Some Open Problems in Birational Geometry

Dedicated to the Memory of John Coates

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John H. Coates (1945–2022) was a great mathematician who had a big impact on the development of number theory. He spent much of his career at Cambridge University. I first met him on his visit to Nottingham in 2004 where I was a PhD student. Later in 2006 I moved to Cambridge. He was very welcoming and I always counted on his generous support. He was a truly gentle and caring person. The last time I talked to him on the phone was in January 2021 and he was in good spirit despite his illness. This text is dedicated to his memory.

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1. Introduction

In each mathematical field open problems are the lifeline that keeps the field fresh and attractive. Birational geometry is one of those fields that has always had plenty of open problems. It is then not surprising that over the years it has attracted many brilliant minds. Some of these problems can be stated in very basic terms but their solutions usually require a great understanding of various tools and techniques some of which are specific to birational geometry itself and some of which are borrowed from other areas of algebraic geometry and mathematics.

We list numerous open problems in birational geometry. This list is only a sample of the many problems in the field. Also note that we do not aim to go much into the history of the problems and related works. We usually only point to one or two sources for more information.

2. PRELIMINARIES

2.1 CONTRACTIONS

By a *contraction* we mean a projective morphism $f\colon X \to Y$ of schemes such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ (f is not necessarily birational). In particular, f is surjective and has connected fibres.

2.2 PAIRS AND SINGULARITIES

A *pair* (X,B) consists of a normal quasi-projective variety X and a $\mathbb{Q}$-divisor $B \geq 0$ such that $K_X + B$ is $\mathbb{Q}$-Cartier. We call B the *boundary divisor*.

Let $\phi\colon W \to X$ be a log resolution of a pair (X,B). Let $K_W + B_W$ be the pullback of $K_X + B$. The *log discrepancy* of a prime divisor D on W with respect to (X,B) is defined as

$$a(D,X,B) := 1 - \mu_D B_W$$

where $\mu_D B_W$ denotes the coefficient of D in B_W.

We say (X,B) is *lc* (resp. *klt*)(resp. *ϵ-lc*) if $a(D,X,B)$ is ≥ 0 (resp. > 0)(resp. $\geq \epsilon$) for every D. This means that every coefficient of B_W is ≤ 1 (resp. < 1)(resp. $\leq 1-\epsilon$). Note that since $a(D,X,B) = 1$ for most prime divisors, we necessarily have $\epsilon \leq 1$.

When we say a pair (X,B) is projective we mean that X is projective.

By a *Calabi-Yau* pair (and log Calabi-Yau pair) we mean a projective pair (X,B) with lc singularities such that $K_X + B \sim_{\mathbb{Q}} 0$.

2.3 B-DIVISORS

A *b-$\mathbb{Q}$-Cartier b-divisor* over a variety X is the choice of a projective birational morphism $Y \to X$ from a normal variety and an $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor M on Y up to the following equivalence: another projective birational morphism $Y' \to X$ from a normal variety and a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor M' define the same b-$\mathbb{Q}$-Cartier b-divisor if there is a common resolution $W \to Y$ and $W \to Y'$ on which the pullbacks of M and M' coincide.

A b-$\mathbb{Q}$-Cartier b-divisor represented by some $Y \to X$ and M is *b-Cartier* if M is b-Cartier, i.e. its pullback to some resolution is Cartier.

2.4 GENERALISED PAIRS

A *generalised pair* consists of

- a normal quasi-projective variety X equipped with a projective morphism $X \to Z$,
- a $\mathbb{Q}$-divisor $B \geq 0$ on X, and
- a b-$\mathbb{Q}$-Cartier b-divisor over X represented by some projective birational morphism $X' \xrightarrow{\phi} X$ and $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor M' on X'

such that M' is nef$/Z$ and K_X+B+M is $\mathbb{Q}$-Cartier, where $M:=\phi_*M'$.

