Why and How Planck Introduced His Celebrated Constant $\hbar$

(The *Almost* Untold Story)

Sergio Cecotti

1. Physics, or *the Art of Reinventing History*

To clarify the spirit and the aims of this article, I start with some general considerations about the tricky relation between physics and the history of the discipline. It is usually said that math is taught *axiomatically* (or in a *structuralist* way [1]) while physics is taught in the order of its historical development, that is, in the sequence:

> Classical mechanics, thermodynamics, electrostatics/magnetostatics, Maxwell electromagnetism, statistical mechanics, special relativity, general relativity, non-relativistic quantum mechanics, ..., quantum gravity (eventually, *we hope*)

The considerable time spent teaching such a long list of *out-of-date theories* is justified with the necessity of developing the *"physical intuition"* of the students, that is, to be sure that they will not think (God forbid!) as mathematicians do. However Steven Weinberg (Nobel Prize for Physics in 1979) stressed that the history we teach is *purely fictional* [2]. We physicists keep *reinventing history* generation after generation to serve the "didactical" purposes of the day, which change with time. We aim to make "intuitive" physics as we understand it *now* – not the way the founders originally understood it 100 or 200 years ago. We try to provide the students with the tools they need to do research on problems which are open *now,* not in the days of Maxwell and Gibbs.

Historical truth may (and indeed *should*) be sacrificed. The fictional history of the physics textbooks presents the development of our science as a *linear process* guided by an ineluctable scientific logic. Physicists are presented as *ideal perfectly-rational beings*, the human counterpart to the perfect gas of macroscopic thermodynamics. The actual history, says Weinberg, was quite different: the path was anything but linear, full of false starts, a lot of stupid ideas were circulating, and the overall process was plagued by the *personal prejudices* of each author. Indeed many correct answers were originally obtained by asking *absolutely wrong* questions.

In no case the textbook fictional history is more radically divergent from the actual facts than in the birth of quantum physics and the way Planck discovered his fundamental constant $\hbar$. Being a physicists and not a historian, I am interested in what Planck's work teaches us doing physics *now*, rather than in the exact chronology of facts. Yes, I will try to present the actual facts with some degree of historical honesty, but also compare them with a few fictional histories useful for *modern purposes,* and I will often add to the statements actually made by physicists around the year 1900, their translation into the present-day language to make them easier to understand by the modern reader with a background in theoretical physics. The main purpose of this survey is to illustrate the *close analogy* between the situation of physics in the decade 1890–1900 and the state-of-the-art of our discipline in the decade 2020–2030, compare what *they did* at the time with what *we are doing* today, and (perhaps) draw some useful lesson for us working now on the frontier of science.

In conclusion: *this note is about physics*. History is a *mere excuse* to talk about physics, not the real topic. My apologies to readers which are interested in ancient facts *per se*.

2. Black Body Radiation: Definitions

The survey article is about the discovery by Planck in 1900 of the correct formula for the black body radiation with the consequent introduction of its fundamental constant h (or the theoretically more natural combination $\hbar \equiv h/2\pi$). I start with the definition of "black body". It is an *ideal* physical system introduced by Robert Kirchhoff in 1860 [3] on the basis of the law of radiation discovered by himself. To be sure I am really referring to the original 19$^{\text{th}}$ century notion, I quote verbatim the (old) definitions from §§. 44, 45 of his classical book [4]

1. **Kirchhoff law:** the ratio E/A of the emissive power to the absorbing power of any body is independent of the nature of the body;
2. **Definition:** a physical system is a *black body* iff $A = 1$ (i.e. it absorbs all incident radiation);
3. **Basic properties:** the emissive power of a black body is independent of its nature. Its emissive power is larger than that of any other body at the same temperature and, in fact, is just equal to the intensity of radiation in the contiguous medium.

As said, the black body is a convenient idealization. It was soon realized that no such a system exists:

Theorem 2.1 (Wien-Lummer [5]). *There is no such a thing in nature as a (perfect) "black body".*

The actual absorption cross section of the body (a function of the frequency ν) measures the deviation from being an ideal black body: it is called the *gray factor*.

Side remark for the *cognoscenti*. The black hole of general relativity comes rather close to be an ideal black body, but of course Wien and Lummer did not know that in 1895. However – even if the black hole horizon *per se* is an *ideal* black body – the full black hole has non-trivial gray factors. I quote from Maldacena-Strominger [6]:

> The black hole emits *blackbody radiation* from the horizon. Potential barriers outside the horizon act as a frequency-dependent filter, reflecting some of the radiation back into the black hole and transmitting some to infinity. [...] In the past, greybody factors have been largely regarded as *annoying factors* which mar the otherwise perfectly thermal blackbody radiation. Now we see that they have an important place in the order of things, and transmit a carefully inscribed message on the quantum structure of black holes.

What Does It Mean in Plain English?

Kirchhoff notion of "black body" may be not easy to grasp, so better translate it in *plain English*. Max Planck himself thought that one needs a plain English (actually plain German) definition, and he wrote it in §§. 51, 52 of his book [4]

Historical plain English: M. Planck, *op. cit.* §§. 51, 52
In a *vacuum bounded by totally reflecting walls* any state of radiation may persist. But as soon as an arbitrarily small quantity of matter is introduced into the vacuum, a stationary state of radiation is gradually established [$\cdots$] possible to change a perfectly arbitrary radiation, which exists at the start in the evacuated cavity with perfectly reflecting walls under consideration, into *black radiation* by the introduction of a minute particle of carbon. The characteristic feature of this process is that the heat of the carbon particle may be just as small as we please.

From this clarification remark by Planck we see that what matters is the electromagnetic *vacuum in finite volume* (the ideal reflecting boundary conditions) and that the black body – the minute particle of carbon – is just a technicality of the way we set that vacuum in thermal equilibrium at temperature T. Its effect on the system is "as small as we please". For clarity we give the translation of Planck's plain German

Translation in *current* plain English
"Black body radiation" is a nickname for the *canonical ensemble* at temperature T of the *free Maxwell theory* in a 3-dimensional cubic box of size L with *reflecting boundary conditions*, i.e. at each wall of the box the normal component of the magnetic field $\vec{B}$ and the parallel components of the electric field $\vec{E}$ vanish

3. The Traditional Fictional History

In 1978 I was an undergraduate and I attended a lecture by Wiki Weisskopf on the exact topic of this article: the discovery of $\hbar$ by Planck. Weisskopf was then an old man who, in his youth, had worked with Bohr, Schrödinger, Heisenberg and Pauli, so he knew the history of the birth of Quantum Physics almost first hand. You would expect that he would give testimony of the authentic story. Instead his lecture was wonderful, very inspiring, but *totally fictional.* He told us the "didactical" history which is the textbook standard since Ehrenfest's wrote it in his 1911 book telling how $\hbar$ would have been discovered not in the real world but in one full of ideal physicists.

