On Eisenstein Congruences and Beyond

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Abstract. Eisenstein congruences play an important role in modern number theory. We survey some topics related to these congruences, starting from the example of Ramanujan's Delta function modulo 691. This paper does not contain any new results, except Theorem 2.4.

Contents

1. Introduction

1.1 The Example of Ramanujan's Delta Function

This paper is a survey about (higher) Eisenstein congruences and their applications in number theory. As far as the author is aware, the first example of Eisenstein congruences was found by Ramanujan in [46]. Consider the formal series $\Delta = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n$ (for $\tau(n)\in\mathbf{Z}$). Ramanujan proved that for all prime p we have

$$\tau(p) \equiv 1 + p^{11} \text{ (modulo } 691).$$

More generally, we have $\tau(n) \equiv \sigma_{11}(n)$ (modulo 691), where $\sigma_m(n) = \sum_{d|n} d^m$. This congruence is a consequence of the Eisenstein congruence

$$\Delta \equiv E_{12} \text{ (modulo 691)}, \tag{1.1}$$

where

$$E_k = -\frac{B_k}{2k} + \sum_{n\geq 1} \sigma_{k-1}(n) q^n$$

is the weight k Eisenstein series of level $\mathrm{SL}_2(\mathbf{Z})$, B_k being the kth Bernoulli number. This congruence can be proved by noticing that E_6^2 is a linear combination of E_{12} and Δ (*cf.* [49, §2.2]). The key fact is that the constant coefficient $-\frac{B_{12}}{24}$ of E_{12} is divisible by 691. A more conceptual argument for this congruence consists in noticing that the modular form E_{12} is cuspidal modulo 691, and thus must coincide with Δ modulo 691.

A different proof using modular symbols has been given by Manin in [31, §1.3]. The key fact used by Manin is a divisibility of special values of an L-function, which we now explain. The L-function of Δ is the Dirichlet series $L(\Delta, s) = \sum_{n\geq 1} \frac{\tau(n)}{n^s}$, which can be shown to extend to an entire function on $\mathbf{C}$. The *critical values* of $L(\Delta, s)$ are the values for $s = 1, 2, \ldots, 11$. Manin proved (in much greater generality) that the numbers

$$r_m(\Delta) := \frac{(m-1)!i^m}{(2\pi)^m} L(\Delta, m)$$

for $m \in \{1, 3, 5, 7, 9, 11\}$ are all proportional up to a rational number, *i.e.* the projective vector $[r_1(\Delta) : r_3(\Delta) : \ldots : r_{11}(\Delta)]$ belongs to $\mathbf{P}^5(\mathbf{Q})$ (*cf.* [31, §1.2]). In our example (for the eigenform Δ), Manin computed

$$\begin{aligned}&\left[r_1(\Delta) : r_3(\Delta) : \ldots : r_{11}(\Delta)\right] \\ &\quad = \left[1 : -\frac{691}{2^2 \cdot 3^4 \cdot 5} : \frac{691}{2^3 \cdot 3^2 \cdot 5 \cdot 7} : -\frac{691}{2^3 \cdot 3^2 \cdot 5 \cdot 7} : \frac{691}{2^2 \cdot 3^4 \cdot 5} : -1\right].\end{aligned} \tag{1.2}$$

Manin then expressed the coefficients $\tau(n)$ of Δ in terms of $r_m(\Delta)$ for $m = 3, 5, 7, 9$ (the "Coefficients Theorem" of [31, §1.3]). Since these values are all divisible by 691 (up to a common transcendental factor called a *period*), Manin's formula shows that $\tau(n) \equiv \sigma_{11}(n)$ (modulo 691).

Thus, we have seen two ways of understanding the Eisenstein congruence (1.1): either by observing that E_{12} is cuspidal modulo 691 (which requires the knowledge of its constant coefficient), or by observing that the odd special values of $L(\Delta, s)$ are (up to a period) divisible by 691. Let us explain why the second observation should be considered as a consequence of the first (at least philosophically).

The Eisenstein series E_{12} also has an associated L-function $L(E_{12}, s)$. One computes easily that $L(E_{12}, s) = \zeta(s)\zeta(s-11)$ where ζ is the Riemann zeta function. A congruence between eigenforms should philosophically give rise to a congruence between special values of L-functions. Therefore, congruence (1.1) should imply

$$r_m(\Delta) \equiv r_m(E_{12}) \text{ (modulo 691)}.$$

This does not quite make sense as the left-hand-side is likely transcendental, but such a congruence should hold after dividing by an appropriate period. To avoid choosing such a period, let us consider the corresponding element in the projective space: one should expect

$$(1.3)\quad \left[r_1(\Delta):r_3(\Delta):\ldots:r_{11}(\Delta)\right]\equiv\left[r_1(E_{12}):r_3(E_{12}):\ldots:r_{11}(E_{12})\right] \text{ (modulo 691)},$$

both sides being in $\mathbf{P}^5(\mathbf{Q})$. The functional equation of ζ gives $r_1(E_{12})=-r_{11}(E_{12})\neq 0$. Furthermore, since $\zeta(s)$ vanishes if s is an even negative integer, one sees that $r_m(E_{12})=0$ for $m\in\{3,5,7,9\}$. Thus, we have $[r_1(E_{12}):r_3(E_{12}):\ldots:r_{11}(E_{12})]=[1:0:\ldots:0:-1]$ in $\mathbf{P}^5(\mathbf{Q})$. We thus expect the Eisenstein congruence (1.1) to yield

$$\left[r_1(\Delta):r_3(\Delta):\ldots:r_{11}(\Delta)\right]\equiv[1:0:\ldots:0:-1] \text{ (modulo 691)}.$$

This is indeed true as (1.2) shows.

The same discussion applies for the even critical values. Manin showed that $[r_2(\Delta):r_4(\Delta):\ldots:r_{10}(\Delta)]$ belongs to $\mathbf{P}^4(\mathbf{Q})$ and computed

$$\left[r_2(\Delta):r_4(\Delta):\ldots:r_{10}(\Delta)\right]=\left[1:-\frac{5^2}{2^4\cdot 3}:\frac{5}{2^2\cdot 3}:-\frac{5^2}{2^4\cdot 3}:1\right].$$

On the other hand, we have for $m\in\{2,\ldots,10\}$ even

$$r_m(E_{12})=\frac{(m-1)!i^m}{(2\pi)^m}\zeta(m)\zeta(m-11)=\frac{B_{12-m}}{12-m}\cdot\frac{B_m}{2m}.$$

One gets $[r_2(E_{12}):r_4(E_{12}):\ldots:r_{10}(E_{12})]=[1/1584:1/28800:1/63504:1/28800:1/1584]$, and one can then check that

$$(1.4)\quad \left[r_2(\Delta):r_4(\Delta):\ldots:r_{10}(\Delta)\right]\equiv\left[r_2(E_{12}):r_4(E_{12}):\ldots:r_{10}(E_{12})\right] \text{ (modulo 691)}.$$

One thus sees that the congruence between modular forms (1.1) is reflected by two different congruences (1.3) and (1.4) between special values of L-functions. More precisely, (1.3) (resp. (1.4)) should be considered as a congruence modulo 691 in the space of even (resp. odd) weight 12 *period polynomials*. An even weight k period polynomial (resp. odd weight k period polynomial) is simply an even degree $k-2$ (resp. odd degree $k-3$) polynomial in $\mathbf{C}[X]$ satisfying some functional equations. We refer to Zagier [73, §2] for the original definition.

Let us note that the space of even weight k period polynomials has an *Eisenstein element* in characteristic 0, *i.e.* an element annihilated by the Hecke operators $T_n-\sigma_{k-1}(n)$, which is simply the polynomial $X^{k-2}-1$ corresponding to $[1:0:\ldots:0:-1]$. However, the space of odd weight k period polynomials does not have any Eisenstein element. Zagier enlarged this latter space so that it contains an Eisenstein element (*cf.* the polynomial p_k^- of [73, §2 Proposition]). Paşol and Popa generalized Zagier's definition to arbitrary weight and level in [44].

Thus, both congruences (1.3) and (1.4) can be considered as Eisenstein congruences in certain (extended) spaces of period polynomials of weight 12. Let us note that (1.4) appears to be more complicated than the "trivial" congruence (1.3) since the coefficients of the former involve Bernoulli numbers, while the latter coefficients are almost all zero modulo 691.

1.2 Some Questions

Ramanujan's Eisenstein congruence raises the following questions:

(Q1) If $k \geq 2$ and $\Gamma = \Gamma_1(N)$ or $\Gamma_0(N)$, do we have "Eisenstein congruences" in weight k and level Γ analogous to (1.1), (1.3) and (1.4) modulo some prime power $\mathfrak{p}^r$ (where $\mathfrak{p}$ is a prime ideal in the ring of integers of a number field)? What can be said about $\mathfrak{p}^r$? More generally, do we have "Eisenstein congruences" for other spaces of automorphic forms, *e.g.* for Bianchi modular forms (GL_2 over an imaginary quadratic field), Shimura curves (non-definite quaternion algebras) or over function fields?