We refer to M' as the *nef part* of the pair. Since a b-$\mathbb{Q}$-Cartier b-divisor is defined birationally, in practice we will often replace X' with a resolution and replace M' with its pullback. When Z is a point we drop it but say the pair is projective.

Now we define generalised singularities. Replacing X' we can assume ϕ is a log resolution of (X,B). We can write

$$K_{X'}+B'+M'=\phi^*(K_X+B+M)$$

for some uniquely determined B'. For a prime divisor D on X' the *generalised log discrepancy* $a(D,X,B+M)$ is defined to be $1-\mu_D B'$.

We say $(X,B+M)$ is *generalised lc* (resp. *generalised klt*)(resp. *generalised ϵ-lc*) if for each D the generalised log discrepancy $a(D,X,B+M)$ is ≥ 0 (resp. >0)(resp. $\geq \epsilon$).

For the basic theory of generalised pairs see [15].

2.5 Minimal Models, Mori Fibre Spaces, and MMP

Let $X\to Z$ be a projective morphism of normal quasi-projective varieties and D be an $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Let Y be a normal quasi-projective variety, projective over Z, and $\phi\colon X \dashrightarrow Y/Z$ be a birational map whose inverse does not contract any divisor. Assume $D_Y:=\phi_*D$ is also $\mathbb{Q}$-Cartier and that there is a common resolution $g\colon W\to X$ and $h\colon W\to Y$ such that $E:=g^*D-h^*D_Y$ is effective and exceptional$/Y$, and $\operatorname{Supp} g_*E$ contains all the exceptional divisors of ϕ.

Under the above assumptions we call Y a *minimal model* of D over Z if D_Y is nef$/Z$. On the other hand, we call Y a *Mori fibre space* of D over Z if there is an extremal contraction $Y\to T/Z$ with $-D_Y$ ample$/T$ and $\dim Y>\dim T$.

If one can run a *minimal model program* (MMP) on D over Z which terminates with a model Y, then Y is either a minimal model or a Mori fibre space of D over Z. If X is a Mori dream space, eg if X is of Fano type over Z, then such an MMP always exists by [12].

3. Open Problems

We work over an algebraically closed field k of characteristic zero unless stated otherwise.

3.1 Minimal Models

One of the central problems in the field concerns existence of minimal models and Mori fibre spaces.

Conjecture 3.2 (Minimal model)**.** *Let (X,B) be a projective lc pair. Then (X,B) has a minimal model or a Mori fibre space.*

When X is a smooth surface and $B = 0$, the statement was settled by the Italian school of algebraic geometry in late 19th century and early 20th century, particularly, by Enriques and Castelnuovo. After contributions of many people in the 1970's-early 1990's notably by Kawamata, Kollár, Mori, Reid, Shokurov, this conjecture was settled in dimension 3. The conjecture was verified by Shokurov in dimension 4. Birkar-Cascini-Hacon-M^{c}Kernan [12] settled the conjecture for pairs of general type with klt singularities, in any dimension. Birkar proved the conjecture in dimension 5 for pairs of non-negative Kodaira dimension. See [10] for some more historical remarks, and [34, 33, 38, 37] for relevant work. Vanishing theorems [27] play an important role in approaching this and related problems.

3.3 Termination

The standard approach to the minimal model conjecture is via running a minimal model program (MMP) on the pair concerned. Running this program needs many ingredients of deep nature. All the necessary ingredients have been worked out except the following.

Conjecture 3.4 (Termination)**.** *Let (X,B) be a projective lc pair. Then every MMP on (X,B) terminates with a minimal model or a Mori fibre space.*

This was proved in dimension 3 by Kawamata [25] and Shokurov, and partial cases in dimension 4 by Birkar and Moraga [31].

3.5 Abundance

One of the top problems in birational geometry and more generally in algebraic geometry is the next problem.

