=1}^{m}\frac{[b_j,\omega_i]}{i\,(i+1)}=\frac{j^2}{j\,(j+1)}+\sum_{i=j+1}^{m}\frac{1}{i\,(i+1)}$$

$$=1-\frac{1}{m+1}=\frac{m}{m+1}. \tag{5.107}$$

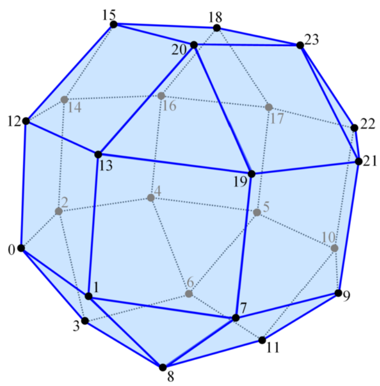
It follows that

$$K_c=K_a=1-\frac{\deg(a)}{2}+\frac{[a,\Omega_2]}{3}=1-\frac{m+1}{2}+\frac{m}{3}=\frac{1}{2}-\frac{m}{6}$$

and

$$K_{b_j}=1-\frac{\deg(b_j)}{2}+\frac{[b_j,\Omega_2]}{3}=\frac{m}{3\,(m+1)}.$$

Example 5.17. Consider a rhombicuboctahedron:



It has 24 vertices, 48 edges, and 26 faces, among them 8 triangular and 18 rectangular.

Let us make it into a digraph G by choosing direction $i\to j$ on an edge (i,j) if $i<j$. Then each face becomes a triangle or square.

For this digraph $|H_2|=1$ and $H_p=\{0\}$ for $p=1$ and $p>2$.

We have $|\Omega_2|=26$ and $\Omega_p=\{0\}$ for $p\geq 3$. Ω_2 is generated by 8 triangles and 18 squares:

$$\Omega_2=\langle e_{023},\ e_{178},\ e_{456},\ e_{9\,10\,11},\ e_{12\,14\,15},\ e_{13\,19\,20},\ e_{16\,17\,18},\ e_{21\,22\,23},$$
$$e_{018}-e_{038},\ e_{0\,1\,13}-e_{0\,12\,13},\ e_{0\,2\,14}-e_{0\,12\,14},\ e_{1\,7\,19}-e_{1\,13\,19},$$
$$e_{236}-e_{246},\ e_{2\,4\,16}-e_{2\,14\,16},\ e_{3\,6\,11}-e_{3\,8\,11},\ e_{4\,5\,17}-e_{4\,16\,17},$$
$$e_{5\,10\,11}-e_{5\,6\,11},\ e_{5\,10\,22}-e_{5\,17\,22},\ e_{7\,8\,11}-e_{7\,9\,11},\ e_{7\,9\,21}-e_{7\,19\,21},$$
$$e_{9\,10\,22}-e_{9\,21\,22},\ e_{12\,13\,20}-e_{12\,15\,20},\ e_{14\,15\,18}-e_{14\,16\,18},$$
$$e_{15\,18\,23}-e_{15\,20\,23},\ e_{17\,22\,23}-e_{17\,18\,23},\ e_{19\,20\,23}-e_{19\,21\,23}\rangle,$$

while the generator of H_2 is a signed sum of all these 2-paths.

This basis in Ω_2 is orthogonal. Hence, we compute the curvature:

$x=$	$0,11,23$	$1,3,4,6,8,9,12,13,$ $15,16,18,20,21$	$2,5,7,14,$ $17,19,22$	10
$[x,\Omega_2]=$	$1+\frac{6}{2}=4$	$1+\frac{4}{2}=3$	$1+\frac{5}{2}=\frac{7}{2}$	$1+\frac{3}{2}=\frac{5}{2}$
$1-\frac{\deg(x)}{2}+\frac{[x,\Omega_2]}{3}=$	$1-\frac{4}{2}+\frac{4}{3}$	$1-\frac{4}{2}+\frac{3}{3}$	$1-\frac{4}{2}+\frac{7/2}{3}$	$1-\frac{4}{2}+\frac{5/2}{3}$
K_x	$=\frac{1}{3}$	$=0$	$=\frac{1}{6}$	$=-\frac{1}{6}$

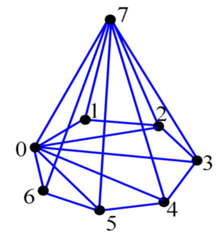
It follows that

$$K_{total}=\tfrac{3}{3}+\tfrac{7}{6}-\tfrac{1}{6}=2.$$

For comparison

$$\chi=|\Omega_0|-|\Omega_1|+|\Omega_2|=24-48+26=2$$
$$=|H_0|-|H_1|+|H_2|.$$

Example 5.18. Consider the following pyramid:



Let us make it into a digraph G by choosing direction $i\to j$ on an edge $i\sim j$ if $i<j$. We have $|\Omega_0|=8,\ |\Omega_1|=18,$

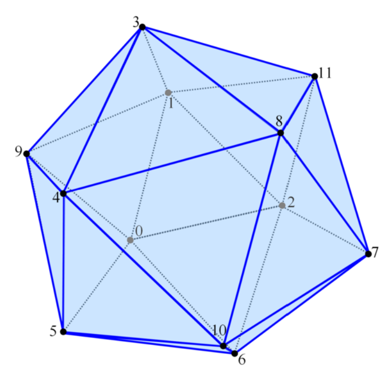
$$\Omega_2=\langle e_{017},e_{027},e_{037},e_{047},e_{057},e_{067},e_{012},e_{023},$$
$$e_{034},e_{045},e_{056},e_{127},e_{237},e_{347},e_{457},e_{567}\rangle$$
$$\Omega_3=\langle e_{0127},e_{0237},e_{0347},e_{0457},e_{0567}\rangle$$
$$\Omega_p=\{0\}\ \text{ for } p\geq 4.$$

Let us compute the curvature:

x	$[x,\Omega_2]$	$[x,\Omega_3]$	$1-\frac{\deg(x)}{2}+\frac{[x,\Omega_2]}{3}-\frac{[x,\Omega_3]}{4}$	$=K_x$
$0,7$	11	5	$1-\frac{7}{2}+\frac{11}{3}-\frac{5}{4}$	$=-\frac{1}{12}$
$1,6$	3	1	$1-\frac{3}{2}+\frac{3}{3}-\frac{1}{4}$	$=\frac{1}{4}$
$2,3,4,5$	5	2	$1-\frac{4}{2}+\frac{5}{3}-\frac{2}{4}$	$=\frac{1}{6}$

It follows that $K_{total}=-\frac{2}{12}+\frac{2}{4}+\frac{4}{6}=1$. For comparison $\chi=8-18+16-5=1$.

Example 5.19. Let us compute the curvature of icosahedron (cf. Example 1.16):



Here we choose arrow $i\to j$ if $i\sim j$ and $i<j$. We have

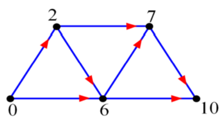
$$|H_1|=0,\quad |H_2|=1,\quad |H_p|=0\quad\text{for } p>2$$
$$|\Omega_0|=12,\quad |\Omega_1|=30,\quad |\Omega_2|=25,\quad |\Omega_3|=6,$$
$$|\Omega_4|=1\quad\text{and}\quad \Omega_p=\{0\}\quad\text{for } p\geq 5.$$

Hence,

$$\chi=|H_0|-|H_1|+|H_2|$$
$$=|\Omega_0|-|\Omega_1|+|\Omega_2|-|\Omega_3|+|\Omega_4|=2.$$

Here are the orthogonal bases in $\Omega_2, \Omega_3, \Omega_4$:

$$\begin{aligned}\Omega_2 = \langle & e_{019}, e_{012}, e_{1211}, e_{026}, e_{059}, e_{056}, e_{5610}, e_{139}, e_{1311}, \\ & e_{267}, e_{6710}, e_{2711}, e_{349}, e_{348}, e_{4810}, e_{3811}, e_{459}, e_{4510}, \\ & e_{7810}, e_{7811}, e_{0111} - e_{0211}, e_{0510} - e_{0610}, \\ & e_{2610} - e_{2710}, e_{3410} - e_{3810}, e_{027} - e_{067} \rangle \\ \Omega_3 = \langle & e_{01211}, e_{05610}, e_{34810}, e_{0267}, \\ & e_{26710}, -e_{06710} + e_{02710} - e_{02610} \rangle \\ \Omega_4 = \langle & e_{026710} \rangle \end{aligned}$$



since the path e_{026710} is "snake like" and, hence, is ∂-invariant.

Computation of the curvature:

$x =$	0	1	2	3, 11
$[x,\Omega_2] =$	$6 + \frac{4}{2} = 8$	$5 + \frac{1}{2} = \frac{11}{2}$	$5 + \frac{4}{2} = 7$	$5 + \frac{2}{2} = 6$
$[x,\Omega_3] =$	$3 + \frac{3}{3} = 4$	1	$3 + \frac{2}{3} = \frac{11}{3}$	1
$[x,\Omega_4] =$	1	0	1	0
$\sum_{p=0}^{4} (-1)^p \frac{[x,\Omega_p]}{p+1}$	$1 - \frac{5}{2} + \frac{8}{3} - \frac{4}{4} + \frac{1}{5}$	$1 - \frac{5}{2} + \frac{11/2}{3} - \frac{1}{4}$	$1 - \frac{5}{2} + \frac{7}{3} - \frac{11/3}{4} + \frac{1}{5}$	$1 - \frac{5}{2} + \frac{6}{3} - \frac{1}{4}$
K_x	$= \frac{11}{30}$	$= \frac{1}{12}$	$= \frac{7}{60}$	$= \frac{1}{4}$

4, 5, 8	6	7	9	10
$5 + \frac{1}{2} = \frac{11}{2}$	$5 + \frac{3}{2} = \frac{13}{2}$	$5 + \frac{3}{2} = \frac{13}{2}$	5	$5 + \frac{6}{2} = 8$
1	$3 + \frac{2}{3} = \frac{11}{3}$	$2 + \frac{2}{3} = \frac{8}{3}$	0	$3 + \frac{3}{3} = 4$
0	1	1	0	1
$1 - \frac{5}{2} + \frac{11/2}{3} - \frac{1}{4}$	$1 - \frac{5}{2} + \frac{13/2}{3} - \frac{11/3}{4} + \frac{1}{5}$	$1 - \frac{5}{2} + \frac{13/2}{3} - \frac{8/3}{4} + \frac{1}{5}$	$1 - \frac{5}{2} + \frac{5}{3}$	$1 - \frac{5}{2} + \frac{8}{3} - \frac{4}{4} + \frac{1}{5}$
$= \frac{1}{12}$	$= -\frac{1}{20}$	$= \frac{1}{5}$	$= \frac{1}{6}$	$= \frac{11}{30}$

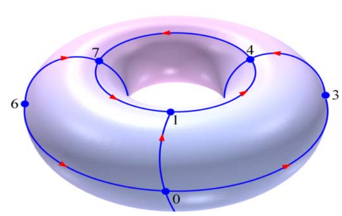
Note that $K_6 = -\frac{1}{20} < 0$.

The total curvature:

$$K_{total} = \frac{11}{30} \cdot 2 + \frac{1}{12} \cdot 4 + \frac{7}{60} + \frac{1}{4} \cdot 2 - \frac{1}{20} + \frac{1}{5} + \frac{1}{6} = 2.$$

Example 5.20. Let us compute the curvature of the 2-torus $G = T \Box T$, where $T = \{0 \to 1 \to 2 \to 0\}$.

Here is the 2-torus G embedded onto a topological torus:



In Example 3.7 we have computed the basis in $\Omega_2(G)$ as follows (see (3.41)):

$$\begin{aligned}\Omega_2(G) = \langle & e_{034} - e_{014},\ e_{145} - e_{125},\ e_{253} - e_{203}, \\ & e_{367} - e_{347},\ e_{478} - e_{458},\ e_{586} - e_{536} \\ & e_{601} - e_{671},\ e_{712} - e_{782},\ e_{820} - e_{860} \rangle.\end{aligned}$$

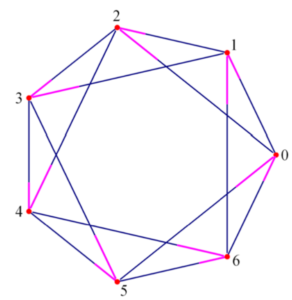
This basis in $\Omega_2(G)$ is orthogonal and $\|\omega\|^2 = 2$ for any element ω of the basis. Besides, for any vertex x, we have $[x, \omega] = 2$ for two of ω, $[x, \omega] = 1$ for two of ω, and $[x, \omega] = 0$ for the rest of ω. Hence,

$$[x, \Omega_2] = \sum_{\omega} \frac{[x, \omega]}{\|\omega\|^2} = \frac{2 \cdot 2 + 2 \cdot 1}{2} = 3$$

and, for any $x \in G$,

$$K_x = 1 - \frac{\deg(x)}{2} + \frac{[x, \Omega_2]}{3} = 1 - \frac{4}{2} + \frac{3}{3} = 0.$$

Example 5.21. Consider the digraph G from Example 4.18.



This digraph has 7 vertices $\{0, \ldots, 6\}$ and 14 arrows as follows:

$$i \to i+1 \text{ and } i \to i+2$$

where addition is considered mod 7.

Fix $p \geq 1$ and consider for any $i = 0, \ldots, 6$ the following ∂-invariant p-path

$$\omega_i = e_{i(i+1)(i+2)\ldots(i+p)}$$

and $(p+1)$-path

$$\varpi_i = e_{i(i+1)(i+2)\ldots(i+p)(i+p+1)}.$$

It was shown in Example 4.18 that $\dim \Omega_p = 14$ and that the space Ω_p has a basis $\langle \omega_i, \partial \varpi_i \rangle_{i=0}^{6}$.

Let us now compute the curvature $K_x^{(N)}$. The sequence $\{\omega_i\}$ is orthonormal, but $\{\partial \varpi_i\}$ is not, which is clear from

$$\partial \varpi_i = \omega_{i+1} + \sum_{q=1}^{p} (-1)^q e_{i\ldots\widehat{i+q}\ldots(i+p+1)} + (-1)^{p+1} \omega_i.$$

Let us replace each $\partial \varpi_i$ with

$$v_i = \partial \varpi_i - (-1)^{p+1} \omega_i - \omega_{i+1} = \sum_{q=1}^{p} (-1)^q e_{i\ldots\widehat{i+q}\ldots(i+p+1)}.$$

Then we obtain that Ω_p has an orthogonal basis $\{\omega_i, v_i\}_{i=0}^{6}$.

By symmetry, $[x, \omega_i]$ is the same for all vertices x and i. Since

$$\sum_{x,i} [x, \omega_i] = 7(p+1),$$

and $\|\omega_i\| = 1$, we obtain

$$\sum_i \frac{[x, \omega_i]}{\|\omega_i\|^2} = p+1.$$

For v_i we have

$$\sum_{x,i} [x, v_i] = 7(p+1)p$$

and $\|v_i\|^2 = p$ whence

$$\sum_i \frac{[x, v_i]}{\|v_i\|^2} = \frac{(p+1)p}{p} = p+1.$$

Hence,

$$[x, \Omega_p] = 2(p+1),$$

which implies that

$$K_x^{(N)} = 1 + \sum_{p=1}^{N} (-1)^p 2 = (-1)^N.$$

Hence, $\{K^{(N)}\}$ is a *periodic* sequence in N.

Problem 5.22. *Describe classes of strongly regular digraphs having a non-trivial periodic sequence* $\{K^{(N)}\}_{N=1}^{\infty}$.

5.5 Computation of $[x, \Omega_2]$

Recall that Ω_2 has always a basis that consists of triangles, double arrows and squares. All different triangles and double arrows in G are always linearly independent and mutually orthogonal. Moreover, they are orthogonal to all squares. However, squares may be not mutually orthogonal in general.

In a special case when G contains no multisquares, are all squares orthogonal (and, hence, linearly independent). Indeed, if two squares are not orthogonal then they must have the same elementary term, say, $e_{abc} - e_{ab'c}$ and $e_{abc} - e_{ab''c}$, which yields a 2-square $a, \{b, b', b''\}, c$ (cf. Subsection 1.5).

Let us introduce the following notation:

$$\begin{aligned}
\deg_{\updownarrow}(x) &= \#\{\text{double arrows } a \rightleftarrows b : x \in \{a,b\}\}, \\
\deg_{\Delta}(x) &= \#\{\text{triangles } e_{abc} : x \in \{a,b,c\}, \\
\deg_{\Box_1}(x) &= \#\{\text{squares } e_{abc} - e_{ab'c} : x \in \{b,b'\}\}, \\
\deg_{\Box_2}(x) &= \#\{\text{squares } e_{abc} - e_{ab'c} : x \in \{a,c\}\}.
\end{aligned}$$

Lemma 5.23. *Assume that G contains no multisquares. Then, for any vertex $x \in G$,*

(5.108)
$$[x, \Omega_2] = 3\deg_{\updownarrow}(x) + \deg_{\Delta}(x) + \frac{1}{2}\deg_{\Box_1}(x) + \deg_{\Box_2}(x).$$

Proof. Let $\{\omega_n\}$ be the sequence of all double arrows, triangles and squares in Ω_2. By hypothesis, the sequence $\{\omega_n\}$ forms an orthogonal basis in Ω_2.