(Q2) Since we know that $\frac{r_3(\Delta)}{r_1(\Delta)}, \dots, \frac{r_9(\Delta)}{r_1(\Delta)}$ are divisible by $p = 691$, can we understand the vector $(\frac{r_3(\Delta)}{p \cdot r_1(\Delta)}, \dots, \frac{r_9(\Delta)}{p \cdot r_1(\Delta)})$ modulo p ("beyond" Eisenstein congruences)? This would perhaps restore the balance between even and odd period polynomials, since (1.4) appears to lie deeper than (1.3). Does this question make sense in more general situations (*e.g.* weight k and level $\Gamma_1(N)$)?

(Q3) What are the applications of Eisenstein congruences in algebraic number theory or arithmetic geometry? For instance, can our Eisenstein congruences for special values of L-functions give some results on the Bloch–Kato conjecture (a generalization of the Birch and Swinnerton–Dyer conjecture for elliptic curves)?

(Q4) On a more technical note, many of the known results (for $\mathrm{GL}_2/\mathbf{Q}$) on Serre's conjecture, the Fontaine–Mazur conjecture, $R = T$ theorems, etc., require the Galois representation attached to our cuspidal eigenform to be residually irreducible. What happens in the reducible case?

The goal of this expository paper is to survey some of the known results about these four questions (which are obviously related to each other). The literature on this topic being very large and the knowledge of the author being limited, we will certainly miss some important results and fail to cite many papers. We apologize for this and do not claim to be exhaustive in our survey.

2. Some Answers

One of the key objects used to study these questions is the *Eisenstein ideal*. This was first defined by Mazur in weight 2 in his seminal 1977 paper "Modular curves and the Eisenstein ideal" [32]. The Eisenstein ideal corresponding to an Eisenstein series E is the annihilator I of E in "the" Hecke algebra $\mathbb{T}$ (over $\mathbf{Z}$). Here, by $\mathbb{T}$ we mean the Hecke algebra acting on all modular forms of a given weight and level (including Eisenstein series). The cuspidal quotient of $\mathbb{T}$ (acting on cuspforms) is denoted by $\mathbb{T}^0$. It is not clear what the correct definition of $\mathbb{T}$ should be: which operators should we include at primes dividing the level N? Some authors consider $\mathbb{T}$ where the Atkin operators U_p (for $p \mid N$) are included (*e.g.* [72] and other papers of that author), while some authors consider instead the Atkin–Lehner involutions w_ℓ when this makes sense (*e.g.* [43, 65] in squarefree level).

There are many results and conjectures related to the above questions. Let us give a brief overview of some of these results and conjectures.

2.1 Overview of Q1

One "measure" of the Eisenstein congruences is the finite abelian group $\mathbb{T}^0/I$. In particular, a prime p is in the support of $\mathbb{T}^0/I$ if and only if there is a cuspidal eigenform f (of our fixed weight and level) congruent to the Eisenstein series E modulo a prime ideal dividing p. In weight 2 and level $\Gamma_0(N)$ for a prime N, Mazur proved that $\mathbb{T}^0/I$ is cyclic of order the numerator of $\frac{N-1}{12}$ (*cf.* [32, Proposition II.9.7]). This number is (up to a factor 2) the constant coefficient at the cusp infinity of the unique Eisenstein series in $M_2(\Gamma_0(N))$, which should be expected since we are considering for which coefficient ring the Eisenstein series is cuspidal.

This result was generalized by Ohta in weight 2 and level $\Gamma_0(N)$ where N is squarefree, after inverting 2 (*cf.* [43, Theorem 3.1.3]). As mentioned above, Ohta includes the Atkin–Lehner involutions in the Hecke algebra $\mathbb{T}$. Yoo proved a theorem similar to Ohta but including the operators U_p instead of the Atkin–Lehner involutions, in some cases after inverting 2 (*cf.* [69, Theorem 1.1] and [70, Theorems 1.3 and 1.4]). In weight 2 and non-squarefree level, the ℓ-primary part of $\mathbb{T}^0/I$ was determined for some choices of I and ℓ by Yoo in [72, Theorem 1.3 (2) and Theorem 4.3].

In weight $k > 2$ and level $\mathrm{SL}_2(\mathbf{Z})$, Kurihara showed that $\mathbb{T}^0/I \otimes \mathbf{Z}_p \simeq \mathbf{Z}_p/B_k \cdot \mathbf{Z}_p$ for odd primes p with $k < p-1$ (*cf.* [21, Lemma 3.1]). This shows that the analogue of Ramanujan's congruence (1.1) holds in level $\mathrm{SL}_2(\mathbf{Z})$ and weight $k < p-1$ if p divides B_k. Ohta considered the Hida theoretic analogue of $\mathbb{T}^0/I$ in [40, Theorem 1.5.5]. His results should (by descent) yield a description of $\mathbb{T}^0/I \otimes \mathbf{Z}_p$ in weight k and level $\Gamma_1(N)$ under the assumptions that p divides N but p does not divide $\varphi(N)$ (the Euler function), combined with some restrictions on the Eisenstein series E. Ohta also computed the index $\mathbb{T}^0/I$ (away from 2) in weight 2 and level $\Gamma_1(N)$ when N is prime [42, Theorem II].

We thus have a good understanding of the generalization of the Eisenstein congruence (1.1). Let us now turn to the two congruences (1.3) and (1.4). We are basically asking whether a congruence between a cuspidal eigenform and an Eisenstein series yields a congruence between special values of L-functions (possibly twisted by Dirichlet characters).

Mazur studied this question in [33]. His setting was as above: weight 2 and level $\Gamma_0(N)$ where N is prime. Mazur gave a congruence formula for $L(f,\chi,1)$ for most *odd* characters χ (*cf.* [33, §7 Proposition]). Vatsal greatly generalized Mazur's result in weight 2 and level $\Gamma_1(N)$ under some conditions on N and the Eisenstein series E (*cf.* [61, Theorem 2.10]). As for Mazur's result, the congruence only holds for "half" of the special values, namely for those values $L(f,\chi,i)$ where i and χ satisfy a certain sign condition depending on E (*cf.* (2.2) below).

Vatsal and Heumann later generalized Vatsal's results to any weight $k \geq 2$ and level $\Gamma_1(N)$ (*cf.* [15, Theorem 5.2]). Their assumptions are $2 \leq k \leq p-1$ and $p \nmid N$, where p is the congruence prime. Again, there is a parity restriction on χ and i. Let us mention that Hirano states in [16, Theorem 0.1] a generalization of the Vatsal–Heumann result by allowing $p \mid N$. The author had a superficial look at Hirano's paper, and it seems to us that the statement of [16, Proposition 2.4] is incorrect. Indeed, the left-hand side of [16, (2.3)] could *a priori* be of dimension > 1 since the localization is only with respect to the *maximal* ideal $\mathfrak{M}_f$ (to get dimension 1, one would need to localize at the *height one prime* ideal $\mathfrak{P}_f$ associated with the cuspidal eigenform f). We have not checked and do not know whether Hirano's proof of [16, Theorem 0.1] still holds.

All the congruences mentioned in the previous two paragraphs (Mazur, Vatsal, Vatsal–Heumann and Hirano) are a generalization of (1.4): they involve some product of (generalized) Bernoulli numbers. Let us give the main ideas behind these results. Let $\Gamma = \Gamma_0(N)$ or $\Gamma = \Gamma_1(N)$ for some $N \geq 1$. Let us suppose we have an Eisenstein congruence $f \equiv E$ (modulo ϖ^r), where f is a cuspidal eigenform in $S_k(\Gamma)$ and ϖ is some uniformizer in a finite extension $\mathcal{O}$ of $\mathbf{Z}_p$. The Eichler–Shimura construction gives rise (after normalization) to a cohomology class $\delta_f^{\alpha} \in H^1(\Gamma, V_{k-2}(\mathcal{O}))^{\pm}$, where $V_{k-2}(R)$ are polynomials of degree $\leq k-2$ in $R[X]$. The sign $\alpha = \pm 1$ corresponds to the eigenvalue for the action of the complex conjugation. Stevens defined similarly a cohomology class δ_E in weight 2 in [58] and [59] (there is only one sign for E, which Vatsal–Heumann denote by $\mathrm{sgn}(E)$). Vatsal and Heumann generalize Stevens' construction to any weight, and construct some cohomology class $\delta_E \in H^1(\Gamma, V_{k-2}(\mathcal{O}))^{\mathrm{sgn}(E)}$. Since the Eisenstein series E is "cuspidal" modulo ϖ^r, the class δ_E modulo ϖ^r belongs to $H^1_{\mathrm{par}}(\Gamma, V_{k-2}(\mathcal{O}/\varpi^r))^{\mathrm{sgn}(E)}$, where H^1_{par} is the parabolic cohomology subgroup.

Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{T}$ corresponding to the congruence $f \equiv E$ (modulo ϖ^r). One can show that there is an isomorphism of $\mathbb{T}^0_{\mathfrak{m}}$-modules

$$H^1_{\mathrm{par}}\left(\Gamma, V_{k-2}\left(\mathcal{O}/\varpi^r\right)\right)^{\mathrm{sgn}(E)} \otimes_{\mathbb{T}^0} \mathbb{T}^0_{\mathfrak{m}} \simeq \mathrm{Hom}\left(\mathbb{T}^0_{\mathfrak{m}}, \mathcal{O}/\varpi^r\right). \tag{2.1}$$

In weight $k=2$ this follows from Wiles' work on Fermat's Last Theorem [67], as noticed by Vatsal in the proof of [61, Theorem 2.10]. The case $k > 2$ can be reduced to the weight 2 case using Hida theory, as sketched at the end of the proof of [15, Theorem 5.2].

By (2.1), we have $\delta_f^{\mathrm{sgn}(E)} \equiv c \cdot \delta_E$ in $H^1_{\mathrm{par}}(\Gamma, V_{k-2}(\mathcal{O}/\varpi^r))^{\mathrm{sgn}(E)}$ for some $c \in (\mathcal{O}/\varpi^r)^{\times}$, since these two classes are residually non trivial and have the same Hecke eigenvalues modulo ϖ^r by assumption. Here, the subscript H^1_{par} means parabolic cohomology. From this, Heumann and Vatsal manage to get a congruence between (normalized) L-values $L(f,\chi,i)$ and $L(E,\chi,i)$. The sign restriction is

$$(-1)^{i-1}\cdot\chi(-1)=\mathrm{sgn}(E). \tag{2.2}$$

For instance, in weight k and level $\mathrm{SL}_2(\mathbf{Z})$, we have $\mathrm{sgn}(E_k)=-1$.

What about congruence (1.3)? To our knowledge, even the obvious generalization in weight k and level $\mathrm{SL}_2(\mathbf{Z})$ is still open in general. The generalization of the space of even polynomials of weight k in a ring R is the space $\mathrm{Symb}_\Gamma(V_{k-2}(R))^{-\mathrm{sgn}(E)}$, where if M is any Γ-module, $\mathrm{Symb}_\Gamma(M)$ is the space of Γ-equivariant homomorphisms $\mathbf{Z}[\mathbf{P}^1(\mathbf{Q})]^0\to M$ (the exponent 0 means divisors of degree zero). The $\mathbb{T}$-modules $\mathrm{Symb}_\Gamma(V_{k-2}(R))^{-\mathrm{sgn}(E)}$ and $H^1(\Gamma,V_{k-2}(R))^{\mathrm{sgn}(E)}$ are dual.

Thus, to generalize (1.3) one would need $H^1(\Gamma,V_{k-2}(\mathcal{O}/\varpi^r))^{\mathrm{sgn}(E)}$ to be locally free of rank one over $\mathbb{T}\otimes\mathcal{O}/\varpi^r$ at $\mathfrak{m}$. It seems to us that by (2.1), $H^1(\Gamma,V_{k-2}(\mathcal{O}))^{\mathrm{sgn}(E)}\otimes_{\mathbb{T}}\mathbb{T}_{\mathfrak{m}}$ should be isomorphic to $\mathrm{Hom}_{\mathbf{Z}_p}(\mathbb{T}_{\mathfrak{m}},\mathcal{O})$ as a $\mathbb{T}_{\mathfrak{m}}$-module. One would need an isomorphism of $\mathbb{T}_{\mathfrak{m}}$-modules $\mathrm{Hom}(\mathbb{T}_{\mathfrak{m}},\mathbf{Z}_p)\simeq\mathbb{T}_{\mathfrak{m}}$, *i.e.* that $\mathbb{T}_{\mathfrak{m}}$ is Gorenstein.

This is a difficult question, which is open in general. It is sufficient that $I\cdot\mathbb{T}^0_{\mathfrak{m}}$ be a principal ideal (*cf.* [10, Lemma 1.11] and [32, Proposition II.15.3]). Mazur proved that $I\cdot\mathbb{T}^0_{\mathfrak{m}}$ is principal in weight 2 and level $\Gamma_0(N)$ if N is prime. In weight k and level $\mathrm{SL}_2(\mathbf{Z})$, Kurihara proved in [21, Theorem 0.4] that, under the condition that $\mathscr{C}(\mathbf{Z}[\zeta_p])(\omega_p^{2-k})=0$, the ideal $I\cdot\mathbb{T}^0_{\mathfrak{m}}$ is principal if and only if $\mathscr{C}(\mathbf{Z}[\zeta_p])(\omega_p^{1-k})$ is cyclic, where $\omega_p:\mathrm{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})\to\mathbf{Z}_p^\times$ is the Teichmüller character and $\mathscr{C}(\mathbf{Z}[\zeta_p])(\omega_p^i)$ is ω_p^i-eigenspace of the p-class group of $\mathbf{Z}[\zeta_p]$. These two conditions on the class group are consequences of the Vandiver conjecture, but remain open.

In conclusion, (1.4) generalizes well because it is related to some modular symbols for which multiplicity-one is known, while the generalization of (1.3), while expected in some cases, remains open because we do not know in general that the Hecke algebra is Gorenstein at Eisenstein primes.

2.2 Overview of Q2

2.2.1 The Case of Weight k and Level 1

Let us first consider the situation in weight k and level $\mathrm{SL}_2(\mathbf{Z})$. We have seen in the previous section that if $p\geq 5$ divides B_k and $k\leq p-1$ then there is a cuspidal eigenform $f\in S_k(\mathrm{SL}_2(\mathbf{Z}),\mathcal{O})$ congruent to E_k modulo ϖ^r (where ϖ is a uniformizer in a finite extension $\mathcal{O}$ of $\mathbf{Z}_p$ and $r\geq 1$ is maximal). Following Zagier [73, §2], one can attach to f two period polynomials

$$r_f^-(X)=\sum_{\substack{1\leq m\leq k-3\\ m\text{ odd}}}\binom{k-2}{m}r_{m+1}(f)X^m$$

and

$$r_f^+(X)=\sum_{\substack{0\leq m\leq k-2\\ m\text{ even}}}\binom{k-2}{m}r_{m+1}(f)X^m$$

where we recall that

$$r_{m+1}(f) = \frac{m! i^{m+1}}{(2\pi)^{m+1}} L(f, m+1).$$

We have $r_f^{\pm}(X) \in \mathrm{Symb}_{\mathrm{SL}_2(\mathbf{Z})}(V_{k-2}(\mathbf{C}))^{\pm}$. Fix embeddings $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ and $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. By the discussion of §2.1, one can find a period $\Omega_f^- \in \mathbf{C}^\times$ such that $\frac{r_f^-(X)}{\Omega_f^-} \in \mathcal{O}[X]$ and

$$\frac{r_f^-(X)}{\Omega_f^-} \equiv r_0^-(X) \text{ (modulo } \varpi^r),$$

where

$$r_0^-(X) = \sum_{\substack{-1 \le m \le k-1 \\ m \text{ odd}}} \frac{B_{m+1}}{(m+1)!} \frac{B_{k-m-1}}{(k-m-1)!} X^m$$

is Zagier's odd Eisenstein extended period polynomial (denoted by $p_k^-(X)$ in [73, §2 Proposition]). Note that $\varpi^r \mid \frac{B_k}{k!}$ by assumption, so $r_0^-(X)$ is indeed a polynomial modulo ϖ^r (there is no term in X^{-1}).

For even period polynomials, we have seen that conjecturally one expects the existence of a period $\Omega_f^+ \in \mathbf{C}^\times$ (which can be taken to be $L(f,1)$) such that $\frac{r_f^+(X)}{\Omega_f^+} \in \mathcal{O}[X]$ and

$$\frac{r_f^+(X)}{\Omega_f^+} \equiv r_0^+(X) \text{ (modulo } \varpi^r), \tag{2.3}$$

where

$$r_0^+(X) = X^{k-2} - 1.$$

As explained in §2.1, (2.3) would hold true if we assume the following consequences of Vandiver's conjecture:

$$\mathscr{C}(\mathbf{Z}[\zeta_p])(\omega_p^{2-k}) = 0 \text{ and } \mathscr{C}(\mathbf{Z}[\zeta_p])(\omega_p^{1-k}) \text{ is cyclic.} \tag{2.4}$$

We shall assume that the two conditions stated in (2.4) hold in what follows.

In particular, for any odd $m \in \{3, \ldots, k-3\}$ we have $\varpi^r \mid \frac{r_m(f)}{L(f,1)}$. Question 2 asks what can be said about the vector

$$\left(\frac{r_m(f)}{\varpi^r \cdot L(f,1)} \text{ (modulo } \varpi) \right)_{m \in \{3, \ldots, k-3\}} \in \mathbf{F}_p^{\frac{k}{2}-2}$$

up to a non-zero scalar (since ϖ was chosen arbitrarily anyway). In other words, can we determine the polynomial

$$r_1^+(X) \equiv \sum_{\substack{2 \le m \le k-4 \\ m \text{ even}}} \binom{k-2}{m} \frac{r_{m+1}(f)}{\varpi^r \cdot L(f,1)} X^m \in \mathbf{F}_p[X]$$

up to non-zero scalar? Sharifi conjectured a beautiful and amazingly simple formula for that polynomial.