Conjecture 3.6 (Abundance)**.** *Let (Y,B_Y) be a projective lc pair with $K_Y + B_Y$ nef. Then $K_Y + B_Y$ is semi-ample, that is, there is a contraction $h\colon Y \to S$ and an ample $\mathbb{Q}$-divisor H on S such that*

$$K_Y + B_Y \sim_{\mathbb{Q}} h^*H.$$

Equivalently, $|m(K_Y + B_Y)|$ is base point free for some $m \in \mathbb{N}$.

When X is a smooth surface and $B = 0$, the statement was settled by the Italian school of algebraic geometry. The conjecture was proved in dimension 3 in a series of papers of Miyaoka and Kawamata, and Keel-Matsuki-M^{c}Kernan. It is also known in any dimension when (Y,B_Y) is klt of general type. Otherwise, it is wide open in dimension ≥ 4. For more details on the 3-dimensional case, see [30, 28].

3.7 Non-vanishing

A problem closely related to the above conjectures is the following.

Conjecture 3.8 (Non-vanishing). *Let (X,B) be a projective lc pair where K_X+B is pseudo-effective. Then the Kodaira dimension $\kappa(K_X+B)\geq 0$. That is,*

$$h^0(X,m(K_X+B))\neq 0$$

for some $m\in\mathbb{N}$.

This is known up to dimension 3. It follows from the abundance conjecture. Conversely, it is expected that it is a first step towards a proof of abundance. Moreover, it is known that it implies the minimal model conjecture [9]. In other words, this problem is at the root of both the minimal model conjecture and the abundance conjecture.

3.9 Finite Generation

The following algebraic looking problem is closely related to the minimal model conjecture. Its local version is related to existence of flips.

Conjecture 3.10 (Finite generation). *Let (X,B) be a projective lc pair. Then*

$$R(K_X+B):=\bigoplus_{m\geq 0}H^0(X,\lfloor m(K_X+B)\rfloor)$$

is a finitely generated k-algebra where k is the ground field.

This follows from the minimal model conjecture and the abundance conjecture combined. It is known in any dimension when (X,B) is klt [12]. An independent proof in the klt case was given by Cascini and Lazić following Siu. The lc case in dimension 4 is a result of Fujino. See [17] and the references therein for more information.

The conjecture in dimension d implies both the minimal model and the abundance conjecture in dimension $d-1$ [21].

Conjecture 3.11 (Finite generation II). *Let X be a normal projective variety and $B_1,\dots,B_r$ be $\mathbb{Q}$-divisors such that (X,B_i) are all klt pairs. Then*

$$R=\bigoplus_{m_1,\dots,m_r\geq 0}H^0\big(X,\big\lfloor\sum m_i(K_X+B_i)\big\rfloor\big)$$

is a finitely generated k-algebra.

When all the B_i are big, the conjecture is known [12]. Even the case when $r=2$ and B_2 is big implies the abundance conjecture in the same dimension [19].

3.12 Generalised Minimal Models

A generalised pair is roughly a pair together with a nef divisor on some birational model. The theory of generalised pairs has been an important tool in many developments in birational geometry in recent years.

Conjecture 3.13. *Let $(X,B+M)$ be a projective generalised lc pair. Then $(X,B+M)$ has a generalised minimal model or a generalised Mori fibre space.*

More precisely, we expect that we can run a minimal model program on the pair which ends with a generalised minimal model or a generalised Mori fibre space. This first appeared as a question in Birkar-Zhang [15, before Lemma 4.4]. It was also highlighted later in work of J. Han and Zh. Li. The conjecture is proved by Moraga in dimension 3, and also in dimenion 4 when K_X+B is pseudo-effective, under mild conditions [31].

3.14 Generalised Abundance

It is expected that any nef divisor on any Calabi-Yau variety X with $h^i(X,\mathcal{O}_X)=0$ for $0<i<\dim X$, is a semi-ample divisor. A much more general form of this is the following abundance conjecture for generalised pairs.