Weisskopf projected on the screen the image of a fireplace similar to this one

and said (I quote from my memories):

> They soon realized that the principles of Physics, as they were known in 1900, were inconsistent with reality. They did not need any fancy precision experiment to see that the classical laws were untenable: to realize that classical Physics is incorrect *it suffices to look at this fire.*

The point he was making is that *fire* cannot exist in a world governed by the classical laws of Physics, so the fact that mankind had discovered fire (already in the Stone Age) what sufficient experimental proof that the world is quantum. Indeed to allow for the existence of fire we need a rather drastic departure from the classical paradigm.

$$
\begin{aligned}
A_1(x,y,z) &= \sum_{k_1,k_2,k_3\in\mathbb{N}^3} A_1(k_1,k_2,k_3)\cos(2\pi k_1 x/L)\sin(2\pi k_2 y/L)\sin(2\pi k_3 z/L)\\
A_2(x,y,z) &= \sum_{k_1,k_2,k_3\in\mathbb{N}^3} A_2(k_1,k_2,k_3)\sin(2\pi k_1 x/L)\cos(2\pi k_2 y/L)\sin(2\pi k_3 z/L)\\
A_3(x,y,z) &= \sum_{k_1,k_2,k_3\in\mathbb{N}^3} A_3(k_1,k_2,k_3)\sin(2\pi k_1 x/L)\sin(2\pi k_2 y/L)\cos(2\pi k_3 z/L)\\
\text{with}\quad & k_1A_1(k_1,k_2,k_3)+k_2A_2(k_1,k_2,k_3)+k_3A_3(k_1,k_2,k_3)=0\quad \textbf{(gauge condition)}
\end{aligned}
$$

Figure 1. The general electromagnetic field satisfying the reflecting boundary conditions.

Why Fire Is Impossible in a Classical World?

Let us explain Weisskopf's claim that you (as an ideal physicist) should discern on-the-spot that the real world is quantum just by looking at a fireplace. I tell the traditional Ehrenfest version of the story (a slightly different viewpoint is offered below).

In Figure 1 I have written the most general electromagnetic field inside a cubic box of size L which at the walls satisfy the totally reflecting boundary conditions. It is written in the old-fashioned *radiation gauge*

$$A_0 = \vec{\nabla}\cdot\vec{A} = 0.$$

The details of the equations in the figure are not important. The formulae are written just to inform the reader that we can write them and study the problem in every minute detail *if so needed.* But we don't need to. The only point of interest is that for all triple of positive[1] integers $(k_1,k_2,k_3)\equiv\vec{k}$ we have *two* real degrees of freedom; we may choose them to be $A_1(\vec{k})$ and $A_2(\vec{k})$.

Plugging in the expression for $\vec{A}(\vec{x})$ in terms of the (chosen) independent degrees of freedom, $A_1(\vec{k})$ and $A_2(\vec{k})$, in the Maxwell Lagrangian we get the Lagrangian governing the dynamics of these independent physical quantities

$$L = \sum_{i=1,2}\sum_{\vec{k}\in\mathbb{N}}\left(\frac{1}{2}\dot{A}_i(\vec{k})^2 - \frac{1}{2}\left(\frac{2\pi\vec{k}}{L}\right)^2 A_i(\vec{k})^2\right)$$

from which we see that our *vacuum bounded by totally reflecting walls* (to use the exact words of Planck) is precisely equivalent to a system of non-interacting harmonic oscillators consisting of *two copies*, $A_1(\vec{k})$ and $A_2(\vec{k})$, of the harmonic oscillator of frequency

$$\nu_{\vec{k}} = |\vec{k}|/L$$

[1] To keep things simple, we are cavalier with the zero modes. They play no role in the argument, and are irrelevant in the computation anyhow.

per wave-number vector $\vec{k} \in \mathbb{N}^3$. Now we apply a fundamental fact of classical Physics, the equipartition theorem.

Theorem 3.1 (Equipartition theorem in *classical* Statistical Mechanics)**.** *In a system of N (classical) harmonic oscillator of frequencies $\nu_1, \dots, \nu_N$ at thermal equilibrium at temperature T, the internal (mean) energy U distributes equally between the several oscillators, independently of their frequency, the energy being equal to $k_B T$ per oscillator.*

Here k_B is the Boltzmann constant.

Proof. For a single oscillator of frequency $\omega/2\pi$

$$U = k_B T^2 \frac{\partial}{\partial T} \log \int_{-\infty}^{+\infty} dp\, dq\, e^{-(p^2+\omega^2 q^2)/(2k_B T)} = k_B T^2 \frac{\partial}{\partial T} \log\left(\frac{\pi k_B T}{\omega}\right) = k_B T$$

so the frequency $\omega/2\pi$ drops out of the formula in the last equality. □

As a corollary, the thermal radiation energy emitted (in electromagnetic waves) at equilibrium by a "black body" of temperature T is *classically*

$$2 \sum_{\vec{k} \in \mathbb{Z}^3} kT = \infty \,!!$$

In other words, a *classical fire* would emit an infinite amount of heat and light (as long its temperature is positive). Therefore a classical fire is impossible. Every (ideal) child who sees a fireplace, should immediately infer that the world follows the rules of Quantum Physics.

THE DISMAL STORY OF THE "NO-CLASSICAL-FIRE" ARGUMENT

However in the real history the *no-classical-fire* argument played no role. The first one to mention that the application of the classical rules leads to the paradox of an infinite energy emission was H.A. Lorentz is his 1908 Rome Lectures. In a reflecting box the electromagnetic field has modes $A_i(\vec{k})$ of arbitrarily *high* frequency $\nu \equiv |\vec{k}|/L \to \infty$, and hence – according to the classical rules – to put the system in thermal equilibrium at every positive temperature ϵ, however small, requires an *infinite* amount of energy. This paradox was dubbed the "ultraviolet catastrophe" by Ehrenfest. At the time it was seen as a major puzzle, but not necessarily as a fundamental problem. It could still be an *apparent* paradox – there are many of them in Theoretical Physics: seemingly inconsistent results which eventually turn out to be in fact logical necessities for the laws to make sense. For instance, initially anomalies in Quantum Field Theory were seen as paradoxes, now we understand them as blessed features of the theory. In 1908 Lorentz did realize that the ultraviolet catastrophe *could* require the rejection of the classical laws, but still hoped that the paradox was merely apparent, and that a better understanding of the classical Physics would solve the conundrum.

Even after 1908 most theoreticians dismissed Lorentz's suggestion that new physical principles may be *possibly* required. As a matter of fact, the scientific community was not aware of the implications of the classical equipartition theorem for at least a full decade after the publication by Planck of his black body radiation formula. This despite the fact that a few months *before* Planck's paper, an article was published in a British journal [7] where the classical formulae for the black body radiation were deduced from the equipartition theorem *almost* correctly. However the author himself *did not believe* in his own work, thinking that there was a subtle flaw in the *proof* of the theorem, not a major problem in the fundamental physical laws used in the proof of the theorem.

