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The μ pattern in words

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In this paper, we study the distribution of the number of occurrences of the simplest frame pattern, called the μ pattern, in words. Given a word $w = w_1 \dots w_n \in \{1, \dots, k\}^n$, we say that a pair $\langle w_i, w_j \rangle$ matches the μ pattern if $i < j$, $w_i < w_j$, and there is no $i < k < j$ such that $w_i \leq w_k \leq w_j$. We say that $\langle w_i, w_j \rangle$ is a trivial μ-match if $w_i + 1 = w_j$ and is a nontrivial μ-match if $w_i + 1 < w_j$. The main goal of this paper is to study the joint distribution of the number of trivial and nontrivial μ-matches in $\{1, \dots, k\}^*$.

1. Introduction

A mesh pattern is a particular type of permutation pattern introduced in [2] by Brändén and Claesson and studied in a series of papers (e.g. see [5] by Kitaev and Liese, and references therein). Consider the visual representation of a permutation as dots on a grid, where the heights of the dots from left to right are in the same relative order as the entries of the permutation. A mesh pattern is represented as a diagram of a permutation where some of the cells determined by the dots are shaded. For example, Figure 1 shows the diagram of a mesh pattern where the underlying permutation is 312. We say that a mesh pattern p with underlying permutation π occurs in a permutation σ if there is a subsequence of σ that is order-isomorphic to π and, further, the shaded areas determined by p and this subsequence of σ contain no elements of σ. In other words, σ has an occurrence of π in the classical sense, plus satisfies additional restrictions given by the positions of the shaded cells.

An example of a permutation that matches the mesh pattern p given in Figure 1 is $\sigma = 6741325$. In Figure 2, one can see that the subsequence 735 matches p.

A particular class of mesh patterns is *boxed patterns* introduced in [1] by Avgustinovich, Kitaev, and Valyuzhenich, who later suggested calling this type of pattern *frame patterns*. In these patterns, all but the boundary cells are shaded. The simplest frame pattern, which is called the μ pattern, has underlying permutation 12 and is defined as follows. Let S_n denote the set

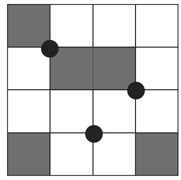

Figure 1: A mesh pattern with underlying permutation 312.

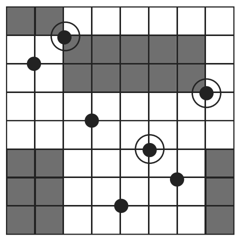

Figure 2: The permutation 6741325 matches the pattern in Figure 1.

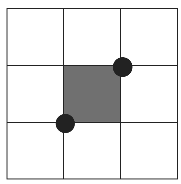

Figure 3: The frame pattern μ.

of all permutations of $\{1, \ldots, n\}$. Given $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in S_n$, we say that a pair $\langle \sigma_i, \sigma_j \rangle$ matches the μ pattern or is a μ-match in σ if $i < j$, $\sigma_i < \sigma_j$, and there is no $i < k < j$ such that $\sigma_i < \sigma_k < \sigma_j$. The μ pattern is shown in Figure 3 using the notation of [2]. Graphically, a μ-match is a pair of increasing dots such that there are no dots within the rectangle created by the original dots.

Analogously, we can define the μ' pattern to be the frame pattern with underlying permutation 21. That is, we say that the pair $\langle \sigma_i, \sigma_j \rangle$ matches the μ' pattern or is a μ'-match in σ if $i < j$, $\sigma_i > \sigma_j$, and there is no $i < k < j$ such that $\sigma_i > \sigma_k > \sigma_j$. For example, if $\sigma = 6741325$ as in Figure 4, then the μ-matches in σ are

$$\langle 6, 7 \rangle, \langle 4, 5 \rangle, \langle 1, 3 \rangle, \langle 1, 2 \rangle, \langle 3, 5 \rangle, \langle 2, 5 \rangle$$

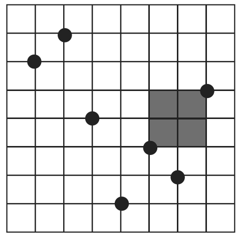

Figure 4: The graph of the permutation 6741325 with the occurrence $\langle 3,5\rangle$ highlighted.

and the μ'-matches in σ are

$$\langle 6,4\rangle, \langle 6,5\rangle, \langle 7,4\rangle, \langle 7,5\rangle, \langle 4,1\rangle, \langle 4,3\rangle, \langle 3,2\rangle.$$

The μ-match $\langle 3,5\rangle$ in the permutation $\sigma = 6741325$ is highlighted in Figure 4; in particular, there are no dots within the shaded rectangle.

We let $N_\mu(\sigma)$ (respectively, $N_{\mu'}(\sigma)$) denote the number of μ-matches (respectively, μ'-matches) in σ. The *reverse* of $\sigma = \sigma_1 \dots \sigma_n \in S_n$, σ^r, is the permutation $\sigma_n\sigma_{n-1}\dots\sigma_1$, and the *complement* of σ, σ^c, is the permutation $(n+1-\sigma_1)(n+1-\sigma_2)\dots(n+1-\sigma_n)$. It is easy to see that $N_\mu(\sigma) = N_{\mu'}(\sigma^r) = N_{\mu'}(\sigma^c)$ and thus, since the reverse and complement are trivial bijections from S_n to itself, studying the distribution of μ-matches in S_n is equivalent to studying the distribution of μ'-matches in S_n.

Avgustinovich, Kitaev, and Valyuzhenich [1] first studied the avoidance of frame patterns including μ and μ' in permutations in the symmetric group S_n. The distribution of μ-matches has also been studied in another setting, namely, Jones, Kitaev, and Remmel [4] studied *cycle-occurrences* of the μ pattern in the cycle structure of permutations.

In this paper, we shall study the distribution of μ-matches in words. For any positive integer k, we let $[k] = \{1,\dots,k\}$. We let $[k]^*$ denote the set of all words over the alphabet $[k]$. We let ϵ denote the empty word and we say ϵ has length 0. If $u = u_1\dots u_s$ and $v = v_1 \dots v_t$ are words in $[k]^*$, we let $uv = u_1\dots u_s v_1 \dots v_t$ denote the concatenation of u and v. We say that a word $u = u_1 \dots u_j$ is a prefix of w if $j \geq 1$ and there is a word v such that $uv = w$, we say that $v = v_1 \dots v_j$ is a suffix of w if $j \geq 1$ and there is a word u such that $uv = w$, and we say that $f = f_1 \dots f_j$ is a factor of w if $j \geq 1$ and there are words u and v such that $ufv = w$. We let $NR([k])$ denote the set of all words $w \in [k]^*$ such that w has no repeated letters, i.e., such that w has no factor of the form ii for $i \in [k]$.

Now suppose that $n \geq 1$ and $w = w_1 \dots w_n \in [k]^n$. Then we let $|w| = n$ denote the length of w. We say that a pair $\langle w_i, w_j \rangle$ is a μ-match in w if $i < j$, $w_i < w_j$, and there is no $i < k < j$ such that $w_i \leq w_k \leq w_j$. We say that $\langle w_i, w_j \rangle$ is a trivial μ-match if $w_i + 1 = w_j$ and is a nontrivial μ-match if $w_i + 1 < w_j$. We then let $\mathrm{triv}_\mu(w)$ denote the number of trivial μ-matches in w and $\mathrm{ntriv}_\mu(w)$ denote the number of nontrivial μ-matches in w. For example, if $w = 123121242416$, then $\mathrm{triv}_\mu(w) = 5$ since w has three $\langle 1, 2 \rangle$-matches, one $\langle 2, 3 \rangle$-match, and one $\langle 3, 4 \rangle$-match. Also, $\mathrm{ntriv}_\mu(w) = 4$ as w has one $\langle 1, 6 \rangle$-match, two $\langle 2, 4 \rangle$-matches, and one $\langle 4, 6 \rangle$-match. Similarly, we say that a pair $\langle w_i, w_j \rangle$ is a μ'-match in w if $i < j$, $w_i > w_j$, and there is no $i < k < j$ such that $w_i \geq w_k \geq w_j$. We say that $\langle w_i, w_j \rangle$ is a trivial μ'-match if $w_i = w_j + 1$ and is a nontrivial μ'-match if $w_i > w_j + 1$. As was the case with permutations, the correspondence that sends $w = w_1 \dots w_n \in [k]^n$ to its reverse $w^r = w_n \dots w_1$ or to its complement $w^c = (k+1-w_1) \dots (k+1-w_n)$ shows that the problem of studying μ-matches in words is equivalent to the problem of studying μ'-matches in words.

The main goal of this paper is to study the generating functions

$$A_\mu^{(k)}(p, q, t) = 1 + \sum_{n \geq 1} A_{n,\mu}^{(k)}(p, q) t^n \text{ and} \tag{1}$$

$$NR_\mu^{(k)}(p, q, t) = 1 + \sum_{n \geq 1} NR_{n,\mu}^{(k)}(p, q) t^n \tag{2}$$

where

$$A_{n,\mu}^{(k)}(p, q) = \sum_{w \in [k]^n} q^{\mathrm{triv}_\mu(w)} p^{\mathrm{ntriv}_\mu(w)} \text{ and} \tag{3}$$

$$NR_{n,\mu}^{(k)}(p, q) = \sum_{w \in NR([k]), |w| = n} q^{\mathrm{triv}_\mu(w)} p^{\mathrm{ntriv}_\mu(w)}. \tag{4}$$

Given a word $w \in [k]^*$, we can write $w = w_1^{j_1} w_2^{j_2} \dots w_s^{j_s}$ where $w_1 \dots w_s$ has no repeated letters. In such a situation, we say that $w_1 \dots w_s$ is the contraction of w and write $\mathrm{cont}(w) = w_1 \dots w_s$. For example, if $w = 1122211233333222444$, then $\mathrm{cont}(w) = 1212324$. It is easy to see that for any $w \in [k]^*$, $\mathrm{triv}_\mu(w) = \mathrm{triv}_\mu(\mathrm{cont}(w))$ and $\mathrm{ntriv}_\mu(w) = \mathrm{ntriv}_\mu(\mathrm{cont}(w))$. Moreover, if $w_1 \dots w_s \in NR([k])$ and $n \geq s$, then the number of $u \in [k]^n$ such that $\mathrm{cont}(u) = w$ equals the number of solutions to $j_1 + \dots + j_s = n$ where each $j_i \geq 1$ which is $\binom{n-1}{s-1}$. Thus it follows that for all $n \geq 1$,

$$A_{n,\mu}^{(k)}(p, q) = \sum_{s=1}^{n} \binom{n-1}{s-1} NR_{s,\mu}^{(k)}(p, q) \text{ and}$$

$$A_{\mu}^{(k)}(p,q,t) = NR_{\mu}^{(k)}(p,q,\frac{t}{1-t}).$$

For this reason, we shall focus on computing the generating functions $NR_{\mu}^{(k)}(p,q,t)$ for the rest of this paper.