Conjecture 3.15 (Lazić-Peternell)**.** *Let (X,B) be a projective klt pair with K_X+B pseudo-effective. Let M be a nef $\mathbb{Q}$-divisor on X. If K_X+B+M is nef, then $K_X+B+M\equiv L$ for some semi-ample $\mathbb{Q}$-divisor L.*

Some cases of the conjecture are proved in dimension 3 by Lazić-Peternell. See [29] and references therein.

3.16 Iitaka Conjecture

One of the old problems that motivated the development of the minimal model program is the following.

Conjecture 3.17 (Iitaka)**.** *Let $f\colon X\to Z$ be a contraction of smooth projective varieties. Then*

$$\kappa(K_X)\geq\kappa(K_F)+\kappa(K_Z)$$

where F is a general fibre of f.

As usual κ denotes the Kodaira dimension. In any dimension, Viehweg proved the conjecture when Z is of general type and Kollár proved the case when F is of general type. Kawamata showed that the conjecture follows from the minimal model and abundance conjectures combined. The conjecture is also known up to dimension 6. See [26, 32, 11] and the references therein.

3.18 Effective Iitaka Fibration

Understanding pluricanonical and anti-pluricanonical systems on varieties are central themes in algebraic geometry. In the case of varieties X of non-negative Kodaira dimension, there is the associated Iitaka fibration $X\dashrightarrow Z$ where $\dim Z=\kappa(K_X)$ and the very general fibres have Kodaira dimension zero. This map is define by the linear system $|mK_X|$ for sufficiently divisible $m\in\mathbb{N}$.

Conjecture 3.19 (Iitaka)**.** *Let X be a smooth projective variety of dimension d and Kodaira dimension $\kappa(K_X) \geq 0$. Then there is a natural number m depending only on d such that the pluricanonical system $|mK_X|$ defines the Iitaka fibration.*

This is known for varieties of general type by a result of Hacon-McKernan and Takayama. A more general result is proved in [23, 24]. Birkar and Zhang essentially reduced the conjecture to the case $\kappa(K_X) = 0$ in [15].

3.20 Boundedness of Complements

Complements were introduced by Shokurov to study anti-pluri-canonical systems on varieties, in particular, Fano varieties. The following is one of the most important problems in birational geometry and more generally algebraic geometry.

Conjecture 3.21 (Shokurov)**.** *Let $d \in \mathbb{N}$ and let $\epsilon \in \mathbb{Q}^{>0}$. Then there is $n \in \mathbb{N}$ satisfying the following. Let $f : X \to Z$ be a Fano contraction where X is of dimension d with ϵ-lc singularities. Then for each $z \in Z$, there is B such that*

- *(X, B) is a klt pair over a neighbourhood of z, and*
- *$n(K_X + B) \sim 0$ over a neighbourhood of z.*

This is known up to dimension 2 by a result of Birkar. It is known in any dimension when Z is a point, in which case, the conjecture is equivalent to the BAB conjecture which was proved by Birkar [6]. The conjecture can be viewed as a relative version of the BAB conjecture. It implies many other results some of which will be mentioned below.

Conjecture 3.22 (Shokurov)**.** *Let $d \in \mathbb{N}$ and let $\Phi \subset [0, 1]$ be a finite set of rational numbers. Then there is $n \in \mathbb{N}$ satisfying the following. Let (X, Δ) be a projective lc pair of dimension d such that $K_X + \Delta + L \sim_{\mathbb{Q}} 0$ for some $L \geq 0$. Then there is $B \geq \Delta$ such that*

- *(X, B) is an lc pair, and*
- *$n(K_X + B) \sim 0$.*

The conjecture was proven by Birkar [7] in any dimension when X is of Fano type, that is, when (X, C) is klt and $-(K_X + C)$ is ample for some C, e.g. when X itself is a klt Fano variety.

A special case of the conjecture is when $L = 0$ in which case the conjecture says that the torsion index of $K_X + \Delta$ is bounded. This is known as the index conjecture for log Calabi-Yau varieties.