A Tale of Two Years: 1900 Versus 2023

The quoted paper by Lord Rayleigh suggests the comparison in the following two boxes

Status of Physics year 1900
(slightly fictional – assuming Lord Rayleigh did believe his own paper)

> We have *two* sacred physical principles: *Maxwell theory* and *Statistical Physics*. Both are confirmed by the experiment to very high accuracy, but when we try to put them together we get *divergent answers* and *inextricable paradoxes* especially for the *Black Body* radiation.
> We need *new physical principles* to reconcile the two.

Status of Physics year 2023

> We have *two* sacred physical principles: *General Relativity* and *Quantum Physics*. Both are confirmed by the experiment to very high accuracy, but when we try to put them together we get *divergent answers* and *inextricable paradoxes* especially for the *Black Hole* radiation.
> We need *new physical principles* to reconcile the two.

To make the analogy even sharper, note that the Black Hole *horizon* satisfies Kirchhoff's definition of Black Body, indeed it is the only true Black Body in nature (although the full physical system is a bit gray – cf. our previous quotation from Maldacena-Strominger). In a sense we are still there, except that now the trouble arises from the high frequency modes of a massless spin-2 boson (the graviton) instead of a massless spin-1 boson (the photon). Let us make a synoptic table comparing the two situations:

1900	**2023**
Maxwell theory (spin-1 massless)	General Relativity (spin-2 massless)
Statistical Physics	Quantum Physics
Black Body	Black Hole
photon gas theory	Quantum Gravity
What they were doing then:	What we are doing now: we use a *Meta-theory* (the swampland program)

In the bottom box on the right-hand side we describe the most promising line of research about Quantum Gravity which is current in the year 2023. By a *Meta-theory* we mean a theory which does not describes any particular physical system but the full class of all meaningful physical theories.[2] In other words, a Meta-theory aims to characterize the consistent "physical theories" out of the huge space of "garbage" mathematical models. A most promising line of research in Quantum Gravity in 2023 is the *swampland program* which was initiated by Cumrun Vafa.[3] The swampland program is the Meta-theory of Quantum Gravity which aims to give general criteria that all consistent quantum theory of gravity should satisfy, without proposing any particular explicit model. The idea is that very few models will pass the swampland tests, so they lead us quite close to the actual theory, allowing us to make several detailed predictions which must be true *whatever the underlying fundamental theory is,* as long as it is consistent.

On the left-hand side of the above table the field "What they were doing then" is blank. Just as now we do not know the basic rules of Quantum Gravity, the theoreticians working on the black body radiation in the decade 1890–1900 did not know the underlying basic theory (which was Quantum Physics). Analogy suggests that at least the most brilliant physicists of the time should have reverted to the use of Meta-theories to make predictions in absence of a theory. We are ready to make the

Time-reversed prediction. *In the years 1890–1900 the most brilliant German/Austrian physicists used a Meta-theory (i.e. the* swampland philosophy*) to study the black body radiation which could not be approached with the (wrong) physical theories known at the time.*

[2] The proper name for the "Meta-theory" would be "Meta-Physics". Unfortunately that name was already taken by a more hand-waving discipline and we use "swampland program" instead.

[3] The original papers are [8, 9]. For recent reviews see [10, 11].

4. Historical Facts: the Context

Let us leave the fictional history for now, and focus on the *actual facts* in their highly non-linear evolution. We start from the general context.

Let us pretend we are theoretical physicists in Berlin in the year 1899. Kirchhoff's definition of the "black body" energy spectral function

$$u(\nu,T)$$

is already thirty years old. We know that $u(\nu,T)$ is the energy density distribution of the Maxwell theory canonical ensemble at temperature T as a function of the frequency ν, and therefore we are fully aware that $u(\nu,T)$ is an *universal* function which enters in the thermal description of *all* electromagnetic phenomena. Thus understanding it is a major problem of Theoretical Physics. Our experimental colleagues in Berlin – in particular Otto Lummer and Ernst Pringsheim – are busy these days to measure $u(\nu,T)$ with increasing accuracy in various regions of the two parameters ν and T because of its technological relevance for the booming German electric-lightning industry which funds rather generously all researches on the black body radiation.

From the experimental side we have at our disposal plentiful of tables with very precise data. What we know *theoretically* about $u(\nu,T)$ in this final year of the 19^{th} century?

We know *for sure* two fundamental facts:

(1) the Stefan-(Boltzmann) law;
(2) the Wien displacement law.

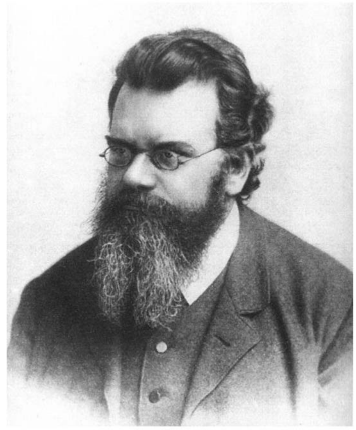
In principle we should also know the formula predicted by the classical Physics[4] (CP)

$$u(\nu,T)=\frac{8\pi\nu^2}{c^3}k_B T \tag{CP}$$

but we – as all other physicists in Berlin – are not aware of it. Yes, in England Lord Rayleigh got the formula $u(\nu,T)\propto T$ (without the correct overall constant), but convinced himself that he did some subtle mistake, given that his result was in total disagreement with the empirical data. The formula (CP) was not deduced correctly from the classical laws, with the proper factor in front, until Albert Einstein wrote it in his 1905 paper on the photoelectric effect for the only purpose of pointing out that *the classical formula is dramatically wrong*, and hence we need a "revolutionary" departure from the classical paradigm.

[4] Here and in the rest of the note c is the speed of light and k the Boltzmann constant.

Josef Stefan



Ludwig Boltzmann

Stefan Law

The first law was introduced by the Austrian (Carinthian Slovenian[5]) physicist Josef Stefan (or **Jožef Štefan** as his name is spelled in Slovenian) in 1879:

Stefan Law (Stefan [12]). *The density of energy of the electromagnetic field at temperature T is*

$$u(T) \equiv \int_0^\infty u(\nu,T)\,d\nu = CT^4$$

for a positive constant C.

Stefan formulated his law on the basis of the existing experimental evidence. Many accurate experiments in Berlin checked its validity, measuring the constant C with high precision. Yet some experimentalists, such as Weber, said: "We are smarter than our colleagues in Berlin, and can make more precise measurements, and see the *tiny deviations* from Stefan's law. But, of course, they were *deadly wrong!* By 1897 perfect accuracy of the law was confirmed (within measurement uncertainties) up to temperatures of 1535 K°.

