ARTICLES

The Early (and Peculiar) History of the Möbius Function

WILLIAM DUNHAM
Bryn Mawr College
Bryn Mawr, PA 19010
bdunham@brynmawr.edu

We begin with this challenge from analysis: for $-1 < x < 1$, determine the exact value of the infinite series

$$\frac{x}{1-x} - \frac{x^2}{1-x^2} - \frac{x^3}{1-x^3} - \frac{x^5}{1-x^5} + \frac{x^6}{1-x^6} - \frac{x^7}{1-x^7} + \frac{x^{10}}{1-x^{10}}$$
$$- \frac{x^{11}}{1-x^{11}} - \frac{x^{13}}{1-x^{13}} + \frac{x^{14}}{1-x^{14}} + \frac{x^{15}}{1-x^{15}} - \frac{x^{17}}{1-x^{17}} - \cdots.$$

The pattern here is anything but clear, as signs flip-flop in strange fashion and certain terms are unexpectedly missing. The sum turns out to be a simple one, but finding it, as we shall see, requires familiarity with something called the "Möbius function."

This function shows up in any comprehensive text on the theory of numbers. After chapters on primes and congruences, on the Euclidean algorithm and Diophantine equations, such a book will sooner or later introduce the Möbius function. It tends to appear alongside its number-theoretic cousins: the sigma, tau, and phi functions. The sigma function, $\sigma(k)$, sums all the divisors of k; the tau function, $\tau(k)$, counts all the divisors of k; and the phi function, $\phi(k)$, counts the numbers less than k and relatively prime to it. These three have an obvious utility in number theory.

The Möbius function, by contrast, seems neither useful nor obvious. It is denoted by $\mu(k)$ and defined, for a whole number k, as follows:

(a) $\mu(1) = 1$.
(b) $\mu(k) = 0$ if k is divisible by the square of some prime.
(c) $\mu(k) = (-1)^r$ if k is the product of r different primes.

Thus, for the first ten numbers, $\mu(1) = 1$; $\mu(2) = \mu(3) = \mu(5) = \mu(7) = -1$ because each is a prime; $\mu(4) = \mu(8) = \mu(9) = 0$ because these are divisible, respectively, by 2^2, 2^2, and 3^2; and $\mu(6) = \mu(10) = (-1)^2 = 1$ because these are the products of two different primes.

At first glance, these values seem uninformative. Unlike the sigma, Möbius's function doesn't *sum* anything. Unlike the tau and phi, Möbius's function doesn't *count* anything. Because every fourth number is divisible by $2^2 = 4$, the Möbius function takes the value zero more than a quarter of the time. And the Möbius function exhibits strange runs, like $\mu(33) = \mu(34) = \mu(35) = 1$ or $\mu(242) = \mu(243) = \mu(244) = \mu(245) = 0$. In such cases, the function seems to conceal numerical information rather

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than to reveal it. Whereas the sigma, tau, and phi functions are "natural" concepts, the Möbius function comes across as something of an oddity. What possible interest could it hold?

Anyone who reads a bit further in that number theory text will discover that the Möbius function is not only important but highly so. The deeper one digs, the more significant the concept becomes. For instance, the prime number theorem can be recast in terms of the Möbius function. The (famously unresolved) Riemann hypothesis can be recast in a similar fashion. And there is even an application to quantum physics called the "free Riemann gas model of supersymmetry" that involves – yes! – the Möbius function. Indeed, this is a concept to be reckoned with.

Where did the idea come from?

August Ferdinand Möbius
(1790–1868)

Here the story holds its surprises, for the function's namesake, August Ferdinand Möbius (1790–1868), did not introduce it via the conditions (a)–(c) above. Rather, the concept appeared in his 1832 paper "*Ueber eine besondere Art von Umkehrung der Riehen*" (On a special kind of inversion of series) about a subject that, at first glance, had nothing to do with number theory [**4**, pp. 105–123].

Möbius began with a real function defined by the power series

$$f(x) = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots,$$

which he sought to invert into an expression for x of the form

$$x = b_1 f(x) + b_2 f(x^2) + b_3 f(x^3) + b_4 f(x^4) + \cdots.$$

The challenge was to determine the values of the b_k based on the known values of the a_k.