3.23 Boundedness of Singularities on Fibrations

Understading singularities on fibrations is an important ingredient of inductive approaches to many problems.

Conjecture 3.24 (Shokurov)**.** *Let $d \in \mathbb{N}$ and let $\epsilon \in \mathbb{Q}^{>0}$. Then there is $\delta \in \mathbb{Q}^{>0}$ satisfying the following. Let (X,B) be a pair and let $f\colon X \to Z$ be a contraction where*

- *(X,B) is ϵ-lc and $\dim X - \dim Z = d$,*
- *$K_X + B \sim_{\mathbb{Q}} 0/Z$, and*
- *$-K_X$ is big over Z.*

Then there is a canonical bundle formula

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$$

where B_Z is the discriminant divisor and M_Z is the moduli divisor such that $(Z, B_Z + M_Z)$ is a generalised δ-lc pair.

The divisor B_Z measures the singularities of the fibres of f and the divisor M_Z measures the variation of the log fibres in their "moduli space". In some formulations of the conjecture instead of saying $(Z, B_Z + M_Z)$ is generalised δ-lc, one says that there is $0 \le N_Z \sim_{\mathbb{Q}} M_Z$ such that $(Z, B_Z + N_Z)$ is a δ-lc pair.

The conjecture is known when $(F, \operatorname{Supp} B_F)$ belongs to a bounded family where F is a general fibre of f and $B_F = B|_F$ [8]. Therefore, the conjecture holds if the horizontal coefficients of B are $\ge t$ for some fixed $t > 0$; this is a consequence of [6, 8]. It is also known when $d \le 1$. Also, a global variant of the conjecture is established in any dimension in [3]. A relatively easy argument reduces the conjecture to the case when Z is a curve.

A special case of the conjecture is due to M$^{\text{c}}$Kernan which says that if $X \to Z$ is a Fano contraction where X is $\mathbb{Q}$-factorial of dimension d with ϵ-lc singularities, then Z has δ-lc singularities.

A consequence of the conjecture is the following simpler-looking problem.

Conjecture 3.25. *Let $d \in \mathbb{N}$ and let $\epsilon \in \mathbb{Q}^{>0}$. Then there is $l \in \mathbb{N}$ satisfying the following. Let $f\colon X \to Z$ be a Fano contraction where X is ϵ-lc of dimension d and Z is a smooth curve. Then for each $z \in Z$, every coefficient of f^*z is $\le l$.*

Conjecture 3.24 can be reduced to this one. This reduction is implicit in the arguments of [8]. In particular, this also shows that Conjecture 3.21 implies Conjecture 3.24.

Mori and Prokhorov proved the latter conjecture in case X is a 3-fold with terminal singularities [35]. The toric case was proved by Birkar and Y. Chen [13].

3.26 Minimal Log Discrepancies

Minimal log discrepancies measure the singularities of a pair around a point. More precisely, given a pair (X,B) and a closed point $x \in X$, the minimal log discrepancy is defined as

$$\operatorname{mld}_x(X,B) = \inf\{a(D,X,B) \mid D \text{ prime divisor over } X \text{ mapping to } x\}$$

where $a(D,X,B)$ denotes the log discrepancy of D with respect to (X,B). By D over X we mean D is a divisor on some log resolution of X.

Conjecture 3.27 (Shokurov)**.** *Let $d \in \mathbb{N}$ and let $\Phi \subset [0,1]$ be a DCC set. Then the set $\{\mathrm{mld}_x(X,B)\}$ satisfies the ACC where (X,B) and x run through all the pairs (X,B) and closed points $x \in X$ such that*

- *(X,B) is of dimension d, and*
- *the coefficients of B are in Φ.*

The conjecture was proved by Alexeev and Shokurov in dimension two. It is also known for toric pairs by a result of Ambro, and known in some other special cases. Otherwise it is wide open.