Why Weber Was Deadly Wrong? Spoiling the historical developments, let us mention the deep reason why Stefan law had to be *exactly correct* (using the modern language).

[5] Until 1918 Slovenia was part of the Austrian half of the Austro-Ungarian empire, and Stefan was an Austrian citizen. Now his birth place is in the Slovenian-speaking territory in Southern Carinthia (one of the Austrian federal states).

Spoiling History: why Stefan law is exactly true (modern view)

Maxwell theory is a Conformal Field Theory. Don't worry, this is just a fancy way of saying: *"light travels at the speed of light"*. While this statement is much less innocent than it may sound, we may take it for granted. An equivalent statement is: the energy-momentum tensor is traceless

$$T^{\mu}{}_{\mu} \equiv T_{00} - \sum_{i=1}^{3} T_{ii} = 0$$

A Conformal Field Theory has no dimensional parameters. Therefore the Physics of a Conformal Field Theory at thermal equilibrium at a temperature T, contains only one *independent* dimensionful parameter, namely T itself, which (in units where the Boltzmann constant $k_B = 1$) has the dimension of an energy. The energy per unit volume $u \equiv U/V \equiv U/L^3$ has the dimension (energy)^4, so we must have

$$u(T) = CT^4$$

for some numerical (but very fundamental!) constant C

Alert! The modern argument is valid only in a quantum world, where, using units where the Planck and Boltzmann constants are both set equal 1, we can measure *lengths* in units of inverse Kelvin degrees (hence volumes in $(\mathrm{K}^\circ)^{-3}$). The actual constant

$$C = \frac{2\pi^5 k_B^4}{12 c^2 h^3}$$

diverges in the classical limit $h \to 0$. As stressed by Weisskopf: *there is no classical fire!*

Boltzmann Theorem or *The Power of Meta-Theories*

The formula $u(T) \propto T^4$ is called the Stefan-*Boltzmann* law because Boltzmann in 1884 gave a *rigorous mathematical derivation* of it in the paper [13]. At this point the reader is supposed to be jumping on his/her chair and yell: *"Wait a minute! How can* that *be possibly true?"*

In 1884 nobody – including the great Ludwig Boltzmann – had the vaguest clue about the correct theory of the Black Body radiation, and the only then existing theory (classical Physics) predicted that $u(T)$ was *linear* in T (with a divergent coefficient) not *quartic!*

Yet, *in the most absolute ignorance of the theory*, Boltzmann not only got the *correct* result, but gave a *mathematically fully rigorous proof* of it, a proof that stands even today to our learned scrutiny. How could he do that?

Was it *Black Magics?* No, Boltzmann used a *Meta-theory* (i.e. the swampland approach in our current parlance) and his success shows the power of *Meta-theories* ($\equiv$ Fundamental Principles) in Physics. Boltzmann's approach to the Black Body radiation fulfills our previous time-reversed prediction that physicists at the end of the nineteenth century should have resorted to swampland program practices to work out the Physics of the Black Body. The extraordinary successes of Boltzmann and his followers in those years is quite encouraging for us working in Quantum Gravity today.

The Meta-theory in question was *Thermodynamics.* Thermodynamics is not a physical theory. It consists of two Principles that *all* physical theories should obey, that is, it is a *general criterion* to distinguish *physical theories* from mathematical models which are just garbage (technically: they belong to the swampland). In the words of Eddington:

> If your theory is contradicted by the experimental data, don't worry: the experiments will turn out to be wrong. But if your theory is contradicted by the Second Principle of Thermodynamics, it is dead without hope.

A theory in disagreement with the Second Principle is in the swampland!

To make a thermodynamical study of a particular physical system you need only one piece of information about it: its *equation of state.* In the case of electromagnetic radiation it suffices to know that *light travels at the speed of light.* Not at all a trivial statement, but Boltzmann was pretty sure of the fact. As the title of his paper implies, this is the only specific ingredient he uses in his swamplandic analysis.

Let us pause a while to review the wonderful proof by Boltzmann of Stefan's law which deserves our thoughtful consideration.

Proof. The two principles of Thermodynamics are encoded in the equation

$$dU = T\,dS - p\,dV, \tag{PP}$$

where U is the internal energy, T the (absolute) temperature, S the entropy, p the pressure, and V the volume. Energy is an extensive quantity proportional to the volume, so

$$U = V\,u, \qquad u \equiv u(T): \text{ energy density.}$$

In any isotropic medium at stationary equilibrium, the components of the energy-momentum tensor are identified with the above thermodynamical quantities by the two relations

$$T_{00} = u, \qquad T_{ij} = p\,\delta_{ij}.$$

The electromagnetic field satisfies the identity $T_{00} = \sum_i T_{ii}$ (light travels at the speed of light), so its equation of state at equilibrium is simply

$$p = \frac{u}{3}. \tag{ES}$$

Now that we have the equation of state (ES) we are in business. We plug it in equation (PP) getting

$$dS \equiv \frac{dU}{T} + \frac{p}{T} dV = \frac{d(uV)}{T} + \frac{1}{3}\frac{u}{T} dV = \frac{V}{T} du + \frac{4}{3}\frac{u}{T} dV \tag{dS}$$

and take its differential using Poincaré identity $d^2 = 0$

$$0 = d^2 S = \frac{u'(T)}{T} dV \wedge dT - \frac{4}{3}\left(\frac{u'(T)}{T} - \frac{u(T)}{T^2}\right) dV \wedge dT,$$

to get a differential equation for the dependence of $u(T)$ on the temperature T

$$T\,\frac{\partial \log u(T)}{\partial T} = 4$$

whose general integral is CT^4. We are entitled to rewrite the overall integration constant C in a fancy way, parametrizing $C \equiv C(h)$ in terms of a new constant h in the form

$$u(T) = C(h)\,T^4 = \frac{2\pi^5 k_B^4}{12c^2h^3}\,T^4$$

where h is an integration constant still to be determined, while k_B is Boltzmann's constant, and c is the speed of light. □

Amazingly simple, elegant, and valid whatever the fundamental theory is (as long as light does travel at the speed of light!). We stress that Planck's constant h emerges from the swamplandic Boltzmann's analysis as an *integrant constant.* Then we may restate Boltzmann's theorem in the light of modern insight (and language)

Theorem 4.1 (Boltzmann 1885)**.** *The family of theories of the Black Body radiation which are consistent with the first and second principles of Thermodynamics are parametrized by one quantity h*

or, equivalently

Classical Physics (which corresponds to a particular value of h, namely zero) admits a one-parameter family of deformations consistent with the fundamental principles of Thermodynamics. Planck's constant is the coordinate on the moduli space of consistent theories.