To simplify things, Möbius set all the a_k equal to 1, thereby turning $f(x)$ into the geometric series

$$f(x) = x + x^2 + x^3 + x^4 + \cdots.$$

This, of course, sums to $x/(1 - x)$ for $-1 < x < 1$. His inversion therefore amounted to writing

$$\begin{aligned} x &= b_1 f(x) + b_2 f(x^2) + b_3 f(x^3) + b_4 f(x^4) + \cdots \\ &= b_1[x + x^2 + x^3 + x^4 + \cdots] + b_2[x^2 + x^4 + x^6 + x^8 + \cdots] \end{aligned}$$

$$+b_3[x^3+x^6+x^9+x^{12}+\cdots]+b_4[x^4+x^8+x^{12}+x^{16}+\cdots]$$
$$+b_5[x^5+x^{10}+x^{15}+x^{20}+\cdots]+\cdots.$$

Collecting powers of x, Möbius arrived at

$$x=b_1x+[b_1+b_2]x^2+[b_1+b_3]x^3+[b_1+b_2+b_4]x^4+[b_1+b_5]x^5$$
$$+[b_1+b_2+b_3+b_6]x^6+[b_1+b_7]x^7+[b_1+b_2+b_4+b_8]x^8+\cdots.$$

Matching coefficients of x, he saw that $b_1=1$. Using this fact and equating coefficients from consecutively higher powers, he got:

$$0=1+b_2,\quad 0=1+b_3,\quad 0=1+b_4+b_4,\quad 0=1+b_5,$$
$$0=1+b_2+b_3+b_6,\ \text{and so on.} \tag{1}$$

Thus,

$$b_2=-1,\ b_3=-1,\ b_4=-1-b_2=0,\ b_5=-1,\ b_6=-1-b_2-b_3=1,$$
$$b_7=-1,\ b_8=-1-b_2-b_4=0,\ \text{etc.}$$

Möbius wished to calculate the b_k by identifying a pattern in these coefficients rather than by solving a string of ever-longer equations. This he did, although here too his approach might strike the modern reader as strangely complicated.

First, Möbius observed, perhaps with some surprise, that "one finds these coefficients … are either -1, 0, or 1." [**4**, p. 110] He then showed that they were generated by the following rules.

Rule 1: If p is prime, the pertinent equation to emerge from (1) is obviously $0=1+b_p$. Hence $b_p=-1$ for all primes p.

Rule 2: If $k=p\cdot q$, where p and q are different primes, then the equation from (1) will be

$$0=1+b_p+b_q+b_{pq}.$$

Rearranging terms and adding $b_p\cdot b_q$ to both sides of the equation, Möbius deduced that

$$-b_{pq}+b_p\cdot b_q=1+b_p+b_q+b_p\cdot b_q=\left(1+b_p\right)\left(1+b_q\right)=0$$

by Rule 1. Thus $b_{pq}=b_p\cdot b_q=(-1)^2=1$. In like fashion, he saw that if k is the product of r different primes, then $b_k=(-1)^r$.

Rule 3: If $k=p^2$ for some prime p, then the corresponding equation from (1) will be

$$0=1+b_p+b_{p^2},$$

and so $b_{p^2}=-(1+b_p)=0$ by Rule 1. Similarly, if $n=p^3$, he got $0=1+b_p+b_{p^2}+b_{p^3}$ and so $b_{p^3}=-(1+b_p)-b_{p^2}=0$. The same outcome holds for any higher power of p.

Rule 4: Finally, Möbius considered the case where k is divisible by the square of a prime. For a simple example, we look at $k=p^2q$ where p and q are different primes. From (1) we have

$$0=1+b_p+b_{p^2}+b_q+b_{pq}+b_{p^2q}.$$

Knowing that $b_{pq}=b_p\cdot b_q$ and adding $b_{p^2}\cdot b_q=0$ to both sides, he got

$$-b_{p^2q}+b_{p^2}\cdot b_q=1+b_p+b_{p^2}+b_q+b_p\cdot b_q+b_{p^2}\cdot b_q$$
$$=\left(1+b_p+b_{p^2}\right)(1+b_q)=0$$

by Rule 1. Thus $b_{p^2q}=b_{p^2}\cdot b_q=0\cdot b_q=0$ by Rule 3.

In this fashion, Möbius determined rules for calculating his coefficients b_k. Of course, these yield precisely what is called $\mu(k)$ in today's textbooks. But it is interesting that, rather than simply *defining* a number theoretic concept, Möbius had *derived* it from the inversion of a geometric series.