It will be useful to consider some refinements of the generating functions $NR_{\mu}^{(k)}(p,q,t)$ depending on the prefix or suffix of the words. That is, given any non-empty words $u,v \in NR([k])$, we let

$$NR_{\mu}^{u,(k)}(p,q,t) = \sum_{n\geq |u|} NR_{n,\mu}^{u,(k)}(p,q)t^n, \tag{5}$$

$$NR_{\mu}^{(k),v}(p,q,t) = \sum_{n\geq |v|} NR_{n,\mu}^{(k),v}(p,q)t^n, \text{ and} \tag{6}$$

$$NR_{\mu}^{u,(k),v}(p,q,t) = \sum_{n\geq |u|+|v|} NR_{n,\mu}^{u,(k),v}(p,q)t^n, \tag{7}$$

where

$$NR_{n,\mu}^{u,(k)}(p,q) = \sum_{w\in u([k]^{n-|u|}),w\in NR([k])} q^{\mathrm{triv}_{\mu}(w)}p^{\mathrm{ntriv}_{\mu}(w)}, \tag{8}$$

$$NR_{n,\mu}^{(k),v}(p,q) = \sum_{w\in ([k]^{n-|v|})v,w\in NR([k])} q^{\mathrm{triv}_{\mu}(w)}p^{\mathrm{ntriv}_{\mu}(w)}, \text{and} \tag{9}$$

$$NR_{n,\mu}^{u,(k),v}(p,q) = \sum_{w\in u([k]^{n-|u|-|v|})v,w\in NR([k])} q^{\mathrm{triv}_{\mu}(w)}p^{\mathrm{ntriv}_{\mu}(w)}. \tag{10}$$

Note if $i,j \in [k]$, then $\langle i,j\rangle$ is a trivial (nontrivial) μ-match if and only if its reverse complement $\langle k+1-j, k+1-i\rangle$ is a trivial (nontrivial) μ-match. It follows that the map that sends any word $w = w_1 \dots w_n$ to its reverse complement $(k+1-w_n)\dots(k+1-w_1)$ shows that for any $i,j \in [k]$,

$$NR_{\mu}^{i,(k)}(p,q,t) = NR_{\mu}^{(k),k+1-i}(p,q,t) \text{ and}$$
$$NR_{\mu}^{i,(k),j}(p,q,t) = NR_{\mu}^{k+1-j,(k),k+1-i}(p,q,t).$$

The computation of $NR_{\mu}^{(k)}(p,q,t)$ is relatively simple for small k. For example, when $k = 2$, then there are no nontrivial μ-matches and the only trivial μ-matches are consecutive. In such a case, it is easy to see that

1. the words of the form $(12)^n$ contribute $\frac{1}{1-qt^2}$ to $NR_{\mu}^{(2)}(p,q,t)$,
2. the words of the form $2(12)^n$ contribute $\frac{t}{1-qt^2}$ to $NR_{\mu}^{(2)}(p,q,t)$,

3. the words of the form $(12)^n 1$ contribute $\frac{t}{1-qt^2}$ to $NR_\mu^{(2)}(p,q,t)$, and
4. the words of the form $2(12)^n 1$ contribute $\frac{t^2}{1-qt^2}$ to $NR_\mu^{(2)}(p,q,t)$.

Thus

$$NR_\mu^{(2)}(p,q,t) = \frac{1+2t+t^2}{1-qt^2}.$$

When $k=3$, then the only nontrivial μ-matches are consecutive occurrences of 13. Indeed, in any word $w \in NR([k])$, only factors of the form iuj where $i+1<j$ and u does not contain any letters s such that $i \le s \le j$ correspond to nontrivial μ-matches. Thus for $k=3$, we can only have a nontrivial μ-match when $i=1$, $j=3$, and $u=\epsilon$. In this case, we shall show that one can compute $NR_\mu^{(3)}(p,q,t)$ by showing that the functions $NR_\mu^{ij,(3)}(p,q,t)$ satisfy some simple recursions for $i \neq j \in [3]$. That is, by knowledge of the first two letters of a word, we can classify all μ-matches involving the first letter of the word, then remove it recursively. The problem with this approach is that it does not generalize. When $k \ge 4$, there are words for which knowledge of the first two letters and, in fact, no finite initial segment is enough to classify all μ-matches involving the first letter. For example, if $k=4$, then for words of the form $w=2(14)^s$, one can always extend w by adding a 3 at the end to ensure that the initial 2 is involved in a trivial μ-match. Similarly if $k \ge 5$ and $v=2(15)^s$, then one can always extend w by adding a 3 at the end to ensure that the initial 2 is involved in a trivial μ-match or by adding a 4 at the end to ensure that the initial 2 is involved in a nontrivial μ-match. Thus it is impossible to determine whether there is a $\langle 2,3\rangle$-match or a $\langle 2,4\rangle$-match by knowledge of the first t letters of a word for any finite t.

Thus we will develop an alternative method to compute $NR_\mu^{(k)}(p,q,t)$ that uses a weighted automaton with C_k states where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the kth Catalan number. This method is useful in that it gives an algorithm for finding $NR_\mu^{(k)}(p,q,t)$ that works even when it is not straightforward to write down recursions, as with $k \ge 4$. The use of automata and the transfer matrix method to find the generating function for the number of occurrences of a pattern in a class of words, or to find the generating function for the number of words that avoid a pattern in class of words, has a long history. See chapter seven of [3] for the history of this method and details of the technique.

The outline of this paper is as follows. In Section 2, we shall show how the functions $NR_\mu^{ij,(3)}(p,q,t)$ satisfy certain simple recursions allowing us to find $NR_\mu^{(3)}(p,q,t)$ and then describe some of the problems of extending this method to find $NR_\mu^{(k)}(p,q,t)$ for $k \geq 4$. In Section 3, we shall describe our automaton method for computing $NR_\mu^{(3)}(p,q,t)$ and give the results for $k=4$ and $k=5$. Finally, in Section 5, we shall look at the problem of computing generating functions for the distribution of consecutive trivial and nontrivial μ-matches in words. In fact, we will show how to compute the generating function

$$CNR_\mu^{(k)}(x_1,\ldots,x_{k-1},t) = 1+\sum_{n\geq 1} t^n \sum_{w\in NR([k]),|w|=n} \prod_{j=1}^{k-1} x_j^{\mathrm{jrise}(w)}$$

where for any word $w = w_1\ldots w_n$ and $1 \leq j \leq k-1$, $\mathrm{jrise}(w) = |\{i : w_{i+1}-w_i = j\}|$. In particular, this will allow us to give a closed form expression for $NR_\mu^{(k)}(0,q,t)$ for all k.

2. A recursive approach for $k = 3, 4$

In this section, we shall first give a straightforward method for finding the generating function $NR_\mu^{(3)}(p,q,t)$ by showing that the functions of the form $NR_\mu^{ij,(3)}(p,q,t)$ satisfy simple recursions. Then we shall describe the difficulties of extending this method to compute the generating functions $NR_\mu^{(k)}(p,q,t)$ for $k \geq 4$.

Case 1. $NR_\mu^{12,(3)}(p,q,t)$.
Any word that starts with 12 is either the word 12 or starts with 121 or 123. All such words have a trivial $\langle 1,2\rangle$-match between w_1 and w_2, and further, w_1 cannot be involved in any other μ-matches. Thus, by paring off w_1, it follows that

$$NR_\mu^{12,(3)}(p,q,t) = qt^2 + qtNR_\mu^{21,(3)}(p,q,t) + qtNR_\mu^{23,(3)}(p,q,t).$$

Case 2. $NR_\mu^{13,(3)}(p,q,t)$.
Any word that starts with 13 is either the word 13 or starts with 131 or 132. All such words have a nontrivial $\langle 1,3\rangle$-match between w_1 and w_2. Furthermore, in words that start 131, w_1 cannot be involved in any other μ-matches because $w_3 = w_1$. In words that start 132, a trivial $\langle 1,2\rangle$-match also exists, and w_1 can be involved in no additional μ-matches. Thus it follows that

$$NR_\mu^{13,(3)}(p,q,t) = pt^2 + ptNR_\mu^{31,(3)}(p,q,t) + pqtNR_\mu^{32,(3)}(p,q,t).$$

This type of reasoning can be used in all the other cases of $NR_\mu^{ij,(3)}(p,q,t)$, for $i \neq j \in [3]$. Rather than giving a detailed reasoning in each case, we shall just list the resulting equations.

$$\begin{aligned}
NR_\mu^{21,(3)}(p,q,t) &= t^2 + tNR_\mu^{12,(3)}(p,q,t) + qtNR_\mu^{13,(3)}(p,q,t),\\
NR_\mu^{23,(3)}(p,q,t) &= qt^2 + qtNR_\mu^{31,(3)}(p,q,t) + qtNR_\mu^{32,(3)}(p,q,t),\\
NR_\mu^{31,(3)}(p,q,t) &= t^2 + tNR_\mu^{12,(3)}(p,q,t) + tNR_\mu^{13,(3)}(p,q,t), \text{ and}\\
NR_\mu^{32,(3)}(p,q,t) &= t^2 + tNR_\mu^{21,(3)}(p,q,t) + tNR_\mu^{23,(3)}(p,q,t).
\end{aligned}$$

Putting these equations together, we obtain the following matrix equation.

$$\begin{bmatrix} -qt^2 \\ -pt^2 \\ -t^2 \\ -qt^2 \\ -t^2 \\ -t^2 \end{bmatrix} = M \begin{bmatrix} NR_\mu^{12,(3)}(p,q,t) \\ NR_\mu^{13,(3)}(p,q,t) \\ NR_\mu^{21,(3)}(p,q,t) \\ NR_\mu^{23,(3)}(p,q,t) \\ NR_\mu^{31,(3)}(p,q,t) \\ NR_\mu^{32,(3)}(p,q,t) \end{bmatrix} \tag{11}$$

where

$$M = \begin{bmatrix} -1 & 0 & qt & qt & 0 & 0 \\ 0 & -1 & 0 & 0 & pt & pqt \\ t & qt & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & qt & qt \\ t & t & 0 & 0 & -1 & 0 \\ 0 & 0 & t & t & 0 & -1 \end{bmatrix}.$$

Thus if we multiply both sides of (11) by M^{-1}, we can can solve for the vector on the right whose components are $NR_\mu^{ij,(3)}(p,q,t)$ for $i \neq j \in [3]$. This gives a refinement of $NR_\mu^{(3)}(p,q,t)$ by the first two letters of a word. One can obtain the generating function $NR_\mu^{(3)}(p,q,t)$ by taking $1 + 3t + \sum_{i\neq j\in[3]} NR_\mu^{ij,(3)}(p,q,t)$ to account for words of length less than two in addition to those counted in the cases above. We have carried out this computation in Mathematica and found that

$$NR_\mu^{(3)}(p,q,t) = \frac{(1+t)^2(1+t-p(q-1)^2t^3)}{1-2qt^2-q^2t^3-pt^2(1+q^2t+2q(q-1)t^2)}. \tag{12}$$

While this approach seems straightforward enough in this case, it does not generalize well. One problem that exists in the cases where $k \geq 4$ is that there are words for which no finite initial segment is enough to classify all μ-matches involving the first letter.