We shall return to this deformation-theoretical viewpoint at the end.

For later reference we notice a corollary to Boltzmann's theorem. Plugging the result $u = CT^4$ in equation (dS) we get

$$dS = 4CVT^2\,dT + \frac{4}{3}CT^3\,dV = d\left(\frac{4C}{3}VT^3\right)$$

so that the entropy is $S = \frac{4}{3}CVT^3$. In particular, in an adiabatic process VT^3 remains constant.

WIEN DISPLACEMENT LAW

The Wien law dwells with the dependence of the Black Body radiation from the frequency ν of the electromagnetic waves, i.e. describes the contribution $u(\nu,T)$ from the oscillators of frequency $\nu = |\vec{k}|/L$ to the energy per unit volume inside our reflecting box in equilibrium at temperature T. By its very definition $u(\nu,T)$ is related to the energy density $u(T)$ by the formula

$$u(T) = \int_0^\infty u(\nu,T)\, d\nu.$$

The idea of Wien [14] was to use Boltzmann's swamplandic approach to the integrand itself. Now $u(\nu,T)$ is a function of two independent variables, ν and T, while the Second Principle yields a single partial differential equation for it. Then the solution now will depend on a boundary condition defined by an unknown function, not just a single constant C as before.

Proof. We increase *adiabatically* the size L of the box by a factor λ:

$$L \to L' \equiv \lambda L, \qquad V \to V' \equiv \lambda^3 V, \qquad \nu \to \nu' \equiv \nu/\lambda,$$

where the last equation is the *Doppler effect* which is a trivial consequence of the formula $\nu = |\vec{k}|/L$ and the fact that $\vec{k}$, being valued in the discrete set $\mathbb{N}^3$, is invariant under all continuous deformations. We already know that in an adiabatic process $VT^3 = \text{const}$, thus $T \to T' \equiv T/\lambda$ and $u \to u' \equiv u/\lambda^4$, so that $V^{4/3}u = \text{const}$. Since this condition holds independently for the contribution to the energy density from each frequency, we have

$$V^{4/3}\, u(\nu,T)\, d\nu = V'^{4/3}\, u(\nu',T')\, d\nu' \equiv V^{4/3}\, \lambda^3\, u(\nu/\lambda, T/\lambda)\, d\nu.$$

Set $\lambda = \nu$ in this identity. We get

$$u(\nu,T) = \nu^3\, \Phi(\nu/T) \qquad \text{where} \quad \Phi(x) \equiv u(1,x^{-1}) \qquad \square$$

In conclusion: independently of the underlying physical law/mechanism, the fundamental principles (Meta-theory) determine the Black Body spectral function $u(\nu,T)$ up to one function $\Phi(x)$ of a single real variable $x \geq 0$. This is real progress, especially because it is an exact result valid independently of the basic theory (as long as it is consistent and predicts that light travels at the speed of light).

Physicists in 1894 had now a well defined problem: find the universal function Φ.

THE QUEST FOR Φ: WIEN'S EDUCATED GUESS VS. EXPERIMENTAL EVIDENCE

Without knowing the theory, Wien could use two traditional strategies to find the function Φ: (*i*) make an educated guess, and (*ii*) fit the experimental data.

Let us start from the educated guess. Wien used a heuristic analogy with Boltzmann statistical theory of perfect gases. In modern terminology he guessed that Φ could be deduced from the canonical ensemble of Statistical Mechanics. We (the posterity) know that that is perfectly correct, but the computation of the canonical distribution requires knowing the fundamental theory. Ignoring the theory, you may only look for rough analogies. The canonical distribution in energy has the form (the notation is slightly modernized, but the ideas were rather well understood in Wien's time)

$$\rho(E,T)\,dE = \phi(E)\,e^{-E/k_B T}\,dE \tag{can}$$

for some non-negative density of states[6] $\phi(E)$. Thus, assuming $\phi(E)$ positive and regular, we have the differential equation

$$\frac{\partial}{\partial T}T^2\frac{\partial}{\partial T}\log\big(E\,\rho(E,T)\big)=0.$$

In modern language Wien made an analogy between the distribution in energy $E\,\rho(E,T)\,dE$ and the distribution in frequency $u(\nu,T)\,d\nu$ and effectively identified the two. Their identification would be exact if the energy of an excited state of each oscillator was an univalued function of its frequency ν (which, of course, is not the case). The identification leads to a differential equation for Φ

$$0=\frac{\partial}{\partial T}T^2\frac{\partial}{\partial T}\log u(\nu,T)=\frac{\partial}{\partial T}T^2\frac{\partial}{\partial T}\log\big(\nu^3\,\Phi(\nu/T)\big)=\frac{\partial}{\partial T}T^2\frac{\partial}{\partial T}\log\Phi(\nu/T)$$

whose general integral is

$$\Phi(\nu/T)=A\,\exp(-B\,\nu/T) \tag{Wien}$$

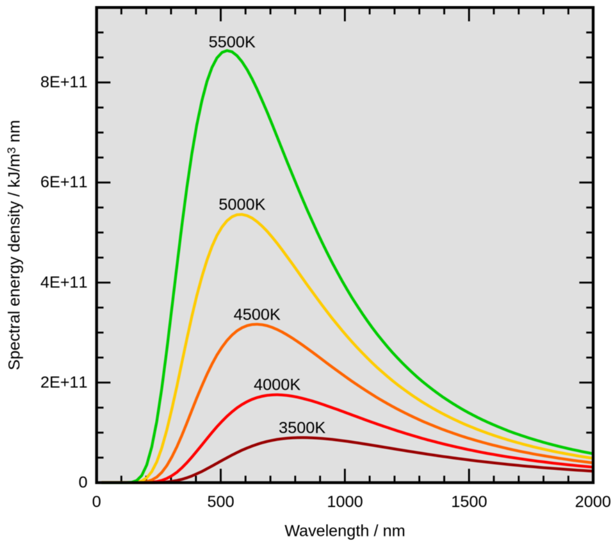
where A and B are two integration constants which are *universal fundamental* constants of Physics, entering in the description of all electromagnetic phenomena at thermal equilibrium.

The equation (Wien) is the celebrated (at the time) *Wien radiation formula* which he published in 1896. Again, the values of A, B are not fixed by the analogy and should be measured by fitting the existing experimental data. With hindsight we now know that $B=h/k_B$, the ratio of two fundamental constants, the Planck and the Boltzmann constants (both introduced by Planck).[7] In particular $B\neq 0$ since otherwise we would conclude that the Black Body radiation is totally independent of the temperature T, which contradicts the Stefan-Boltzmann law $u(T)\propto T^4$. But $B\neq 0$ means $h\neq 0$ i.e. the world is quantum. Thus the educated guess – *if correct!* – would predict that the world is quantum, not classical.