We should say a word about the modern notation. This is not due to Möbius, who, as we just saw, used "b_k". Rather, it was the mathematician Franz Mertens (1840–1927) who, in an 1874 paper, introduced "μ", the Greek counterpart of "m," to denote this function [**3**]. We can read this choice as homage to "Möbius" ... with the fortunate coincidence that it also celebrated "Mertens"!

Returning to the original inversion problem, we restate Möbius's conclusion as

$$x = b_1 f(x) + b_2 f(x^2) + b_3 f(x^3) + b_4 f(x^4) + \cdots = \sum_{k=1}^{\infty} b_k f(x^k) = \sum_{k=1}^{\infty} \mu(k) f(x^k),$$

or simply

$$x = \sum_{k=1}^{\infty} \frac{\mu(k)\, x^k}{1 - x^k}$$

because, as noted,

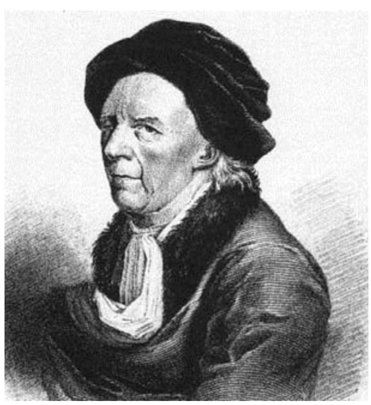
$$f(x) = \frac{x}{1-x}\ .$$

This, by the way, answers our opening question, for the infinite series

$$\frac{x}{1-x} - \frac{x^2}{1-x^2} - \frac{x^3}{1-x^3} - \frac{x^5}{1-x^5} + \frac{x^6}{1-x^6} - \frac{x^7}{1-x^7} + \frac{x^{10}}{1-x^{10}} - \frac{x^{11}}{1-x^{11}} - \cdots$$
$$= \sum_{k=1}^{\infty} \frac{\mu(k)x^k}{1-x^k} = x.$$

In short, our infinite series sums to x. What could be simpler?

So, it is tempting to conclude that the Möbius function first appeared in this paper from 1832 and was subsequently given the name of the author. Alas, that conclusion needs revision, for in 1748 Leonhard Euler (1707–1783) had stumbled upon the same idea, although in a very different fashion. This, by the way, was half a century before August Ferdinand Möbius was even born.



Leonhard Euler
(1707–1783)

Euler's discovery appeared in Chapter XV of his classic text, *Introductio in analysin infinitorum*. The title of this chapter translates as "On Series Which Arise from Products," and, as we shall see, it was aptly chosen [**1**, pp. 228–255].

First, for an arbitrary whole number n, we introduce the infinite quotient

$$M_n = \frac{1}{\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\cdots},$$

where the terms in the denominators run through the primes. (Note: Euler, who never employed subscripts, simply called this "M," but the subscript will prove useful below.) Because $1 + a + a^2 + a^3 + \cdots = 1/(1-a)$, he expressed this as an infinite product of infinite series:

$$M_n = \left(1+\frac{1}{2^n}+\frac{1}{4^n}+\frac{1}{8^n}+\cdots\right)\cdot\left(1+\frac{1}{3^n}+\frac{1}{9^n}+\frac{1}{27^n}+\cdots\right)\cdot\left(1+\frac{1}{5^n}+\frac{1}{25^n}+\cdots\right)$$
$$\cdot\left(1+\frac{1}{7^n}+\frac{1}{49^n}+\cdots\right)\cdots.$$

Upon multiplying these series, Euler concluded that

$$M_n = 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\frac{1}{8^n}+\frac{1}{9^n}+\frac{1}{10^n}+\cdots=\sum_{k=1}^{\infty}\frac{1}{k^n},$$

where every whole number k appears in one and only one denominator. This follows from the unique factorization of whole numbers into primes, the so-called fundamental theorem of arithmetic.

Perhaps more relevant to our story is the behavior of the reciprocal $1/M_n$, an infinite product that we shall denote by Q_n. Clearly,

$$Q_n = \left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\left(1-\frac{1}{7^n}\right)\cdots,$$

an expression whose product Euler sought to determine. We first observe that the primes occurring in these various denominators are different, so there is no way that something like

$$\frac{1}{9^n}=\frac{1}{3^n}\cdot\frac{1}{3^n} \quad \text{or} \quad \frac{1}{12^n}=\frac{1}{2^n}\cdot\frac{1}{2^n}\cdot\frac{1}{3^n}$$

could appear in the product. In other words, when the binomials of Q_n are multiplied, the resulting terms must look like $1/a^n$ where a is *not* divisible by the square of any prime. Such a number is called "square-free."