In order to state his conjecture, let us introduce some notation. If A is a commutative ring, let $K_2(A)$ be Quillen's second K-group of A. If x and y are in $A^\times$, one can define the Steinberg symbol $\{x,y\} \in K_2(A)$. It is bilinear in x and y and satisfies the relation $\{x,y\}=0$ whenever $x+y=1$ (this is called the Steinberg relation). If A is a field, these Steinberg symbols and relations describe completely $K_2(A)$, namely we have an isomorphism

$$K_2(A) \simeq A^\times \otimes_{\mathbf{Z}} A^\times / \langle \{\{x,y\} : x+y=1\} \rangle.$$

If $A = \mathcal{O}_{K,S}$ is the S-ring of integers in a number field K, where S is a finite set of primes, then we have an exact sequence

$$0 \to K_2(\mathcal{O}_{K,S}) \to K_2(K) \to \bigoplus_{\mathfrak{p} \notin S} \mathbf{F}_{\mathfrak{p}}^\times \to 0, \tag{2.5}$$

the map $K_2(K) \to \mathbf{F}_{\mathfrak{p}}^\times$ being the *tame symbol*, *i.e.* it sends $\{x,y\}$ to $(-1)^{v_{\mathfrak{p}}(x)v_{\mathfrak{p}}(y)} \frac{x^{v_{\mathfrak{p}}(y)}}{y^{v_{\mathfrak{p}}(x)}}$ modulo $\mathfrak{p}$.

Let $\zeta_p \in \overline{\mathbf{Q}}$ be a primitive pth root of unity. There is an action of $\mathrm{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})$ on $K_2(\mathbf{Z}[\frac{1}{p},\zeta_p])$ and one can show that under our assumptions on class groups, there is an isomorphism $\varphi : (K_2(\mathbf{Z}[\frac{1}{p},\zeta_p]) \otimes \mathbf{Z}/p\mathbf{Z})(\omega_p^{2-k}) \xrightarrow{\sim} \mathbf{Z}/p\mathbf{Z}$.

We can now state Sharifi's conjecture (*cf.* [54, Conjecture 3]).

Conjecture 2.1 (Sharifi). *One can normalize φ such that we have in $\mathbf{F}_p[X]$*

$$r_1^+(X) = \sum_{\substack{2 \le m \le k-4 \\ m \text{ even}}} \binom{k-2}{m} \varphi(\{\eta_{m+1}, \eta_{k-m-1}\}) X^m$$

where if j is odd, $\eta_j = \prod_{a=1}^{p-1}(1-\zeta_p^a)^{a^{j-1}} \in \mathbf{Z}[\frac{1}{p},\zeta_p]^\times$.

Since $r_1^+(X)$ is non-zero modulo p, Conjecture 2.1 implies the following purely algebraic conjecture:

Conjecture 2.2. *The Steinberg symbols $\{\prod_{a=1}^{p-1}(1-\zeta_p^a)^{a^m}, \prod_{a=1}^{p-1}(1-\zeta_p^a)^{a^{k-2-m}}\}$ for even $m \in \{2,\dots,k-4\}$ generate $(K_2(\mathbf{Z}[\frac{1}{p},\zeta_p]) \otimes \mathbf{Z}/p\mathbf{Z})(\omega_p^{2-k})$.*

As far as we know, Conjecture 2.2 remains open. This is a refined version of an earlier conjecture of McCallum and Sharifi [37, Conjecture 5.3], which was made before Sharifi discovered the relation between Steinberg products and Eisenstein congruences. The best result we currently have toward Conjecture 2.1 is the following

Theorem 2.3. *Under the assumption (2.4), Conjectures 2.1 and 2.2 are equivalent.*

Let us give the idea of the proof. One easily sees that for any prime ℓ, there exists $c_\ell \in \mathbf{F}_p$ such that

$$(T_\ell - \ell^{k-1} - 1)(r_1^+) = c_\ell \cdot r_0^+, \tag{2.6}$$

where T_ℓ is the ℓth Hecke operator.

Furthermore, there exists ℓ such that $c_\ell \neq 0$, *i.e.* r_1^+ is not annihilated by the Eisenstein ideal I but instead by I^2. We call r_1^+ a *higher Eisenstein element*. The notion of higher Eisenstein elements was introduced in my thesis in Mazur's setting [25]. The vector consisting of the c_ℓ's is unique up to scaling; let us fix some particular choice. Under (2.4), the relation (2.6) (for all ℓ) characterizes uniquely r_1^+ modulo a multiple of r_0^+. Let us note that we do not have a simple formula for the coefficients c_ℓ. They may be expressed using Merel's type formulae on Hecke operators acting on Manin symbols [38], but it is unclear whether the resulting formula can be simplified.

Let us assume Conjecture 2.2. To prove Conjecture 2.1, it is enough to prove that the polynomial $\sum_{\substack{2\leq m\leq k-4 \\ m \text{ even}}} \binom{k-2}{m}\varphi(\{\eta_{m+1},\eta_{k-m-1}\})X^m$ is annihilated by I^2 (it cannot be annihilated by I because otherwise it would be proportional to r_0^+, which is clearly not the case since it is non-zero and its coefficient in X^{k-2} is zero). It suffices to show that the image of

$$\sum_{\substack{2\leq m\leq k-4 \\ m \text{ even}}} \binom{k-2}{m}\varphi\big(\{\eta_{m+1},\eta_{k-m-1}\}\big)X^m$$

in $H^1_{\mathrm{par}}(\mathrm{SL}_2(\mathbf{Z}), V_{k-2}(\mathbf{F}_p))$ is annihilated by I. This has been proved by Fukaya–Kato in [12, Theorem 5.2.3]. Fukaya and Kato actually work in level $\Gamma_1(p)$ and weight 2, but it is well-known that one can always pass from modular symbols of level $\mathrm{SL}_2(\mathbf{Z})$ and weight k modulo p to modular symbols of level $\Gamma_1(p)$ and weight 2 modulo p (with nebentype ω_p^{k-2}).

There is actually a whole sequence of higher Eisenstein elements, $r_0^+, r_1^+, \ldots, r_g^+$ in $\mathbf{F}_p[X]$ for some $g \geq 1$. They satisfy $(T_\ell - \ell^{k-1} - 1)(r_i^+) = c_\ell \cdot r_{i-1}^+$ modulo the subgroup generated by $r_0^+, \ldots, r_{i-2}^+$, for all primes ℓ and $1 \leq i \leq g$. The element r_i^+ is uniquely determined by this property, modulo the subgroup generated by $r_0^+, \ldots, r_{i-1}^+$. These elements are the key to go beyond the Eisenstein congruence (2.3). It would be very interesting to know r_2^+, when it exists.

Similarly, there exists a sequence of higher Eisenstein elements $r_0^-, r_1^-, \ldots, r_g^-$; these are extended period polynomials (with coefficients in X^{k-1} and X^{-1}). We do not know what the coefficients of r_1^- are. Presumably, these should be linear combinations of products of Bernoulli numbers and Steinberg symbols. We still have some non-trivial information about r_1^-. To our knowledge, although this result is not deep it is new.

Theorem 2.4. *Assume that Conjecture 2.1 is true. Write*

$$r_1^- = \sum_{\substack{-1\leq m\leq k-1 \\ m \text{ odd}}} a_m \cdot X^m$$

for $a_m \in \mathbf{F}_p$. *Then we have in* $\mathbf{F}_p$*:*

$$(4-2k)\cdot a_{k-1} + \sum_{\substack{1\le m\le k-3 \\ m \text{ odd}}} a_m$$
$$= \sum_{\substack{m+n\ge k-2 \\ m \text{ even } \ge 2, n \text{ odd } \ge 1}} \binom{n}{k-2-m} \varphi(\{\eta_{m+1},\eta_{k-m-1}\}) \frac{B_{n+1}}{(n+1)!}\frac{B_{k-n-1}}{(k-n-1)!}.$$

This common quantity is zero if and only if $g \ge 2$.

Note that φ is only unique up to a scalar, but that scaling it also scales the coefficients a_m by the same factor (because the coefficients c_ℓ above will also be scaled by that factor).

Let us sketch a proof. If R is a commutative ring in which $(k-1)!$ is invertible, let $\hat{W}_{k-2}(R)$ be Zagier's space of extended period polynomials of weight k (*cf.* [73, §2 Theorem]). It decomposes into even and odd parts: $\hat{W}_{k-2}(R) = \hat{W}_{k-2}(R)^+ \bigoplus \hat{W}_{k-2}(R)^-$. We have $\hat{W}_{k-2}(R)^+ \subset R[X]_{k-2}$ while $\hat{W}_{k-2}(R)^- \subset X^{-1}\cdot R[X]_k$. The spaces $\hat{W}_{k-2}(R)^{\pm}$ carry an action of Hecke operators.