[6] Classically $\phi(E)$ is the volume of the hypersurface in phase space with energy E. In general $\phi(E)\,dE$ is just a measure which may be totally discontinuous with respect to the Lebesgue (as it happens for a quantum system with a purely discrete spectrum).

[7] The constant A is not a new fundamental one but a combination of other constants of nature, see below.

The experimental data known in the Summer of 1900[8]



confirmed Wien's formula to very high accuracy and the values of A and B were determined with good precision.

However new and more precise measures in September-October 1900 by Heinrich Rubens and Ferdinand Kurlbaum in Berlin showed that the Wien radiation formula is *not exactly correct*: there is a significant deviation when the wavelength is large (i.e. ν small, the infrared region of the spectrum) or equivalently T large. More precisely Rubens and Kurlbaum found a linear grow of $u(\nu,T) \propto T$ for large T. Their finding was physically intuitive: as $T \to \infty$ the energy density $u(\nu,T)$ at fixed frequency ν cannot go to the finite constant $A\nu^3$ as predicted by Wien's formula; it should grow indefinitely.

Why We Have This Intuition? Because our intuition is mostly "classical" and a linear growth with T is the classical answer as Lord Rayleigh had shown a few months before in May 1900, although the fully correct classical formula (CP) was written down only in 1905 by Einstein. Classical Physics is *not totally wrong*: it just happens to work only in the classical limit, which in the Black Body case is $\nu \to 0$ or equivalently $T \to \infty$. Said differently: the problem with Wien's formula was that it is correct only in the "*extremal* quantum regime": high frequencies and low temperatures.

The stage is now set, and it is time for our protagonist to arrive on the scene.

[8] The picture shows the results of modern experiments. The original curves in 1900 are almost indistinguishable from the present ones.

5. Planck Enters the Scene

Karl Ernst Ludwig Ma(r)x Planck (Max Planck for short)[9] was the successor of Kirchhoff as professor of Theoretical Physics at the University of Berlin (full professor since 1892).

Planck was a specialist in Thermodynamics and was greatly interested in its Second Law and its applications in Physics and Chemistry. In his view entropy was the most fundamental and deep quantity of Physics. His main work in the early 1890s was not in Theoretical Physics, but rather chemical thermodynamics. It should be pointed out that Planck was a *conservative* scientist, a strictly thermodynamical scholar and a *strong opponent* of the modern probabilistic (i.e. statistical mechanical) interpretation of entropy proposed a few year earlier by Boltzmann. Planck did *not* believe in the "atomic hypothesis" of Boltzmann. For him the world was continuous, no atoms or other discrete structures. Least of all Planck could accept Boltzmann's idea that entropy was "merely" an *emerging* quantity from the underlying microscopic dynamics, not a fundamental decree by God.

While his name is associated to the most radical revolution in the history of human thinking, Planck himself was an extremely conservative physicist for *all* his life, even after his epoch-making discoveries. He was *an unwilling and unintended revolutionary.*

To illustrate the enduring influence of the work Planck did in the 1890's, let me stress the following fact: in almost all subjects of Physics each author uses his own conventions and notations, so that half of each paper goes in fixing notations and conventions. But there is *one exception*: in Thermodynamics the symbols

$$V, p, T, S, U, F, H, \dots$$

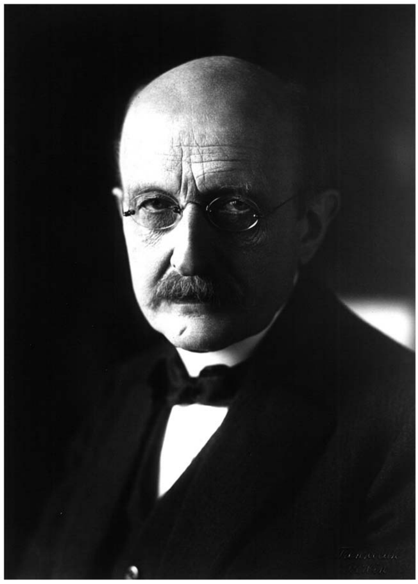
have the same identical meaning for everybody, and no one ever dares to use different symbols for the thermodynamical quantities. *Why?* It is the result of the authority and long-lasting influence of the treatise *Vorlesungen über Thermodynamics* that Planck published in 1897 which sets the standards for Thermodynamics still in use today.

Planck and the Black Body Radiation

Around 1895 Planck starts working on the Black Body radiation problem. At the time he strongly believed that the Wien formula was *exact*. His aim was to give a conceptual proof of it from the basic principles of Thermodynamics. At the beginning Planck wrote a few very wrong papers on the subject which we shall gloss over.

Then in 1899 he published in *Annalen der Physik* a thermodynamical argument leading to a differential relation between the entropy S_ν and the energy U_ν of each

[9] His name was misspelled in "Marx" in the birth certificate.



Karl Ernst Ludwig Ma(r)x Planck

oscillator of frequency ν present in the reflecting cavity. The relation between U_ν and the spectral density is given by the equation

$$u(\nu,T) = \frac{8\pi\nu^2}{c^3}U_\nu.$$

Planck's differential relation was

$$\frac{\partial^2 S_\nu}{\partial U_\nu^2} = -\frac{1}{a_\nu U_\nu},$$

where a_ν depends only on ν. Of course Planck was perfectly aware that the basic relation $\partial S/\partial U = 1/T$ implies

$$\left(\frac{\partial^2 S}{\partial U^2}\right)^{-1} \equiv \frac{\partial U}{\partial(1/T)},$$

thus his differential relation can be rewritten in a form which (in my view) is more natural:

$$T^2\frac{\partial u(\nu,T)}{\partial T} = a_\nu u(\nu,T).$$

This is a first order linear ODE instead of a non-linear second order one. But Planck was *too fond* of entropy to write his equations in terms of simpler quantities. I am pretty sure he used the simpler form in his actual computations, but thought that an equation *to be a fundamental law of Physics* should be written as a statement about the divine entropy, not on any lesser quantity.

Integrating either one of the two equivalent equations, one gets

$$u(\nu,T) = C_\nu \exp(-a_\nu/T),$$

where C_ν is an integration constant depending only on ν; then Wien's displacement law implies that $a_\nu = a\,\nu$ and $C_\nu = C\,\nu^3$ for a suitable constant C.

After five years of assiduous work Planck was finally happy: *he had proven Wien's formula from first principles!* (I suspect that his "first principle" argument was just a version of the heuristic guess we attributed to Wien, just with improved language and notations).