Any double arrow $a \rightleftarrows b$ induces two independent elements e_{aba} and e_{bab} of Ω_2. Clearly, we have

$$[x, e_{aba}] + [x, e_{bab}] = \begin{cases} 3, & x \in \{a,b\} \\ 0, & \text{otherwise.} \end{cases}$$

Hence,

$$\sum_{\omega_n \text{ is a double arrow}} \frac{[x, \omega_n]}{\|\omega\|^2} = 3\deg_{\updownarrow}(x). \tag{5.109}$$

For a triangle $e_{abc} \in \Omega_2$ we have

$$[x, e_{abc}] = \begin{cases} 1, & x \in \{a,b,c\} \\ 0, & \text{otherwise} \end{cases}$$

and, hence,

$$\sum_{\omega_n \text{ is a triangle}} \frac{[x, \omega_n]}{\|\omega\|^2} = \deg_{\Delta}(x). \tag{5.110}$$

For a square $e_{abc} - e_{ab'c} \in \Omega_2$ we have

$$[x, e_{abc} - e_{ab'c}] = \begin{cases} 2, & x \in \{a,c\} \\ 1, & x \in \{b,b'\} \\ 0, & \text{otherwise} \end{cases}.$$

Hence,

$$\sum_{\omega_n \text{ is a square}} \frac{[x, \omega_n]}{\|\omega\|^2} = \frac{1}{2}\deg_{\Box_1}(x) + \deg_{\Box_2}(x).$$

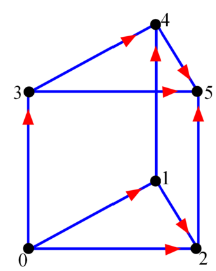
Since $\{\omega_n\}$ is an orthogonal basis that consists of all double arrows, triangles and squares, we obtain

$$\begin{aligned}
[x, \Omega_2] &= \sum_n \frac{[x, \omega_n]}{\|\omega_n\|^2} \\
&= 3\deg_{\updownarrow}(x) + \deg_{\Delta}(x) + \frac{1}{2}\deg_{\Box_1}(x) + \deg_{\Box_2}(x).
\end{aligned}$$

□

Example 5.24. For the prism as shown here we have:

$$\begin{aligned}
&\deg_{\Delta}(x) = 1 \text{ for all } x; \\
&\deg_{\Box_1}(0) = 0,\ \deg_{\Box_2}(1) = 2 \\
&\deg_{\Box_1}(1) = 1,\ \deg_{\Box_2}(1) = 1 \\
&\deg_{\Box_1}(2) = 2,\ \deg_{\Box_2}(2) = 0 \\
&\deg_{\Box_1}(3) = 2,\ \deg_{\Box_2}(3) = 0 \\
&\deg_{\Box_1}(4) = 1,\ \deg_{\Box_2}(4) = 1 \\
&\deg_{\Box_1}(5) = 0,\ \deg_{\Box_2}(5) = 2.
\end{aligned}$$



Consequently, we obtain by (5.108)

$$[x, \Omega_2] = \begin{cases} 3, & x = 0, 5 \\ \frac{5}{2}, & x = 1, 4 \\ 2, & x = 2, 3 \end{cases}.$$

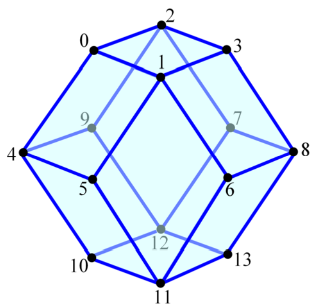
Since $\Omega_3 = \langle e_{0125} - e_{0145} + e_{0345} \rangle$, $\Omega_4 = \{0\}$ and

$$[x, \Omega_3] = \frac{1}{3}\begin{cases} 3, & x = 0, 5 \\ 2, & x = 1, 4 \\ 1, & x = 2, 3 \end{cases},$$

it follows that

$$K_x = 1 - \frac{\deg(x)}{2} + \frac{[x, \Omega_2]}{3} - \frac{[x, \Omega_3]}{4} = \begin{cases} \frac{1}{4}, & x = 0, 5 \\ \frac{1}{6}, & x = 1, 4 \\ \frac{1}{12}, & x = 2, 3 \end{cases}.$$

Example 5.25. Consider a rhombic dodecahedron:



The arrows along the edges point in direction of the higher vertex number. The faces give rise to 12 squares forming a basis in space Ω_2, and $\Omega_p = \{0\}$ for all $p \geq 3$.

For $x \in \{0, 13\}$ we have $\deg(x) = 3$,

$$\deg_{\square_1}(x) = 0, \ \deg_{\square_2}(x) = 3,$$

whence $[x, \Omega_2] = 3$ and

$$K_x = 1 - \tfrac{3}{2} + \tfrac{3}{3} = \tfrac{1}{2}.$$

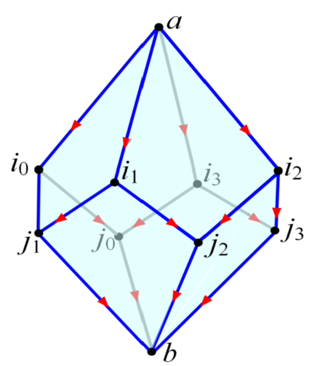
For $x \in \{3,5,6,7,9,10\}$ we have $\deg(x) = 3$, $\deg_{\square_1}(x) = 2$, $\deg_{\square_2}(x) = 1$, whence $[x, \Omega_2] = 2$ and

$$K_x = 1 - \tfrac{3}{2} + \tfrac{2}{3} = \tfrac{1}{6}.$$

Finally, for $x \in \{1,2,4,8,11,12\}$ we have $\deg(x) = 4$, $\deg_{\square_1}(x) = 2$, $\deg_{\square_2}(x) = 2$, whence $[x, \Omega_2] = 3$ and

$$K_x = 1 - \tfrac{4}{2} + \tfrac{2}{3} = 0.$$

Example 5.26. Consider a trapezohedron T_m as in Subsection 2.1.



By Proposition 2.1, the space Ω_2 is spanned by $2m$ squares as follows:

$$\Omega_2 = \left\langle e_{ai_{k-1}j_k} - e_{ai_kj_k}, e_{i_kj_kb} - e_{i_kj_{k+1}b} \right\rangle_{m=0}^{m-1};$$

also, $\Omega_3 = \langle \tau_m \rangle$, where

$$\tau_m = \sum_{k=0}^{m-1} \left(e_{ai_kj_kb} - e_{ai_kj_{k+1}b} \right),$$

and $\Omega_p = \{0\}$ for all $p \geq 4$.

For all vertices we have $\deg_\Delta(x) = 0$. For $x \in \{a, b\}$ we have $\deg_{\square_1}(x) = 0$, $\deg_{\square_2}(x) = m$, whence $[x, \Omega_2] = m$. Since $\deg(x) = m$ and

$$[x, \Omega_3] = \frac{[x, \tau_m]}{\|\tau_m\|^2} = \frac{m}{m} = 1,$$

we obtain

$$K_a = K_b = 1 - \tfrac{m}{2} + \tfrac{m}{3} - \tfrac{1}{4} = \tfrac{3}{4} - \tfrac{m}{6}.$$

For all other vertices $x \in \{i_k, j_k\}$ we have

$$\deg_{\square_1}(x) = 2, \ \deg_{\square_2}(x) = 1,$$

whence $[x, \Omega_2] = 2$. Since $\deg(x) = 3$ and

$$[x, \Omega_3] = \frac{[x, \tau_m]}{\|\tau_m\|^2} = \frac{2}{m},$$

we obtain

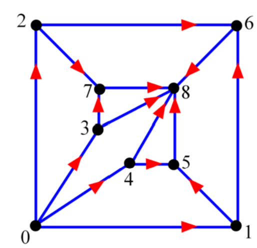
$$K_x = 1 - \tfrac{3}{2} + \tfrac{2}{3} - \tfrac{1/m}{4} = \tfrac{1}{6} - \tfrac{1}{4m}.$$

The total curvature

$$K_{total} = 2\left(\tfrac{3}{4} - \tfrac{m}{6}\right) + 2m\left(\tfrac{1}{6} - \tfrac{1}{4m}\right) = 1$$

matches the Euler characteristic $\chi = 1$.

Example 5.27. Consider a broken cube from Example 2.9. Then we have:



Ω_2 is spanned by 6 squares and 2 triangles,

$$\Omega_3 = \langle e_{0158} - e_{0168} + e_{0268} - e_{0278} + e_{0378} - e_{0458} \rangle$$

and $\Omega_p = \{0\}$ for $p \geq 4$.

For $x = 0$ we have $\deg_{\square_1}(0) = 0$, $\deg_{\square_2}(0) = 4$, $\deg_\Delta(0) = 0$ whence $[0, \Omega_2] = 4$.

Since $\deg(0) = 4$ and $[0, \Omega_3] = 1$, it follows that

$$K_0 = 1 - \tfrac{4}{2} + \tfrac{4}{3} - \tfrac{1}{4} = \tfrac{1}{12}.$$

For $x \in \{1,2,6\}$ we have $\deg_{\square_1}(x) = 2$, $\deg_{\square_2}(0) = 1$, $\deg_\Delta(x) = 0$ whence $[x, \Omega_2] = 2$. Since $\deg(x) = 3$ and $[x, \Omega_3] = \frac{1}{3}$, it follows that

$$K_x = 1 - \tfrac{3}{2} + \tfrac{2}{3} - \tfrac{1/3}{4} = \tfrac{1}{12}.$$

For $x \in \{3,4\}$ we have $\deg_{\square_1}(x) = 2$, $\deg_{\square_2}(x) = 0$, $\deg_\Delta(x) = 1$ whence $[x, \Omega_2] = 2$. Since $\deg(x) = 3$ and $[x, \Omega_3] = \frac{1}{6}$, it follows that

$$K_x = 1 - \tfrac{3}{2} + \tfrac{2}{3} - \tfrac{1/6}{4} = \tfrac{1}{8}.$$

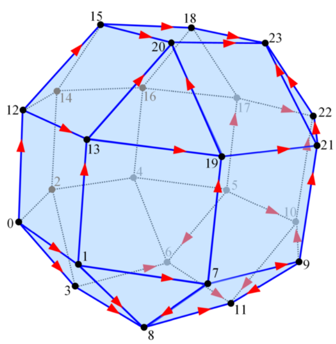
For $x \in \{5,7\}$ we have $\deg_{\square_1}(x) = 1$, $\deg_{\square_2}(x) = 1$, $\deg_{\Delta}(x) = 1$ whence $[x,\Omega_2] = 5/2$. Since $\deg(x) = 3$ and $[x,\Omega_3] = \frac{1}{3}$, it follows that

$$K_x = 1 - \tfrac{3}{2} + \tfrac{5/2}{3} - \tfrac{1/3}{4} = \tfrac{1}{4}.$$

Finally, for $x = 8$ we have $\deg_{\square_1}(8) = 0$, $\deg_{\square_2}(8) = 3$, $\deg_{\Delta}(8) = 2$ whence $[8,\Omega_2] = 5$. Since $\deg(8) = 5$ and $[8,\Omega_3] = 1$, it follows that

$$K_8 = 1 - \tfrac{5}{2} + \tfrac{5}{3} - \tfrac{1}{4} = -\tfrac{1}{12}.$$

Example 5.28. Consider again a rhombicuboctahedron (see Example 5.17).



We have for all vertices

$$\deg(x) = 4 \text{ and } \deg_{\Delta}(x) = 1.$$

All squares are linearly independent and $\Omega_3 = \{0\}$ (cf. Example 5.17).

For $x = 11$: $\deg_{\square_1}(x) = 0$, $\deg_{\square_2}(x) = 3$,

$$[x,\Omega_2] = 4, \quad K_x = 1 - \frac{4}{2} + \frac{4}{3} = \frac{1}{3}.$$

For $x = 19$: $\deg_{\square_1}(x) = 1$, $\deg_{\square_2}(x) = 2$,

$$[x,\Omega_2] = \frac{7}{2}, \quad K_x = 1 - \frac{4}{2} + \frac{7/2}{3} = \frac{1}{6}.$$

For $x = 13$: $\deg_{\square_1}(x) = 2$, $\deg_{\square_2}(x) = 1$,

$$[x,\Omega_2] = 3, \quad K_x = 1 - \frac{4}{2} + \frac{3}{3} = 0.$$

For $x = 10$ we have $\deg_{\square_1}(x) = 3$, $\deg_{\square_2}(x) = 0$, whence $[x,\Omega_2] = \frac{5}{2}$ and

$$K_x = 1 - \tfrac{4}{2} + \tfrac{5/2}{3} = -\tfrac{1}{6}.$$

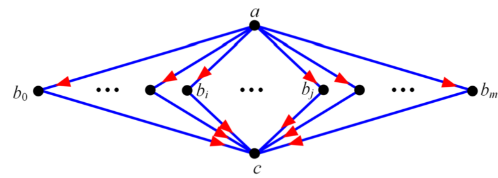
Consider now a general case when G may contain multisquares. Fix a semi-arrow $a \rightharpoonup c$ and denote by $\{b_i\}_{i=0}^m$ the sequence of all vertices b_i such that $a \to b_i \to c$. Let $m \geq 1$. Then we have an m-square

$$\sigma = \{a, \{b_i\}_{i=0}^m, c\} \tag{5.111}$$

that gives rise the following to the following family of squares

$$\left\{e_{ab_ic} - e_{ab_jc} : 0 \leq i < j \leq m\right\} \tag{5.112}$$

(cf. Subsection 1.5 and Example 5.16).



An m-square

The family (5.112) contains m linearly independent squares, for example, they are

$$\left\{e_{ab_0c} - e_{ab_ic}\right\}_{i=1}^m. \tag{5.113}$$

As in Example 5.16, let $\{\omega_i\}_{i=1}^m$ be an orthogonalization of the sequence (5.113). Using the computations (5.106) and (5.107) of Example 5.16 we obtain

$$\sum_{i=1}^m \frac{[x,\omega_i]}{\|\omega_i\|^2} = \begin{cases} m, & x \in \{a,c\} \\ \frac{m}{m+1}, & x \in \{b_i\}_{i=0}^m \\ 0, & \text{otherwise.} \end{cases} \tag{5.114}$$

For any m-square σ as in (5.111), denote

$$[x,\sigma] = \begin{cases} m, & x \in \{a,c\} \\ \frac{m}{m+1}, & x \in \{b_i\}_{i=0}^m \\ 0, & \text{otherwise,} \end{cases} \tag{5.115}$$

so that

$$[x,\sigma] = \sum_{i=1}^m \frac{[x,\omega_i]}{\|\omega_i\|^2}. \tag{5.116}$$

Proposition 5.29. *For any vertex $x \in G$, we have*

$$[x,\Omega_2] = 3\deg_{\updownarrow}(x) + \deg_{\Delta}(x) + \sum_{\substack{\sigma \text{ is an } m\text{-square} \\ m \geq 1}} [x,\sigma]. \tag{5.117}$$

Proof. Indeed, each m-square contributes m linearly independent elements to Ω_2, and different multiple squares give rise to mutually orthogonal elements. Hence, using in each multiple square an orthogonal basis and adding to them all double arrows and triangles, we obtain an orthogonal basis in Ω_2. Hence, combining (5.101), (5.109), (5.110) and (5.116), we obtain (5.117). □

Let us prove the following identity for $[x,\sigma]$ that may be useful for computer assisted computations.

Lemma 5.30. *Let $s_{ij} = e_{ab_ic} - e_{ab_jc}$ be all squares in an m-square σ as in (5.112). Then we have, for all x,*

$$[x,\sigma] = \frac{1}{m+1} \sum_{0 \leq i < j \leq m} [x, s_{ij}]. \tag{5.118}$$

Proof. Indeed, if $x \in \{a,c\}$ then $[x,s_{ij}] = 2$ and the number of terms in the above sum is $\frac{m(m+1)}{2}$, so that the right hand side of (5.118) equals to m as well as the left hand side. If $x = b_k$ then

$$[x,s_{ij}] = \begin{cases} 1, & i = k \text{ or } j = k, \\ 0, & \text{otherwise} \end{cases}$$

and the number of 1's in the sum (5.118) is m, so that the right hand side of (5.118) equals to $\frac{m}{m+1}$ as well as the left hand side.

Finally, if x does not belong to $\{a, c, b_k\}$ then the both sides of (5.118) vanish. □

For any vertex x denote

$$\deg_{m\square_1}(x) = \#\left\{m\text{-squares } \left\{a, \left\{b_j\right\}, c\right\} : x \in \left\{b_j\right\}\right\}$$

and

$$\deg_{m\square_2}(x) = \#\left\{m\text{-squares } \left\{a, \left\{b_j\right\}, c\right\} : x \in \{a, c\}\right\}.$$

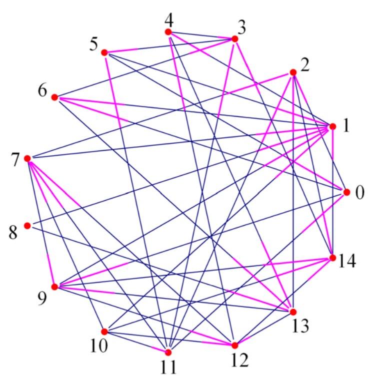
Corollary 5.31. *For any $x \in G$ we have*

$$[x, \Omega_2] = 3\deg_{\updownarrow}(x) + \deg_{\Delta}(x) + \sum_{m \geq 1}\left(\frac{m}{m+1}\deg_{m\square_1}(x) + m\deg_{m\square_2}(x)\right). \tag{5.119}$$

Proof. Indeed, this follows from (5.115) and (5.117). □

Clearly, the identity (5.108) is a particular case of (5.119) in the case when all m-squares are 1-squares.

Example 5.32. Consider a randomly generated digraph:



We have $|\Omega_0| = 15$, $|\Omega_1| = 39$, $|\Omega_2| = 28$, $|\Omega_3| = 4$, $\Omega_p = \{0\}$ for $p \geq 4$, $|H_1| = 2$, $|H_2| = 1$, $H_p = \{0\}$ for $p \geq 3$.

In particular,

$$\begin{aligned}\chi &= |H_0| - |H_1| + |H_2| \\ &= |\Omega_0| - |\Omega_1| + |\Omega_2| - |\Omega_3| = 0.\end{aligned}$$

Here are the bases in Ω_2, Ω_3:

$$\begin{aligned}\Omega_2 = \langle & e_{13214} - e_{131214}, e_{13214} - e_{13914}, e_{0214} - e_{0914}, \\ & e_{143} - e_{163}, e_{1413} - e_{1613}, e_{506} - e_{516}, e_{7214} - e_{7914}, \\ & e_{914} - e_{9124}, e_{1014} - e_{10124}, e_{1072} - e_{10112}, \\ & e_{10113} - e_{10143}, e_{1109} - e_{1179}, e_{1151} - e_{1171}, \\ & e_{1243} - e_{12143}, e_{1271} - e_{12141}, e_{791}, e_{91214}, e_{9141}, \\ & e_{1071}, e_{10117}, e_{10127}, e_{101214}, e_{10141}, e_{1102}, e_{1135}, \\ & e_{1150}, e_{1172}, e_{13912}\rangle \\ \Omega_3 = \langle & e_{101172}, e_{1391214}, e_{101271} - e_{1012141}, \\ & e_{110214} - e_{110914} + e_{117914} - e_{117214}\rangle.\end{aligned}$$

Note that the above basis in Ω_2 is not orthogonal: it contains a 2-square

$$\sigma = \{13 \to \{2, 9, 12\} \to 14\}$$

that corresponds to two squares

$$e_{13214} - e_{131214} \quad \text{and} \quad e_{13214} - e_{13914},$$

while all other squares in the above basis of Ω_2 are 1-squares.