As an example of the kind of analysis involved, consider the case $NR_\mu^{21,(4)}(p,q,t)$. Words that start with 21 are either the word 21, which contributes a factor of t^2 to $NR_\mu^{21,(4)}(p,q,t)$, or they begin with 212, 213, or 214. Words that begin with 212 and 213 contribute $tNR_\mu^{12,(4)}(p,q,t)$ and $qtNR_\mu^{13,(4)}(p,q,t)$, respectively, by accounting for all μ-matches involving the first letter and then removing it. Words that begin with 214 are more complicated to count, because the weight depends on whether or not there is a 3 that appears later in the word. To determine the contribution of such words to $NR_\mu^{21,(4)}(p,q,t)$, we must consider several cases. We classify each word starting with 214 by the first occurrence of a 2 or 3, so that each word starting with 214 falls into exactly one of the cases below.

Case 1. $214\{14\}^*$.
The first three letters of such a word contribute p^2t^3 as $\langle 2,4\rangle$ and $\langle 1,4\rangle$ are nontrivial μ-matches. Since the pair 14 is then repeated zero or more times, and each occurrence contributes a factor of pt^2, the contribution to $NR_\mu^{21,(4)}(p,q,t)$ of the words in this case is

$$\frac{p^2t^3}{1-pt^2}. \tag{13}$$

Case 2. $214\{14\}^*1$.
By the exact same reasoning as the previous case, this case contributes a factor of

$$\frac{p^2t^4}{1-pt^2}. \tag{14}$$

Since there is an additional last letter that is not involved in any μ-matches, the power of t is one greater than in Case 1.

Case 3. $\underline{2}1\underline{4\{14\}}^*2\ldots$.
For a word of this form, we will remove all the underlined letters so that we can give its contribution in terms of $NR_\mu^{12,(4)}(p,q,t)$. In order to do that, of course, we must account for all μ-matches involving the underlined letters. The first letter is involved in a $\langle 2,4\rangle$-match so it contributes a factor of pt. The second underlined letter is involved in a $\langle 1,4\rangle$-match so it contributes an additional pt. Each 14 pair contributes a factor of pt^2 as before, so that

the contributions of the words in this case is

$$\frac{p^2t^2}{1-pt^2}NR_{\mu}^{12,(4)}(p,q,t). \tag{15}$$

Case 4. $2\underline{1}\,\underline{4\{14\}^*}3\ldots$.
This is the same as the previous case except the first letter is now involved in a trivial $\langle 2,3\rangle$-match that introduces an additional q, and the word that remains after we remove letters begins with 13. Thus the contribution of the words in this case is

$$\frac{qp^2t^2}{1-pt^2}NR_{\mu}^{13,(4)}(p,q,t). \tag{16}$$

Case 5. $\underline{214\{14\}^*}12\ldots$.
As with Case 3, we remove the underlined letters. The first three letters contribute a factor of p^2t^3, and each 14 factor contributes pt^2. Thus the contribution of the words in this case is

$$\frac{p^2t^3}{1-pt^2}NR_{\mu}^{12,(4)}(p,q,t). \tag{17}$$

Case 6. $\underline{214\{14\}^*}13\ldots$.
These words are just like the words of Case 5 except there is an extra factor of q coming from the $\langle 2,3\rangle$ match. Thus the contribution of the words in this case is

$$\frac{qp^2t^3}{1-pt^2}NR_{\mu}^{13,(4)}(p,q,t). \tag{18}$$

Putting (13), (14), (15), (16), (17), and (18) together with our initial observations, we obtain the following equation.

$$\begin{aligned} &NR_{\mu}^{21,(4)}(p,q,t) \\ &\quad = t^2 + \frac{p^2t^3(1+t)}{1-pt^2} + (t + \frac{p^2t^2(1+t)}{1-pt^2})NR_{\mu}^{12,(4)}(p,q,t) + \\ &\qquad (qt + \frac{qp^2t^2(1+t)}{1-pt^2})NR_{\mu}^{13,(4)}(p,q,t). \end{aligned}$$

We were able to complete a case-by-case analysis in this way for the case $k=4$. However, even in the case $k=5$, this type of analysis seemed too complex to be feasible. Even if it were feasible to perform this kind of case-by-case analysis for larger k, there is another problem with this method, namely that it relies upon being able to invert a large symbolic matrix whose entries are rational functions in p, q, and t. If we classify words by the first two letters, as above, this means there are $k(k-1)$ possibilities for w_1w_2 and so the matrix M that must be inverted is a square matrix of dimension $k(k-1)$. Using more than the first two letters might make the equations easier to write down, but then the matrix would be too large to invert. We describe an alternative approach in the next section.

3. A weighted automaton approach

In this section, we describe an alternative way of computing the generating functions $NR_\mu^{(k)}(p,q,t)$ using state transition diagrams or finite automata. For the case $k=3$, we will compare this method to the one used in the previous section and show that it is preferable because it is more easily generalized to larger k. While it also involves inverting a large matrix, the main benefit is that there is an algorithm for determining the entries of this matrix, so it does not involve any complicated casework as with the method of the previous section.

Given a word $w = w_1 \ldots w_n$ in $NR([k])$, we say that there is a *potential* $\langle i,j\rangle$-match in w if there is an $\langle i,j\rangle$-match in $w' = wj$ that is not present in w. In other words, w has a potential $\langle i,j\rangle$-match in w if adding a j to the end of w would create an $\langle i,j\rangle$-match. Given a word $w = w_1 \ldots w_n$ with a potential $\langle i,j\rangle$-match and an extension of w, $w' = ww_{n+1}$, we say that w_{n+1} *completes* the $\langle i,j\rangle$-match if $w_{n+1} = j$, *kills* the $\langle i,j\rangle$-match if $i < w_{n+1} < j$, and does not change the status of the $\langle i,j\rangle$-match if $w_{n+1} > j$ or $w_{n+1} \le i$.

To each word $w = w_1 \ldots w_n$ in $NR([k])$, we associate a *state matrix* M_w which is a $0,1$-valued $k \times k$ matrix. If $w = \epsilon$, then $M_w = 0_{k\times k}$. If $w \ne \epsilon$, then

(i) $M_w(i,j) = 0$ if $i \ge j$,
(ii) $M_w(i,j) = 1$ if $i < j$ and there is a potential $\langle i,j\rangle$-match in w, and
(iii) $M_w(i,j) = 0$ otherwise.

It is easy to see that M_w is an upper triangular matrix with 0's on the diagonal. Note that several different words may have the same state matrix. For example, if $k=4$,

$$M_{134} = M_{2134} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

since the only potential μ-match in either 134 or 2134 is a $\langle 1,2\rangle$-match.

Next we present a simple algorithm for obtaining the state matrix M_w for a word $w = w_1w_2\dots w_n$. Let $w^{(0)} = \epsilon$ and $w^{(\ell)} = w_1\dots w_\ell$ for $\ell \in \{1,2,\dots,n\}$. We will create a sequence of matrices $M_{w^{(0)}}, M_{w^{(1)}}, M_{w^{(2)}}, \dots, M_{w^{(n)}}$ resulting in our desired state matrix $M_w = M_{w^{(n)}}$. Set $M_{w^{(0)}} = 0_{k\times k}$ and for $\ell = 1,2,\dots,n$, if the addition of the letter w_ℓ to $w^{(\ell-1)}$ creates a potential $\langle i,j\rangle$-match, set $M_{w^{(\ell)}}(i,j) = 1$. Similarly, if it kills or completes any potential $\langle i,j\rangle$-match, set $M_{w^{(\ell)}}(i,j) = 0$. The state matrix M_w is simply $M_{w^{(n)}}$ because it has 1's exactly in the positions (i,j) where w has a potential $\langle i,j\rangle$-match. For example, the computation of M_{134} would produce the following sequence of matrices

$$\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \overset{1}{\to} \begin{bmatrix} 0&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix} \overset{13}{\to} \begin{bmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{bmatrix} \overset{134}{\to} \begin{bmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}.$$

The next lemma gives some properties of this algorithm.

Lemma 1. *Let $w = w_1\dots w_n$ be a word in $NR([k])$ and $\ell \in \{1,2,\dots,n\}$.*

(a) Row w_ℓ of $M_{w^{(\ell)}}$ has all 1's to the right of the main diagonal. For $i < w_\ell$, row i of $M_{w^{(\ell)}}$ has 0's in column w_ℓ and to the right. For $i > w_\ell$, row i of $M_{w^{(\ell)}}$ is the same as row i of $M_{w^{(\ell-1)}}$.

(b) To the right of the main diagonal, each row of the matrix $M_{w^{(\ell)}}$ is some number of 1's followed by some number of 0's.

(c) In each row, the leftmost entry that changes from a 1 in $M_{w^{(\ell-1)}}$ to a 0 in $M_{w^{(\ell)}}$ indicates a μ-match that has been completed by the addition of the letter w_ℓ.

(d) The matrix $M_{w^{(\ell)}}$ completely determines the last letter w_ℓ of all associated nonempty words.