A few months later Rubens and Kurlbaum published their results showing that for large T the function $u(\nu,T)$ grows linearly, that is, in Planck "entropic" way of writing

$$\frac{\partial^2 S_\nu}{\partial U_\nu^2} = -\frac{1}{b_\nu U_\nu^2} \quad \text{for large } U_\nu$$

b_ν being a constant depending only on ν. Summarizing: the experiments were consistent with the expressions[10]

$$\text{(ED)} \qquad T^2\frac{\partial u(\nu,T)}{\partial T} = \begin{cases} a_\nu\, u(\nu,T) & \text{for } u(\nu,T) \text{ small} \quad \text{(IR)} \\ b'_\nu\, u(\nu,T)^2 & \text{for } u(\nu,T) \text{ large} \quad \text{(UV)} \end{cases}$$

The new data were a major blow to Planck: *the derivation of Wien's formula from the very first principles on which he had worked* so hard *for five years* was deadly wrong!!

Planck had to accept the crude fact that his "first-principle" argument had been purely *ad hoc* instead than fundamental. In the 3-pages paper [15], Planck tried to *guess* the correct formula *phenomenologically* from the data, without any claim of doing something "educated". He just wrote the *very simplest* expression interpolating the experimental data (ED):

$$T^2\frac{\partial u(\nu,T)}{\partial T} = a_\nu\, u(\nu,T) + b'_\nu\, u(\nu,T)^2,$$

or, in the way he preferred to write it to emphasize the role of entropy,

$$\frac{\partial^2 S_\nu}{\partial U_\nu^2} = \frac{\alpha_\nu}{U_\nu(\beta_\nu + U_\nu)} \qquad \left[\begin{array}{l} \alpha_\nu, \beta_\nu \text{ constants} \\ \text{depending only on } \nu \end{array}\right.$$

Integrating the equation, and using Wien's displacement law and the (now experimentally known) classical asymptotics as boundary conditions, one gets

$$(\bigstar) \qquad u(\nu,T) = \frac{C\nu^3}{e^{a\nu/T} - 1}$$

[10] b'_ν is a constant depending only on ν which is related to b_ν.

where C and a are fundamental constants which Planck determines in the follow-up paper [16] in the following form, giving also their numerical values

$$C = \frac{8\pi h}{c^3}, \quad a = \frac{h}{k_B}, \quad h = 6.55 \cdot 10^{-27} \text{ erg} \cdot \text{sec}, \quad k_B = 1.346 \cdot 10^{-16} \text{ erg/deg}$$

Here h is the *Planck constant* and k_B is the Boltzmann constant also introduced by Planck in this work, although (of course) it was implicit in Boltzmann's papers.

The Black Body formula (★) was announced by Planck in a meeting of the Berlin Academy of Science on December 14 1900, a day which is now officially regarded as the birthday of Quantum Theory.

Planck thought of his formula as a mere phenomenological guess. He introduced it just as the simplest interpolation between the two experimentally know regimes for small and large ν. In the words of the historian Jammer [17]:

> Never in the history of Physics was there such an inconspicuous mathematical interpolation with such far-reaching physical and philosophical consequences.

PLANCK IS NOW CONFRONTED WITH A NEW CHALLENGE

Anyhow, Planck's formula was confirmed by the experiments with very good accuracy, and Planck convinced himself that the formula was *exactly right,* even if he had no meaningful derivation for it. This state-of-affairs gave him a new challenge: *produce a first-principle derivation of his own formula!*

He started to work hard on a derivation of his formula, which was the topic of his second paper [16] that we already quoted above in relation with the numerical values of the various constants.

There was one aspect of his formula which Planck considered *nice and suggestive:* the fact that the entropy can be written in the form

$$(S_\nu) \qquad S_\nu = k_B \left\{ \left(1 + \frac{U_\nu}{h\nu}\right) \log\left(1 + \frac{U_\nu}{h\nu}\right) - \frac{U_\nu}{h\nu} \log \frac{U_\nu}{h\nu} \right\}$$

which he compared with the Boltzmann equation for the entropy

$$S = k_B \log W$$

where W is the statistical number of states. As stressed before, Planck *did not believe* in Statistical Mechanics, and even less in Boltzmann's probabilistic interpretation of entropy. But Boltzmann formula was the *only* inspiration Planck got, and he resorted to it, hoping that at the end of the computation he could get rid of the undesired equation.

Planck interpreted the equation in the sense opposite to Boltzmann's. For Boltzmann entropy is *emergent* from the microscopic dynamics of atoms, while for Planck entropy is the fundamental physical quantity, *an absolute Divine decree.* For him Bolzmann's equation is merely the definition of an *auxiliary* function

$W \equiv \exp(S/k_B)$ whose only meaning is to give a convenient parametrization: W has typically better analytical properties than S and its use is a technical trick to simplify the computations, not a fundamental principle.

In his second paper Planck shows that the function $W_\nu \equiv \exp(S_\nu/k_B)$ defined by the entropy (S_ν) of an oscillator of frequency ν is indeed equal to a statistical count of states *provided* one assumes that a system of several oscillators of frequency ν can exchange energy only in *integral multiples* of $h\nu$ for a certain constant h whose value he determines in the same paper.

We stress that for Planck the quantization of energy *was not a physical reality.* It was a mere technical trick to model a difficult continuous problem with a discrete one, and make the counting mathematically well defined. The number of states W_ν was not an intrinsic property for him: just a simple hand-waving way to model some yet to be understood deeper (and more traditional!) Physics.

Nothing illustrates what he was really thinking better than his own words. In a letter to Robert Wood, written as late as in 1931 (when Quantum Mechanics had already been set in its final form mainly by Dirac), he says:

> To summarize, all what happened can be described as simply an act of desperation… had been wrestling unsuccessfully for six years (since 1894) with the problem of equilibrium between radiation and matter […] [the] approach was opened to me by maintaining the two laws of thermodynamics […] they must, it seems to me, be upheld in all circumstances. For the rest, I was ready to sacrifice every one of my previous convictions about physical laws. Boltzmann had explained how thermodynamic equilibrium is established by means of a statistical equilibrium, and if such an approach is applied to the equilibrium between matter and radiation, one finds that the continuous loss of energy into radiation can be prevented by assuming that energy is forced, at the onset, to remain together in certain quanta. This was a purely formal assumption and I really did not give it much thought except that no matter what the cost, I must bring about a positive result

Thus on December 14 1900 the quantum revolution started, but nobody at the time seemed to notice it, least of all Planck. He and his contemporaries where happy with the impressive accuracy of the experimental predictions of the formula. The quantum jumps of energy were considered a trivial technicality that did not merit attention. Nobody, and *certainly not Planck*, realized that his new radiation law necessitated a break with classical Physics. Until the Rome lectures by Lorentz in 1908, no one mentioned the fact that classical Physics predicted a "ultraviolet catastrophe", and that "new Physics" was required otherwise no fire could possibly exist!

As a matter of fact, for the first decade of the 20^{th} century – with a *single* distinguish exception – no physicist believed that the quanta were for real. Planck himself never fully surrendered to the idea, even later when quantization became mainstream in the scientific community.