For the vertex $x = 13$ we have then

$$\deg_{2\square_1}(x) = 0, \quad \deg_{2\square_2}(x) = 1$$

as well as

$$\deg_{\Delta}(x) = 1, \ \deg_{\square_1}(x) = 0, \ \deg_{\square_2}(x) = 1,$$

whence by (5.119)

$$\begin{aligned}[13, \Omega_2] &= \deg_{\Delta}(x) + \frac{1}{2}\deg_{\square_1}(x) + \deg_{\square_2}(x) + \frac{2}{3}\deg_{2\square_1}(x) \\ &\quad + 2\deg_{\square_2}(x) \\ &= 1 + 1 + 2 = 4\end{aligned}$$

Since also $\deg(13) = 6$ and $[13, \Omega_3] = 1$, we obtain

$$K_{13} = 1 - \frac{6}{2} + \frac{4}{3} - \frac{1}{4} = -\frac{11}{12}.$$

Since the vertex $x = 2$ we have

$$\deg_{2\square_1}(x) = 1, \quad \deg_{2\square_2}(x) = 0$$

and

$$\deg_{\Delta}(x) = 2, \ \deg_{\square_1}(x) = 2, \ \deg_{\square_2}(x) = 1,$$

whence

$$[2, \Omega_2] = 2 + \frac{2}{2} + 1 + \frac{2}{3} = \frac{14}{3}.$$

Since also $\deg(2) = 5$ and $[2, \Omega_3] = \frac{3}{2}$, we obtain

$$K_2 = 1 - \frac{5}{2} + \frac{14/3}{3} - \frac{3/2}{4} = -\frac{23}{72}.$$

Computation of the curvature at all other vertices yields

$$\{K_x\}_{x=0}^{14} = \{-\tfrac{7}{24}, -\tfrac{1}{12}, -\tfrac{23}{72}, -\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, -\tfrac{1}{3}, \tfrac{1}{6}, 0, \tfrac{13}{72}, \tfrac{2}{3}, \tfrac{1}{6}, \tfrac{1}{18}, -\tfrac{11}{12}, \tfrac{13}{24}\}.$$

5.6 Curvature of n-Cube

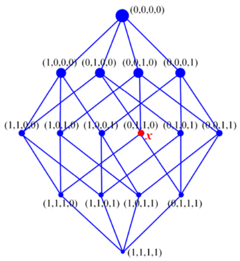
We use the notation of Subsection 3.4 where n-cube was defined. The purpose of this section is to prove the following statement.

Theorem 5.33. *For any vertex x in n-cube we have*

$$K_x(n\text{-cube}) = \frac{1}{(n+1)\binom{n}{|x|}}.$$

For example, in a 4-cube that is shown here, for the marked vertex x we have $|x| = 2$ and

$$K_x = \frac{1}{5\binom{4}{2}} = \frac{1}{30}.$$



Let us first prove some lemmas about binomial coefficients.

Lemma 5.34. *We have for all $M \geq l \geq 0$*

$$\sum_{j=0}^{l} \binom{M}{j} (-1)^j = (-1)^l \binom{M-1}{l}. \tag{5.120}$$

Proof. Induction in M. For $M = l$ we have

$$\sum_{j=0}^{l} \binom{l}{j} (-1)^j = (1-1)^l = 0 = (-1)^l \binom{l-1}{l}.$$

Induction step from M to $M+1$. We have

$$\begin{aligned}
\sum_{j=0}^{l} \binom{M+1}{j} (-1)^j &= \sum_{j=0}^{l} \left(\binom{M}{j} + \binom{M}{j-1} \right) (-1)^j \\
&= (-1)^l \binom{M-1}{l} + \sum_{j=1}^{l} \binom{M}{j-1} (-1)^j \\
&= (-1)^l \binom{M-1}{l} - \sum_{i=0}^{l-1} \binom{M}{i} (-1)^i \\
&= (-1)^l \binom{M-1}{l} - (-1)^{l-1} \binom{M-1}{l-1} \\
&= (-1)^l \binom{M}{l}.
\end{aligned}$$

□

Lemma 5.35. *We have for all $N \geq 0$ and $M \geq 1$*

$$\sum_{l=0}^{N} \binom{N}{l} \frac{(-1)^l}{l+M} = \frac{1}{M\binom{N+M}{M}} \tag{5.121}$$

Proof. We start with the identity

$$\sum_{l=0}^{N} \binom{N}{l} (-z)^l = (1-z)^N$$

for all $z \in \mathbb{R}$, whence

$$\sum_{l=0}^{N} \binom{N}{l} (-z)^{l+M-1} = (-1)^{M-1} (1-z)^N z^{M-1}.$$

Integrating this identity from 0 to 1, we obtain

$$\begin{aligned}
-\sum_{l=0}^{N} \binom{N}{l} \frac{(-z)^{l+M}}{l+M} \Bigg|_0^1 &= (-1)^{M-1} B(N+1, M) \\
&= (-1)^{M-1} \frac{\Gamma(N+1)\Gamma(M)}{\Gamma(N+M+1)} \\
&= (-1)^{M--1} \frac{N!(M-1)!}{(N+M)!} \\
&= (-1)^{M-1} \frac{1}{M\binom{N+M}{M}}
\end{aligned}$$

while the left hand side is equal to

$$-\sum_{l=0}^{N} \binom{N}{l} \frac{(-1)^{l+M}}{l+M} = (-1)^{M+1} \sum_{l=0}^{N} \binom{N}{l} \frac{(-1)^l}{l+M},$$

which proves the claim. □

Lemma 5.36. *We have*

$$K_m := \sum_{k=0}^{m} \sum_{l=0}^{n-m} \binom{m}{k} \binom{n-m}{l} \frac{(-1)^{k+l}}{\binom{k+l}{l}(k+l+1)} = \frac{1}{(m+1)\binom{n+1}{m+1}}.$$

Proof. Set

$$\begin{aligned}
S_{m,l} &= \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{k+l}}{\binom{k+l}{l}(k+l+1)} \\
&= l! \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^{k+l}}{(k+1)\ldots(k+l)(k+l+1)} \\
&= l! \sum_{k=0}^{m} \frac{(-1)^{k+l} m(m-1)\ldots(m-k+1)}{(k+l+1)!} \\
&= \frac{l!}{(m+l+1)\ldots(m+1)} \\
&\quad \times \sum_{k=0}^{m} \frac{(-1)^{k+l}(m+l+1)\ldots.(m+1)m(m-1)\ldots(m-k+1)}{(k+l+1)!} \\
&= -\frac{1}{(l+1)\binom{m+l+1}{l+1}} \sum_{k=0}^{m} \binom{m+l+1}{k+l+1} (-1)^{k+l+1} \\
&= -\frac{1}{(l+1)\binom{m+l+1}{l+1}} \sum_{j=l+1}^{m+l+1} \binom{m+l+1}{j} (-1)^j \\
&= \frac{1}{(l+1)\binom{m+l+1}{l+1}} \sum_{j=0}^{l} \binom{m+l+1}{j} (-1)^j
\end{aligned}$$

By (5.120) with $M = m+l+1$ we obtain

$$\sum_{j=0}^{l} \binom{m+l+1}{j} (-1)^j = (-1)^l \binom{m+l}{l}$$

whence

$$\begin{aligned}
S_{m,l} &= \frac{(-1)^l}{(l+1)\binom{m+l+1}{l+1}} \binom{m+l}{l} \\
&= \frac{(-1)^l \, l!m!}{(m+l+1)!} \frac{(m+l)!}{l!m!} \\
&= \frac{(-1)^l}{m+l+1}.
\end{aligned}$$

Therefore, by (5.121) with $N = n-m$ and $M = m+1$,

$$\begin{aligned}
K_m &= \sum_{l=0}^{n-m} \binom{n-m}{l} S_{m,l} = \sum_{l=0}^{n-m} \binom{n-m}{l} \frac{(-1)^l}{m+l+1} \\
&= \frac{1}{(m+1)\binom{n+1}{m+1}}.
\end{aligned}$$

□

Proof of Theorem 5.33. Fix a vertex x of the n-cube and non-negative integers k,l,p such that

$$k+l=p.$$

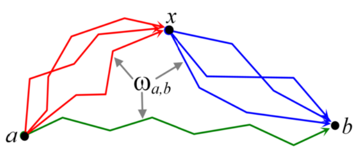
Let a and b be two vertices in the n-cube such

$$a \preceq x \preceq b, \quad |x|-|a|=k, \quad \text{and} \quad |b|-|x|=l. \tag{5.122}$$

The cube $D_{a,b}$ has dimension $|b|-|a|=p$, and for any ∂-invariant p-path $\omega_{a,b}$ between a and b (cf. (3.43)), we have

$$\left\|\omega_{a,b}\right\|^2=p! \quad \text{and} \quad \left[x,\omega_{a,b}\right]=k!l!.$$

Indeed, $\omega_{a,b}$ is an alternating sum of all the elementary allowed paths from a to b, and the number of the elementary allowed paths from a to b going through x is $k!l!$, because the number of such paths from a to x is equal to $k!$ and that from x to b is equal to $l!$.



Hence, we have for such $\omega_{a,b}$

$$\frac{\left[x,\omega_{a,b}\right]}{\left\|\omega_{a,b}\right\|^2}=\frac{k!l!}{p!}=\frac{1}{\binom{k+l}{k}}.$$

Set $m=|x|$ and observe that the number of vertices $a\preceq x$ with $|x|-|a|=k$ is equal to $\binom{m}{k}$. Indeed, in the binary representations $a=(a_1,\ldots a_n,)$ and $x=(x_1,\ldots x_n,)$, we have $a_i\le x_i$ and $\sum_i (x_i-a_i)=k$ which is only possible if $a_i=0$ at k out of m positions where $x_i=1$.

Similarly, the number of the vertices $b\succeq x$ with $|b|-|x|=l$ is equal to $\binom{n-m}{l}$. Hence, the number of pairs a,b satisfying (5.122) is equal to

$$\binom{m}{k}\binom{n-m}{l}.$$

By Proposition 3.9, all p-paths $\omega_{a,b}$ with $a\preceq b$ form an orthogonal basis in Ω_p (n-cube). If x does not satisfy the condition $a\preceq x\preceq b$ then we have

$$\left[x,\omega_{a,b}\right]=0.$$

Hence, we obtain

$$\begin{aligned}[x,\Omega_p] &= \sum_{\substack{a\preceq x\preceq b\\ |b|-|a|=p}} \frac{\left[x,\omega_{a,b}\right]}{\left\|\omega_{a,b}\right\|} \\ &= \sum_{\substack{k+l=p\\ a\preceq x\preceq b\\ |x|-|a|=k,\ |b|-|x|=l}} \frac{\left[x,\omega_{a,b}\right]}{\left\|\omega_{a,b}\right\|} = \sum_{k+l=p}\binom{m}{k}\binom{n-m}{l}\frac{1}{\binom{k+l}{k}},\end{aligned}$$

which implies by Lemma 5.36 that

$$\begin{aligned}K_x &= \sum_{p\ge 0}\frac{(-1)^p}{p+1}[x,\Omega_p] \\ &= \sum_{k=0}^{m}\sum_{l=0}^{n-m}\binom{m}{k}\binom{n-m}{l}\frac{(-1)^{k+l}}{\binom{k+l}{l}(k+l+1)} \\ &= \frac{1}{(m+1)\binom{n+1}{m+1}} \\ &= \frac{m!\,(n-m)!}{(n+1)!} \\ &= \frac{1}{(n+1)\binom{n}{m}}.\end{aligned}$$

□

Note that the number of vertices x with $|x|=m$ is equal to $\binom{n}{m}$ whence

$$K_{total}=\sum_{m=0}^{n}\frac{1}{(n+1)\binom{n}{m}}\binom{n}{m}=\sum_{m=0}^{n}\frac{1}{n+1}=1,$$

as expected because $\chi=1$.

5.7 Curvature of a Join

The main result of this section is Proposition 5.39 below. Recall that a join $Z=X*Y$ of two digraphs was defined in Subsection 3.6.

Let us first prove two lemmas. Everywhere $\langle\cdot,\cdot\rangle$ denotes the natural inner product in all spaces $\Lambda_*(X)$, $\Lambda_*(Y)$ and $\Lambda_*(Z)$.

Lemma 5.37 ([29, Lemma 3.10])**.** *If $u,u'\in\Lambda_*(X)$ and $v,v'\in\Lambda_*(Y)$ then*

$$\left\langle uv,u'v'\right\rangle_Z=\left\langle u,u'\right\rangle_X\left\langle v,v'\right\rangle_Y. \tag{5.123}$$

Proof. Indeed, due to bilinearity it suffices to prove (5.123) if u,u',v,v' are elementary paths, say

$$u=e_{i_0\ldots i_p},\quad u'=e_{i'_0\ldots i'_{p'}},\ v=e_{j_0\ldots j_q},\ v'=e_{j'_0\ldots j'_{q'}}.$$

Then

$$\begin{aligned}\left\langle uv,u'v'\right\rangle_Z &= \langle e_{i_0\ldots i_p j_0\ldots j_q}, e_{i'_0\ldots i'_{p'} j'_0\ldots j'_{q'}}\rangle = \delta_{i_0\ldots i_p j_0\ldots j_q}^{i'_0\ldots i'_{p'} j'_0\ldots j'_{q'}} \\ &= \delta_{i_0\ldots i_p}^{i'_0\ldots i'_{p'}}\,\delta_{j_0\ldots j_q}^{j'_0\ldots j'_{q'}} = \langle e_{i_0\ldots i_p}, e_{i'_0\ldots i'_{p'}}\rangle\langle e_{j_0\ldots j_q}, e_{j'_0\ldots j'_{q'}}\rangle \\ &= \left\langle u,u'\right\rangle_X\left\langle v,v'\right\rangle_Y.\end{aligned}$$

□

Lemma 5.38. *Let $Z=X*Y$ be the join of two digraphs X and Y. Then, for all $x\in X$ and $r\ge 0$ we have*

$$[x,\Omega_r(Z)]=[x,\Omega_r(X)]+\sum_{\substack{p+q=r-1,\\ p,q\ge 0}}[x,\Omega_p(X)]\dim\Omega_q(Y). \tag{5.124}$$

Proof. Let $\mathcal{B}_p(X)$ be an orthonormal basis in $\Omega_p(X)$ and $\mathcal{B}_q(Y)$ be an orthonormal basis in $\Omega_q(Y)$, for all $p,q\ge 0$. By Theorem 3.12, we obtain the following basis in $\Omega_r(Z)$: it consists of all elements of $\mathcal{B}_r(X)$, $\mathcal{B}_r(Y)$ as well as of the elements of the form

$$\left\{uv: u\in\mathcal{B}_p(X), v\in\mathcal{B}_q(Y), p+q=r-1,\ p,q\ge 0\right\}. \tag{5.125}$$

Note that the set (5.125) is empty if $r=0$, so it makes sense to consider it only if $r \geq 1$. This basis in also orthonormal due to the identity (5.123). Therefore, we obtain, for any $x \in X$ and any $r \geq 0$

$$[x,\Omega_r(Z)] = \sum_{u\in\mathcal{B}_r(X)} (T_x u, u) + \sum_{v\in\mathcal{B}_r(Y)} (T_x v, v) + \sum_{\substack{p+q=r-1,\\ p,q\geq 0}} \sum_{\substack{u\in\mathcal{B}_p(X)\\ v\in\mathcal{B}_q(Y)}} (T_x(uv), uv).$$

Since $T_x v = 0$ and $T_x(uv) = (T_x u)v$, we obtain

$$(T_x(uv), uv) = ((T_x u)v, uv) = (T_x u, u)(v,v) = (T_x u, u)$$

and

$$\sum_{\substack{u\in\mathcal{B}_p(X)\\ v\in\mathcal{B}_q(Y)}} (T_x(uv), uv) = [x,\Omega_p(X)] \dim\Omega_q(Y),$$

whence (5.124) follows. □

Proposition 5.39. *Let $Z = X * Y$ be the join of two digraphs X and Y. Assume that $\Omega_N(X)$ and $\Omega_N(Y)$ vanish for large enough N. Then, for any $x \in X$, we have*

$$K_x(Z) = K_x(X) - \sum_{p\geq 0} (-1)^p C_p(Y)[x,\Omega_p(X)], \tag{5.126}$$

where

$$C_p(Y) = \sum_{q\geq 0} \frac{(-1)^q}{p+q+2} \dim\Omega_q(Y).$$

A similar formula holds for $K_y(Z)$ for $y \in Y$:

$$K_y(Z) = K_y(Y) - \sum_{q\geq 0} (-1)^q C_q(X)[y,\Omega_q(Y)],$$

where

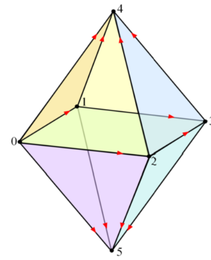
$$C_q(X) = \sum_{p\geq 0} \frac{(-1)^p}{p+q+2} \dim\Omega_p(X).$$

Proof. It follows from (5.124) that

$$\begin{aligned} K_x(Z) &= \sum_{r\geq 0} (-1)^r \frac{[x,\Omega_r(Z)]}{r+1} \\ &= K_x(X) + \sum_{p,q\geq 0} \frac{(-1)^{p+q+1}}{p+q+2} [x,\Omega_p(X)] \dim\Omega_q(Y) \\ &= K_x(X) \\ &\quad - \sum_{p\geq 0} (-1)^p \left(\sum_{q\geq 0} \frac{(-1)^q}{p+q+2} \dim\Omega_q(Y) \right) [x,\Omega_p(X)], \end{aligned}$$

which was to be proven. □

Example 5.40. Consider on octahedron Z based on a square:



We have

$$Z = X * Y$$

where X is the following square:

$$X = \{0 \to 1 \to 3, 0 \to 2 \to 3\}$$

and $Y = \{4,5\}$.