There is also the following related conjecture.

Conjecture 3.28 (Ambro)**.** *Let (X,B) be a pair. Then $\mathrm{mld}_x(X,B)$ viewed as a function on the set of closed points of X is lower semi-continuous.*

Ambro proved the conjecture in dimension 3.

See [18] and the references therein for more on these conjectures which also discusses the conjectures for generalised pairs. Shokurov proved that the two conjectures together imply the termination of flips conjecture.

3.29 Boundedness of Certain Rationally Connected Varieties

The next problem is a generalisation of the BAB conjecture.

Conjecture 3.30 (McKernan-Prokhorov)**.** *Let $d \in \mathbb{N}$ and let $\epsilon \in \mathbb{Q}^{>0}$. Consider projective pairs (X,B) where*

- *(X,B) is ϵ-lc of dimension d,*
- *$-(K_X+B)$ is nef, and*
- *X is rationally connected.*

Then such X form a bounded family.

A slightly weaker form of this is known up to dimension 3 [14] which crucially relies on [3].

3.31 Boundedness of Calabi-Yau Varieties

Boundedness of Fano varieties [6] and canonically polarised varieties [22] are central results in birational geometry and moduli theory. One can ask whether Calabi-Yau varieties and pairs also form bounded families under natural conditions. It is well-known that K3 surfaces do not form a bounded family but they are bounded topologically.

Conjecture 3.32 (Yau)**.** *For fixed d, consider smooth projective varieties X of dimension d with $K_X \equiv 0$. Then such X are topologically bounded.*

See page 3 of [42]. Also see [41] and the references therein for some related results.

In any fixed dimension, boundedness of Calabi-Yau varieties with klt singularities and with an ample Weil divisor of bounded volume is a result of Birkar [5].

3.33 Semi-ampleness of Moduli Divisors

Let (X,B) be a projective lc pair and $f\colon X \to Z$ be a contraction such that $K_X + B \sim_{\mathbb{Q}} 0/Z$. Then the so-called canonical bundle formula says that we write

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$$

where B_Z is the discriminant divisor and M_Z is the moduli divisor. The divisor B_Z measures the singularities of the fibres of f and the divisor M_Z measures the variation of the log fibres in their "moduli space". Given any birational contraction $\phi\colon Z' \to Z$ one can similarly define a discriminant divisor $B_{Z'}$ and moduli divisor $M_{Z'}$ so that

$$K_{Z'} + B_{Z'} + M_{Z'} = \phi^*(K_Z + B_Z + M_Z).$$

Actually $\phi_* B_{Z'} = B_Z$ and $\phi_* M_{Z'} = M_Z$. An important result of Kawamata and Ambro [1] (in the klt case which was extended to the lc case by Fujino and Gongyo) says that $M_{Z'}$ is nef and abundant when Z' is a high enough resolution and that $M_{Z''}$ is the pullback of $M_{Z'}$ for any birational contraction $Z'' \to Z'$. But more is expected.

Conjecture 3.34. *Let $d \in \mathbb{N}$ and let $\Phi \subset [0,1]$ be a finite set of rational numbers. Then there is $n \in \mathbb{N}$ satisfying the following. Let (X,B) be a projective lc pair of dimension d and $f\colon X \to Z$ be a contraction such that $K_X + B \sim_{\mathbb{Q}} 0/Z$. Assume the horizontal coefficients of B are in Φ. Then we have an adjunction formula*

$$K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$$

and there is a resolution $Z' \to Z$ so that $nM_{Z'}$ is base point free.

This conjecture is widely open. See [36, §7 and 8] for more details, where the conjecture is verified when $\dim X - \dim Z = 1$. Another version of the conjecture just says that $M_{Z'}$ is semi-ample without fixing n. This is also wide open but it is known in some very special cases, e.g. $\dim Z = 1$ and when the general fibres of f are abelian varieties, see [20] and the reference therein.