6. Einstein the *Fervent* Revolutionary

The one exception was Albert Einstein. While Planck was a conservative which *never* intended to make a revolution (and never fully accepted it even after he did it), Einstein wanted *ardently* to be a revolutionary who breaks away from traditional thinking and introduces brand new foundations of science. He was fully aware that his ideas were a *radical departure* from the established laws of Physics, and he enjoyed a lot the feeling of being the one who reshapes Science from scratch.

In 1905 Einstein published the three foundational papers of Modern Physics. Einstein himself thought that the most revolutionary paper of the three was the one on the photoelectric effect (for which he was later awarded the Nobel Prize). In a letter of May 1905 to his friend Habicht, he referred to his paper on *"radiation and the energetic properties of light"* as *"very revolutionary."* The paper under preparation on special relativity was more modestly referred to as *"an electrodynamics of moving bodies with the use of a modification of the ideas of space and time"*.

In his paper Einstein proposes that the light is *actually* made by *quanta* (later called *photons*) of energy $h\nu$, transforming what for Planck had been a mere computational trick into the fundamental fabric of Nature.

In addition, Einstein made *extremely clear* that the Planck formula was *at odds* with classical Physics. As already said, he computed the classical formula

$$u(\nu,T) = 8\pi\nu^2 k_B T/c^3,$$

correctly for the first time, for the only purpose of contrasting it with the Planck one which Einstein emphasized was very far from what classical Physics will predict.

Unfortunately nobody else shared his view at the time. His idea of quanta of light was too radical a revolution for his contemporaries. The situation did not improve even after Robert Millikan in 1916 confirmed experimentally the equation predicted by Einstein in 1905 for the maximal energy of a photon-produced electron as a function of the frequency of the incoming light. Despite the perfect experimental agreement, no physicists would believe that the photons were for real. I quote from the book by Kragh [18]:

> None of the experimentalists concluded in favor of Einstein's *"bold, not to say reckless hypothesis,"* as Millikan called it in 1916. What Millikan had confirmed was Einstein's equation, not his theory [...] It was possible to derive the experimentally confirmed equation without the light-quantum hypothesis, and when these more or less classical (and, indeed, more or less ad hoc) alternatives turned out to be *untenable*, there was always the possibility to declare the photoelectric effect unexplained for the time being. *This is what happened.*

What was Planck's reaction to Einstein hypothesis of "photons" as real objects?

We know that as late as in 1913 Planck was utterly against the idea. One episode is very illuminating about what he was thinking at the time about the whole photoelectric theory by Einstein. In 1913 Albert Einstein was proposed for membership in the prestigious Prussian Academy of Sciences. Planck was a member of the panel which had to present the candidate Einstein to the Academy for approval and write the *laudatio* to motivate his nomination. Yes, the panel praised Einstein, but also stated that the candidate

> may sometimes have missed the target in his speculations, as, for example, in his hypothesis of light-quanta.

Thus even in 1913 Planck (and the other Prussian academicians) still thought that quanta were quite unreasonable speculations.

7. More Fictional History

We return to the deformation-theoretical viewpoint that was briefly mentioned in §. **IV**.

I am told that Ed Witten once said:

> The way they arrived to the quantum theory is *very funny*. At the end of the 19^{th} century, after the work of Hamilton, Liouville, and Jacobi, they should have known that the classical laws of Physics admit *a canonical one-parameter space of deformations*. How is it possible that nobody asked what is the value of the deformation parameter $\hbar$ in the real world?

and then he added

> We must ascertain that we are not *doing the same mistake twice*. Are our *present* law of Physics *rigid*, or is their *universal space of deformations non-trivial*?

I leave the crucial question to the young readers who are tomorrow's scientists.

Side remark for the *cognoscenti*. Although the fact that classical Physics had a one-parameter space of deformations was implicit in the work of the classical mechanicians, it was only stated explicitly by Moyal in 1926 and the rigorous proof of the underlying mathematical statement was given only recently by Kontsevitch. See the book [19].

References

[1] N. Bourbaki, *Éléments de mathématique,* 30 volumes, Springer. MR0580296

[2] S. Weinberg, Dynamics and algebraic symmetries, in *Lectures on Elementary Particles and Quantum Field Theory.* Vol. 1, S. Deser (ed.), Massachusetts Institute of Technology Press, Cambridge, pp. 283–393.

[3] G. Kirchhoff, Pogg. Ann. **109** (1860) 275.

[4] M. Planck, *The Theory of Heat Radiation*, first edition (in German) 1908. MR0111466

[5] W. Wien and O. Lummer, *Methode zur Prüfung des Strahlungsgesetzes absolut schwarzer Körper,* Annalen der Physik **56** (1895) 451.

[6] J. M. Maldacena and A. Strominger, *Black hole grey body factors and d-brane spectroscopy,* Phys. Rev. D **55** (1997) 861–870, arXiv:hep-th/9609026 [hep-th]. MR1435261

[7] Lord Rayleigh, *Remarks upon the Law of Complete Radiation*, in The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **XLIX**, January–June 1900, pp. 539–541

[8] C. Vafa, *The string landscape and the swampland,* arXiv:hep-th/0509212.

[9] H. Ooguri and C. Vafa, *On the geometry of the string landscape and the swampland,* Nucl. Phys. B **766** (2007) 21–33, arXiv:hep-th/0605264. MR2302899

[10] D. Brennan, F. Carta and C. Vafa, *The string landscape, the swampland, and the missing corner,* arXiv:1711.00864.

[11] E. Palti, *The swampland: introduction and review,* arXiv:1903.06239. MR3963651

[12] J. Stefan, Wien. Berichte **79** (1879) 391.

[13] L. Boltzmann, *Ableitung des Stefan'schen Gesetzes, betreffend die Abhängigkeit der Wärmestrahlung von der Temperatur aus der electromagnetischen Lichttheorie* [Derivation of Stefan's law, concerning the dependency of heat radiation on temperature from the electromagnetic theory of light] Annalen der Physik und Chemie **258** (1884) 291–294.

[14] W. Wien, Wiedemann's Annal. **52** (1894) 132.

[15] M. Planck, *On an improvement of Wien's equation for the spectrum,* Verhandl Dtsch. phys. Ges. **2** (1900) 202.

[16] M. Planck, *On the law of distribution of energy in the normal spectrum,* Annalen der Physik **4** (1901) 553.

[17] M. Jammer, *The Conceptual Development of Quantum Mechanics* (1966).

[18] H. Kragh *Quantum Generations. A History of Physics in the Twentieth Century* (1999).

[19] N. Moshayedi, *Kontsevich's Deformation Quantization and Quantum Field Theory*, Lecture Notes in Mathematics, vol. **2311**. MR4485383

Sergio Cecotti

cecotti@bimsa.cn

BIMSA, Beijing (China)