Since $\Omega_q(Y)$ is non-trivial only for $q=0$ and $\dim\Omega_0(Y) = 2$, we obtain

$$C_p(Y) = \frac{2}{p+2}.$$

As we have computed in Example 5.9,

$$[0,\Omega_2(X)] = [3,\Omega_2(X)] = 1, \quad [1,\Omega_2(X)] = [2,\Omega_2(X)] = \frac{1}{2}$$

and

$$K_0(X) = K_3(X) = \frac{1}{3}, \qquad K_1(X) = K_2(X) = \frac{1}{6}.$$

Hence, we obtain by (5.126), for $x = 0$ or 3,

$$\begin{aligned} K_x(Z) &= \frac{1}{3} - \sum_{p\geq 0} (-1)^p \frac{2}{p+2} [x,\Omega_p(X)] \\ &= \frac{1}{3} - 1 + \frac{2}{3}\cdot 2 - \frac{2}{4}\cdot 1 = \frac{1}{6}, \end{aligned}$$

and for $x = 1$ or 2,

$$\begin{aligned} K_x(Z) &= \frac{1}{6} - \sum_{p\geq 0} (-1)^p \frac{2}{p+2} [x,\Omega_p(X)] \\ &= \frac{1}{6} - 1 + \frac{2}{3}\cdot 2 - \frac{2}{4}\cdot\frac{1}{2} = \frac{1}{4}. \end{aligned}$$

Next, we have

$$C_q(X) = \sum_{p\geq 0} \frac{(-1)^p}{p+q+2} \dim\Omega_p(X) = \frac{4}{q+2} - \frac{4}{q+3} + \frac{1}{q+4}.$$

Since $[y,\Omega_0(Y)] = 1$, $\Omega_q(Y) = \{0\}$ for $q \geq 1$, and $K_y(Y) = 1$, we obtain, for $y = 4$ or 5,

$$K_y(Z) = 1 - C_0(X)[y,\Omega_0(Y)] = 1 - \left(\frac{4}{2} - \frac{4}{3} + \frac{1}{4}\right) = \frac{1}{12}.$$

5.8 Strongly Regular Digraphs

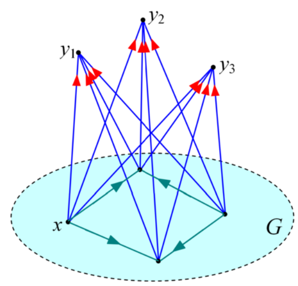
Recall that a graph is called regular if $\deg(x)$ is constant.

Definition. We say that a digraph G is *strongly regular* if the function $x \mapsto [x, \Omega_p]$ is constant for any p (in particular, G is regular because $\deg(x) = [x, \Omega_1]$ is constant).

For a strongly regular digraph G the function $x \mapsto K_x$ is constant, and we set

$$K(G) := K_x = \frac{\chi(G)}{|V|}.$$

Recall the definition of m-suspension $\mathrm{sus}_m G$: it is obtained by adding to G new m vertices $\{y_1, \ldots, y_m\}$ and all arrows $x \to y_i$ $\forall x \in G$.



In other words, $\mathrm{sus}_m G = G * Y$ where

$$Y = \{y_1, \ldots, y_m\}.$$

Theorem 5.41. *Let G be a strongly regular digraph, such that for some $k, m \in \mathbb{N}$ and any $p \geq 0$*

$$(\text{binom }(k,m)) \qquad \dim \Omega_p(G) = \binom{k}{p+1} m^{p+1}.$$

Then $\mathrm{sus}_m G$ is strongly regular, and for all $p \geq 0$

$$(\text{binom }(k+1,m)) \qquad \dim \Omega_p(\mathrm{sus}_m G) = \binom{k+1}{p+1} m^{p+1}.$$

Proof. We have

$$|X| = \dim \Omega_0(X) = \binom{k}{1} n = kn.$$

Since for any $x \in X$

$$\sum_{x \in X} [x, \Omega_p(X)] = [\mathbf{1}, \Omega_p(X)] = (p+1) \dim \Omega_p(X),$$

it follows that

$$[x, \Omega_p(X)] = \frac{(p+1)\dim \Omega_p(X)}{|X|} = \frac{p+1}{kn} \binom{k}{p+1} n^{p+1}$$
$$= \binom{k-1}{p} n^p.$$

Since $\dim \Omega_0(Y) = n$ and $\Omega_q(Y) = \{0\}$ for all $q \geq 1$, we obtain from (5.124) that, for $r \geq 1$,

$$[x, \Omega_r(Z)] = [x, \Omega_r(X)] + n [x, \Omega_{r-1}(X)]$$
$$= \binom{k-1}{r} n^r + n \binom{k-1}{r-1} n^{r-1} = \binom{k}{r} n^r.$$

In the same way, for any $y \in Y$ and $r \geq 1$,

$$[y, \Omega_r(Z)] = [y, \Omega_r(Y)] + \sum_{\substack{p+q=r-1,\\ p,q \geq 0}} [y, \Omega_q(Y)] \dim \Omega_p(X)$$
$$= \dim \Omega_{r-1}(X) = \binom{k}{r} n^r.$$

It follows that, for all $z \in Z$,

$$[z, \Omega_r(Z)] = \binom{k}{r} n^r.$$

Consequently, we have

$$\dim \Omega_r(Z) = \frac{|Z| [z, \Omega_r(Z)]}{r+1} = \frac{|X| + |Y|}{r+1} \binom{k}{r} n^r = \frac{kn+n}{r+1} \binom{k}{r} n^r$$
$$= \binom{k+1}{r+1} n^{r+1}.$$

Finally, for $r = 0$ we obtain

$$\dim \Omega_0(Z) = kn + n = (k+1) n = \binom{k+1}{0+1} n^{0+1}.$$

□

5.9 Digraphs of Constant Curvature

For the digraph G as in Theorem 5.41 we have

$$\chi(G) = \sum_{p \geq 0} (-1)^p \dim \Omega_p = \sum_{p=0}^{k-1} (-1)^p \binom{k}{p+1} m^{p+1}$$
$$= -\sum_{j=1}^{k} (-1)^j \binom{k}{j} m^j = 1 - (1-m)^k.$$

It follows that

$$K(G) = \frac{\chi(G)}{|V|} = \frac{\chi(G)}{\dim \Omega_0} = \frac{1-(1-m)^k}{km}.$$

Of course, the same formula is true for $K(\mathrm{sus}_m G)$ with k replaced by $k+1$:

$$K(\mathrm{sus}_m G) = \frac{1-(1-m)^{k+1}}{(k+1)m}$$

Example 5.42. We have seen that a triangle (= 2-simplex) is strongly regular and

$$\dim \Omega_0 = 3, \ \dim \Omega_1 = 3, \ \dim \Omega_2 = 1, \ \dim \Omega_p = 0 \ \text{ for } p \geq 3$$

that is, the sequence $\{\dim \Omega_p\}_{p \geq 0}$ is the sequence $\binom{3}{p+1}$ that satisfies (binom$(3,1)$). The 1-suspension of an n-simplex is an $(n+1)$-simplex. Hence, we obtain by induction that the n-simplex is strongly regular and satisfies (binom $(n+1, 1)$). In particular,

$$K(n\text{-simplex}) = \frac{1}{n+1}.$$

For any $m \in \mathbb{N}$ denote by D_m a digraph with m vertices and no arrows. Then

$$\dim \Omega_0 (D_m) = m = \binom{1}{p+1} m^{p+1} \quad \text{for } p = 0,$$
$$\dim \Omega_p (D_m) = 0 = \binom{1}{p+1} m^{p+1} \quad \text{for } p \geq 1,$$

so that (binom$(1,m)$) is satisfied. Clearly, D_m is strongly regular.

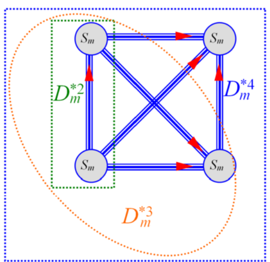
For any $k \in \mathbb{N}$ define digraph D_m^{*k} as the k-th join power of D_m, that is,

$$D_m^{*1} = D_m$$

and

$$D_m^{*(k+1)} = D_m^{*k} * D_m = \mathsf{sus}_m D_m^{*k}.$$

Here are digraphs $D_m^{*1}, D_m^{*2}, D_m^{*3}, D_m^{*4}$:



In fact, D_m^{*k} is a digraph version of a complete k-partite graph $K_{m,m,\ldots,m}$ where the index m repeats k times, that can also be denoted by $\overrightarrow{K}_{m,m,\ldots,m}$.

Using Theorem 5.41, by obtain by induction that D_m^{*k} is strongly regular and satisfies (binom (k,m)).

Hence, D_m^{*k} has a constant curvature

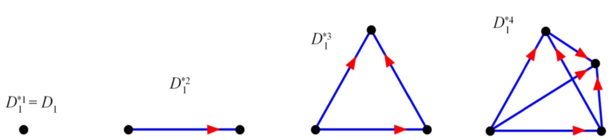
$$K(D_m^{*k}) = \frac{1-(1-m)^k}{km}. \tag{5.127}$$

One can show that the only non-trivial Betti number of D_m^{*k} is $\beta_{k-1} = (m-1)^k$ (see [7]).

Example 5.43. For $m = 1$ we have by (5.127)

$$K(D_1^{*k}) = \frac{1}{k}.$$

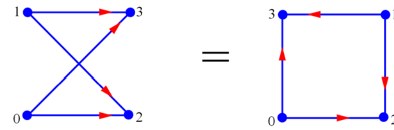
Clearly, D_1^{*k} is a $(k-1)$-simplex:



Example 5.44. For $m = 2$ we have by (5.127)

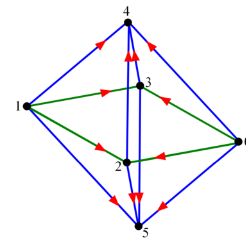
$$K(D_2^{*k}) = \begin{cases} 0, & k \text{ even}, \\ \frac{1}{k}, & k \text{ odd}. \end{cases}$$

For example, D_2^{*2} is a diamond: that is an analogue of 1-sphere. We have $K(D_2^{*2}) = 0$.



We can regard $D_2^{*(k+1)}$ as a digraph analogue of a k-sphere $\mathbb{S}^k$ because $D_2^{*(k+1)}$ is obtained from D_2^{*k} by 2-suspension, similarly to how $\mathbb{S}^k$ is obtained from $\mathbb{S}^{k-1}$. Besides, the only non-trivial Betti number of $D_2^{*(k+1)}$ is $\beta_k = 1$ like the Betti numbers for $\mathbb{S}^k$.

Here is D_2^{*3}, that is an octahedron, based on a diamond:



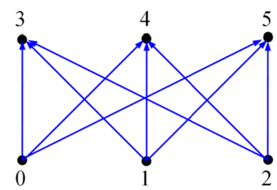
It is an analogue of 2-sphere; it has constant curvature $\frac{1}{3}$.

D_2^{*4} is an analogue of 3-sphere; it has constant curvature 0.

Example 5.45. For $m = 3$ we have by (5.127)

$$K(D_3^{*k}) = \frac{1-(-2)^k}{3k} = \frac{1}{3k} \begin{cases} 1-2^k, & k \text{ even}, \\ 1+2^k, & k \text{ odd}. \end{cases}$$

Here is D_3^{*2} that is a directed version of $K_{3,3}$:



We have $K(D_3^{*2}) = -\frac{1}{2}$ and $K(D_3^{*3}) = 1$.

5.10 Cartesian Product and Curvature

Recall that a Cartesian product $X \square Y$ of two digraphs was defined in Subsection 3.2.

Theorem 5.46. *Let X be any digraph with a finite chain sequence $\{\Omega_p\}$ and Y be a cyclic digraph $\{0 \to 1 \to 2 \to \ldots \to 0\}$ of at least 3 vertices. Then, with respect to the natural inner product $\langle \cdot, \cdot \rangle$, we have*

$$K_z (X \square Y) = 0 \quad \textit{for any } z \in X \square Y.$$

In particular, we have $K(T^{\square n}) = 0$. Recall that in Example 5.20 we have computed directly that $K(T^{\square 2}) = 0$.

Proof. Let $Y = (V, E)$. Then

$$\Omega_0 (Y) = \langle e_a : a \in V \rangle, \quad \Omega_1 (Y) = \{e_{ab} : ab \in E\},$$
$$\Omega_p (Y) = \{0\} \text{ for } p > 2.$$

We have

$$K_x(X) = \sum_{p \geq 0} (-1)^p \frac{[x, \Omega_p]}{p+1}.$$

Denote by $\mathcal{B}_p(X)$ an orthogonal basis in $\Omega_p(X)$ so that

$$[x,\Omega_p] = \sum_{\omega\in\mathcal{B}_p(X)} \frac{[x,\omega]}{\|\omega\|^2}.$$

We have by Theorem 3.5

$$\mathcal{B}_p(Z) = \{u\times e_a,\ v\times e_{ab}\ : u\in\mathcal{B}_p(X),\ v\in\mathcal{B}_{p-1}(X),\ a\in V,\ ab\in E\}.$$

This basis is orthogonal due to the identity

$$\langle u\times\omega, u'\times\omega'\rangle_Z = \binom{p+q}{p}\langle u,u'\rangle_X \langle\omega,\omega'\rangle_Y, \tag{5.128}$$

where $u\in\Omega_p(X)$, $u'\in\Omega_{p'}(X)$, $\omega\in\Omega_q(Y)$, $\omega'\in\Omega_{q'}(Y)$ (see [29, Lemma 4.13]).

Hence, we have

$$[z,\Omega_p(Z)] = \sum_{\substack{u\in\mathcal{B}_p(X)\\ a\in V}} \frac{[z,u\times e_a]}{\|u\times e_a\|^2} + \sum_{\substack{v\in\mathcal{B}_{p-1}(X)\\ ab\in E}} \frac{[z,v\times e_{ab}]}{\|v\times e_{ab}\|^2}.$$

Let $u=\sum u^{i_0\ldots i_p} e_{i_0\ldots i_p}$ so that

$$u\times e_a = \sum_{i_0\ldots i_p} u^{i_0\ldots i_p} e_{i_0\ldots i_p}\times e_a.$$

We have for $z=(x,y)$

$$[z, e_{i_0\ldots i_p}\times e_a] = [(x,y), e_{(i_0a)(i_1a)\ldots(i_pa)}] = [x,e_{i_0\ldots i_p}][y,a],$$

whence

$$\sum_{a\in V}[z,e_{i_0\ldots i_p}\times e_a] = [x,e_{i_0\ldots i_p}].$$

It follows that

$$\begin{aligned}\sum_{a\in V}[z,u\times e_a] &= \sum_{a\in V}\sum_{i_0\ldots i_p}(u^{i_0\ldots i_p})^2[z,e_{i_0\ldots i_p}\times e_a]\\ &= \sum_{i_0\ldots i_p}\sum_{a\in V}(u^{i_0\ldots i_p 2})[z,e_{i_0\ldots i_p}\times e_a]\\ &= \sum_{i_0\ldots i_p}(u^{i_0\ldots i_p})^2[x,e_{i_0\ldots i_p}] = [x,u].\end{aligned}$$

Since also $\|u\times e_a\| = \|u\|$, we obtain

$$\sum_{u\in\mathcal{B}_p(X)}\sum_{a\in V}\frac{[z,u\times e_a]}{\|u\times e_a\|^2} = \sum_{u\in\mathcal{B}_p(X)}\frac{[x,u]}{\|u\|^2} = [x,\Omega_p(X)].$$

Now let us handle the term $[z,v\times e_{ab}]$. Let $v=\sum_{i_0\ldots i_p} v^{i_0\ldots i_{p-1}} e_{i_0\ldots i_{p-1}}$ so that

$$v\times e_{ab} = \sum_{i_0\ldots i_p} v^{i_0\ldots i_{p-1}} e_{i_0\ldots i_{p-1}}\times e_{ab}.$$

We have

$$e_{i_0\ldots i_{p-1}}\times e_{ab} = \sum_{k=0}^{p-1}(-1)^{p-1-k} e_{(i_0a)(i_1a)\ldots(i_ka)(i_kb)\ldots.(i_{p-1}b)}.$$

Note that

$$[(x,y), e_{(i_0a)(i_1a)\ldots(i_ka)(i_kb)\ldots.(i_{p-1}b)}] = \begin{cases}[x,e_{i_0\ldots i_k}], & y=a\\ [x,e_{i_k\ldots i_{p-1}}], & y=b\\ 0, & \text{otherwise.}\end{cases}$$

Considering all arrows $ab\in E$, there is exactly one $a=y$ and exactly one $b=y$. It follows that

$$\begin{aligned}\sum_{ab\in E}[(x,y), e_{(i_0a)(i_1a)\ldots(i_ka)(i_kb)\ldots.(i_{p-1}b)}] &= [x,e_{i_0\ldots i_k}] + [x,e_{i_k\ldots i_{p-1}}]\\ &= [x,e_{i_0\ldots i_{p-1}}] + 1_{\{x=i_k\}}\end{aligned}$$

and

$$\begin{aligned}\sum_{ab\in E}[z,e_{i_0\ldots i_{p-1}}\times e_{ab}] &= \sum_{k=0}^{p-1}([x,e_{i_0\ldots i_{p-1}}]+\mathbf{1}_{\{x=i_k\}})\\ &= (p+1)[x,e_{i_0\ldots i_{p-1}}].\end{aligned}$$

We obtain that

$$\begin{aligned}\sum_{ab\in E}[z,v\times e_{ab}] &= \sum_{i_0\ldots i_p}\sum_{ab\in E}(v^{i_0\ldots i_{p-1}})^2[z,e_{i_0\ldots i_{p-1}}\times e_{ab}]\\ &= (p+1)\sum_{i_0\ldots i_p}(v^{i_0\ldots i_{p-1}})^2[x,e_{i_0\ldots i_{p-1}}]\\ &= (p+1)[x,v].\end{aligned}$$

Since

$$\left\|e_{i_0\ldots i_{p-1}}\times e_{ab}\right\|^2 = p,$$

we have

$$\|v\times e_{ab}\|^2 = \sum_{i_0\ldots i_p}(v^{i_0\ldots i_{p-1}})^2 p = p\|v\|^2,$$

whence

$$\sum_{ab\in E}\frac{[z,v\times e_{ab}]}{\|v\times e_{ab}\|^2} = \frac{p+1}{p}\frac{[x,v]}{\|v\|^2}$$

and

$$\sum_{v\in\mathcal{B}_{p-1}(X)}\sum_{ab\in E}\frac{[z,v\times e_{ab}]}{\|v\times e_{ab}\|^2} = \frac{p+1}{p}[x,\Omega_{p-1}(X)].$$

We obtain

$$[z,\Omega_p(Z)] = [x,\Omega_p(X)] + \frac{p+1}{p}[x,\Omega_{p-1}(X)],$$

whence it follows that

$$\begin{aligned}K_z - 1 &= \sum_{p\geq 1}(-1)^p\frac{[z,\Omega_p(Z)]}{p+1}\\ &= \sum_{p\geq 1}(-1)^p\frac{[x,\Omega_p(X)]}{p+1} + \sum_{p\geq 1}(-1)^p\frac{[x,\Omega_{p-1}(X)]}{p}\\ &= (K_x-1) - K_x = -1,\end{aligned}$$

that is, $K_z = 0$. ☐

5.11 Some Problems

Problem 5.47. *How to compute $K(X\Box Y)$ for general digraphs X, Y?*

Problem 5.48. *Is $|\Omega_2| = 25$ true for an icosahedron (see Example 5.19) with any numbering of the vertices?*

Problem 5.49. *Let a digraph G be determined by a triangulation of $\mathbb{S}^2$ (see Subsection 1.10). Assume that $\deg(x) \le 4$ for all $x \in G$. Is it true that $K_x \ge 0$ for all $x \in G$?*

We have verified above that $K_x \ge 0$ for the following triangulations of $\mathbb{S}^2$: simplex, bipyramid, octahedron, but with specific orientations of edges (the question remains open when the numbering of vertices is arbitrary). All these digraphs have $\deg(x) \le 4$. We have seen that $K_x < 0$ can occur for icosahedron with $\deg(x) = 5$ and for a pyramid with $\deg(x) = 7$.