Thus, upon multiplying these binomials, Euler found that

$$Q_n = 1-\frac{1}{2^n}-\frac{1}{3^n}-\frac{1}{5^n}+\frac{1}{6^n}-\frac{1}{7^n}+\frac{1}{10^n}-\frac{1}{11^n}-\frac{1}{13^n}+\frac{1}{14^n}+\frac{1}{15^n}-\cdots$$

The pattern here should look familiar. The signs obey Rules 1 to 4 of the Möbius function. Euler put it this way:

> "We note that the terms with primes, or products of three different primes, or any product of an odd number of different primes, appear with a negative sign. Those terms which are the product of two, four, six, or any even number of different primes, appear with a positive sign." [**1**, p. 230]

(To this we might add that terms divisible by the square of a prime do not appear at all.) As an example of the pattern, Euler observed that " … the term $1/30^n$ appears with a negative sign, because 30 is the product of three different primes."[5]

This description exactly matches the rules Möbius would later generate. In modern notation, what Euler had found was

$$Q_n = \frac{1}{M_n} = \sum_{k=1}^{\infty} \frac{\mu(k)}{k^n}.$$

Next, Euler introduced an infinite quotient differing from M_n above only in the signs of the terms. We shall write it as

$$N_n = \frac{1}{\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right)\cdots}.$$

Letting R_n be the reciprocal of N_n, we see that

$$\begin{aligned} R_n &= \left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{3^n}\right)\left(1+\frac{1}{5^n}\right)\left(1+\frac{1}{7^n}\right) \\ &= 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{6^n}+\frac{1}{7^n}+\frac{1}{10^n}+\frac{1}{11^n}+\frac{1}{13^n}+\frac{1}{14^n}+\frac{1}{15^n}+\cdots \end{aligned}$$

This is the same series as that of Q_n above but with plus signs throughout. As before, only square-free terms appear in the denominators, so we can express this result as

$$R_n = \sum_{k=1}^{\infty} \frac{[\mu(k)]^2}{k^n}.$$

At this point, it is worth observing that Euler's derivations conformed to the fashion of the eighteenth century, when modern analytic rigor still lay far over the mathematical horizon. Later mathematicians would tidy up the logic of his results, but in this, as in so many other cases throughout his career, Euler's analytic intuition did not fail him.

Having generated these formulas, he was ready to specify values of n. To follow his line of attack, we recall three particular series whose sums were familiar to Euler.

- Since (at least) the previous century, the harmonic series was known to diverge. In Euler's day, this divergence was expressed as

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots = \infty.$$

- In 1734, Euler had evaluated the sum of reciprocals of the squares as

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}.$$

- Likewise, Euler had summed the reciprocals of the 4^{th} powers to get

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

Armed with these results, he returned to M_n, Q_n, and R_n above. For $n = 1$,

$$M_1 = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\cdots = \infty$$

and so its reciprocal, Q_1, is given by

$$Q_1 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} - \frac{1}{11} - \frac{1}{13} + \frac{1}{14} + \frac{1}{15} - \cdots = \frac{1}{\infty} = 0.$$

In modern notation, this becomes the critical formula

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0.$$

Next, Euler let $n = 2$ to get

$$M_2 = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6},$$

and so

$$\begin{aligned} Q_2 &= 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \frac{1}{13^2} + \frac{1}{14^2} + \frac{1}{15^2} - \cdots \\ &= \frac{1}{M_2} = \frac{6}{\pi^2}, \end{aligned}$$

which we would write as

$$\sum_{k=1}^{\infty} \frac{\mu\,(k)}{k^2} = \frac{6}{\pi^2}.$$

This is a remarkable result, but Euler had one more trick up his sleeve. For $n = 4$, he knew that