Proof. (a) After reading w_ℓ, row w_ℓ of $M_{w^{(\ell)}}$ has all 1's to the right of the main diagonal because if $j > w_\ell$, then every $\langle w_\ell, j\rangle$ is a potential μ-match. For $i < w_\ell$, row i has 0's in column w_ℓ and to the right because w_ℓ completes any potential $\langle i, w_\ell\rangle$-match and kills any potential $\langle i,j\rangle$-match for $j > w_\ell$. For $i > w_\ell$, row i does not change because the addition of the letter w_ℓ does not change the status of any potential μ-matches of the form $\langle i,j\rangle$.

(b) By part (a), reading a letter w_ℓ can do one of three things to any row of the matrix, to the right of the main diagonal: set it to all 1's, set it to all 0's to the right of a certain column, or not change it. The only time matrix entries are set to 0, it is the case that all other matrix entries in the same row and to the right are also set to 0. Therefore each row, to the right of the main diagonal, is a sequence of 1's followed by a sequence of 0's.

(c) Suppose that for some $i < w_\ell$, $M_{w^{(\ell-1)}}(i, w_\ell) = 1$. This means there is a potential $\langle i, w_\ell\rangle$-match in $w^{(\ell-1)}$ that is completed by the addition of the letter w_ℓ, making $M_{w^{(\ell)}}(i, w_\ell) = 0$. Since $i < w_\ell$, apply part (a) to conclude that in row i, $M_{w^{(\ell)}}$ has 0's in column w_ℓ and to the right. That is, the entry in position (i, w_ℓ) of $M_{w^{(\ell-1)}}$ is the leftmost entry in row i to change from a 1 to a 0.

Now suppose that for some $i < w_\ell$, $M_{w^{(\ell-1)}}(i, w_\ell) = 0$. Then by part (b), since each row to the right of the main diagonal is a sequence of 1's followed by a sequence of 0's, we can conclude that every entry to the right of column w_ℓ in $M_{w^{(\ell-1)}}$ is a 0. By part (a), only the entries to the right of column w_ℓ in row i are set to 0, for $i < w_\ell$. This says that if $i < w_\ell$ and $M_{w^{(\ell-1)}}(i, w_\ell) = 0$, then nothing in row i changes from a 1 to a 0. In this case, no $\langle i, w_\ell\rangle$-match is completed by the addition of the letter w_ℓ.

In row w_ℓ, nothing changes from a 1 to a 0, since part (a) says that row w_ℓ of $M_{w^{(\ell)}}$ has all 1's to the right of the main diagonal. In any row $i > w_\ell$, no entries change, according to part (a). Thus, in each row, the leftmost entry that changes from a 1 in $M_{w^{(\ell-1)}}$ to a 0 in $M_{w^{(\ell)}}$ indicates a μ-match that has been completed by the addition of the letter w_ℓ.

(d) By part (a), if w_ℓ is the last letter of a word, then row w_ℓ of $M_{w^{(\ell)}}$ contains all 1's to the right of the main diagonal, which we will call a *full row*. Further, if row i of $M_{w^{(\ell)}}$ is a full row, this means all the letters after the last occurrence i in $w^{(\ell)}$ are less than i. Otherwise, there would be some zeros in row i. Thus if $M_{w^{(\ell)}}$ has at least one full row, say in rows $i_1 < i_2 < \cdots < i_j$, then the last letters of w are $i_j \cdots i_2 i_1$. Also, if the matrix has no full rows, the associated nonempty words w must end in $w_\ell = k$, the largest letter in the alphabet, because ending in any other letter would create a full row. Thus, it is easy to determine the last letter w_ℓ from the matrix $M_{w^{(\ell)}}$, that is, w_ℓ equals the index of the first full row or k if no full rows exist. □

We use these state matrices M_w to make a state transition diagram, or a weighted directed graph $G = (V, E)$. The vertex set is the set of all possible state matrices $V = \{M_w : w \in NR([k])\}$ and there is an edge from M_w to $M_{w'}$ if $w' = w w_{n+1}$. Also include a separate vertex for the empty

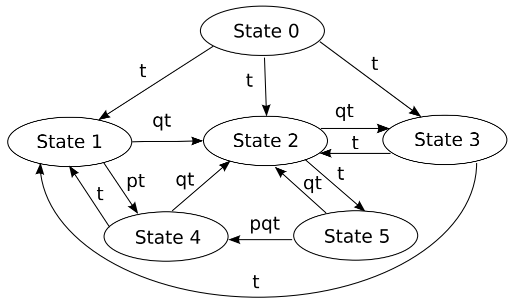

Figure 5: The state diagram for the case $k = 3$.

word. Lemma 1 part (d) says that all nonempty words associated with a given vertex of the graph G end in the same letter. This means our edge set is well-defined and each vertex has $k-1$ outgoing edges because any of $k-1$ letters could be appended to the end of the word without a consecutive repeat. The vertex for the empty word will have k outgoing edges because any of $\{1, 2, \ldots, k\}$ can be appended to ϵ. To procedurally find all the vertices to which a given vertex M_w should point, for each possible $w_{n+1} \neq w_n \in [k]$, find the state matrix $M_{ww_{n+1}}$ using Lemma 1 part (a). That is, when adding a letter w_{n+1}, set row w_{n+1} to be all 1's to the right of the main diagonal. For $i < w_{n+1}$, set row i to have 0's in column w_{n+1} and to the right. For $i > w_{n+1}$, leave row i unchanged.

Assign to each edge in the graph a weight that represents the change in weight from w to ww_{n+1}. Since each additional letter introduces a t into the generating function $NR_\mu^{(k)}(p, q, t)$, each edge will be weighted with at least a t. Further, if adding w_i completes a μ-match, then the edge weight would also reflect that by including p's, q's, or both. For example, the edge between M_{13} and M_{134} would be weighted qt since the last 4 completes a trivial $\langle 3, 4\rangle$-match. Part (c) of Lemma 1 determines the edge weight in a simple way. Start each edge with a weight of t. In going from M_w to $M_{ww_{n+1}}$, note the leftmost position in each row where a matrix entry changes from a 1 to a 0. For each such position on the superdiagonal, multiply the edge weight by q for completing a trivial μ-match, and for each such position elsewhere, multiply the edge weight by p for completing a nontrivial μ-match.

For example, the state diagram associated with $k = 3$ is given in Figure 5, where the state labeled 0 represents the empty word and the other states represent words with a given state matrix M_w, according to the following table.

State	1	2	3	4	5
M_w	$\begin{bmatrix}0&1&1\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}$	$\begin{bmatrix}0&1&1\\0&0&1\\0&0&0\end{bmatrix}$

Next, we use the state transition diagram to find the generating function $NR_\mu^{(k)}(p,q,t)$. Label the states of the diagram, excluding the state for the empty word, with $1, 2, \ldots, N$ where $N = N(k)$ is the number of state matrices. Label the state for the empty word with 0, as in Figure 5. For $i \in \{0, 1, \ldots, N\}$, let

$$S_i^{(k)}(p,q,t) = \sum_{w \text{ in state } i} q^{\mathrm{triv}_\mu(w)} p^{\mathrm{ntriv}_\mu(w)} t^{|w|}.$$

Since the empty word is the only word in state 0, it is easy to see that $S_0^{(k)}(p,q,t) = 1$ for all k. Also, since each word $w \in NR([k])$ is in exactly one state, it follows that

$$NR_\mu^{(k)}(p,q,t) = \sum_{i=0}^{N} S_i^{(k)}(p,q,t) = 1 + \sum_{i=1}^{N} S_i^{(k)}(p,q,t). \tag{19}$$

For each $i \in \{1, 2, \ldots, N\}$, let $In(i)$ be the set of states $j \in \{0, 1, \ldots, N\}$ such that there is an edge from state j to state i with edge weight $e_{j\to i}$. For each $i \in \{1, 2, \ldots, N\}$, we have the equation

$$S_i^{(k)}(p,q,t) = \sum_{j \in In(i)} S_j^{(k)}(p,q,t) e_{j\to i} \tag{20}$$

that says all words in state i come from adding a letter to a word in some state j. The edge weight $e_{j\to i}$ accounts for increases in length and number of μ-matches from adding this letter. Using the fact that $S_0^{(k)}(p,q,t) = 1$, this gives a system of N linear equations in variables $S_i^{(k)}(p,q,t)$ for $i \in \{1, 2, \ldots, N\}$. Solving this system involves inverting a square $N \times N$ matrix. By (19), we can easily compute the generating function $NR_\mu^{(k)}(p,q,t)$ from such a solution.

For example, in the case $k = 3$ with the states labeled as in Figure 5, the system of linear equations from (20) gives

$$\begin{bmatrix} -t \\ -t \\ -t \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & t & t & 0 \\ qt & -1 & t & qt & qt \\ 0 & qt & -1 & 0 & 0 \\ pt & 0 & 0 & -1 & pqt \\ 0 & t & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} S_1^{(3)}(p,q,t) \\ S_2^{(3)}(p,q,t) \\ S_3^{(3)}(p,q,t) \\ S_4^{(3)}(p,q,t) \\ S_5^{(3)}(p,q,t) \end{bmatrix}. \tag{21}$$

Call the left-hand side vector T_3, the matrix M_3, and the right-hand side vector S_3, and let $1^{(5)}$ be the 1×5 row vector of all 1's. Then by (19),

$$NR_\mu^{(3)}(p,q,t) = 1 + 1^{(5)} S_3 = 1 + 1^{(5)} M_3^{-1} T_3.$$

In the case $k = 3$, one can use Mathematica to invert M_3 and compute $NR_\mu^{(3)}(p,q,t)$ directly. For larger k, however, we can take a shortcut using determinants and minors that eliminates the need to compute the full matrix inverse. Note that for any k, the right-hand side vector S_k will have ith component $S_i^{(k)}$, for $i = 1, 2, \ldots, N(k)$. Also, the left-hand side vector T_k will have entries which are all $-t$ or 0, and so we can label the states 1 through $N(k)$ so that the first component of $T(k)$ is a $-t$. Now, take the system of equations that results from our original system $T_k = M_k S_k$ by subtracting the first equation from any other equation which has a $-t$ on the left-hand side. This results in an equivalent system of equations $\overline{T_k} = \overline{M_k} S_k$ with the same solution S_k but now $\overline{T_k}$ is a column vector with first component $-t$ and all other components 0.

Since $\overline{M_k}$ arises from M_k by subtracting the first row from some other rows, an operation which preserves the determinant, we have $\det(\overline{M_k}) = \det(M_k)$.