Problem 5.50. *Denote $D = \max_{x\in G} \deg(x)$. Is it true that $|K_x| \le C_D$ for some constant C_D depending only on D? What about upper bounds for $|K_x^{(2)}|$ and $|K_x^{(3)}|$?*

Note that K_x can be take arbitrarily large positive and negative values. For example, for a strongly regular digraph satisfying (binom (k,m)), we have

$$K_x = \frac{1-(1-m)^k}{km},$$

while $D = \frac{2\dim\Omega_1}{\dim\Omega_0} = (k-1)m$. In this case one can verify that $|K_x| \le e^{0.3D}$.

Problem 5.51. *What can be said about the curvature of random digraphs?*

Problem 5.52. *Let $\mathcal{S}$ be a simplicial complex and $G_{\mathcal{S}}$ be its Hasse diagram (see Subsection 1.9). Is there any relation of $K_x(G_{\mathcal{S}})$ to properties of $\mathcal{S}$? For example, we have*

$$K_{total}(G_{\mathcal{S}}) = \chi(G_{\mathcal{S}}) = \chi_{simp}(\mathcal{S}).$$

Can one give an explicit formula for computing $K_\sigma(G_{\mathcal{S}})$ for any simplex $\sigma \in \mathcal{S}$?

6. Hodge Laplacian on Digraphs

In this section $\mathbb{K} = \mathbb{R}$. Let us fix an arbitrary inner product $\langle\cdot,\cdot\rangle$ in each of the spaces $\mathcal{R}_p$ so that we have an inner product also in all Ω_p. In all examples we use the natural inner product.

6.1 Definition and Spectral Properties of Δ_p

For the operator $\partial : \Omega_p \to \Omega_{p-1}$, consider the adjoint operator $\partial^* : \Omega_{p-1} \to \Omega_p$. By the definition of an adjoint operator, we have

$$\langle \partial u, v\rangle = \langle u, \partial^* v\rangle \quad \text{for all } u \in \Omega_p \text{ and } v \in \Omega_{p-1}.$$

Definition. Define the *Hodge-Laplace operator* $\Delta_p : \Omega_p \to \Omega_p$ by

$$\Delta_p u = \partial^*\partial u + \partial\partial^* u. \tag{6.129}$$

The pairs ∂^*, ∂ and ∂, ∂^* appearing in (6.129) are the following operators:

$$\Omega_{p-1} \underset{\partial^*}{\overset{\partial}{\leftrightarrows}} \Omega_p \quad \text{and} \quad \Omega_p \underset{\partial^*}{\overset{\partial}{\leftrightarrows}} \Omega_{p+1}.$$

Proposition 6.1. *The operator Δ_p is self-adjoint and non-negative definite.*

Proof. We have for all $u, v \in \Omega_p$

$$\langle \Delta_p u, v\rangle = \langle \partial^*\partial u + \partial\partial^* u, v\rangle = \langle \partial u, \partial v\rangle + \langle \partial^* u, \partial^* v\rangle = \langle u, \Delta_p v\rangle$$

so that Δ_p is self-adjoint, and

$$\langle \Delta_p u, u\rangle = \|\partial u\|^2 + \|\partial^* u\|^2 \ge 0, \tag{6.130}$$

so that $\Delta_p \ge 0$. □

Hence, the spectrum of Δ_p is real, non-negative and consists of a finite sequence of eigenvalues.

Proposition 6.2. *Denote $D = \max_{i\in V} \deg(i)$. If $\langle\cdot,\cdot\rangle$ is the natural inner product then $\operatorname{spec}\Delta_0 \subset [0, 2D]$.*

Proof. By the variational principle, it suffices to prove that for all $u \in \Omega_0$

$$\frac{\langle \Delta_0 u, u\rangle}{\|u\|^2} \le 2D.$$

Since $\partial u = 0$, we have by (6.130)

$$\langle \Delta_0 u, u\rangle = \|\partial^* u\|^2.$$

Since for any $i \to j$

$$\langle \partial^* u, e_{ij}\rangle = \langle u, \partial e_{ij}\rangle = \langle u, e_j - e_i\rangle = u^j - u^i,$$

it follows that

$$\|\partial^* u\|^2 = \sum_{i\to j}(u^j - u^i)^2 \le 2\sum_{i\to j}(u^j)^2 + 2\sum_{i\to j}(u^i)^2$$
$$= 2\sum_i \deg(i)(u^i)^2 \le 2D\|u\|^2, \tag{6.131}$$

whence the claim follows. □

The bottom eigenvalue of Δ_0 is always 0 because if all $u^k = 1$ then by (6.131) $\partial^* u = 0$ and, hence, $\Delta_0 u = \partial\partial^* u = 0$. If G a complete bipartite graph $K_{D,D}$, then G is D-regular and $2D$ is the top eigenvalue of Δ_0.

For a general p, *the multiplicity of 0 as an eigenvalue of Δ_p is equal to the Betti number β_p* as we will see below in Corollary 6.7.

Problem 6.3. *Find reasonable upper bounds for $\operatorname{spec}\Delta_p$. The question amounts to obtaining an upper bound for the Rayleigh quotient for non-zero $u \in \Omega_p$:*

$$\frac{\|\partial u\|^2 + \|\partial^* u\|^2}{\|u\|^2} \le ?$$

Problem 6.4. *Find estimates of the eigenvalues of Δ_p in terms of geometric and combinatorial properties of G.*

6.2 Harmonic Paths

A path $u \in \Omega_p$ is called *harmonic* if $\Delta_p u = 0$.

Lemma 6.5 ([23, Lemma 3.2])**.** *A path $u \in \Omega_p$ is harmonic if and only if $\partial u = 0$ and $\partial^* u = 0$.*

Proof. Indeed, if $\partial u = 0$ and $\partial^* u = 0$ then by (6.129) we have $\Delta_p u = 0$. Conversely, if $\Delta_p u = 0$ then we obtain by (6.130) that

$$\|\partial u\|^2 + \|\partial^* u\|^2 = \langle \Delta_p u, u \rangle = 0,$$

whence $\|\partial u\| = \|\partial^* u\| = 0$. □

Denote by $\mathcal{H}_p$ the set of all harmonic paths in Ω_p, so that $\mathcal{H}_p$ is a subspace of Ω_p.

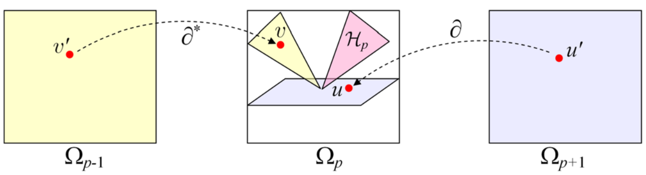
Theorem 6.6 (Hodge decomposition [23, Lemma 3.3])**.** *The space Ω_p is an orthogonal sum:*

$$\Omega_p = \partial\Omega_{p+1} \bigoplus \partial^*\Omega_{p-1} \bigoplus \mathcal{H}_p. \tag{6.132}$$

Proof. If $u \in \partial\Omega_{p+1}$ and $v \in \partial^*\Omega_{p-1}$ then $u = \partial u'$ and $v = \partial^* v'$, and we have

$$\langle u, v \rangle = \langle \partial u', \partial^* v' \rangle = \langle \partial^2 u', v' \rangle = 0,$$

so that the subspaces $\partial\Omega_{p+1}$ and $\partial^*\Omega_{p-1}$ are orthogonal.



Denote by K the orthogonal complement of $\partial\Omega_{p+1} \bigoplus \partial^*\Omega_{p-1}$ in Ω_p. Then we have

$$w \in K \Leftrightarrow \langle w, u \rangle = 0 \ \forall u \in \partial\Omega_{p+1} \ \text{ and } \ \langle w, v \rangle = 0 \ \forall v \in \partial^*\Omega_{p-1},$$

that is,

$$\begin{aligned} w \in K &\Leftrightarrow \langle w, \partial u' \rangle = 0 \ \forall u' \in \Omega_{p+1} \ \text{and} \ \langle w, \partial^* v' \rangle = 0 \ \forall v' \in \Omega_{p-1} \\ &\Leftrightarrow \langle \partial^* w, u' \rangle = 0 \ \forall u' \in \Omega_{p+1} \ \text{and} \ \langle \partial w, v' \rangle = 0 \ \forall v' \in \Omega_{p-1} \\ &\Leftrightarrow \partial^* w = 0 \ \text{ and } \ \partial w = 0 \\ &\Leftrightarrow w \in \mathcal{H}_p. \end{aligned}$$

Hence, $K = \mathcal{H}_p$ which finishes the proof. □

Corollary 6.7 ([23, Corollary 3.4])**.** *There is a natural linear isomorphism*

$$H_p \cong \mathcal{H}_p. \tag{6.133}$$

In particular, $\dim \mathcal{H}_p = \beta_p$*; that is, the multiplicity of* 0 *as an eigenvalue of* Δ_p *is equal to the Betti number* β_p*.*

Proof. Observe that $Z_p := \ker \partial|_{\Omega_p}$ is the orthogonal complement of $\partial^*\Omega_{p-1}$ in Ω_p because, for any $u \in \Omega_p$,

$$\begin{aligned} u \in Z_p &\Leftrightarrow \partial u = 0 \Leftrightarrow \langle \partial u, v \rangle = 0 \ \forall v \in \Omega_{p-1} \\ &\Leftrightarrow \langle u, \partial^* v \rangle = 0 \ \forall v \in \Omega_{p-1} \Leftrightarrow u \perp \partial^*\Omega_{p-1}. \end{aligned}$$

Since by (6.132)

$$\Omega_p = \partial\Omega_{p+1} \bigoplus \mathcal{H}_p \bigoplus \partial^*\Omega_{p-1}$$

we obtain

$$Z_p = (\partial^*\Omega_{p-1})^\perp = \partial\Omega_{p+1} \bigoplus \mathcal{H}_p \tag{6.134}$$

whence $\mathcal{H}_p \cong Z_p / \partial\Omega_{p+1} = H_p$. □

Remark 6.8. It follows from this argument that $\mathcal{H}_p$ is an orthogonal complement of B_p in Z_p and that any homology class $\omega \in H_p$ has a unique harmonic representative $u \in \mathcal{H}_p$. In addition, u minimizes the norm $\|\cdot\|$ among all representatives of ω.

6.3 Matrix of Δ_p

Let $\{\alpha_i\}$ be an orthonormal basis in Ω_p, $\{\beta_m\}$ be an orthonormal basis in Ω_{p-1} and $\{\gamma_n\}$ be an orthonormal basis in Ω_{p+1}:

$$\underset{\{\beta_m\}}{\Omega_{p-1}} \ \underset{\partial}{\overset{\partial^*}{\rightleftarrows}} \ \underset{\{\alpha_i\}}{\Omega_p} \ \underset{\partial}{\overset{\partial^*}{\rightleftarrows}} \ \underset{\{\gamma_n\}}{\Omega_{p+1}} \ .$$

The operator $\partial : \Omega_p \to \Omega_{p-1}$ has in the bases $\{\alpha_i\}$ and $\{\beta_m\}$ the matrix representation

$$B = \left(\langle \beta_m, \partial\alpha_i \rangle\right)_{m,i}, \tag{6.135}$$

where m is the row index and i is the column index.

Similarly, the operator $\partial^* : \Omega_p \to \Omega_{p+1}$ has the matrix representation

$$C = \left(\langle \gamma_n, \partial^*\alpha_i \rangle\right)_{n,i} = \left(\langle \partial\gamma_n, \alpha_i \rangle\right)_{n,i}, \tag{6.136}$$

where n is the row index and i is the column index. Since $\Delta_p = \partial^*\partial + (\partial^*)^* \partial^*$, we obtain the matrix representation of Δ_p in the basis $\{\alpha_i\}$:

$$\text{matrix of } \Delta_p = B^T B + C^T C. \tag{6.137}$$

More explicitly, the (i, j)-entry of the matrix of Δ_p in the basis $\{\alpha_i\}$ is given by

$$\left\langle \Delta_p \alpha_i, \alpha_j \right\rangle = \sum_m \langle \partial\alpha_i, \beta_m \rangle \left\langle \partial\alpha_j, \beta_m \right\rangle + \sum_n \langle \alpha_i, \partial\gamma_n \rangle \left\langle \alpha_j, \partial\gamma_n \right\rangle, \tag{6.138}$$

where i is the row index and j is the column index.

Example 6.9. Recall that $\Omega_{-1} = \{0\}, \Omega_0 = \{e_i : i \in V\}$ and $\Omega_1 = \langle e_{kl} : k \to l \rangle$. Assuming that $\langle \cdot, \cdot \rangle$ is the natural inner product, we obtain by (6.138) that the matrix of Δ_0 is

$$\begin{aligned} \left\langle \Delta_0 e_i, e_j \right\rangle &= \sum_{k \to l} \langle e_i, \partial e_{kl} \rangle \left\langle e_j, \partial e_{kl} \right\rangle \\ &= \sum_{k \to l} \langle e_i, e_l - e_k \rangle \left\langle e_j, e_l - e_k \right\rangle \\ &= \sum_{k \to l} (\delta_{il} - \delta_{ik}) \left(\delta_{jl} - \delta_{jk}\right) \\ &= \sum_{k \to i} \delta_{ij} + \sum_{i \to l} \delta_{ij} - \mathbf{1}_{\{i \to j\}} - \mathbf{1}_{\{j \to i\}} \end{aligned}$$

$$= \deg(i)\delta_{ij} - \mathbf{1}_{\{i\to j\}} - \mathbf{1}_{\{j\to i\}}.$$

If G has no double arrow then

$$\text{the matrix of } \Delta_0 = \operatorname{diag}(\deg(i)) - \mathbf{1}_{\{i\sim j\}},$$

where $\mathbf{1}_{\{i\sim j\}}$ is the adjacency matrix of G. Hence, in this case Δ_0 is the usual unnormalized Laplacian (= Kirchhoff operator) on functions on V. Consequently, we have

$$\operatorname{trace}\Delta_0 = \sum_{i\in V} \deg(i) = 2|E|. \tag{6.139}$$

6.4 Examples of Computation of the Matrix of Δ_1

In this section, we denote by V and E respectively the numbers of vertices and arrows of the digraph in question.

Let us compute Δ_1 for the natural inner product. We use the orthonormal bases $\{e_m\}$ in Ω_0 and $\{e_{ij} : i\to j\}$ in Ω_1. Let $\{\gamma_n\}$ be an orthonormal basis in Ω_2.

The matrix of Δ_1 has dimensions $E\times E$ and, by (6.138), its entries are

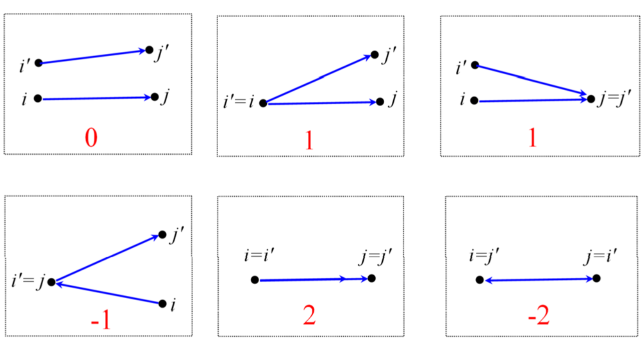
(6.140)
$$\langle \Delta_1 e_{ij}, e_{i'j'}\rangle = \sum_m \langle \partial e_{ij}, e_m\rangle\langle \partial e_{i'j'}, e_m\rangle + \sum_n \langle e_{ij}, \partial\gamma_n\rangle\langle e_{i'j'}, \partial\gamma_n\rangle$$

for all arrows $i\to j$ and $i'\to j'$.

For the first sum in (6.140) we have

$$\begin{aligned}\sum_m \langle \partial e_{ij}, e_m\rangle \langle \partial e_{i'j'}, e_m\rangle &= \sum_m \langle e_j - e_i, e_m\rangle\langle e_{j'} - e_{i'}, e_m\rangle \\ &= \sum_m (\delta_{jm} - \delta_{im})(\delta_{j'm} - \delta_{i'm}) \\ &= \delta_{jj'} - \delta_{ij'} - \delta_{ji'} + \delta_{ii'} =: [ij, i'j'].\end{aligned}$$

The values of $[ij, i'j']$ are shown here:



Hence, in the case $p = 1$, we have

$$B^T B = \left([ij, i'j']\right). \tag{6.141}$$

In particular, diagonal entries of $B^T B$ are equal to 2.

Example 6.10. Consider a 1-torus $T = \{0\to 1\to 2\to 0\}$. In this case we have $\Omega_1 = \langle e_{01}, e_{12}, e_{20}\rangle$ and

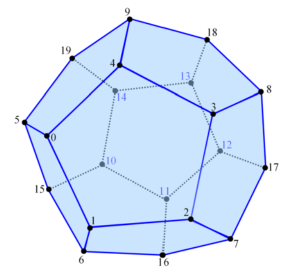
$$\text{the matrix of } \Delta_1 = B^T B = \left([ij, i'j']\right)$$

$$= \begin{pmatrix} & e_{01} & e_{12} & e_{20} \\ e_{01} & [01,01] & [01,12] & [01,20] \\ e_{12} & [12,01] & [12,12] & [12,20] \\ e_{20} & [20,01] & [20,12] & [20,20]\end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2\end{pmatrix}.$$

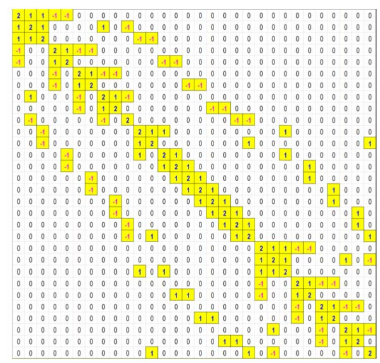
The eigenvalues of Δ_1 are $(0,3,3)$.