3.35 Finiteness of Minimal Models

In dimension two if a pair has a minimal model, then the minimal model is unique. In higher dimension uniqueness does not hold but the following conjecture says that there should be finitely many in some sense.

Conjecture 3.36 (Kawamata)**.** *Let* (X,B) *be a projective klt pair. Then* (X,B) *has only finitely many* $\mathbb{Q}$*-factorial minimal models, up to isomorphism.*

Here we need to consider the minimal models (Y,B_Y) up to isomorphism forgetting the birational map $X \dashrightarrow Y$. The conjecture is open from dimension 3.

3.37 The Cone Conjecture

Conjecture 3.36 is related to another more refined problem on the geometry of cones on a Calabi-Yau pair. First we need to recall some notation. Let X be a normal projective variety. Let $N^1(X)$ be the space of $\mathbb{R}$-Cartier divisors on X modulo numerical equivalence. We denote the closed cone in $N^1(X)$ generated by nef divisors on X as $\overline{A}(X)$ and the closed cone generated by movable divisors as $\overline{M}(X)$. The cone generated by effective divisors is denoted $B^e(X)$ (this is not necessarily closed). Define

$$A^e(X) = \overline{A}(X) \cap B^e(X) \quad \text{and} \quad M^e(X) = \overline{M}(X) \cap B^e(X).$$

Now assume (X,B) is a projective pair. The group of automorphisms of X mapping B to itself is denoted by $\mathrm{Aut}(X,B)$. A pseudo-automorphism of X is a birational map $X \dashrightarrow X$ which is an isomorphism in codimension one. The group of pseudo-automorphisms mapping B to itself is denoted by $\mathrm{PsAut}(X,B)$.

A small $\mathbb{Q}$-factorial modification (SQM) of X is a birational map $X \dashrightarrow X'$ which is an isomorphism in codimension one and where X' is a normal $\mathbb{Q}$-factorial variety.

Conjecture 3.38 (Morrison-Kawamata-Totaro)**.** *Let* (X,B) *be a* $\mathbb{Q}$*-factorial projective klt Calabi-Yau pair. Then*

1. *the number of* $\mathrm{Aut}(X,B)$*-equivalence classes of faces of the cone* $A^e(X)$ *corresponding to birational contractions or fiber space structures is finite. Moreover, there exists a rational polyhedral cone* Π *which is a fundamental domain for the action of* $\mathrm{Aut}(X,B)$ *on* $A^e(X)$ *in the sense that*
   - $A^e(X) = \bigcup_{g \in \mathrm{Aut}(X,B)} g_*\Pi$,
   - $\mathrm{Int}\,\Pi \cap g_*\,\mathrm{Int}\,\Pi = \emptyset$ *unless* $g_* = 1$;
2. *the number of* $\mathrm{PsAut}(X,B)$*-equivalence classes of chambers* $A^e(X')$ *in the cone* $M^e(X)$ *corresponding to SQMs* X' *of* X *is finite. Equivalently, the number of isomorphism classes of SQMs of* X *(ignoring the birational identification with* X*) is finite. Moreover, there exists a rational polyhedral cone* Π' *which is a fundamental domain for the action of* $\mathrm{PsAut}(X,B)$ *on* $M^e(X)$*.*

The conjecture was settled in dimension two by Totaro [39]. The conjecture also makes sense in the relative setting. Kawamata proved the relative version for 3-folds with terminal singularities over a positive dimensional base when $B = 0$.

The conjecture was first proposed by Morrison for varieties, then generalised to the relative setting by Kawamata for varieties and extended to pairs in the relative setting by Totaro. For more on relevant results and history see [39].

3.39 Boundedness of Stein Degree

Let $S \to Z$ be a projective morphism between varieties and let $S \to V \to Z$ be the Stein factorisation. We define the Stein degree of S over Z to be $\mathrm{sdeg}(S/Z) := \deg(V/Z)$. If $S \to Z$ is not surjective, this degree is 0 by convention.