Recall that

$$\left(\overline{M_k}\right)^{-1} = \frac{1}{\det\left(\overline{M_k}\right)} \begin{bmatrix} \overline{C}_{1,1} & \ldots & \overline{C}_{k,1} \\ \overline{C}_{1,2} & \ldots & \overline{C}_{k,2} \\ \vdots & \ldots & \vdots \\ \overline{C}_{1,k} & \ldots & \overline{C}_{k,k} \end{bmatrix}$$

where $\overline{C}_{i,j}$ is (i,j)th cofactor of $\overline{M_k}$.

It follows from equation (19) that

$$NR_\mu^{(k)}(p,q,t) = 1 + 1^{(N(k))} M_k^{-1} T_k$$

$$
\begin{aligned}
&= 1 + 1^{(N(k))}\left(\overline{M_k}\right)^{-1}\overline{T_k}\\
&= 1 + \frac{1}{\det\left(\overline{M_k}\right)} 1^{(N(k))} \begin{bmatrix} \overline{C}_{1,1} & \dots & \overline{C}_{k,1} \\ \overline{C}_{1,2} & \dots & \overline{C}_{k,2} \\ \vdots & \dots & \vdots \\ \overline{C}_{1,k} & \dots & \overline{C}_{k,k} \end{bmatrix} \begin{bmatrix} -t \\ 0 \\ \vdots \\ 0 \end{bmatrix}\\
&= 1 + \frac{-t\sum_{j=1}^{k} \overline{C}_{1,j}}{\det\left(\overline{M_k}\right)}\\
&= 1 + \frac{-t\sum_{j=1}^{k} \overline{C}_{1,j}}{\det\left(M_k\right)}, \qquad (22)
\end{aligned}
$$

which Mathematica can compute for $k \le 5$. When $k = 3$, we obtain

$$NR_{\mu}^{(3)}(p,q,t) = \frac{(1+t)^2(1+t-p(q-1)^2t^3)}{1-2qt^2-q^2t^3-pt^2(1+q^2t+2q(q-1)t^2)},$$

which is the same generating function found in (12). When $k = 4$, we obtain

$$NR_{\mu}^{(4)}(p,q,t) = \frac{P_4(x,y,t)}{Q_4(x,y,t)}$$

where

$$
\begin{aligned}
&P_4(x,y,t) = (1+t)^2\times\\
&\quad \big(1+2t+(1-2x-y)t^2+\\
&\quad \left(-4x-6y+6xy-2x^2y+2y^2-2xy^2\right)t^3+\\
&\quad \big(-y(4+y)-x^3y\left(3+y+y^2\right)+x^2\left(1+5y+y^2\right)+\\
&\qquad x\left(-2+3y+y^2+y^3\right)\big)\,t^4+\\
&\quad \big(2x^2+8xy-4x^2y+6y^2-26xy^2+28x^2y^2-\\
&\qquad 8x^3y^2-4y^3+4xy^3+6x^2y^3-6x^3y^3\big)\,t^5+\\
&\quad \big(y^2(6+y)+x^4y\left(4-2y-3y^2\right)-x^3y\left(8-6y+y^2+y^3\right)-\\
&\qquad 3xy\left(-2+4y+3y^2+y^3\right)+x^2\left(1-y+2y^2+12y^3+4y^4\right)\big)\,t^6-\\
&\quad 2y\big(x^5y^2-(-1+y)y^2+2xy\left(1-6y+2y^2\right)+x^3\left(1+9y-11y^2-2y^3\right)+\\
&\qquad x^2\left(1-8y+21y^2-2y^3\right)+x^4\left(-1-3y+y^3\right)\big)\,t^7+\\
&\quad y\,\big(-4y^2+x^6(-1+y)y^2+xy\left(-6+19y+2y^2\right)-\\
&\qquad x^2\left(2-10y+13y^2+2y^3\right)-x^5\left(1-2y+y^2+6y^3\right)-
\end{aligned}
$$

$$
\begin{aligned}
&x^3\left(1+16y^2+6y^3\right)+x^4\left(3-6y+16y^2+11y^3\right)\right)t^8+\\
&2(-1+x)xy^2\left(2x(1-6y)y+x^3(2-5y)y+\right.\\
&\quad\left.x^4(-1+y)y+3y^2+x^2\left(-1+11y^2\right)\right)t^9+\\
&(-1+x)y^2\left(x^5(-1+y)y-y^2+xy(-2+7y)+\right.\\
&\quad\left.x^2\left(-1+7y-15y^2\right)+x^4\left(1+y-4y^2\right)+x^3\left(-2-y+10y^2\right)\right)t^{10}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
Q_4(x,y,z) = {} & 1+(-5x-4y)t^2-2(x(1+y)(x+y))t^3+\\
&\left(8x^2-x^3+18xy-4x^2y-3x^3y+6y^2-4xy^2-x^3y^2-x^3y^3\right)t^4+\\
&2x\left(7xy(1+y)+2y^2(1+y)-2x^2\left(-1+y+2y^2+y^3\right)\right)t^5+\\
&\left(8x(-3+y)y^2-4y^3+x^4\left(2+4y-2y^2\right)+\right.\\
&\quad\left.x^2y\left(-24+29y+2y^2+y^3\right)-x^3\left(5-9y+3y^2+3y^3\right)\right)t^6+\\
&2x\left(-y^3(1+y)+2xy^2\left(-5-4y+y^2\right)+\right.\\
&\quad\left.x^3\left(-1+5y+2y^2\right)+x^2y\left(-9+y+8y^2\right)\right)t^7+\\
&\left(2x(7-2y)y^3+y^4+x^6y^4+x^2y^2\left(25-40y+4y^2\right)+\right.\\
&\quad x^3y\left(12-35y+14y^2-5y^3\right)-\\
&\quad\left.x^5\left(1+y-2y^2+y^3+3y^4\right)+x^4\left(1-6y+6y^2+8y^3+5y^4\right)\right)t^8+\\
&2x^2y\left(2x^4y^3+4y^2(1+y)-3xy\left(-3+y+4y^2\right)-\right.\\
&\quad\left.x^3\left(2-y+y^2+6y^3\right)+x^2\left(3-7y+2y^2+10y^3\right)\right)t^9+\\
&xy\left(-3y^3+6x^5y^3+2xy^2(-5+8y)-3x^2y\left(3-10y+9y^2\right)-\right.\\
&\quad\left.x^4\left(-1+2y+5y^2+18y^3\right)+x^3\left(-2+11y-10y^2+25y^3\right)\right)t^{10}+\\
&2x^3y^2\left(2x^3y^2-2y(1+y)+x^2\left(2-3y-7y^2\right)+x\left(-2+4y+8y^2\right)\right)t^{11}+\\
&x^2y^2\left(x+y-3xy+x^2y\right)^2t^{12}.
\end{aligned}
$$

When $k=5$, we are able to compute $NR_\mu^{(5)}(p,q,t)$ despite not being able to do so with the method of Section 2. In this case, $NR_\mu^{(5)}(p,q,t)=\frac{P_5(p,q,t)}{Q_5(p,q,t)}$ where $P_5(x,y,t)$ and $Q_5(p,q,t)$ are degree 36 polynomials in t. We shall not give the polynomials $P_5(p,q,t)$ and $Q_5(p,q,t)$ here because they would take several pages to even write down.

We end this section with a rather unexpected result about the size $N(k)$ of the square matrix M_k, or the number of state matrices.

Theorem 2. *The number of state matrices $N(k)$ is the k^{th} Catalan number C_k.*

Proof. Since $N(k)$ is the number of state matrices for nonempty words over $[k]$, let $N_j(k)$ be the number of state matrices for words over $[k]$ ending in j. Then clearly

$$N(k) = \sum_{j=1}^{k} N_j(k). \tag{23}$$

Suppose w is a word ending with j whose associated state matrix is M_w. Then, by Lemma 1 part (a), row j of M_w has all 1's to the right of the main diagonal and for any $m \geq 1, p \geq 0$, M_w has a 0 in row $j - m$ and column $j + p$. Since every entry of M_w on the diagonal and in the lower triangular part is 0 by definition, this leaves a $(j - 1) \times (j - 1)$ submatrix in the upper left corner and a $(k - j) \times (k - j)$ submatrix in the lower right corner with some entries yet undetermined. It is clear that the presence of letters $i \geq j$ in the word w has no bearing on the μ-matches in w of the form $\langle a, b\rangle$ with $a, b \in [j - 1]$. Thus, the submatrix in the upper left corner of M_w is just the state matrix for the word $w' \in NR([j - 1])$ that comes from removing all letters $i \geq j$ in w. Similarly, the presence of letters $i \leq j$ in w has no impact on the μ-matches in w involving only the letters in $\{j + 1, j + 2, \ldots, k\}$. Thus the lower right submatrix is just the state matrix for the word $w'' \in NR([k-j])$ that comes from removing all letters $i \leq j$ and then subtracting j from all the remaining letters. It follows that each word w ending in j can be decomposed uniquely into the ordered pair $(M_{w'}, M_{w''})$ from which it is possible to reconstruct M_w. Thus, for each $j \in [k]$,

$$N_j(k) = N(j - 1)N(k - j).$$

Combining this with equation (23) gives

$$N(k) = \sum_{j=1}^{k} N(j - 1)N(k - j),$$

which proves that $N(k)$ satisfies the Catalan recurrence. Further, it easy to see that in the case $k = 1$, there is only one state matrix, the 1×1 zero matrix, which proves that $N(k)$ is the k^{th} Catalan number. $\square$

The state transition method for finding $NR_{\mu}^{(k)}(p, q, t)$ therefore depends on being able to invert a $C_k \times C_k$ square matrix, as compared to a $k(k-1) \times k(k-1)$ matrix using the case-by-case analysis of the previous section. Even if we use the shortcut that requires us to only compute the determinant and the cofactors from the first row, this becomes difficult as k increases, since C_k grows rapidly. The advantage of this automaton method is not in the computation size, but rather the main benefit is that it gives a completely algorithmic way to set up a system of equations whose solution would lead to $NR_\mu^{(k)}(p,q,t)$. When compared with the approach of Section 2, whereby it took a lot of careful analysis to write down this system for $k=4$ and doing so for $k=5$ was infeasible, the automaton method is much more straightforward.