$$M_4 = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{90},$$

and so

$$\frac{M_2}{M_4} = \frac{\pi^2/6}{\pi^4/90} = \frac{15}{\pi^2}.$$

But the Ms and the Qs are reciprocals of one another, and so

$$\begin{aligned} \frac{15}{\pi^2} = \frac{M_2}{M_4} = \frac{Q_4}{Q_2} &= \frac{\left(1 - \frac{1}{2^4}\right)\left(1 - \frac{1}{3^4}\right)\left(1 - \frac{1}{5^4}\right)\left(1 - \frac{1}{7^4}\right)\cdots}{\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\left(1 - \frac{1}{7^2}\right)\cdots} \\ &= \left(1 + \frac{1}{2^2}\right)\left(1 + \frac{1}{3^2}\right)\left(1 + \frac{1}{5^2}\right)\left(1 + \frac{1}{7^2}\right)\cdots = R_2, \end{aligned}$$

because

$$\frac{\left(1 - \frac{1}{p^4}\right)}{\left(1 - \frac{1}{p^2}\right)} = \frac{\left(1 + \frac{1}{p^2}\right)\cdot\left(1 - \frac{1}{p^2}\right)}{\left(1 - \frac{1}{p^2}\right)} = \left(1 + \frac{1}{p^2}\right).$$

Consequently,

$$\frac{15}{\pi^2} = R_2 = \sum_{k=1}^{\infty} \frac{[\mu\,(k)]^2}{k^2}.$$

In words, this means that if we sum the reciprocals of the squares of the *square-free* whole numbers, we get

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{10^2}+\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{14^2}+\frac{1}{15^2}+\cdots$$
$$=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{25}+\frac{1}{36}+\frac{1}{49}+\frac{1}{100}+\frac{1}{121}+\frac{1}{169}+\frac{1}{196}+\frac{1}{225}+\cdots=\frac{15}{\pi^2}.$$

These formulas involving the Möbius function can be spotted–albeit without the "μ" notation–in Euler's *Introductio* [**2**].

$$0 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} - \frac{1}{11} - \frac{1}{13} + \frac{1}{14} + \frac{1}{15}\ \&c.$$

$$\frac{6}{\pi\pi} = 1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{6^2} - \frac{1}{7^2} + \frac{1}{10^2} - \frac{1}{11^2} - \&c.$$

$$\frac{15}{\pi^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{10^2} + \frac{1}{11^2} + \&c.$$

From this last expression, it follows that the sum the reciprocals of the squares of those integers that are *not* square-free will be

$$\sum_{k=1}^{\infty}\frac{1}{k^2}-\sum_{k=1}^{\infty}\frac{[\mu(k)]^2}{k^2}=\frac{\pi^2}{6}-\frac{15}{\pi^2}.$$

That is,

$$\frac{1}{16}+\frac{1}{64}+\frac{1}{81}+\frac{1}{144}+\frac{1}{256}+\frac{1}{324}+\frac{1}{400}+\frac{1}{576}+\frac{1}{625}+\cdots=\frac{\pi^4-90}{6\pi^2}.$$

Look at this sum for a moment. It is exact. It is strange. It is astonishing. We have surely found our way into analytic territory where intuition is of no use whatever.

These wonderful results are examples of Euler being Euler, manipulating symbols with a gusto that can take one's breath away. In so doing, he not only anticipated the Möbius function but generated formulas more sophisticated than anything its namesake would discover eight decades later. Euler was, yet again, far ahead of his time.

With this, we conclude our story of a familiar number theoretic concept and its most peculiar ancestry. This tale reminds us–if we need reminding–that the history of mathematics can provide a host of unexpected rewards.

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Summary. The Möbius function is a fixture of modern courses in number theory. It is usually traced back to an 1832 paper by August Ferdinand Möbius where the function unexpectedly arose in answer to an analytic, rather than a number theoretic, question. But perhaps more unexpected is that the function can be found in Leonhard Euler's classic text, *Introductio in analysis infinitorum*, from 1748. Besides presenting the origins of what might be called the "Euler/ Möbius" function, this article is a reminder that the history of mathematics holds its share of surprises.

WILLIAM DUNHAM (MR Author ID: 238997) is a research associate in mathematics at Bryn Mawr College. He has authored four books on the history of mathematics, among which are *Euler: The Master of Us All* (MAA, 1999) and *The Calculus Gallery* (Princeton, 2005), and he co-edited, with Don Albers and Jerry Alexanderson, *The G. H. Hardy Reader* (MAA, 2015). From 2014–2016, Dunham was the MAA's George Pólya Lecturer.