There is a perfect Hecke equivariant bilinear pairing $\bullet : \hat{W}_{k-2}^+(\mathbf{F}_p) \times \hat{W}_{k-2}^-(\mathbf{F}_p) \to \mathbf{F}_p$ defined by the formula

$$\sum_{\substack{0\le m\le k-2 \\ m \text{ even}}} a_m\cdot X^m \bullet \sum_{\substack{-1\le n\le k-1 \\ n \text{ odd}}} b_n\cdot X^n \mapsto \sum_{\substack{m+n\ge k-2 \\ m \text{ even and } n \text{ odd}}} \delta_n \cdot \frac{m!n!}{(m+n-k+2)!(k-2)!}\cdot a_m\cdot b_n$$

where $\delta_n = 1$ except if $n=k-1$ in which case $\delta_n = 2$. This definition is taken from a formula of Kohnen and Zagier (generalizing an earlier formula of Haberland) expressing the Petersson product of two modular forms as a pairing between their (extended) period polynomials (*cf.* [20, p. 246]). The Hecke property satisfied by the higher Eisenstein elements shows that the pairing $r_i^+ \bullet r_j^-$ depends only on $i+j$, is zero if $i+j<g$ and non-zero if $i+j=g$.

We have checked that the identity of Theorem 2.4 holds using the tables of Sharifi and McCallum [37] available on Sharifi's webpage [51] for $p \le 1129$. It turns out that for $p\le 1129$, we have $g\ge 2$ only when $(p,k) = (547,486)$. This example has already been noticed by Calegari in his computations on Galois deformations [4, p. 68]. When $g\ge 2$, there exist higher Eisenstein elements r_2^- and r_2^+. We do not know any explicit result concerning these elements, and in particular concerning the pairing $r_1^+\bullet r_2^- = r_2^+\bullet r_1^-$.

2.2.2 Sharifi's Conjecture in Weight 2

Sharifi's conjecture has been stated more generally in weight $k=2$ and level $\Gamma_1(N)$. We refer the reader to [52, Conjecture 5.8] for the original conjecture, and to [53, Conjecture 4.3.5] for a slightly more general formulation. One should be able to formulate the conjecture in weight $k>2$ using Hida theory, but we are not aware of such a formulation in the literature, except in the case of level $N=1$ considered above. Let us briefly explain what the conjecture says (in weight 2).

Let $H_1(X_1(N), \text{cusps}, \mathbf{Z})$ be the singular homology relative to the cusps of the (compact) modular curve $X_1(N)$ of level $\Gamma_1(N)$. If α and β are in $\mathbf{P}^1(\mathbf{Q})$, let $\{\alpha,\beta\}\in$

$H_1(X_1(N),\text{cusps},\mathbf{Z})$ be the class of the closed geodesics in the upper-half plane between α and β (this is called a *modular symbol*). If $(u,v) \in (\mathbf{Z}/N\mathbf{Z})^2$ is such that $\gcd(u,v,N)=1$, let $\xi(u,v) \in H_1(X_1(N),\text{cusps},\mathbf{Z})$ be the modular symbol $\{\frac{b}{d},\frac{a}{c}\}$ where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ is such that $(c,d) \equiv (u,v)$ (modulo N); one can check that this does not depend on the choice of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Manin proved that the elements $\xi(u,v)$ (called *Manin symbols*) generate $H_1(X_1(N),\text{cusps},\mathbf{Z})$, and also determined all the relations satisfied by these elements [30, §1.9 Theorem].

Using Manin's presentation of $H_1(X_1(N),\text{cusps},\mathbf{Z})$, Sharifi defined a map

$$\varpi : H_1\left(X_1(N),C^o,\mathbf{Z}\right) \to K_2\left(\mathbf{Z}\left[\zeta_N,\frac{1}{N}\right]\right) \otimes \mathbf{Z}\left[\frac{1}{2}\right]$$

where C^o is the subset of cusps in $X_1(N)$ not lying over the cusp ∞ of $X_0(N)$. This map is characterized by the beautiful formula

$$\varpi\left(\xi(u,v)\right) = \left\{1-\zeta_N^u, 1-\zeta_N^v\right\}$$

for $u,v \in (\mathbf{Z}/N\mathbf{Z}) - \{0\}$. Sharifi conjectured the following:

Conjecture 2.5 (Sharifi)**.** *The map ϖ is annihilated by the Eisenstein ideal I generated by the Hecke operators $T_\ell - \ell - \langle \ell \rangle$ for primes $\ell \nmid N$, where $\langle \ell \rangle$ is the ℓth diamond operator. (There is also a conjecture regarding the Hecke operators for $\ell \mid N$.)*

Much is now known about this conjecture. Fukaya and Kato proved it after tensoring with $\mathbf{Z}_p$ when $p \geq 5$ divides N (*cf.* [12, Theorem 5.2.3]). More recently, Sharifi and Venkatesh proved that the restriction of ϖ to $H_1(X_1(N),\mathbf{Z})$ is annihilated by I (*cf.* [53, Theorem 4.3.7]). Their proof uses the K-theory of $\mathbb{G}_m^2$ and its relation with motivic cohomology.

The result of Sharifi and Venkatesh should yield a generalization of Theorem 2.3 in weight 2 and level $\Gamma_1(N)$, under the condition that the Hecke algebra $\mathbb{T}_\mathfrak{m}$ is Gorenstein at an Eisenstein maximal ideal $\mathfrak{m}$ (without this condition, we do not even have a generalization of (1.3), as explained above). The Gorenstein property has been proved in certain specific situations (*e.g.* by Skinner and Wiles [55] and Ohta [41, Theorem 3.3.2]), but does not hold in general.

Let us describe one situation where Skinner and Wiles prove that $\mathbb{T}_\mathfrak{m}$ is a complete intersection, and *a fortiori* Gorenstein. Let $N \geq 1$ be prime to p and such that $p \nmid \varphi(N)$. Let $\varphi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}^\times$ be a primitive even Dirichlet character, where $\mathcal{O}$ is a finite unramified extension of $\mathbf{Z}_p$ (one can also view ϕ as having values in $\mathbf{C}$). Let $\chi = \varphi \cdot \omega_p : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}^\times$. Assume that p does not divide the χ-eigenspace of the class group of the extension $\mathbf{Q}_\chi$ of $\mathbf{Q}$ cut out by χ (this is equivalent to the fact that a certain generalized Bernoulli number is prime to p). There is an Eisenstein series $E_{1,\varphi}$ whose ℓth Fourier coefficient is $1+\ell\varphi(\ell)$ (for a prime ℓ). The L-function of $E_{1,\varphi}$ is $\zeta(s)L(\varphi,s-1)$. Let I be the corresponding Eisenstein ideal, generated by the Hecke operators $T_\ell - 1 - \ell\langle \ell \rangle$ and $\langle \ell \rangle - \varphi(\ell)$ for primes $\ell \nmid Np$, and by $U_\ell - 1$ for $\ell \mid Np$. There is a unique maximal ideal $\mathfrak{m}$ of $\mathbb{T}$

containing I and p. Skinner and Wiles proved that $\mathbb{T}_{\mathfrak{m}}$ is a complete intersection, by identifying $\mathbb{T}_{\mathfrak{m}}$ with a minimal universal deformation ring [55, Theorem 6.1] (*cf.* also the discussion of [41, Remark 3.3.3] for the Hecke operators U_ℓ).

In this particular situation, one has an Eisenstein congruence similar to (1.3). Namely, assume that there exists $f \in S_2(\Gamma_1(Np))$ such that

$$f \equiv E_{1,\varphi} \text{ (modulo } \pi^r) \tag{2.7}$$

for some uniformizer π in a finite extension of $\mathbf{Z}_p$ and some $r \geq 1$. Let ψ be an even Dirichlet character of conductor m, and assume for simplicity that $\gcd(m, Np) = 1$. We have $L(E_{\varphi,1}, \psi, 1) = L(\psi, 1)L(\psi\varphi, 0) = 0$ if $\psi \neq 1$. If $\psi = 1$, then $L(E_{1,\varphi}, \psi, 1) \neq 0$. There exists a period Ω_f^+ such that for all even $\psi \neq 1$ of conductor m prime to Np, we have

$$\frac{\tau(\overline{\psi})L(f,\psi,1)}{\Omega_f^+} \equiv 0 \text{ (modulo } \pi^r)$$

and

$$\frac{L(f,1)}{\Omega_f^+} \equiv 1 \text{ (modulo } \pi^r).$$

Here, $\tau(\psi)$ is the Gauss sum associated with ψ.