In addition, our automaton method makes it easy to compute $NR_{n,\mu}^{(k)}(p,q)$ for any n. Recall the system of linear equations $T_k = M_k S_k$, written out in matrix form for the case $k=3$ in (21). Notice that $1^{(C_k)}(-T_k)$ gives the sum of the weights of all words of length 1. In other words, $1^{(C_k)}(-T_k) = NR_{1,\mu}^{(k)}(p,q)t^1$. Next, notice that the adjacency matrix for the graph without state 0 is given by $A_k = M_k + I$ where I is the identity matrix of size C_k. Since there are no edges leading in to state 0, powers of this smaller adjacency matrix are sufficient to find $NR_{n,\mu}^{(k)}(p,q)t^n$. It follows that

$$NR_{n,\mu}^{(k)}(p,q)t^n = 1^{(C_k)}A_k^{n-1}(-T_k),$$

or

$$NR_{n,\mu}^{(k)}(p,q) = \frac{1^{(C_k)}A_k^{n-1}(-T_k)}{t^n}.$$

4. Consecutive μ-matches

In general, it is much easier to study consecutive μ-matches, which are the topic of this section. Given a word $w = w_1 \dots w_n \in [k]^n$ and $j \in [k-1]$, we let $\mathrm{jrise}(w) = |\{i : w_{i+1} - w_i = j\}|$ and $\mathrm{rise}(w) = |\{i : w_i < w_{i+1}\}|$. In this section, we shall study the generating function

$$CNR^{(k)}(x_1,\dots,x_{k-1},t) = 1 + \sum_{n\geq 1} CNR_n^{(k)}(x_1,\dots,x_{k-1})t^n$$

where

$$CNR_n^{(k)}(x_1,\dots,x_{k-1}) = \sum_{w\in NR([k]),|w|=n} \prod_{j=1}^{k-1} x_j^{\mathrm{jrise}(w)}.$$

For any $1 \leq i \leq k$, we let

$$CNR^{i,(k)}(x_1,\dots,x_{k-1},t) = \sum_{n\geq 1} CNR_n^{i,(k)}(x_1,\dots,x_{k-1})t^n$$

where

$$CNR_n^{i,(k)}(x_1,\ldots,x_{k-1}) = \sum_{w\in i[k]^{n-1}, w\in NR([k])} \prod_{j=1}^{k-1} x_j^{\mathrm{jrise}(w)}.$$

Thus

$$CNR^{(k)}(x_1,\ldots,x_{k-1},t) = 1 + \sum_{i=1}^{k} CNR^{i,(k)}(x_1,\ldots,x_{k-1},t).$$

First we show that the generating functions $CNR^{i,(k)}(x_1,\ldots,x_{k-1},t)$ satisfy simple recursions. That is, we can classify the words in $NR([k])$ that start with s as either (i) the single letter s itself, (ii) sw where w starts with one of $1,2,\ldots,s-1$, or (iii) sw where w starts with $s+j$ for some $1 \le j \le k-s$. It follows that for $s = 1,2,\ldots,k$,

$$\begin{aligned}(24)\quad CNR^{s,(k)}(x_1,\ldots,x_{k-1},t) =& t + \left(\sum_{i=1}^{s-1} tCNR^{i,(k)}(x_1,\ldots,x_{k-1},t)\right) + \\ & \sum_{j=1}^{k-s} x_j tCNR^{s+j,(k)}(x_1,\ldots,x_{k-1},t),\end{aligned}$$

or, equivalently,

$$\begin{aligned}(25)\quad -t = & \left(\sum_{i=1}^{s-1} tCNR^{i,(k)}(x_1,\ldots,x_{k-1},t)\right) - CNR^{s,(k)}(x_1,\ldots,x_{k-1},t) + \\ & \sum_{j=1}^{k-s} x_j tCNR^{s+j,(k)}(x_1,\ldots,x_{k-1},t).\end{aligned}$$

The way in which this relates to the μ-matches discussed in this paper can be seen by considering the specialization $NR_\mu^{(k)}(0,q,t)$. Setting $p=0$ in (2) and (4) gives

$$NR_\mu^{(k)}(0,q,t) = 1 + \sum_{n\ge 1} NR_{n,\mu}^{(k)}(0,q)t^n,$$

where

$$NR_{n,\mu}^{(k)}(0,q) = \sum_{w\in NR([k]), |w|=n, \mathrm{ntriv}_\mu(w)=0} q^{\mathrm{triv}_\mu(w)}.$$

Similarly, if $s \in [k]$, setting $p=0$ in (5) and (8) gives a refinement by the first letter

$$NR_{\mu}^{s,(k)}(0,q,t) = \sum_{n\geq 1} NR_{n,\mu}^{s,(k)}(0,q)t^n,$$

where

$$NR_{n,\mu}^{s,(k)}(0,q) = \sum_{w\in s[k]^{n-1}, w\in NR([k]), \mathrm{ntriv}_{\mu}(w)=0} q^{\mathrm{triv}_{\mu}(w)}.$$

If we consider, then, only words in $NR([k])$ that have no nontrivial μ-matches, it is clear that if such a word contains the letter s, the next letter must be one of $\{1,2,\ldots,s,s+1\}$ to avoid nontrivial μ-matches. Thus, we can classify the words in $NR([k])$ starting with s and having no nontrivial μ-matches as either (i) the single letter s, (ii) sw where w starts with one of $1,2,\ldots,s-1$, or (iii) sw where w starts with $s+1$. Note that in case (ii), the first s is not involved in any trivial μ-matches because from one letter to the next, we must either decrease or go up by exactly one. This means there must be another s in w before any $s+1$, so the first s is not involved in any trivial μ-matches. In case (iii), there is a trivial μ-match between s and the first letter of w, which is consecutive. Thus, it follows that

$$NR_{\mu}^{s,(k)}(0,q,t) = t + \left(\sum_{i=1}^{s-1} tNR_{\mu}^{i,(k)}(0,q,t)\right) + qtNR_{\mu}^{s+1,(k)}(0,q,t). \tag{26}$$

Notice that this recursion is the same as in equation (24) if we set $x_1 = q$ and $x_i = 0$ for $1 < i \leq k-1$. Therefore, for any $s \in [k]$,

$$NR_{\mu}^{s,(k)}(0,q,t) = CNR^{s,(k)}(q,0,\ldots,0,t), \tag{27}$$

from which it follows that

$$NR_{\mu}^{(k)}(0,q,t) = CNR^{(k)}(q,0,\ldots,0,t). \tag{28}$$

Thus the generating function $CNR^{(k)}(x_1,\ldots,x_{k-1},t)$ can be viewed as a refinement of the generating function $NR_{\mu}^{(k)}(0,q,t)$, which explains its relevance to our study of the distribution of μ matches in words.

We have shown that the generating functions $CNR^{i,(k)}(x_1,\ldots,x_{k-1},t)$ satisfy simple recursions. For example, if we write equations (25) down in matrix form in the case $k = 6$, we have

$$\begin{bmatrix} -t \\ -t \\ -t \\ -t \\ -t \\ -t \end{bmatrix} = \begin{bmatrix} -1 & x_1t & x_2t & x_3t & x_4t & x_5t \\ t & -1 & x_1t & x_2t & x_3t & x_4t \\ t & t & -1 & x_1t & x_2t & x_3t \\ t & t & t & -1 & x_1t & x_2t \\ t & t & t & t & -1 & x_1t \\ t & t & t & t & t & -1 \end{bmatrix} \begin{bmatrix} CNR^{1,(6)}(\overline{x},t) \\ CNR^{2,(6)}(\overline{x},t) \\ CNR^{3,(6)}(\overline{x},t) \\ CNR^{4,(6)}(\overline{x},t) \\ CNR^{5,(6)}(\overline{x},t) \\ CNR^{6,(6)}(\overline{x},t) \end{bmatrix} \tag{29}$$

where $\overline{x} = x_1, x_2, x_3, x_4, x_5$.

Thus let $T^{(k)} = [T_{j,1}]$ be the column vector of length k such that $T_{j,1} = -t$ for all j, $N^{(k)} = [N_{j,1}]$ be the column vector of length k such that $N_{j,1} = CNR^{j,(k)}(x_1, \dots, x_{k-1}, t)$ for all j, and $M^{(k)} = ||m_{i,j}||$ be the $k \times k$ matrix whose entries are equal to t below the diagonal, -1 on the diagonal, x_it on the ith superdiagonal. Then we will have

$$T^{(k)} = M^{(k)}N^{(k)}$$

so that

$$N^{(k)} = (M^{(k)})^{-1}T^{(k)}. \tag{30}$$

Our first result gives an expression for $CNR^{(k)}(x_1, \dots, x_{k-1}, t)$ in terms of the determinant of $M^{(k)}$. That is, we have the following theorem.

Theorem 3.

$$CNR^{(k)}(x_1, \dots, x_{k-1}, t) = \frac{(-1)^k(1+t)^k}{\det(M^{(k)})}. \tag{31}$$

Proof. First we consider the system of equations that results from the system of equations (25) by subtracting the first equation from each of the remaining equations so that we can use a shortcut as in (22). If we do this in the case of $k = 6$, then we can write the resulting set of equations in matrix form as

$$\begin{bmatrix} -t \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & x_1t & x_2t & x_3t & x_4t & x_5t \\ 1+t & -(1+x_1t) & (x_1-x_2)t & (x_2-x_3)t & (x_3-x_4)t & (x_4-x_5)t \\ 1+t & t-x_1t & -(1+x_2t) & (x_1-x_3)t & (x_2-x_4)t & (x_3-x_5)t \\ 1+t & t-x_1t & t-x_2t & -(1+x_3)t & (x_1-x_4)t & (x_2-x_5)t \\ 1+t & t-x_1t & t-x_2t & t-x_3t & -(1+x_4t) & (x_1-x_5)t \\ 1+t & t-x_1t & t-x_2t & t-x_3t & t-x_4t & -(1+x_5)t \end{bmatrix} N^{(6)}.$$

In general, this will result in matrix equation

$$\overline{T^{(k)}} = \overline{M^{(k)}}N^{(k)}$$

where $\overline{T^{(k)}} = [\overline{t}_{j,1}]$ is a column vector of length k such that $\overline{t}_{1,1} = -t$ and $\overline{t}_{j,1} = 0$ for $j \geq 2$ and $\overline{M^{(k)}} = ||\overline{m}_{i,j}||$ is the $k \times k$ matrix such that
(i) the elements in row 1 are $-1, x_1 t, \ldots, x_{k-1} t$, reading from left to right, and
(ii) for each $i > 1$, the elements in the ith row are

$$1+t, t-x_1 t, \ldots, t-x_{i-2}t, -(1+x_{i-1}t), (x_1-x_i)t, (x_2-x_{i+1})t, \ldots, (x_{k-i}-x_{k-1})t.$$