Recall that a log Calabi-Yau fibration $(X,B) \to Z$ consists of an lc pair (X,B) and a contraction $X \to Z$ such that $K_X + B \sim_{\mathbb{Q}} 0/Z$.

Conjecture 3.40 (Birkar)**.** *Let $d \in \mathbb{N}$ and let $t \in \mathbb{R}^{>0}$. Let $(X,B) \to Z$ be a log Calabi-Yau fibration of dimension d and let S be a horizontal/Z component of B whose coefficient in B is $\geq t$. Then $\mathrm{sdeg}(S/Z)$ is bounded from above depending only on d,t.*

The case $t = 1$ is a recent result of Birkar. This case has important applications to boundedness of stable minimal models and to existence of their moduli spaces. See [2] for more details.

A variant of the conjecture is of interest in arithmetic geometry, e.g. over number fields.

Conjecture 3.41 (Birkar)**.** *Let $d \in \mathbb{N}$ and let $t \in \mathbb{R}^{>0}$. Let k be a field of characteristic zero. Let (X,B) be a log Calabi-Yau pair over k of dimension d and let S be a component of B whose coefficient in B is $\geq t$. Then*

$$\mathrm{sdeg}(S/\operatorname{Spec} k) = \dim_k H^0(S, \mathcal{O}_S)$$

is bounded from above depending only on d,t.

3.42 Boundedness of Irrationality

An important problem in birational geometry is to check whether a variety is rational or if not how far it is from being rational or at least from being rationally connected. There are different ways to measure the degree of irrationality or irrational-connectedness.

The following problem was motivated by the study of degenerations of Fano varieties.

Conjecture 3.43 (Birkar-Loginov)**.** *Let $d \in \mathbb{N}$ and let $t \in \mathbb{R}^{>0}$. Suppose that $f\colon (X,B) \to Z$ is a Fano type log Calabi-Yau fibration where $\dim X = d$. Assume S is a component of B with coefficient $\geq t$ contracted to a point on Z. Then there is a rational map $S \dashrightarrow T$ where the general fibres are rationally connected and T is a smooth projective variety with bounded gonality.*

By gonality of T we mean the least possible degree of dominant rational maps $T \dashrightarrow \mathbb{P}^{\dim T}$. Without the Fano type assumption the answer to the question is expected to be negative. For example, it is conjectured that gonality of K3 surfaces F are not bounded but they appear as fibres of the log Calabi-Yau fibration $F \times Z \to Z$ where the morphism is projection and Z is a smooth curve.

The conjecture was confirmed when $\dim X = 3$ and $\dim Z \geq 1$ by Birkar and Loginov. See [4] for more details.

3.44 Rational Points on Varieties

Let k be a number field. For a variety over k and for a field extension $k \subset k'$ we denote the set of k'-rational points of X by $X(k')$. It is expected that there is a close connection between the geometry of X and the properties of the sets $X(k')$.

Conjecture 3.45 (Bombieri-Lang)**.** *Let X be a smooth projective variety of general type over a number field k. Then $X(k')$ is not Zariski dense for any finite field extension $k \subset k'$.*

This is known in dimension one by a result of Faltings.

In the opposite direction we have:

Conjecture 3.46 (Campana)**.** *Let X be a smooth projective variety over a number field k. If X is Calabi-Yau or Fano, then $X(k')$ is Zariski dense for some finite extension $k \subset k'$.*

This is known for Fano surfaces and most Fano 3-folds but it seems it is not known in full generality for Calabi-Yau surfaces. See [40] for more details.

Campana proposes a more general conjecture. He defines special varieties and expects that the latter conjecture holds for them. See [16] for more details.

One can also consider pairs with singularities:

Conjecture 3.47. *Let (X,B) be a klt Calabi-Yau pair over a number field k. Then $X(k')$ is Zariski dense for some finite extension $k \subset k'$.*

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