The analogue of Conjecture 2.1 is the following. There should exist a surjective group homomorphism $\alpha : (K_2(\mathbf{Z}[\zeta_{Np}]) \otimes_{\mathbf{Z}} \mathcal{O})(\varphi) \to \mathbf{Z}/p\mathbf{Z}$ such that

$$\frac{\tau(\overline{\psi})L(f,\psi,1)}{\pi^r\Omega_f^+}$$
$$\equiv \alpha\left(\mathrm{Norm}\left\{\prod_{a\in(\mathbf{Z}/m\mathbf{Z})^\times}(1-\zeta_m^a)^{\psi^{-1}(a)}, \prod_{a\in(\mathbf{Z}/mNp\mathbf{Z})^\times}(1-\zeta_{mNp}^a)^{(\varphi^{-1}\psi)(a)}\right\}\right) \text{ (modulo } \pi)$$

where $\mathrm{Norm} : K_2(\mathbf{Z}[\zeta_{mNp}]) \to K_2(\mathbf{Z}[\zeta_{Np}])$ is the norm map and (φ) means the φ-eigenspace. In particular, we expect that the Eisenstein congruence (2.7) implies $(K_2(\mathbf{Z}[\zeta_{Np}]) \otimes \mathcal{O})(\varphi) \neq 0$. Although we have not checked it, this may follow from Ohta's computation of $\mathbb{T}^0/I$ [40, Theorem 1.5.5]. A similar statement has been proved by Ohta in level $\Gamma_1(N)$ when N is prime.

Given the results of Fukaya–Kato or Sharifi–Venkatesh, we expect that a result similar to Theorem 2.3 holds. We have not checked the details though. Let us briefly explain the appearance of the norm in the formula. The special value $\frac{\tau(\overline{\psi})L(f,\psi,1)}{\pi^r\Omega_f^+}$ is related to the modular symbol

$$\Theta_\psi = \sum_{a\in(\mathbf{Z}/m\mathbf{Z})^\times} \psi^{-1}(a)\left\{\infty, \frac{a}{m}\right\} \in H_1\big(X_1(N), \mathcal{O}\big)^+$$

(the '+' being for the action of the complex conjugation). We want to compute $\varpi(\Theta_\psi)$, but we cannot do so directly because Θ_ψ cannot be explicitly written as a combination of Manin symbols $\xi(u,v)$.

Nevertheless, we can go up to level $\Gamma_1(mNp)$ and express $W_{mNp}(\Theta_\psi)$ as a combination of Manin symbols there, where W_{mNp} is the Atkin–Lehner involution. One can then use Sharifi's conjecture at level mNp and descend using the norm. We need a certain compatibility between the maps ϖ when changing levels; this has been done in [68] and [26]. This whole procedure has been carried out in at level $\Gamma_1(N)$ when N is prime in [27].

2.2.3 Mazur's Case

The phenomenon of "higher Eisenstein congruences" for modular symbols discovered by Sharifi had actually been noticed in a very special case by Mazur [32], although Mazur does not phrase it in terms of K-theory and cup products. Mazur's setting is that of weight 2 and level $\Gamma_0(N)$. This situation has been studied extensively and we have many additional results in this case. We thus spend some time to describe these results in more detail.

Assume that N is prime and let $p \geq 5$ be a prime dividing $N-1$. In this case, there is only one Eisenstein series

$$E_{2,N} = \frac{N-1}{24} + \sum_{n\geq 1}\Bigg(\sum_{\substack{d\mid n\\ \gcd(d,N)=1}} d\Bigg)q^n.$$

Let $I \subset \mathbb{T}$ be the corresponding Eisenstein ideal. Mazur proved (*cf.* [32, Proposition II.18.8]) that there is a unique maximal ideal $\mathfrak{m}$ containing I and p. He also proved that there is a group isomorphism

$$\mathcal{M} : H_1\big(X_0(N),\mathbf{Z}_p\big)^+/I\cdot H_1\big(X_0(N),\mathbf{Z}_p\big)^+ \xrightarrow{\sim} (\mathbf{Z}/N\mathbf{Z})^\times \otimes \mathbf{Z}_p \tag{2.8}$$

sending a Manin symbol $\xi(u,v)$ to $uv^{-1}\otimes 1$ (where u, $v \in (\mathbf{Z}/N\mathbf{Z})^\times$).

The map $\mathcal{M}$ defined in (2.8) can actually be deduced from Sharifi's map ϖ as follows: there is a commutative diagram

$$\begin{array}{ccc} H_1(X_1(N),C^o,\mathbf{Z}_p)^+ & \xrightarrow{\quad\varpi\quad} & K_2(\mathbf{Z}[\zeta_N,\tfrac{1}{N}])\otimes\mathbf{Z}_p \\ \big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\partial} \\ H_1(X_0(N),\mathbf{Z}_p)^+/I\cdot H_1(X_0(N),\mathbf{Z}_p)^+ & \xrightarrow{\ \mathcal{M}\ } & (\mathbf{Z}/N\mathbf{Z})^\times\otimes\mathbf{Z}_p \end{array}$$

where $\pi : H_1(X_1(N),C^o,\mathbf{Z}_p)^+ \to H_1(X_0(N),\mathbf{Z}_p)^+$ is induced by the forgetful map and $\partial : K_2(\mathbf{Z}[\zeta_N,\frac{1}{N}]) \to (\mathbf{Z}/N\mathbf{Z})^\times \otimes \mathbf{Z}_p$ is the tame symbol map as in (2.5).

Fix a surjective group homomorphism $\log : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{Z}/p\mathbf{Z}$ (which is similar to the choice of φ just above Conjecture 2.1). By duality, the group homomorphism

$$\log\circ\mathcal{M} : H_1\big(X_0(N),\mathbf{Z}/p\mathbf{Z}\big)^+/I\cdot H_1\big(X_0(N),\mathbf{Z}/p\mathbf{Z}\big)^+ \to \mathbf{Z}/p\mathbf{Z}$$

gives an element $e_1^- \in H_1(X_0(N),\mathbf{Z}/p\mathbf{Z})^-$ annihilated by I. One can lift e_1^- to $H_1(Y_0(N),\mathbf{Z}/p\mathbf{Z})^-$. This lift is unique up to a multiple of e_0^-, where e_0^- is the class of a little loop around the cusp ∞. Note that e_0^- is annihilated by I. The element e_1^- satisfies the property (in $H_1(Y_0(N),\mathbf{Z}/p\mathbf{Z})^-$)

$$(T_\ell - \ell - 1)\big(e_1^-\big) = (\ell-1)\log(\ell)\cdot e_0^- \tag{2.9}$$

for all primes $\ell \nmid N$.

The relation (2.9) is the analogue of (2.6) in weight k and level 1, with the notable difference that the coefficient c_ℓ in Mazur's case is very explicit: it is simply $(\ell-1)\log(\ell)$. Using the so called *winding homomorphism*, Mazur deduces that there is a group isomorphism $I/I^2 \otimes \mathbf{Z}/p\mathbf{Z} \xrightarrow{\sim} \mathbf{Z}/p\mathbf{Z}$ given by

$$T_\ell - \ell - 1 \mapsto (\ell-1)\log(\ell) \tag{2.10}$$

(*cf.* [32, Proposition II.18.9]). This is a beautiful formula: it essentially gives higher Eisenstein congruences for cuspforms.

Mazur attributes this formula in the case $N=11$ to Serre [32, p. 139]. If $N=11$ and $p=5$, we have $S_2(\Gamma_0(11)) = \mathbf{C}\cdot f$ where $f = q\prod_{n\geq 1}(1-q^n)^2\prod_{n\geq 1}(1-q^{11n})^2$. We know that $f \equiv E_{2,11}$ (modulo p). Formula (2.10) tells us that one can normalize $\log : (\mathbf{Z}/11\mathbf{Z})^\times \to \mathbf{Z}/5\mathbf{Z}$ so that for all prime $\ell \neq 11$, we have

$$a_\ell(f) \equiv \ell + 1 + p\cdot(\ell-1)\log(\ell) \text{ (modulo } p^2), \tag{2.11}$$

where $a_\ell(f)$ is the ℓth Fourier coefficient of f. As Mazur notes, it turns out that we need to choose $\log(-3) = 2$. For a thorough discussion of this example and its history, see [62, §1.1.1].

A key remark is that (2.11) may be rewritten as

$$a_\ell(f) \equiv \chi(\ell)^{-1} + \ell\chi(\ell) \text{ (modulo } p^2), \tag{2.12}$$

where $\chi : (\mathbf{Z}/11\mathbf{Z})^\times \to (\mathbf{Z}/25\mathbf{Z})^\times$ is given by $\chi(\ell) = 1 + 5\log(\ell)$. This implies that the Galois representation $J_0(11)[25]$ is *reducible*, *i.e.* its trace is the sum of two characters. This is remarkable, because the Eisenstein congruence $f \equiv E_{2,11}$ (modulo p) *a priori* only tells us that $J_0(11)[5]$ is reducible. To our knowledge, this remark was first made by Calegari in [4, p. 68]. Wake called this phenomenon *extra-reducibility* in [62]. He also generalized (2.12) in weight k (under some assumptions) (*cf.* [62, Corollary 5.2.5]).