It follows that

$$N^{(k)} = \left(\overline{M^{(k)}}\right)^{-1} \overline{T^{(k)}}.$$

Hence, if $1^{(k)}$ is the row vector of length k consisting of all 1's, then

$$\begin{aligned} CNR^{(k)}(x_1, \ldots, x_{k-1}, t) &= 1 + 1^{(k)} N^{(k)} \\ &= 1 + 1^{(k)} \left(\overline{M^{(k)}}\right)^{-1} \overline{T^{(k)}}. \end{aligned}$$

Since $\overline{M^{(k)}}$ arises from $M^{(k)}$ by taking -1 times the first row and adding it to each of the other rows, we will have

$$\det\left(\overline{M^{(k)}}\right) = \det(M^{(k)}).$$

Using the formula

$$\left(\overline{M^{(k)}}\right)^{-1} = \frac{1}{\det\left(\overline{M^{(k)}}\right)} \begin{bmatrix} \overline{C}_{1,1} & \ldots & \overline{C}_{k,1} \\ \overline{C}_{1,2} & \ldots & \overline{C}_{k,2} \\ \vdots & \ldots & \vdots \\ \overline{C}_{1,k} & \ldots & \overline{C}_{k,k} \end{bmatrix}$$

where $\overline{C}_{i,j}$ is (i,j)th cofactor of $\overline{M^{(k)}}$, it follows that

$$\begin{aligned} CNR^{(k)}(x_1, \ldots, x_{k-1}, t) &= 1 + 1^{(k)} \left(\overline{M^{(k)}}\right)^{-1} \overline{T^{(k)}} \\ &= 1 + \frac{-t \sum_{j=1}^{k} \overline{C}_{1,j}}{\det\left(\overline{M^{(k)}}\right)} \\ &= \frac{\det\left(\overline{M^{(k)}}\right) - t \sum_{j=1}^{k} \overline{C}_{1,j}}{\det\left(\overline{M^{(k)}}\right)}. \end{aligned}$$

However, if we expand $\det(\overline{M^{(k)}})$ about the first row, we see that

$$\det\left(\overline{M^{(k)}}\right) = (-1)\overline{C}_{1,1} + \sum_{j=2}^{k} x_{j-1}t\overline{C}_{1,j}$$

so that

$$CNR^{(k)}(x_1,\ldots,x_{k-1},t) = \frac{-(1+t)\overline{C}_{1,1} + \sum_{j=2}^{k}(x_{j-1}t-t)\overline{C}_{1,j}}{\det\left(\overline{M^{(k)}}\right)}. \tag{32}$$

Now let $\overline{U^{(k)}}$ be the matrix that arises from $\overline{M^{(k)}}$ by replacing the first row of $\overline{M^{(k)}}$ by $[-(1+t), x_1t-t, x_2t-t, \ldots x_{k-1}t-t]$. For example, in the case where $k=6$,

$$\overline{U^{(6)}} = \begin{bmatrix} -(1+t) & x_1t-t & x_2t-t & x_3t-t & x_4t-t & x_5t-t \\ 1+t & -(1+x_1t) & (x_1-x_2)t & (x_2-x_3)t & (x_3-x_4)t & (x_4-x_5)t \\ 1+t & t-x_1t & -(1+x_2t) & (x_1-x_3)t & (x_2-x_4)t & (x_3-x_5)t \\ 1+t & t-x_1t & t-x_2t & -(1+x_3)t & (x_1-x_4)t & (x_2-x_5)t \\ 1+t & t-x_1t & t-x_2t & t-x_3t & -(1+x_4t) & (x_1-x_5)t \\ 1+t & t-x_1t & t-x_2t & t-x_3t & t-x_4t & -(1+x_5)t \end{bmatrix}.$$

Computing the determinant of $\overline{U^{(k)}}$ by expanding about the first row, we see that

$$\det\left(\overline{U^{(k)}}\right) = -(1+t)\overline{C}_{1,1} + \sum_{j=2}^{k}(x_{j-1}t-t)\overline{C}_{1,j},$$

so that (32) becomes

$$CNR^{(k)}(x_1,\ldots,x_{k-1},t) = \frac{\det\left(\overline{U^{(k)}}\right)}{\det\left(\overline{M^{(k)}}\right)}.$$

However it is easy to compute $\det(\overline{U^{(k)}})$. That is, adding the first row of $\overline{U^{(k)}}$ to each of the remaining rows will result in a matrix $\overline{V^{(k)}}$ whose entries below the diagonal are 0, whose entries on the diagonal are $-(1+t)$, and whose entries on the ith superdiagonal are x_it-t. For example, in the case

$k = 6$,

$$\overline{V^{(6)}} = \begin{bmatrix} -(1+t) & x_1t-t & x_2t-t & x_3t-t & x_4t-t & x_5-t \\ 0 & -(1+t) & x_1t-t & x_2t-t & x_3t-t & x_4t-t \\ 0 & 0 & -(1+t) & x_1t-t & x_2t-t & x_3t-t \\ 0 & 0 & 0 & -(1+t) & x_1t-t & x_2t-t \\ 0 & 0 & 0 & 0 & -(1+t) & x_1t-t \\ 0 & 0 & 0 & 0 & 0 & -(1+t) \end{bmatrix}.$$

Thus

$$\det\left(\overline{U^{(k)}}\right) = \det\left(\overline{V^{(k)}}\right) = (-(1+t))^k = (-1)^k(1+t)^k.$$

Hence

$$CNR^{(k)}(x_1,\ldots,x_{k-1},t) = \frac{\det\left(\overline{U^{(k)}}\right)}{\det\left(\overline{M^{(k)}}\right)} = \frac{(-1)^k(1+t)^k}{\det\left(\overline{M^{(k)}}\right)} = \frac{(-1)^k(1+t)^k}{\det\left(M^{(k)}\right)}.$$

□

Next we will show how to get a closed form expression for $\det(\overline{M^{(k)}})$. We will start with a simple case that has special relevance, namely, where we set $x_1 = q$ and $x_i = 0$ for $i \geq 2$ so that $CNR^{(k)}(q,0,\ldots,0,t) = NR_\mu^{(k)}(0,q,t)$. Let $M^{(k)}(q)$ be the matrix that arises from $M^{(k)}$ by setting $x_1 = q$ and $x_i = 0$ for $i \geq 2$. For example, in the case $k = 6$,

$$M^{(6)}(q) = \begin{bmatrix} -1 & qt & 0 & 0 & 0 & 0 \\ t & -1 & qt & 0 & 0 & 0 \\ t & t & -1 & qt & 0 & 0 \\ t & t & t & -1 & qt & 0 \\ t & t & t & t & -1 & qt \\ t & t & t & t & t & -1 \end{bmatrix}.$$

Theorem 4. *For all $k \geq 1$,*

$$\det(M^{(k)}(q)) = (-1)^k + \sum_{\ell=1}^{k-1}(qt)^{k-\ell}\sum_{s=1}^{\ell}(-1)^{k-s}\binom{\ell}{s}\binom{k-\ell-1}{s-1}t^s. \tag{33}$$

Proof. Note that if $M^{(k)}(q) = ||m_{i,j}||_{i,j=1,\ldots,k}$, then

$$\det(M^{(k)}(q)) = \sum_{\sigma=\sigma_1\ldots\sigma_k \in S_k} \operatorname{sgn}(\sigma)\prod_{i=1}^{k} m_{i,\sigma_i}.$$

First, note that the only permutations σ that contribute to this sum are those for which $\sigma_i \leq i+1$ for all $1 \leq i \leq k$. We classify such permutations according to the set T of rows i such that $\sigma_i \neq i+1$, which is the set of rows where σ does not pick the element on the superdiagonal. Clearly any such T must be of the form $\{t_1 < \cdots < t_\ell\} \subseteq [k]$ where $t_\ell = k$. But in such a case, we know that $\sigma_1 = 2,\ \sigma_2 = 3, \ldots, \sigma_{t_1-1} = t_1$ and $\sigma_{t_1} \neq t_1+1$, which means that $\sigma_{t_1} = 1$. Thus σ must contain the cycle $(1,2,\ldots,t_1-1,t_1)$. But then $\sigma_{t_1+1} = t_1+2,\ \sigma_{t_1+2} = t_1+3, \ldots, \sigma_{t_2-1} = t_2$ and $\sigma_{t_2} \neq t_2+1$, which means that $\sigma_{t_2} = t_1+1$. Thus σ must contain the cycle $(t_1+1, t_1+2, \ldots, t_2-1, t_2)$. Continuing on in this way, it is easy to see that T corresponds to the permutation

$$\sigma^{(T)} = (1,2,\ldots,t_1-1,t_1)(t_1+1,t_1+2,\ldots,t_2-1,t_2)\ldots(t_{\ell-1}+1,t_{\ell-1}+2,\ldots,t_\ell-1,t_\ell).$$

Note that if $t_0 = 0$, then

$$\mathrm{sgn}(\sigma^{(T)}) = (-1)^{\sum_{j=0}^{\ell-1} t_{j+1}-t_j-1} = (-1)^{t_\ell - t_0 - \ell} = (-1)^{k-\ell}.$$

Now a cycle $(t_j+1, t_j+2, \ldots, t_{j+1}-1, t_{j+1})$ gives rise to a factor of $(qt)^{t_{j+1}-t_j-1} m_{t_{j+1},t_j+1}$ in $\prod_{i=1}^k m_{i,\sigma_i^{(T)}}$. Hence T corresponds to the term

$$\begin{aligned}\mathrm{sgn}(\sigma^{(T)}) \prod_{i=1}^k m_{i,\sigma_i^{(T)}} &= (-1)^{k-\ell}(qt)^{\sum_{j=0}^{\ell-1} t_{j+1}-t_j-1} \prod_{j=0}^{\ell-1} m_{t_{j+1},t_j+1} \\ &= (-qt)^{k-\ell} \prod_{j=0}^{\ell-1} m_{t_{j+1},t_j+1}.\end{aligned}$$