Example 6.11. Consider a dodecahedron (as in Example 5.7):



We have $V = 20$, $E = 30$, $\Omega_2 = \{0\}$ and $|H_1| = 11$. In particular, $C^T C = 0$ and, hence, $\Delta_1 = B^T B$.

The matrix of Δ_1 is shown here:



The eigenvalues of Δ_1 are:

$$\left(0_{11},\ 2_5,\ 3_4,\ 5_4,\ \left(3\pm\sqrt{5}\right)_3\right),$$

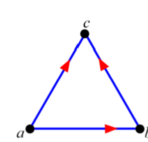
where the subscripts show multiplicity.

For a general digraph G with $\Omega_2 \neq \{0\}$, let us compute the entry $\langle e_{ij}, \partial\gamma_n\rangle$ of the matrix C assuming that $\gamma_n = \gamma$ is a triangle or square (note that although Ω_2 always has a basis of triangles and squares, the squares in this basis do not have to be orthogonal). If $\gamma = e_{abc}$ is a triangle then we have

$$\langle e_{ij}, \partial\gamma\rangle = \langle e_{ij}, e_{ab} + e_{bc} - e_{ac}\rangle = [ij, \gamma],$$

where

$$[ij,\gamma] := \begin{cases} 1, & \text{if } ij \in \{ab, bc\} \\ -1 & \text{if } ij = ac \\ 0, & \text{otherwise.}\end{cases}$$

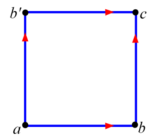


If $\gamma = \frac{e_{abc} - e_{ab'c}}{\sqrt{2}}$ is a (normalized) square then

$$\langle e_{ij}, \partial\gamma\rangle = \frac{1}{\sqrt{2}}\langle e_{ij}, e_{ab} + e_{bc} - e_{ab'} - e_{b'c}\rangle = \frac{1}{\sqrt{2}}[ij,\gamma],$$

where

$$[ij,\gamma] = \begin{cases} 1, & \text{if } ij \in \{ab, bc\} \\ -1 & \text{if } ij \in \{ab', b'c\} \\ 0, & \text{otherwise.} \end{cases}$$



Example 6.12. Let G be a triangle $\{0 \to 1 \to 2, 0 \to 2\}$. Then $\Omega_1 = \langle e_{01}, e_{12}, e_{02} \rangle$ and

$$B^T B = ([ij, i'j']) = \begin{pmatrix} & e_{01} & e_{12} & e_{02} \\ e_{01} & [01,01] & [01,12] & [01,20] \\ e_{12} & [12,01] & [12,12] & [12,20] \\ e_{02} & [02,01] & [02,12] & [02,02] \end{pmatrix} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The basis $\{\gamma_n\}$ of Ω_2 consists of a single triangle $\gamma = e_{012}$ so that

$$C = \begin{pmatrix} & e_{01} & e_{12} & e_{02} \\ e_{012} & [01,\gamma] & [12,\gamma] & [02,\gamma] \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix},$$

$$C^T C = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix},$$

$$\text{matrix of } \Delta_1 = B^T B + C^T C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Example 6.13. Let G be a square $\{0 \to 1 \to 3, 0 \to 2 \to 3\}$. Then $\Omega_1 = \langle e_{01}, e_{02}, e_{13}, e_{23} \rangle$ and

$$B^T B = ([ij, i'j']) = \begin{pmatrix} & e_{01} & e_{02} & e_{13} & e_{23} \\ e_{01} & [01,01] & [01,02] & [01,13] & [01,23] \\ e_{02} & [02,01] & [02,02] & [02,13] & [02,23] \\ e_{13} & [12,01] & [13,02] & [13,13] & [13,23] \\ e_{23} & [23,01] & [23,02] & [23,13] & [23,23] \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 & 0 \\ 1 & 2 & 0 & -1 \\ -1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 2 \end{pmatrix}.$$

The basis $\{\gamma_n\}$ of Ω_2 consists of a single square $\gamma = \frac{1}{\sqrt{2}}(e_{013} - e_{023})$ so that

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} & e_{01} & e_{02} & e_{13} & e_{23} \\ \gamma & [01,\gamma] & [02,\gamma] & [13,\gamma] & [23,\gamma] \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 1 & -1 \end{pmatrix},$$

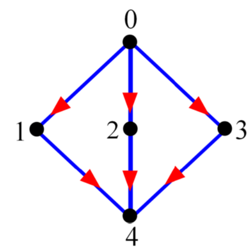
$$C^T C = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

Hence,

$$\text{matrix of } \Delta_1 = B^T B + C^T C = \begin{pmatrix} \frac{5}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \end{pmatrix},$$

and the eigenvalues of Δ_1 are $(2_3, 4)$.

Example 6.14. Consider the following digraph:



Here $V = 5,\ E = 6,\ |\Omega_2| = 2$ and

$$\Omega_2 = \langle e_{014} - e_{024}, e_{014} - e_{034} \rangle.$$

However, this basis is *not* orthogonal.

Orthogonalization gives an orthonormal basis for Ω_2:

$$\gamma_1 = \frac{1}{\sqrt{2}}(e_{014} - e_{024}),$$
$$\gamma_2 = \frac{1}{\sqrt{6}}(e_{014} + e_{024} - 2e_{034}).$$

Since

$$\partial\gamma_1 = \frac{1}{\sqrt{2}}(e_{01} + e_{14} - e_{02} - e_{24}),$$
$$\partial\gamma_2 = \frac{1}{\sqrt{6}}(e_{01} + e_{04} + e_{02} + e_{24} - 2e_{03} - 2e_{34}),$$

we obtain

$$C = (\langle e_{ij}, \partial\gamma_n \rangle) = \begin{pmatrix} & e_{01} & e_{14} & e_{02} & e_{24} & e_{03} & e_{34} \\ \partial\gamma_1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \partial\gamma_2 & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}$$

and

$$C^T C = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}.$$

Now we compute $B^T B$:

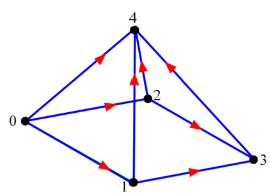
$$B^T B = \left([e_{ij}, e_{i'j'}]\right) = \begin{pmatrix} 2 & -1 & 1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \end{pmatrix},$$

whence

$$\text{matrix of } \Delta_1 = B^T B + C^T C$$
$$= \begin{pmatrix} \frac{8}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & \frac{8}{3} \end{pmatrix}.$$

The eigenvalues of Δ_1 are $(2_4, 3, 5)$.

Example 6.15. Consider the following pyramid:



For this digraph $V = 5,\ E = 8,\ |\Omega_2| = 5$, and

$$\Omega_2 = \langle e_{014}, e_{024}, e_{134}, e_{234}, e_{013} - e_{023} \rangle .$$

We have then

$$B^T B = \left([ij, i'j']\right)$$
$$= \begin{pmatrix} & e_{01} & e_{02} & e_{13} & e_{23} & e_{04} & e_{14} & e_{24} & e_{34} \\ e_{01} & 2 & 1 & -1 & 0 & 1 & -1 & 0 & 0 \\ e_{02} & 1 & 2 & 0 & -1 & 1 & 0 & -1 & 0 \\ e_{13} & -1 & 0 & 2 & 1 & 0 & 1 & 0 & -1 \\ e_{23} & 0 & -1 & 1 & 2 & 0 & 0 & 1 & -1 \\ e_{04} & 1 & 1 & 0 & 0 & 2 & 1 & 1 & 1 \\ e_{14} & -1 & 0 & 1 & 0 & 1 & 2 & 1 & 1 \\ e_{24} & 0 & -1 & 0 & 1 & 1 & 1 & 2 & 1 \\ e_{34} & 0 & 0 & -1 & -1 & 1 & 1 & 1 & 2 \end{pmatrix},$$

$$C = \begin{pmatrix} & e_{01} & e_{02} & e_{13} & e_{23} & e_{04} & e_{14} & e_{24} & e_{34} \\ e_{014} & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ e_{024} & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\ e_{134} & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ e_{234} & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \\ \frac{1}{\sqrt{2}}(e_{013} - e_{023}) & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C^T C = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -1 & 1 & 0 & 0 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & -1 & 0 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} & 0 & -1 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} & 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & 0 & 2 & -1 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 & 2 \end{pmatrix},$$

$$\text{matrix of } \Delta_1 = B^T B + C^T C$$
$$= \begin{pmatrix} \frac{7}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{7}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{7}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{7}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 4 \end{pmatrix}.$$

The eigenvalues of Δ_1 are $(3_5,\ 5_3)$.

Example 6.16. Let G be an $(n-1)$-simplex, that is, the vertices are $\{0, 1, \ldots, n-1\}$ and

$$i \to j \Leftrightarrow i < j.$$

Let us show that

$$A := \text{matrix of } \Delta_1 = \operatorname{diag}(n).$$

Let ij and $i'j'$ be two arrows. Then the $(ij, i'j')$-entry of A is

$$\begin{aligned} A_{ij,i'j'} &= \left(B^T B\right)_{ij,i'j'} + \left(C^T C\right)_{ij,i'j'} \\ &= [ij, i'j'] + \sum_n [ij, \gamma_n] [i'j', \gamma_n], \end{aligned} \tag{6.142}$$

where $\{\gamma_n\}$ is an orthonormal basis of Ω_2, which we may take to consist of all triangles in G.

If $ij = i'j'$ then $[ij, i'j'] = 2$. Since the arrow ij belongs to $(n-2)$ triangles γ_n, we obtain

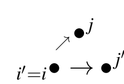
$$A_{ij,ij} = 2 + (n-2) = n,$$

that is, all the diagonal entries of Δ_1 are equal to n. It remains to show that if $ij \neq i'j'$ then

$$A_{ij,i'j'} = 0. \tag{6.143}$$

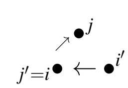
If ij and $i'j'$ have no common vertex then they cannot belong to the same triangle γ_n and, hence, all the terms in (6.142) vanish.

Suppose $i' = i$ and $j' \neq j$:



Then $[ij, i'j'] = 1$ while $[ij, \gamma_n][i'j', \gamma_n]$ is nonzero only when γ_n is the triangle formed by i, j, j'. In this case the arrows ij and $i'j'$ have opposite orientations with respect to γ_n, whence $[ij, \gamma_n][i'j', \gamma_n] = -1$ and (6.143) follows.

Suppose $j' = i$ and $i' \neq j$:



Then $[ij, i'j'] = -1$ while $[ij, \gamma_n][i'j', \gamma_n]$ is nonzero only when γ_n is the triangle $i'ij$. In this case the arrows ij and $i'j'$ have the same orientation with respect to γ_n, whence $[ij, \gamma_n][i'j', \gamma_n] = 1$ and again (6.143) follows.

The cases $j = i'$ and $j = j'$ are similar.

Problem 6.17. *Describe all the digraphs for which Δ_1 has only one eigenvalue.*

Problem 6.18. *Devise a program for computing the matrix and spectrum of Δ_1 for large digraphs.*

6.5 Trace of Δ_1

Recall that by (6.139)

$$\operatorname{trace}\Delta_0 = \sum_{i\in V} \deg(i) = 2E,$$

where E denotes the number of arrows. Here is a similar result for the trace of Δ_1.

Theorem 6.19. *Let T be the number of triangles in Ω_2, S be the number of linearly independent squares in Ω_2, and D be the number of double arrows $a \rightleftarrows b$. Then*

$$\operatorname{trace}\Delta_1 = 2E + 3T + 2S + 4D. \tag{6.144}$$

By a square here we mean an allowed 2-path $e_{abc} - e_{ab'c}$ such that $a \neq c$ and $a \not\to c$.

For example, for the pyramid from Example 6.15 we have $E = 8$, $T = 4$, $S = 1$ and $D = 0$, whence

$$\operatorname{trace}\Delta_1 = 2\cdot 8 + 3\cdot 4 + 2\cdot 1 = 30,$$

which matches the sum of the eigenvalues as well as the sum of the diagonal values of the matrix of Δ_1 as determined there.

Proof. Let $\{\gamma_n\}$ be an orthogonal basis for Ω_2. Let us first prove that

$$\operatorname{trace}\Delta_1 = 2E + \sum_n \frac{\|\partial\gamma_n\|^2}{\|\gamma_n\|^2}. \tag{6.145}$$

By (6.137), $\operatorname{trace}\Delta_1 = \operatorname{trace} B^T B + \operatorname{trace} C^T C$. As we have seen above (see (6.141)), all the diagonal entries of $B^T B$ are equal to 2 so that

$$\operatorname{trace} B^T B = 2E.$$

Let us compute $\operatorname{trace} C^T C$. Without loss of generality assume that the basis $\{\gamma_n\}$ is orthonormal basis. Let $\{\alpha_i\}$ be the sequence of all arrows. Since $\{\alpha_i\}$ is an orthonormal basis for Ω_1, we have by (6.136)

$$C = \left(\langle\partial\gamma_n, \alpha_i\rangle\right)_{n,i}$$

and, hence,

$$\left(C^T C\right)_{ij} = \sum_n \langle\partial\gamma_n, \alpha_i\rangle \langle\partial\gamma_n, \alpha_j\rangle.$$

It follows that

$$\operatorname{trace} C^T C = \sum_i \sum_n \langle\partial\gamma_n, \alpha_i\rangle^2 = \sum_n \sum_i \langle\partial\gamma_n, \alpha_i\rangle^2 = \sum_n \|\partial\gamma_n\|^2,$$

whence (6.145) follows.

As we know, Ω_2 has a basis $\{\gamma_n\}$ that consists of triangles, squares and double arrows. The only non-orthogonal pairs in this basis are pairs of squares containing the same elementary 2-path, like $e_{abc} - e_{ab'c}$ and $e_{abc} - e_{ab''c}$. Assume first that the entire basis $\{\gamma_n\}$ is orthogonal (which is equivalent to absence of multisquares).

A double arrow $a \rightleftarrows b$ gives two elements of the basis $\{\gamma_n\}$: e_{aba} and e_{bab}. If $\gamma_n = e_{aba}$ then

$$\|\gamma_n\|^2 = 1, \quad \partial\gamma_n = e_{ba} + e_{ab}, \quad \|\partial\gamma_n\|^2 = 2$$

and

$$\frac{\|\partial\gamma_n\|^2}{\|\gamma_n\|^2} = 2.$$

The same is true for $\gamma_n = e_{bab}$ so that each double arrow contributes 4 to the sum

$$\sum_n \frac{\|\partial\gamma_n\|^2}{\|\gamma_n\|^2}. \tag{6.146}$$

If γ_n is a triangle e_{abc} then

$$\|\gamma_n\|^2 = 1, \quad \partial\gamma_n = e_{bc} - e_{ac} + e_{ab}, \quad \|\partial\gamma_n\|^2 = 3,$$

whence

$$\frac{\|\partial\gamma_n\|^2}{\|\gamma_n\|^2} = 3,$$

so that each triangle contributes 3 to the sum (6.146).

If γ_n is a square $e_{abc} - e_{ab'c}$ then

$$\|\gamma_n\|^2 = 2, \quad \partial\gamma_n = e_{ab} + e_{bc} - e_{ab'} - e_{b'c}, \quad \|\partial\gamma_n\|^2 = 4,$$

so that

$$\frac{\|\partial\gamma_n\|^2}{\|\gamma_n\|^2} = 2,$$

so that each square contributes 2 to the sum (6.146). Hence, we obtain that the sum (6.146) is equal to $3T + 2S + 4D$, which proves (6.144) in this case.

In the general case G may contain multisquares. Assume that G contains the following m-square

$$a, \ \{b_k\}_{k=0}^m, \ c$$

which gives rise to m linearly independent squares:

$$e_{ab_0c} - e_{ab_1c}, \ e_{abc} - e_{ab_2c}, \ \ldots, \ e_{abc} - e_{ab_mc}. \tag{6.147}$$

The sequence (6.147) is not orthogonal, and its orthogonalization gives the following sequence:

$$\begin{aligned}
\omega_1 &= e_{ab_0c} - e_{ab_1c} \\
\omega_2 &= e_{ab_0c} + e_{ab_1c} - 2e_{ab_2c} \\
&\ldots \\
\omega_k &= e_{ab_0c} + \ldots + e_{ab_{k-1}c} - ke_{ab_kc} \\
&\ldots \\
\omega_m &= e_{ab_0c} + \ldots + e_{ab_{m-1}c} - me_{ab_mc}
\end{aligned}$$

(cf. Example 5.16). We have

$$\partial \omega_k = \left(e_{ab_0} + e_{b_0c}\right) + \ldots + \left(e_{ab_{k-1}} + e_{b_{k-1}c}\right) - k\left(e_{ab_k} + e_{b_kc}\right)$$
$$\|\partial \omega_k\|^2 = 2k + 2k^2, \quad \|\omega_k\|^2 = k + k^2,$$

whence

$$\frac{\|\partial \omega_k\|^2}{\|\omega_k\|^2} = 2.$$

Hence, each ω_k contributes 2 to the sum (6.146), which completes the proof. □

Since the sum of all eigenvalues is $\operatorname{trace}\Delta_1$ and the eigenvalue 0 has the multiplicity β_1, we obtain that the average of the positive eigenvalues is

$$\lambda_{average} = \frac{\operatorname{trace}\Delta_1}{E - \beta_1}.$$

6.6 An Upper Bound on $\lambda_{\max}(\Delta_1)$

Denote by $\lambda_{\max}(A)$ the maximal eigenvalue of a symmetric operator A. Recall that, by Proposition 6.2,

$$\lambda_{\max}(\Delta_0) \le 2 \max_i \deg(i).$$

For any arrow $i \to j$ in G denote by $\deg_\Delta(ij)$ the number of triangles containing the arrow $i \to j$, and by $\deg_\square(ij)$ the number of squares containing $i \to j$.