This extra-reducibility has been used in a crucial way in my work on higher Eisenstein elements, but in a different form. Namely, we reinterpret (2.12) by saying that

$$f \equiv \frac{E_{1,\chi} + E_{\chi^{-1},1}}{2} \text{ (modulo } p^2)$$

for some standard Eisenstein series $E_{1,\chi}$ and $E_{\chi^{-1},1}$ in $M_2(\Gamma_1(11))$. Thus, our extra-reducibility is in some sense explained by the modular curve $X_1(11)$. This is not surprising, as Mazur's proof of (2.8) uses the Shimura covering $X_1(N) \to X_0(N)$. Of course all this discussion is not specific to $N=11$; it can be generalized to any prime N as above.

Let us conclude this paragraph by mentioning that even in Mazur's case, Sharifi's conjecture yields more information than the extra-reducibility considered above. Namely, one can use Sharifi's map $\varpi : H_1(X_1(N), C^o, \mathbf{Z}_p)^+ \to K_2(\mathbf{Z}[\zeta_N, \frac{1}{N}]) \otimes \mathbf{Z}_p$ to construct the second higher Eisenstein element $e_2^- \in H_1(Y_0(N), \mathbf{Z}/p\mathbf{Z})^-$ (when it exists). The element e_2^- is uniquely determined modulo the subgroup generated by e_0^- and e_1^-, and satisfies the equation

$$(T_\ell - \ell - 1)(e_2^-) = (\ell - 1)\log(\ell) \cdot e_1^- \text{ (modulo } (\mathbf{Z}/p\mathbf{Z}) \cdot e_0^-).$$

We refer to [28, Theorem 1.12] for a formula for e_2^-. The element e_2^- does not arise from any extra-reducibility; it is thus in some sense more mysterious than e_1^-.

We should really view e_2^- as the true analogue of r_1^+ (*cf.* (2.6)) in weight k and level 1 (where there is no extra-reducibility). Although the higher Eisenstein element e_1^+ has been determined in [25] using the extra-reducibility, we do not know a formula for e_2^+ (when it exists, *i.e.* when e_2^- exists). This is similar to the fact that the higher Eisenstein element r_1^- in weight k and level 1 is not explicitly known.

2.3 Overview of Q3

Eisenstein congruences have been used to prove many important results in number theory. We survey a few of these results, but given the vast literature on the subject our coverage is far from exhaustive.

Possibly one of the first applications is the pioneering work of Mazur and Tate [35]. They prove that no elliptic curve over $\mathbf{Q}$ has a rational point of order 13, which is equivalent to the fact that $X_1(13)$ has no rational points except its 6 rational cusps. To do so, they rely on the fact that $J_1(13)$ has a rational point of order 19 (in the cuspidal group). Although they do not use Eisenstein ideals explicitly, this rational point is explained by an Eisenstein congruence modulo 19. These ideas were used extensively later by Mazur in [32] to determine the possible rational torsion of elliptic curves over $\mathbf{Q}$. Mazur's torsion theorem is crucial in the proof of Fermat's last theorem, as was noted by Serre in his proof of [50, Proposition 6].

Another pioneering work using Eisenstein congruences is the work of Ribet on class groups of cyclotomic fields [47]. Ribet proved that if p is an odd prime and $\chi : (\mathbf{Z}/p\mathbf{Z})^\times \to \mathbf{Z}_p^\times$ is an odd character, then the χ^{-1}-eigenspace of the p-class group of $\mathbf{Q}(\zeta_p)$ is non-trivial if the generalized Bernoulli number $B_{1,\chi}$ is not coprime to p (the converse is an older result of Herbrand).

Ribet's idea is quite simple: the condition that $B_{1,\chi}$ is not coprime to p implies that there is an Eisenstein congruence between a cuspidal eigenform and the Eisenstein series $E_{1,\omega_p^{-1}\chi}$. One can use this cusp form to construct a Galois representation $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathbf{F}_p)$ which is reducible and cuts out an everywhere unramified $\mathbf{Z}/p\mathbf{Z}$-extension L of $\mathbf{Q}(\zeta_p)$ such that $\mathrm{Gal}(\mathbf{Q}(\zeta_p)/\mathbf{Q})$ acts by χ^{-1} on $\mathrm{Gal}(L/\mathbf{Q}(\zeta_p))$. Class field theory then yields the result on the class group of $\mathbf{Q}(\zeta_p)$.

One may summarize Ribet's idea by saying that an Eisenstein congruence relates the geometry/arithmetic of $\mathrm{GL}_2/\mathbf{Q}$ (*e.g.* cuspidal eigenforms) to the arithmetic of $\mathrm{GL}_1/\mathbf{Q}$ (*e.g.* class groups). This is also a motivating idea in Sharifi's conjecture (*cf.* [13, §1.2])

Ribet's idea has been vastly generalized and is a central tool for people working in Iwasawa theory. For instance, Mazur and Wiles proved the Iwasawa main conjecture for cyclotomic fields using these kinds of ideas [36]. Wiles then generalized these techniques to prove the Iwasawa main conjecture for totally real fields [66]. Skinner and Urban have used Eisenstein congruences for the unitary group $\mathrm{GU}(2,2)$ to prove a divisibility in the Iwasawa main conjecture for elliptic curves [57].

In another direction, Ribet's idea has been used by Calegari and Emerton in [5] to prove the following beautiful result. Let N and p be primes ≥ 5 with $p \mid N-1$, as in Mazur's setting. If the quantity $\prod_{k=1}^{\frac{N-1}{2}} k^k$ is a pth power modulo N, then the p-class group of $\mathbf{Q}(N^{1/p})$ is not cyclic. Their proof relies on two ingredients: a criterion of Merel for the rank of $\mathbb{T}^0_{\mathfrak{m}}$ to be ≥ 2 [39] and a careful analysis of the Galois representation $J_0(N)[I^3,p]$ (which is not reducible as noted before). Calegari and Emerton's result has recently been proved using only classical algebraic number theory in [24] and then refined in [48].

Lang and Wake [22] have also recently used Eisenstein congruences to prove that if N and p are primes ≥ 5 such that p divides $N+1$, then p divides the class number of $\mathbf{Q}(N^{1/p})$. They prove that there is an Eisenstein congruence in weight 2 and level $\Gamma_0(N^2)$. Although unlike Mazur's case there is no extra-reducibility associated with this Eisenstein congruence, they show that after restricting their Galois representation to $\mathbf{Q}(N^{1/p})$ the representation becomes extra-reducible.

We have seen in §2.2 that Eisenstein (higher) congruences provide (higher) congruences for special values of L-functions. The Bloch–Kato conjecture relates these special values to Selmer groups, in a way generalizing the Birch and Swinnerton-Dyer conjecture for elliptic curves. One may thus ask whether Eisenstein congruences can be used to prove results toward the Bloch–Kato conjecture. Mazur proved that this is indeed the case [33].

For the sake of simplicity, let us consider the simplest application of Mazur's result. Denote by E the elliptic curve over $\mathbf{Q}$ given by $X_0(11)$. Let $D<0$ be a fundamental discriminant coprime with 11 so that 11 is inert in $\mathbf{Q}(\sqrt{D})$. We denote by $E^{(D)}$ the quadratic twist of E by $\mathbf{Q}(\sqrt{D})$. Mazur then proved that p divides the algebraic part of $L(E^{(D)},1)$ if and only if the p-part of the Selmer group of $E^{(D)}$ (over $\mathbf{Q}$) is non-trivial. This is predicted by the BSD conjecture for $E^{(D)}$.

Mazur's proof is rather indirect: he first proves that p divides the algebraic part of $L(E^{(D)},1)$ if and only if p divides $h(D)$ (the class number of $\mathbf{Q}(\sqrt{D})$). His proof uses a congruence similar to (1.4), involving products of Bernoulli numbers. Mazur then proves that p divides $h(D)$ if and only if the p-part of the Selmer group of $E^{(D)}$ is non-trivial. This can be shown using Galois cohomology. The key idea is that the residual Galois representation modulo p is reducible, so the Selmer group is related to a class group.

Very recently, the author and Jun Wang tackled the same situation as Mazur but for $D > 0$ and 11 splitting in $\mathbf{Q}(\sqrt{D})$ (*cf.* [27]). This was left open in Mazur's paper. The main reason is that the "trivial" Eisenstein congruence of the type (1.3) is not good enough: it only gives

$$L^{\mathrm{alg}}\big(E^{(D)},1\big) \equiv 0 \text{ (modulo 5)},$$

where $L^{\mathrm{alg}}(E^{(D)},1)$ is the "algebraic part" of $L(E^{(D)},1)$.

One needs a "higher" congruence for the special values $L(E^{(D)},1)$ modulo 25. Such a congruence can be obtained using Sharifi's conjecture, and more precisely the element e_2^- considered at the end of §2.2. It takes the form

$$L^{\mathrm{alg}}\big(E^{(D)},1\big) \equiv 5\cdot(*)\cdot\big(h(D)\cdot\log\big(u(D)\big)\big)^2 \text{ (modulo 25)}, \tag{2.13}$$

where $(*)$ is a non-zero easy factor, $u(D)$ is a fundamental unit of $\mathbf{Q}(\sqrt{D})$ and $\log : \mathcal{O}_{\mathbf{Q}(\sqrt{D})}^{\times} \to \mathbf{Z}/5\mathbf{Z}$ is a discrete logarithm modulo 11. The author gave another proof of (2.13) in using Mazur's element e_1^- and a formula of Waldspurger and Popa for the special value $L(E/\mathbf{Q}(\sqrt{D}),1)$ (*cf.* [23, Equation (26)]).

Using Galois cohomological methods, Wang and I then proved that $h(D)\cdot\log(u(D)) \equiv 0$ (modulo 5) if and only if the 5-part of the Selmer group of $E^{(D)}$ is non-trivial. Putting this together with (2.13), we get that 25 divides $L^{\mathrm{alg}}(E^{(D)},1)$ if and only if the 5-part of the Selmer group of $E^{(D)}$ is non-trivial. This is again predicted by the BSD conjecture (*cf.* the introduction of [27]).

We hope to have illustrated the fact that Eisenstein (higher) congruences have important applications to conjectures on special values of L-functions. Let us mention that, similarly, Eisenstein congruences have been used to get results for the Iwasawa main conjecture of residually reducible elliptic curves over $\mathbf{Q}$ (the results of Skinner and Urban assume in particular that the elliptic curve is residually irreducible). See for instance the work of Greenberg and Vatsal [14], and more recently Castella–Grossi–Lee–Skinner [6] and Skinner–Grossi [7].

2.4 Overview of Q4

Let us first consider the question of Serre's conjecture. The classical conjecture of Serre [50] starts with an absolutely irreducible odd Galois representation $\overline{\rho} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$. It tells us that $\overline{\rho}$ is modular and predicts explicitly the "minimal" weight and level for a newform f giving rise to $\overline{\rho}$. The conjecture has been proved by Khare and Wintenberger [18] (building on the work of many others).

If $\overline{\rho}$ is reducible, we have to be careful. One reason is that in the residually reducible case, the reduction of the Galois representation $\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(K)$ attached to a newform f depends on the choice of a lattice $T \subset K^2$ where K is a finite extension of $\mathbf{Q}_p$. One may consider the semi-simplification $\overline{\rho}^{\mathrm{ss}}$ and ask for the "minimal" weight and level of a newform f giving rise to $\overline{\rho}^{\mathrm{ss}}$.

Serre's original conjecture must be modified in the reducible case. For instance, one may consider $\overline{\rho}^{\mathrm{ss}} = 1 \oplus \overline{\chi}_p$, where $\overline{\chi}_p$ is the reduction modulo p of the p-adic cyclotomic character. As recalled above, Mazur proved that for N prime, there exists a newform $f \in S_2(\Gamma_0(N))$ such that $\overline{\rho}_f^{\mathrm{ss}} \simeq 1 \oplus \overline{\chi}_p$ if and only if p divides the numerator of $\frac{N-1}{12}$. But there are no newforms of weight 2 and level 1. Therefore, there is no "minimal" level in this case. Following Ribet, we call the set of such N's the *non-optimal levels* of $\overline{\rho}^{\mathrm{ss}}$.

Yoo and Ribet partly generalized Mazur's result to square-free level, but their results are not complete. The situation is quite complicated in non-prime square-free level. Let us give an example of their results (*cf.* [71, Theorem 2.3]). Assume that $p \geq 5$. For primes N_1 and N_2 with $N_1 \equiv 1$ (modulo p), there exists a newform $f \in S_2(\Gamma_0(N_1N_2))$ with $\overline{\rho}_f^{\mathrm{ss}} \simeq 1 \oplus \overline{\chi}_p$ and $U_{N_i} f = f$ for $i = 1, 2$ if and only if either $N_2 \equiv 1$ (modulo p) or N_2 is a pth power modulo N_1.

The question of non-optimal levels has been considered for certain levels in weight $k > 2$ by Billerey and Menares in [2] and [3].

Let us now consider $R = T$ type theorems. Mazur defined under some conditions the universal deformation ring attached to a residual Galois representation $\overline{\rho} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ (*cf.* [34]). Let us note that R fails (without additional deformation conditions) to exist if $\overline{\rho}$ is reducible and semi-simple. By adding certain deformation conditions at bad places and on the determinant, one can often prove that R is isomorphic to a certain Hecke algebra T. This was first proved under some assumptions, including the irreducibility of the restriction of $\overline{\rho}$ to $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\zeta_p))$, by Wiles and Taylor–Wiles in their proof of Fermat's last theorem [67, 60].

To our knowledge, the first $R = T$ theorem in the case where $\overline{\rho}$ is reducible but not semi-simple is due to Skinner and Wiles [55, Theorem 6.1]. Skinner and Wiles assume that their deformations are ordinary at p. Their proof relies on the numerical criteria of Wiles and Lenstra [29]. They compute directly the congruence module and the cohomology groups involved in the numerical criterion.

Later, Calegari and Emerton proved an $R = T$ theorem in Mazur's situation [5] (weight 2 and level $\Gamma_0(N)$ for a prime N such that p divides the numerator of $\frac{N-1}{12}$). As explained above, this situation is the "non-minimal" case. Their residual representation is $\overline{\rho} = 1 \oplus \overline{\chi}_p$, so in particular the universal deformation ring R does not exist without modification. They rigidify the deformation problem by adding the data of a line fixed by the inertia at N. The local condition at p is "finite flat". Their proof again consists in checking the numerical criterion "by hand" (using class field theory).

Shortly after, Calegari proved various $R = T$ theorems in minimal cases. For instance, if $k < p - 1$ and $p \mid\mid B_k$ then Calegari proves that $R = T$ where T is the Eisenstein completion of the Hecke algebra of weight k and level 1. In that situation, R turns out to be a DVR [4, Lemma 4.8], and the surjective map $R \to T$ must be an isomorphism. As Calegari noted, his results are not covered by Skinner and Wiles since he considers either non-ordinary deformations or locally split residual representations.

The result of Calegari and Emerton is very specific to Mazur's situation. It is not obvious how to extend it to more general situations (*e.g.* weight 2 and squarefree level). Wake and Wang-Erickson developed extensively what seems like the most natural setting to study residually reducible representations. Namely, instead of considering deformations of $\overline{\rho}$, they simplify the problem and only deform the trace of $\overline{\rho}$, *i.e.* the pseudo-representation attached to $\overline{\rho}$. Wake and Wang-Erickson had to consider pseudo-deformation conditions, and in particular the "finite flat" condition, whose definition is not obvious a priori. They develop the general theory of pseudo-deformation conditions in [63] and the finite-flat condition in [64, §2.3]. They rely crucially on the notion of *generalized matrix algebras* (GMA) due to Bellaïche and Chenevier [1, 8], which are a kind of matrix representation of a pseudo-presentation but with entries in modules rather than rings.

Wake and Wang-Erickson applied their pseudo-deformation techniques to prove a pseudo-deformation version of the $R = T$ theorem of Calegari–Emerton. They also applied it in some cases to certain Eisenstein ideals of weight 2 and squarefree level [65]. Wake, Wang-Erickson and Hsu proved $R = T$ in the same situation (weight 2 and squarefree level) but under different assumptions. They require a numerical condition which turns out to force the rank of R and T over $\mathbf{Z}_p$ to be 3 (*cf.* [17, Theorem 1.3.3]).

Wake used similar techniques in higher weight and prime level $\Gamma_0(N)$ [62], under the assumptions that $p \geq 5$, $k > 2$ is even, $p \nmid \zeta(1-k)$ and $p \mid N-1$. Let us note that Wake does not prove an $R = T$ theorem in this case, but instead a weaker version for reducible deformations. Deo improved Wake's result and managed to prove $R = T$ under certain assumptions [9, Theorem B]. Deo actually considers both deformations and pseudo-deformations in his work, as he needs to know when a pseudo-deformation arises from an actual deformation (of a Galois representation).

It is clear that $R = T$ type results in the residually reducible case constitute a fertile area of current research, as many cases are still unexplored (*e.g.* more general weights and levels).

Theorems of the type $R = T$ are useful in proving modularity lifting theorems, and in particular in proving results toward the Fontaine–Mazur conjecture. The Fontaine–Mazur conjecture basically states that an odd Galois representation $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ satisfying certain natural conditions should be associated (up to twist) with a cuspidal eigenform. This conjecture has been proved in many cases by Kisin [19] and Emerton [11], but they require that the restriction of $\overline{\rho}$ to $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}(\zeta_p))$ be irreducible (and in particular that $\overline{\rho}$ be irreducible).

In the case where $\overline{\rho}$ is reducible, Skinner–Wiles were able in the ordinary case to prove (most of the time) the Fontaine–Mazur conjecture [56, §1 Theorem]. Unlike in their previous work [55], they do not prove an $R = T$ theorem in general, but instead rely on a base change to a totally real field. Recently, Pan proved the residually reducible Fontaine–Mazur conjecture in most of the non-ordinary cases, and also in the ordinary cases left open by Skinner and Wiles (*cf.* [45, Theorem 1.0.2]).

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