Theorem 6.20. *Assume that there is an orthogonal basis $\{\gamma_n\}$ for Ω_2 that consists of triangles and squares. Then*

(6.148)
$$\lambda_{\max}(\Delta_1) \le 2 \max_i \deg(i) + 3 \max_{i\to j} \deg_\Delta(ij) + 2 \max_{i \to j} \deg_\square(ij).$$

Proof. Recall that

$$\lambda_{\max}(\Delta_1) = \sup_{u \in \Omega_1 \setminus \{0\}} \left(\frac{\|\partial u\|^2}{\|u\|^2} + \frac{\|\partial^* u\|^2}{\|u\|^2} \right).$$

Since the operators $\partial : \Omega_1 \to \Omega_0$ and $\partial^* : \Omega_0 \to \Omega_1$ are dual, they have the same norm. The norm of the latter was estimated in the proof of Proposition 6.2 (cf. (6.131)), whence we obtain the same estimate for the norm of the former, that is, for any non-zero $u \in \Omega_1$,

$$\frac{\|\partial u\|^2}{\|u\|^2} \le 2 \max_{i \in V} \deg(i).$$

Let us prove that

$$\frac{\|\partial^* u\|^2}{\|u\|^2} \le 3 \max_{i \to j} \deg_\Delta(ij) + 2 \max_{i \to j} \deg_\square(ij). \tag{6.149}$$

Let $u = \sum_{i \to j} u^{ij} e_{ij}$ and, hence,

$$\|u\|^2 = \sum_{i \to j} (u^{ij})^2$$

Using the basis $\{\gamma_n\}$ in Ω_2, we obtain

$$\|\partial^* u\|^2 = \sum_n \frac{\langle \partial^* u, \gamma_n\rangle^2}{\|\gamma_n\|^2} = \sum_n \frac{\langle u, \partial\gamma_n\rangle^2}{\|\gamma_n\|^2}.$$

If γ_n is a triangle e_{abc} then $\|\gamma_n\| = 1$,

$$\langle u, \partial\gamma_n\rangle = \langle u, e_{bc} - e_{ac} + e_{ab}\rangle = u^{bc} - u^{ac} + u^{ab},$$
$$\langle u, \partial\gamma_n\rangle^2 \le 3\left((u^{bc})^2 + (u^{ac})^2 + (u^{ab})^2\right).$$

Summing up over all triangles γ_n and using that any arrow $i \to j$ occurs in $\deg_\Delta(ij)$ triangles, we obtain

$$\sum_{n:\gamma_n \text{ is a triangle}} \frac{\langle u, \partial\gamma_n\rangle^2}{\|\gamma_n\|^2} \le 3 \sum_{i \to j} (u^{ij})^2 \deg_\Delta(ij)$$
$$\le 3\|u\|^2 \max_{i \to j} \deg_\Delta(ij). \tag{6.150}$$

Let now γ_n be a square $e_{abc} - e_{ab'c}$ (such that $a \not\to c$). Then $\|\gamma_n\|^2 = 2$,

$$\langle u, \partial\gamma_n\rangle = \langle u, e_{ab} + e_{bc} - e_{ab'} + e_{b'c}\rangle = u^{ab} + u^{bc} - u^{ab'} - u^{b'c},$$
$$\langle u, \partial\gamma_n\rangle^2 \le 4\left((u^{ab})^2 + (u^{bc})^2 + (u^{ab'})^2 + (u^{b'c})^2\right).$$

Summing up over all squares γ_n and using that any arrow $i \to j$ occurs in $\deg_\square(ij)$ squares, we obtain

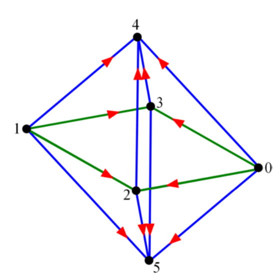
$$\sum_{n:\gamma_n \text{ is a square}} \frac{\langle u, \partial\gamma_n\rangle^2}{\|\gamma_n\|^2} \le 2 \sum_{i \to j} (u^{ij})^2 \deg_\square(ij)$$
$$\le 2\|u\|^2 \max_{i \to j} \deg_\square(ij). \tag{6.151}$$

Adding up (6.150) and (6.151), we obtain (6.149). □

Problem 6.21. *How sharp is the upper bound on $\lambda_{\max}(\Delta_1)$ in (6.148)? Is it attained on some digraphs? Extend (6.148) to the general case when a basis of triangles and squares requires orthogonalization.*

6.7 Examples of Computations of spec Δ_1

Example 6.22. Consider an octahedron based on a diamond:



For this digraph $V = 6$, $E = 12$, $|\Omega_2| = 8$. The space Ω_2 is generated by 8 triangles:

$$\Omega_2 = \langle e_{024}, e_{025}, e_{034}, e_{035}, e_{124}, e_{125}, e_{134}, e_{135}\rangle.$$

Hence, $T = 8$, $S = 0$, and we obtain

$$\operatorname{trace}\Delta_1 = 2E + 3T = 48.$$

Since $\beta_1 = 0$, it follows that

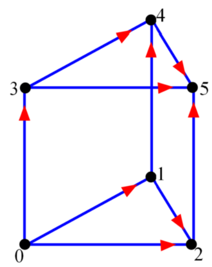
$$\lambda_{average} = \frac{\operatorname{trace}\Delta_1}{E - \beta_1} = \frac{48}{12} = 4.$$

The eigenvalues of Δ_1 are

$$(2_3, 4_6, 6_3),$$

where the subscript denotes the multiplicity.

Example 6.23. Consider a prism as in Example 5.24:



Since $E = 9,\ T = 2,\ S = 3$, we have

$$\operatorname{trace}\Delta_1 = 2E + 3T + 2S = 30$$

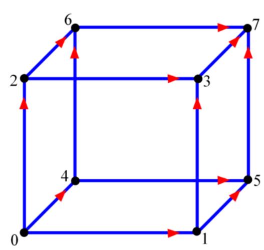
and

$$\lambda_{average} = \frac{\operatorname{trace}\Delta_1}{E - \beta_1} = \frac{30}{9}.$$

The eigenvalues of Δ_1 are

$$(2,\ (\tfrac{5}{2})_2,\ 3_3,\ 4,\ 5_2).$$

Example 6.24. Consider a 3-cube:



We have $V = 8,\ E = 12,\ |\Omega_2| = 6,\ H_p = \{0\}$ for $p \geq 1$. Space Ω_2 is generated by 6 squares, so that

$$S = 6 \quad \text{and} \quad T = 0.$$

Hence, we obtain by (6.144)

$$\operatorname{trace}\Delta_1 = 2E + 2S = 2 \cdot 12 + 2 \cdot 6 = 36.$$

Since $\beta_1 = 0$, we obtain

$$\lambda_{average} = \frac{\operatorname{trace}\Delta_1}{E - \beta_1} = 3.$$

In fact, the eigenvalues of Δ_1 on a 3-cube are

$$(2_6, 3_2, 4_3, 6).$$

Example 6.25. Let G be the n-cube, that is,

$$G = I^{n\Box} = \underbrace{I\Box I\Box \ldots \Box I}_{n \text{ times}}$$

where $I = \{0 \to 1\}$ (see Subsection 3.4). Then

$$V = 2^n, \quad E = n2^{n-1}, \quad S = |\Omega_2| = 2^{n-3} n(n-1)$$

and $T = 0$. Hence,

$$\operatorname{trace}\Delta_1 = 2E + 2S = 2^{n-2} n(n+3)$$

and

$$\lambda_{average} = \frac{\operatorname{trace}\Delta_1}{E - \beta_1} = \frac{2^{n-2} n(n+3)}{n2^{n-1}} = \frac{n+3}{2}.$$

For example, for the 4-cube we obtain

$$\operatorname{trace}\Delta_1 = 2^2 \cdot 4 \cdot 7 = 112.$$

The eigenvalues of Δ_1 on the 4-cube are

$$(2_{10}, 3_8, 4_9, 6_4,\ 8).$$

For the 5-cube we obtain

$$\operatorname{trace}\Delta_1 = 2^3 \cdot 5 \cdot 8 = 320.$$

The eigenvalues of Δ_1 on the 5-cube are

$$(2_{15}, 3_{20}, 4_{25}, 5_4, 6_{10}, 8_5, 10).$$

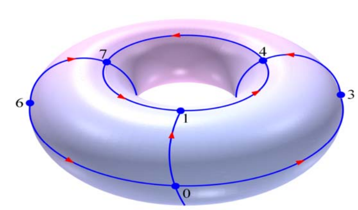
Problem 6.26. *Determine the full spectrum of* Δ_1 *on the n-cube. In particular, prove that*

$$\lambda_{\max} = 2n \quad \textit{and} \quad \lambda_{\min} = 2_{\frac{n(n+1)}{2}}.$$

Prove that $\operatorname{spec}\Delta_1$ *consists of all even integers from* 2 *to* $2n$ *and of all odd integers from* 3 *to* n.

The difficulty here is that the method of separation of variables does not work for Δ_1 on Cartesian products.

Example 6.27. Consider the 2-torus $G = T\Box T$ where $T = \{0 \to 1 \to 2 \to 0\}$.



Here $V = 9,\ E = 18,\ |\Omega_2| = 9,\ |H_1| = 2$. Space Ω_2 is generated by 9 squares, whence

$$\operatorname{trace}\Delta_1 = 2 \cdot 18 + 2 \cdot 9 = 54.$$

The eigenvalues of Δ_1 on the 2-torus are

$$(0_2,\ (\tfrac{3}{2})_4,\ 3_8,\ 6_4).$$

For the 3-torus $G = T^{\square 3}$ we have

$$E = 81, \quad S = |\Omega_2| = 81, \quad |H_1| = 3,$$

whence

$$\operatorname{trace}\Delta_1 = 2\cdot 81 + 2\cdot 81 = 324.$$

The eigenvalues of Δ_1 on the 3-torus are

$$(0_3, (\tfrac{3}{2})_{12}, 3_{30}, (\tfrac{9}{2})_{16}, 6_{12}, 9_8).$$

For the n-torus $G = T^{\square n}$ we have

$$E = n3^n, \ \ S = |\Omega_2| = \frac{n(n-1)}{2}3^n, \ \ |H_1| = n,$$

whence

$$\operatorname{trace}\Delta_1 = 2E + 2S = n(n+1)3^n$$

and

$$\lambda_{average} = (n+1)\frac{3^n}{3^n-1}.$$

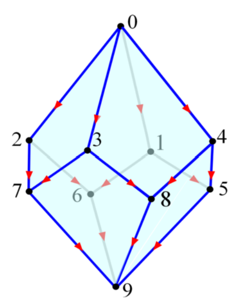
Problem 6.28. *Compute the full spectrum of Δ_1 for the n-torus. In particular, prove that*

$$\lambda_{\max} = (3n)_{2^n}.$$

In fact, $\lambda_{\min} = 0_n$, which is a consequence of $\beta_1 = n$.

Example 6.29. Consider a trapezohedron T_m (see Subsection 2.1 and Proposition 2.1).

For example, T_4 is shown here:



We have $V = 2m+2$, $E = 4m$, while Ω_2 is generated by $S = 2m$ squares. It follows that on T_m

$$\operatorname{trace}\Delta_1 = 2E + 2S = 12m.$$

Since $\beta_1 = 0$, we obtain

$$\lambda_{average} = \frac{\operatorname{trace}\Delta_1}{E-\beta_1} = \frac{12m}{4m} = 3.$$

In the case $m = 2$ the eigenvalues of Δ_1 are as follows:

$$(2, 3_5, \tfrac{7}{2} \pm \tfrac{1}{2}\sqrt{17}),$$

where

$$\lambda_{\min} = \frac{7}{2} - \frac{1}{2}\sqrt{17} = 1.438\ldots \quad \text{and}$$
$$\lambda_{\max} = \frac{7}{2} + \frac{1}{2}\sqrt{17} = 5.561\ldots.$$

In the case $m = 3$ the trapezohedron T_3 coincides with a 3-cube, and as was already shown above, the eigenvalues of Δ_1 are:

$$(2_6, 3_2, 4_3, 6).$$

In the case $m = 4$ the characteristic polynomial of Δ_1 is

$$(z-2)(z-3)^4(z-5)(z^2-9z+16)(z^2-4z+\tfrac{7}{2})^2(z^2-6z+7)^2,$$

and the eigenvalues of Δ_1 are

$$\{2, 3_4, 5, \ \tfrac{9}{2} \pm \tfrac{1}{2}\sqrt{17}, \ (2 \pm \tfrac{1}{2}\sqrt{2})_2, \ (3 \pm \sqrt{2})_2\},$$

with

$$\lambda_{\min} = 2 - \frac{1}{2}\sqrt{2} = 1.292\ldots \quad \text{and}$$
$$\lambda_{\max} = \frac{9}{2} + \frac{1}{2}\sqrt{17} = 6.561\ldots.$$

In the case $m = 5$ the characteristic polynomial of Δ_1 is

$$(z-2)(z-\tfrac{5}{2})^4(z-6)(z^2-10z+20)(z^2-7z+11)^2$$
$$\times (z^2-5z+5)^2(z^2-4z+\tfrac{11}{4})^2,$$

and the eigenvalues of Δ_1 are

$$\{2, (\tfrac{5}{2})_4, 6, 5 \pm \sqrt{5}, (\tfrac{7}{2} \pm \tfrac{1}{2}\sqrt{5})_2, (\tfrac{5}{2} \pm \tfrac{1}{2}\sqrt{5})_2, \ (2 \pm \tfrac{1}{2}\sqrt{5})_2\},$$

where

$$\lambda_{\min} = 2 - \frac{1}{2}\sqrt{5} = 0.881\ldots \quad \text{and} \quad \lambda_{\max} = 5 + \sqrt{5} = 7.236\ldots.$$

In the case $m = 6$ the characteristic polynomial of Δ_1 is

$$(z-2)^5(z-3)^7(z-4)^2(z-7)(z-8)(z^2-3z+\tfrac{3}{2})^2(z^2-6z+6)^2,$$

and the eigenvalues of Δ_1 are

$$(2_5, 3_7, 4_2, 7, 8, (\tfrac{3}{2} \pm \tfrac{1}{2}\sqrt{3})_2, (3 \pm \sqrt{3})_2),$$

where

$$\lambda_{\min} = \frac{3}{2} - \frac{1}{2}\sqrt{3} = 0.633\ldots \quad \text{and} \quad \lambda_{\max} = 8.$$

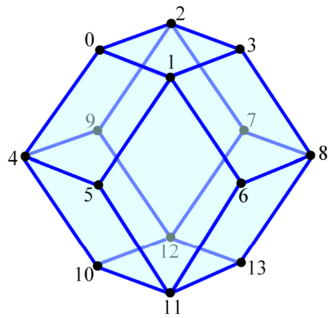
In the case $m = 7$ the characteristic polynomial of Δ_1 is

$$(z-2)(z-8)(z^2-12z+28)(z^3-6z^2+\tfrac{41}{4}z-\tfrac{29}{8})^2$$
$$\times (z^3-10z^2+31z-29)^2(z^3-7z^2+\tfrac{63}{4}z-\tfrac{91}{8})^2$$
$$\times (z^3-8z^2+19z-13)^2.$$

It has eigenvalues 2 and 8, and all other eigenvalues are irrational.

Problem 6.30. *Determine the full spectrum of Δ_1 on the trapezohedron T_m for any m. In particular, what are $\lambda_{\min}$ and $\lambda_{\max}$?*

Example 6.31. Consider a rhombic dodecahedron as in Example 5.25. The arrows go along edges from smaller numbers to larger ones.



Here $V = 14$, $E = 24$, $S = 12$, $T = 0$. It follows that

$$\operatorname{trace}\Delta_1 = 2E + 2S = 72,$$
$$\lambda_{average} = \frac{\operatorname{trace}\Delta_1}{E - \beta_1} = \frac{72}{24} = 3.$$

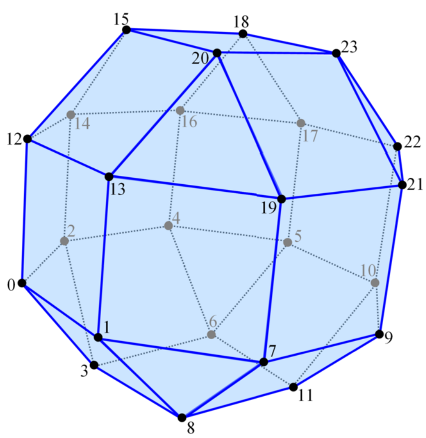
The characteristic polynomial of Δ_1 is

$$(z-1)^3 (z-2)^3 (z-3)^9 (z-4)^2 (z-7) (z^2 - 7z + 8)^3,$$

and the eigenvalues of Δ_1 are

$$(1_3, 2_3, 3_9, 4_2, 7, (\tfrac{7}{2} \pm \tfrac{\sqrt{17}}{2})_3).$$

Example 6.32. Consider a rhombicuboctahedron (see also Examples 5.17 and 5.28).



Here $V = 24$, $E = 48$, $|\Omega_2| = 26$. Ω_2 is generated by 8 triangles and 18 squares so that $T = 8$, $S = 18$. Hence, we obtain

$$\operatorname{trace}\Delta_1 = 2E + 3T + 2S = 156.$$

Since $\beta_1 = 0$, we have

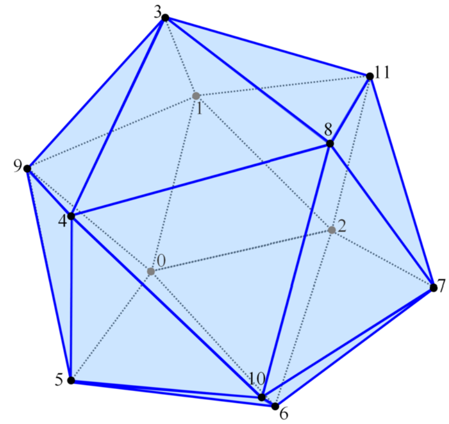
$$\lambda_{average} = \frac{\operatorname{trace}\Delta_1}{E - \beta_1} = \frac{156}{48} = 3.25.$$

A computation of the eigenvalues of Δ_1 gives

$$\lambda_{\min} = 0.518\ldots \quad \text{and} \quad \lambda_{\max} = 7_2.$$

There are many multiple eigenvalues: 1_3, 2_3, 3_3, 4_4, 5_6, etc. The full spectrum of Δ_1 is shown here:

Example 6.33. Consider the icosahedron as in Examples 1.16, 5.19.



We have here $V = 12$, $E = 30$, $|\Omega_2| = 25$. The space Ω_2 is generated by 20 triangles and 5 squares (cf. Example 5.19). Hence, $T = 20$, $S = 5$ and

$$\operatorname{trace}\Delta_1 = 2E + 3T + 2S = 130.$$

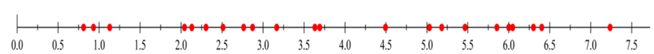
Since $\beta_1 = 0$, we have

$$\lambda_{average} = \frac{\operatorname{trace}\Delta_1}{E - \beta_1} = \frac{130}{30} = 4.333\ldots$$

Computation shows that

$$\lambda_{\min} = 0.810\ldots \quad \text{and} \quad \lambda_{\max} = (5 + \sqrt{5})_3.$$

Other multiple eigenvalues are 6_5 and $(5 - \sqrt{5})_3$. The full spectrum of Δ_1 is shown here:

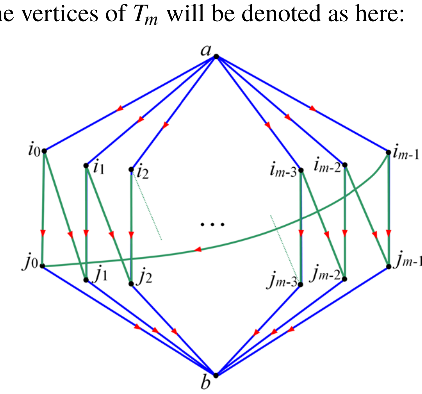


6.8 Eigenvalues of Δ_1 on Trapezohedron

Here we give a partial answer to Problem 6.30. Recall that the trapezohedra T_m were defined in Subsection 2.1.

Proposition 6.34. *For any $m \geq 2$, the operator Δ_1 on the trapezohedron T_m has eigenvalues $\lambda = 2$ and $\lambda = m + 1$.*

Proof. The vertices of T_m will be denoted as here:



Consider the following 1-paths on T_m:

$$v = e_{i_0 j_1} + e_{i_1 j_2} + \ldots + e_{i_{m-1} j_0} - \left(e_{i_0 j_0} + e_{i_1 j_1} + \ldots + e_{i_{m-1} j_{m-1}}\right)$$

$$= \sum_{k=0}^{m-1} (e_{i_{k-1} j_k} - e_{i_k j_k}),$$

where the index k is regarded $\bmod m$, and

$$u = e_{ai_0} + e_{ai_1} + \ldots + e_{ai_{m-1}} - \left(e_{j_0 b} + e_{j_1 b} + \ldots + e_{j_{m-1} b}\right)$$
$$= \sum_{k=0}^{m-1} (e_{ai_k} - e_{j_k b}).$$

The 1-paths u and v are obviously allowed and, hence, ∂-invariant. We will prove that

$$\Delta_1 v = 2v \quad \text{and} \quad \Delta_1 u = (m+1)\, u,$$

which will settle the claim. We have clearly

$$\partial v = \sum_{k=0}^{m-1} (e_{j_k} - e_{i_{k-1}} - e_{j_k} + e_{i_k}) = 0,$$

and, hence, $\partial^* \partial v = 0$.

In order to compute $\partial^* v \in \Omega_2$ we use the following orthogonal basis in Ω_2 that consists of all $2m$ squares in T_m:

$$\varphi_k = e_{ai_{k-1} j_k} - e_{ai_k j_k} \quad \text{and} \quad \psi_k = e_{i_k j_k b} - e_{i_k j_{k+1} b},$$

where $k = 0, \ldots, m-1$ (cf. Proposition 2.1). We have for any k

$$\langle \partial^* v, \varphi_k \rangle = \langle v, \partial \varphi_k \rangle = \left\langle v, e_{i_{k-1} j_k} + e_{ai_{k-1}} - e_{i_k j_k} - e_{ai_k} \right\rangle = 2,$$
$$\langle \partial^* v, \psi_k \rangle = \langle v, \partial \psi_k \rangle = \left\langle v, e_{j_k b} + e_{i_k j_k} - e_{j_{k+1} b} - e_{i_k j_{k+1}} \right\rangle = -2,$$

which together with $\|\varphi_k\|^2 = \|\psi_k\|^2 = 2$ implies that

$$\partial^* v = \sum_{k=0}^{m-1} (\varphi_k - \psi_k).$$

Hence, we obtain

$$\begin{aligned}
\Delta_1 v = \partial \partial^* v &= \sum_{k=0}^{m-1} (\partial \varphi_k - \partial \psi_k) \\
&= \sum_{k=0}^{m-1} (e_{i_{k-1} j_k} + e_{ai_{k-1}} - e_{i_k j_k} - e_{ai_k}) \\
&\quad - \sum_{k=0}^{m-1} (e_{j_k b} + e_{i_k j_k} - e_{j_{k+1} b} - e_{i_k j_{k+1}}) \\
&= 2 \sum_{k=0}^{m-1} (e_{i_{k-1} j_k} - e_{i_k j_k}) = 2v.
\end{aligned}$$

Next, let us compute $\partial^* u$. We have for any k,

$$\langle \partial^* u, \varphi_k \rangle = \langle u, \partial \varphi_k \rangle = \left\langle u, e_{i_{k-1} j_k} + e_{ai_{k-1}} - e_{i_k j_k} - e_{ai_k} \right\rangle = 0,$$
$$\langle \partial^* u, \psi_k \rangle = \langle u, \partial \psi_k \rangle = \left\langle u, e_{j_k b} + e_{i_k j_k} - e_{j_{k+1} b} - e_{i_k j_{k+1}} \right\rangle = 0,$$

whence $\partial^* u = 0$ and, hence, $\partial \partial^* u = 0$. It remains to compute $\partial^* \partial u$. We have

$$\partial u = \sum_{k=0}^{m-1} (e_{i_k} - e_a - e_b + e_{j_k}) = \sum_{k=0}^{m-1} (e_{i_k} + e_{j_k}) - m(e_a + e_b).$$

For any 0-path e_i and any 1-path $e_{\alpha\beta}$ we have

$$\left\langle \partial^* e_i, e_{\alpha\beta} \right\rangle = \left\langle e_i, \partial e_{\alpha\beta} \right\rangle = \left\langle e_i, e_\beta - e_\alpha \right\rangle = \delta_{i\beta} - \delta_{i\alpha}$$

whence

$$\partial^* e_i = \sum_{\alpha \to \beta} \left(\delta_{i\beta} - \delta_{i\alpha}\right) e_{\alpha\beta} = \sum_{\alpha \to i} e_{\alpha i} - \sum_{i \to \beta} e_{i\beta}.$$

It follows that

$$\partial^* e_{i_k} = e_{ai_k} - e_{i_k j_k} - e_{i_k j_{k+1}},$$
$$\partial^* e_{j_k} = e_{i_{k-1} j_k} + e_{i_k j_k} - e_{j_k b},$$
$$\partial^* e_a = -\sum_{k=0}^{m-1} e_{ai_k}, \quad \partial^* e_b = \sum_{k=0}^{m-1} e_{j_k b},$$

whence

$$\begin{aligned}
\Delta_1 u = \partial^* \partial u &= \sum_{k=0}^{m-1} (e_{ai_k} - e_{i_k j_k} - e_{i_k j_{k+1}} + e_{i_{k-1} j_k} + e_{i_k j_k} - e_{j_k b}) \\
&\quad + m \sum_{k=0}^{m-1} (e_{ai_k} - e_{j_k b}) \\
&= (m+1) \sum_{k=0}^{m-1} (e_{ai_k} - e_{j_k b}) = (m+1)\, u,
\end{aligned}$$

which finishes the proof. □

6.9 Spectrum of Δ_p on Join

In this section we use the augmented chain complex (3.46):

$$(6.152) \quad \mathbb{K} \overset{\partial}{\leftarrow} \Omega_0 \overset{\partial}{\leftarrow} \Omega_1 \overset{\partial}{\leftarrow} \ldots \overset{\partial}{\leftarrow} \Omega_{p-1} \overset{\partial}{\leftarrow} \Omega_p \overset{\partial}{\leftarrow} \ldots$$

Denote by $\widetilde{\Delta}_p$ the Hodge Laplacian associated with this complex. Of course, $\widetilde{\Delta}_p$ coincides with Δ_p for $p \geq 1$ but is different for $p = -1$ and $p = 0$.

For example, we have for the chain complex (6.152)

$$\langle \partial^* e, e_i \rangle = \langle e, \partial e_i \rangle = \langle e, e \rangle = 1$$

so that

$$\partial^* e_i = \sigma := \sum_{k \in V} e_k$$

whence

$$\widetilde{\Delta}_{-1} e = \partial \partial^* e = \partial \sigma = |V|\, e.$$

In particular,

$$\operatorname{spec} \widetilde{\Delta}_{-1} = \{|V|\}.$$

In the case $p = 0$ we have

$$\widetilde{\Delta}_0 e_i = \partial^* \partial e_i + \partial \partial^* e_i = \partial^* e + \Delta_0 e_i = \Delta_0 e_i + \sigma,$$

that is,

$$(\widetilde{\Delta}_0 e_i)^j = (\Delta_0 e_i)^j + 1.$$

Therefore, the matrix of $\widetilde{\Delta}_0$ is obtained from the matrix of Δ_0 by adding 1 to *each* entry. For any $u \in \Omega_0$ we have

$$\widetilde{\Delta}_0 u = \Delta_0 u + \left(\sum_{k \in V} u^k \right) \sigma.$$

The advantage of using the chain complex (6.152) lies in the following statements.

Lemma 6.35 ([23, Lemma 5.5]). *Let X, Y be two digraphs. Then, for $u \in \Omega_p(X)$, $v \in \Omega_q(Y)$ and $r = p + q + 1$, we have*

$$\widetilde{\Delta}_r(u * v) = (\widetilde{\Delta}_p u) * v + u * \widetilde{\Delta}_q v. \tag{6.153}$$

Theorem 6.36. *Let X, Y be two digraphs. We have for any $r \geq 0$*

$$\operatorname{spec} \widetilde{\Delta}_r(X * Y) = \bigsqcup_{\{p,q \geq -1 : p+q = r-1\}} \left(\operatorname{spec} \widetilde{\Delta}_p(X) + \operatorname{spec} \widetilde{\Delta}_q(Y) \right). \tag{6.154}$$

Here we denote by $\operatorname{spec} A$ a sequence of all the eigenvalues of the operator A counted with multiplicities. The sum of two such sequences consists of all pairwise sums of the elements of the sequences, and the disjoint union of sequences means the union of all sequences, summing up the multiplicities. In particular, if one of the sequences is empty then its sum with another sequence is also empty.

Proof of Theorem 6.36. Observe that if $u \in \Omega_p(X)$ and $v \in \Omega_q(Y)$ are eigenvectors such that

$$\widetilde{\Delta}_p u = \lambda u \quad \text{and} \quad \widetilde{\Delta}_q v = \mu v,$$

then we have by (6.153) for $r = p + q + 1$:

$$\widetilde{\Delta}_r(u * v) = (\widetilde{\Delta}_p u) * v + u * \widetilde{\Delta}_q v = (\lambda + \mu)(u * v),$$

that is, $u * v$ is an eigenvector of $\widetilde{\Delta}_r$ on $X * Y$ with the eigenvalue $\lambda + \mu$.

In each $\Omega_p(X)$ there is a basis that consists of eigenvectors of $\widetilde{\Delta}_p$; denote by $\{u_k\}$ the union of all such bases of $\Omega_p(X)$ across all $p \geq -1$, with the corresponding eigenvalues $\{\lambda_k\}$. Let $\{v_l\}$ be a similar sequence on Y with the eigenvalues $\{\mu_l\}$. By Theorem 3.12, we have, for any $r \geq -1$,

$$\Omega_r(X * Y) \cong \bigoplus_{\{p,q \geq -1 : p+q = r-1\}} \left(\Omega_p(X) \otimes \Omega_q(Y) \right),$$

that is, $\Omega_r(X * Y)$ has a basis

$$\{u_k * v_l : |u_k| + |v_l| = r - 1\}.$$

The elements of this basis are the eigenvectors of $\widetilde{\Delta}_r$ on $X * Y$ with eigenvalues $\lambda_k + \mu_l$, whence (6.154) follows. ☐

In particular, for $r = 0$ we have

$$\begin{aligned}
\operatorname{spec} \widetilde{\Delta}_0(X * Y) &= \left(\operatorname{spec} \widetilde{\Delta}_{-1}(X) + \operatorname{spec} \widetilde{\Delta}_0(Y) \right) \\
&\quad \sqcup \left(\operatorname{spec} \widetilde{\Delta}_0(X) + \operatorname{spec} \widetilde{\Delta}_{-1}(Y) \right) \\
&= \left(\{|X|\} + \operatorname{spec} \widetilde{\Delta}_0(Y) \right) \\
&\quad \sqcup \left(\operatorname{spec} \widetilde{\Delta}_0(X) + \{|Y|\} \right)
\end{aligned} \tag{6.155}$$

and for $r = 1$

$$\begin{aligned}
\operatorname{spec} \widetilde{\Delta}_1(X * Y) &= \left(\operatorname{spec} \widetilde{\Delta}_{-1}(X) + \operatorname{spec} \widetilde{\Delta}_1(Y) \right) \\
&\quad \sqcup \left(\operatorname{spec} \widetilde{\Delta}_1(X) + \operatorname{spec} \widetilde{\Delta}_{-1}(Y) \right) \\
&\quad \sqcup \left(\operatorname{spec} \widetilde{\Delta}_0(X) + \operatorname{spec} \widetilde{\Delta}_0(Y) \right).
\end{aligned}$$

Since $\widetilde{\Delta}_1 = \Delta_1$, we conclude that

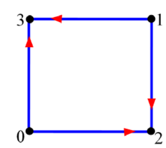
$$\begin{aligned}
\operatorname{spec} \Delta_1(X * Y) &= (\{|X|\} + \operatorname{spec} \Delta_1(Y)) \\
&\quad \sqcup (\operatorname{spec} \Delta_1(X) + \{|Y|\}) \\
&\quad \sqcup \left(\operatorname{spec} \widetilde{\Delta}_0(X) + \operatorname{spec} \widetilde{\Delta}_0(Y) \right).
\end{aligned} \tag{6.156}$$

6.10 Spectrum of Δ_1 on Digraph Spheres

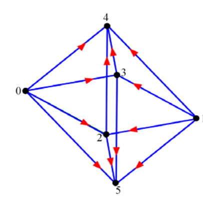
Consider a family $\{S^n\}_{n=0}^{\infty}$ of digraphs that is defined inductively as follows: $S^0 = \{\cdot, \cdot\}$ and

$$S^{n+1} = \mathsf{sus}_2 S^n.$$

For example, S^1 is a diamond and S^2 the octahedron (see also Example 3.10):



S^1 is a diamond



S^2 is an octahedron

The digraph S^n can be regarded as an analogue of an n-sphere. In the notation of Subsection 5.9, we have $S^n = D_2^{*(n+1)}$.

Proposition 6.37. *We have for all $n \geq 0$*

$$\operatorname{spec} \Delta_1(S^n) = \left\{ 2(n-1)_{\frac{n(n+1)}{2}},\ (2n)_{n(n+1)},\ 2(n+1)_{\frac{n(n+1)}{2}} \right\}. \tag{6.157}$$

Example 6.38. For example, we have

$$\operatorname{spec} \Delta_1(S^1) = \{0, 2_2, 4\}$$

and

$$\operatorname{spec} \Delta_1(S^2) = \{2_3, 4_6,\ 6_3\}$$

as we have seen above. For $n = 3$ we obtain from (6.157)

$$\operatorname{spec} \Delta_1(S^3) = \{4_6,\ 6_{12},\ 8_6\}.$$

Proof of Proposition 6.37. Let us first prove by induction that

$$\operatorname{spec} \widetilde{\Delta}_0(S^n) = \left\{ (2n)_{n+1}, (2n+2)_{n+1} \right\}. \tag{6.158}$$

For $n = 0$ we have

$$\operatorname{spec} \widetilde{\Delta}_0(S^0) = \{0, 2\}$$

which verifies (6.158) for $n=0$. For the induction step from $n-1$ to n, let us observe that $S^n = S^0 * S^{n-1}$, $\left|S^0\right| = 2$ and $\left|S^{n-1}\right| = 2n$, so that we obtain by (6.155)

$$\begin{aligned}
\operatorname{spec}\widetilde{\Delta}_0(S^n) &= \left(\left\{\left|S^0\right|\right\} + \operatorname{spec}\widetilde{\Delta}_0(S^{n-1})\right) \\
&\quad \sqcup \left(\operatorname{spec}\widetilde{\Delta}_0(S^0) + \left\{\left|S^{n-1}\right|\right\}\right) \\
&= \left(\{2\} + \operatorname{spec}\widetilde{\Delta}_0(S^{n-1})\right) \sqcup (\{0,2\} + \{2n\}) \\
&= \left(\{2\} + \operatorname{spec}\widetilde{\Delta}_0(S^{n-1})\right) \sqcup (\{2n, 2n+2\}).
\end{aligned}$$

By the induction hypothesis we have

$$\operatorname{spec}\widetilde{\Delta}_0(S^{n-1}) = \{(2n-2)_n, (2n)_n\}, \tag{6.159}$$

whence

$$\begin{aligned}
\operatorname{spec}\widetilde{\Delta}_0(S^n) &= \{(2n)_n, (2n+2)_n\} \sqcup \{2n, 2n+2\} \\
&= \{(2n)_{n+1}, (2n+2)_{n+1}\},
\end{aligned}$$

which was to be proved.

Let us prove (6.157). For $n=0$ we have

$$\operatorname{spec}\Delta_1(S^0) = \emptyset,$$

which matches (6.157). For the induction step from $n-1$ to n, we obtain by (6.156) and (6.159)

$$\begin{aligned}
\operatorname{spec}\Delta_1(S^n) &= \left(\left\{\left|S^0\right|\right\} + \operatorname{spec}\Delta_1(S^{n-1})\right) \\
&\quad \sqcup \left(\operatorname{spec}\Delta_1(S^0) + \left\{\left|S^{n-1}\right|\right\}\right) \\
&\quad \sqcup \left(\operatorname{spec}\widetilde{\Delta}_0(S^0) + \operatorname{spec}\widetilde{\Delta}_0(S^{n-1})\right) \\
&= \left(\{2\} + \operatorname{spec}\Delta_1(S^{n-1})\right) \\
&\quad \sqcup \left(\{0,2\} + \{(2n-2)_n, (2n)_n\}\right) \\
&= \left(\{2\} + \operatorname{spec}\Delta_1(S^{n-1})\right) \\
&\quad \sqcup \{(2n-2)_n, (2n)_{2n}, (2n+2)_n\}.
\end{aligned}$$

Using the induction hypothesis

$$\operatorname{spec}\Delta_1(S^{n-1}) = \left\{2(n-2)_{\frac{n(n-1)}{2}},\ 2(n-1)_{n(n-1)},\ (2n)_{\frac{n(n-1)}{2}}\right\}$$

we obtain

$$\begin{aligned}
\operatorname{spec}\Delta_1(S^n) &= \left\{2(n-1)_{\frac{n(n-1)}{2}},\ (2n)_{n(n-1)},\ 2(n+1)_{\frac{n(n-1)}{2}}\right\} \\
&\quad \sqcup \{2(n-1)_n, (2n)_{2n}, 2(n+1)_n,\} \\
&= \left\{2(n-1)_{\frac{n(n+1)}{2}},\ (2n)_{n(n+1)},\ 2(n+1)_{\frac{n(n+1)}{2}}\right\},
\end{aligned}$$

which finishes the proof. □

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