Now m_{t_{j+1},t_j+1} is equal to t if $t_{j+1} > t_j+1$ and is equal to -1 if $t_{j+1} = t_j+1$. Thus, shifting indices,

$$\prod_{j=0}^{\ell-1} m_{t_{j+1},t_j+1} = \prod_{j=1}^{\ell} (t\chi(t_j > t_{j-1}+1) - \chi(t_j = t_{j-1}+1))$$

where for any statement A, $\chi(A) = 1$ if A is true and $\chi(A) = 0$ if A is false. Thus $T = [k]$ corresponds to the term $(-1)^k$ and if $|T| = \ell < k$, then T corresponds to the term $(-qt)^{k-\ell} \prod_{j=1}^{\ell}(t\chi(t_j > t_{j-1}+1) - \chi(t_j = t_{j-1}+1))$. Hence

$$\det(M^{(k)}(q)) =$$

$$(-1)^k+\sum_{\ell=1}^{k-1}(-qt)^{k-\ell}\sum_{\{k\}\subseteq\{t_1<\cdots<t_\ell\}\subseteq[k]}\prod_{j=1}^{\ell}(t\chi(t_j>t_{j-1}+1)-\chi(t_j=t_{j-1}+1)).$$

Next consider the term $\sum_{\{k\}\subseteq\{t_1<\cdots<t_\ell\}\subseteq[k]}\prod_{j=1}^{\ell}(t\chi(t_j > t_{j-1}+1) - \chi(t_j = t_{j-1}+1))$. Note that choosing a valid set T, that is, one such that $\{k\} \subseteq T \subseteq [k]$, is equivalent to choosing a sequence $u = (u_1, \ldots, u_\ell)$ where each $u_j \geq 0$, by setting $u_j = t_j - t_{j-1} - 1$. Since $\sum_{j=1}^{\ell} t_j - t_{j-1} - 1 = \sum_{j=1}^{\ell} u_j = k - \ell$, we can now determine a sequence u by instead choosing a set $S = \{i_1, \ldots, i_s\} \subseteq [\ell]$ and a sequence $a = (a_1, \ldots, a_s)$ where $a_1+\cdots+a_s = k-\ell$ and each $a_i \geq 1$. We interpret S and A in the following way: set $u_{i_1} = a_1$, $u_{i_2} = a_2$, …, $u_{i_s} = a_s$ and set $u_j = 0$ if $j \in [\ell] - S$. Thus $u_j > 0$, or equivalently $t_j > t_{j-1}+1$, for s values of j, giving a factor of t^s. Similarly, $u_j = 0$, or equivalently $t_j = t_{j-1}+1$, for $\ell - s$ values of j, giving a factor of $(-1)^{\ell-s}$. Since we can choose the set S in any of $\binom{\ell}{s}$ ways, it follows that

$$\sum_{\{k\}\subseteq\{t_1<\cdots<t_\ell\}\subseteq[k]}\prod_{j=1}^{\ell}(t\chi(t_j>t_{j-1}+1)-\chi(t_j=t_{j-1}+1)) =$$
$$\sum_{s=1}^{\ell}(-1)^{\ell-s}t^s\binom{\ell}{s}|\{a_1+\cdots+a_s=k-\ell : a_i \geq 1\}|.$$

But it is well known that the number of solutions to $a_1 + \cdots + a_s = k - \ell$ where a_i are positive integers is the composition number $\binom{k-\ell-1}{s-1}$. Thus

$$\begin{aligned}\det(M^{(k)}(q)) &= (-1)^k+\sum_{\ell=1}^{k-1}(-qt)^{k-\ell}\sum_{s=1}^{\ell}(-1)^{\ell-s}t^s\binom{\ell}{s}\binom{k-\ell-1}{s-1}\\ &= (-1)^k+\sum_{\ell=1}^{k-1}(qt)^{k-\ell}\sum_{s=1}^{\ell}(-1)^{k-s}\binom{\ell}{s}\binom{k-\ell-1}{s-1}t^s,\end{aligned}$$

which is what we wanted to prove. □

We then have the following corollary

Corollary 5. *For all $k \geq 2$,*

$$NR_{\mu}^{(k)}(0,q,t) = \frac{(1+t)^k}{1+\sum_{\ell=1}^{k-1}(qt)^{k-\ell}\sum_{s=1}^{\ell}(-1)^s\binom{\ell}{s}\binom{k-\ell-1}{s-1}t^s}. \tag{34}$$

We can apply the same technique to compute $\det(M^{(k)})$ in general. First let $P^{(k)}$ be the matrix that arises from $M^{(k)}$ by adding -1 times column i to column $i-1$ for $i = 2,\ldots,k$. For example, in the case where $k = 6$, $M^{(6)}$ is given in equation (29) and

$$P^{(6)} = \begin{bmatrix} -(1+x_1t) & (x_1-x_2)t & (x_2-x_3)t & (x_3-x_4)t & (x_4-x_5)t & x_5t \\ t+1 & -(1+x_1t) & (x_1-x_2)t & (x_2-x_3)t & (x_3-x_4)t & x_4t \\ 0 & t+1 & -(1+x_1t) & (x_1-x_2)t & (x_2-x_3)t & x_3t \\ 0 & 0 & t+1 & -(1+x_1t) & (x_1-x_2)t & x_2t \\ 0 & 0 & 0 & t+1 & -(1+x_1t) & x_1t \\ 0 & 0 & 0 & 0 & t+1 & -1 \end{bmatrix}.$$

It is easy to see that $P^{(k)}$ is a matrix whose entries in the last column are $x_{k-1}t, x_{k-2}t, \ldots, x_1t, -1$, reading from top to bottom and whose remaining entries are equal to
(a) $-(1+x_1t)$ on the diagonal,
(b) $t+1$ on the subdiagonal,
(c) 0 below the subdiagonal, and
(d) $(x_i - x_{i+1})t$ on the ith superdiagonal.
Now let $R^{(k)}$ denote the transpose of $P^{(k)}$ and $V^{(k-1)}$ denote the $(k-1) \times (k-1)$ matrix which results from $R^{(k)}$ by removing the last row and column. For example,

$$R^{(6)} = \begin{bmatrix} -(1+x_1t) & t+1 & 0 & 0 & 0 & 0 \\ (x_1-x_2)t & -(1+x_1t) & t+1 & 0 & 0 & 0 \\ (x_2-x_3)t & (x_1-x_2)t & -(1+x_1t) & t+1 & 0 & 0 \\ (x_3-x_4)t & (x_2-x_3)t & (x_1-x_2)t & -(1+x_1t) & t+1 & 0 \\ (x_4-x_5)t & (x_3-x_4)t & (x_2-x_3)t & (x_1-x_2)t & -(1+x_1t) & t+1 \\ x_5t & x_4t & x_3t & x_2t & x_1t & -1 \end{bmatrix}$$

and

$$V^{(5)} = \begin{bmatrix} -(1+x_1t) & t+1 & 0 & 0 & 0 \\ (x_1-x_2)t & -(1+x_1t) & t+1 & 0 & 0 \\ (x_2-x_3)t & (x_1-x_2)t & -(1+x_1t) & t+1 & 0 \\ (x_3-x_4)t & (x_2-x_3)t & (x_1-x_2)t & -(1+x_1t) & t+1 \\ (x_4-x_5)t & (x_3-x_4)t & (x_2-x_3)t & (x_1-x_2)t & -(1+x_1t) \end{bmatrix}.$$

If one expands the determinant of $R^{(k)}$ about the last row, then it is easy to see that

(35)

$$\det\left(M^{(k)}\right) = \det\left(R^{(k)}\right)$$
$$= -\det\left(V^{(k-1)}\right) + \sum_{i=1}^{k-1}(-1)^{2k-i}x_i t(1+t)^i \det\left(V^{(k-i-1)}\right)$$

where by convention we set $\det(V^{(0)}) = 1$. Thus to complete our formula for $\det(M^{(k)})$, we need only find a formula for $\det(V^{(k)})$. But $V^{(k)} = ||v_{i,j}||$ has a form similar to the matrix $M^{(k)}(q)$ in that it is a matrix whose diagonal entries are constant, whose superdiagonal entries are constant, and whose entries above the superdiagonal are 0. That is,

$$\det\left(V^{(k)}\right) = \sum_{\sigma=\sigma_1\ldots\sigma_k\in S_k} \operatorname{sgn}(\sigma)\prod_{i=1}^{k} v_{i,\sigma_i}$$

and we classify the permutations σ according to the set T of rows i such that $\sigma_i \neq i+1$ which is the set of rows where σ does not pick the element on the superdiagonal. Again, any such T must be of the form $\{t_1 < \cdots < t_\ell\} \subseteq [k]$ where $t_\ell = k$, and by the same reasoning as before, T corresponds to the permutation

$$\sigma^{(T)} =$$
$$(1,2,\ldots,t_1-1,t_1)(t_1+1,t_1+2,\ldots,t_2-1,t_2)\ldots(t_{\ell-1}+1,t_{\ell-1}+2,\ldots,t_\ell-1,t_\ell).$$

Thus

$$\operatorname{sgn}(\sigma^{(T)})\prod_{i=1}^{k} v_{i,\sigma_i^{(T)}} =$$
$$(-1)^{k-\ell}(1+t)^{\sum_{j=0}^{\ell-1} t_{j+1}-t_j-1}\prod_{j=0}^{\ell-1} v_{t_{j+1},t_j+1} = (-(1+t))^{k-\ell}\prod_{j=1}^{\ell} v_{t_j,t_{j-1}+1}.$$

In this case, $T = [k]$ corresponds to the term $(-1)^k(1+x_1t)^k$ and if $|T| = \ell <$

k, then T corresponds to the term $(-1)^{k-\ell}(1+t)^{k-\ell}\prod_{j=1}^{\ell} v_{t_j,t_{j-1}+1}$. Hence

(36)

$$\det\left(V^{(k)}\right) = (-1)^k(1+x_1t)^k + \sum_{\ell=1}^{k-1}(-1)^{k-\ell}(1+t)^{k-\ell}\sum_{\{k\}\subseteq\{t_1<\cdots<t_\ell\}\subseteq[k]}\prod_{j=1}^{\ell} v_{t_j,t_{j-1}+1}.$$

Next consider the term $\sum_{\{k\}\subseteq\{t_1<\cdots<t_\ell\}\subseteq[k]}\prod_{j=1}^{\ell} v_{t_j,t_{j-1}+1}$. Choosing a set T is equivalent to choosing a sequence $u=(u_1,\ldots,u_\ell)$ where each $u_j \geq 0$, by setting $u_j = t_j - t_{j-1} - 1$. Since $\sum_{j=1}^{\ell} t_j - t_{j-1} - 1 = \sum_{j=1}^{\ell} u_j = k-\ell$, we can instead choose a set $S=\{i_1,\ldots,i_s\}\subseteq[\ell]$ and a sequence $a=(a_1,\ldots,a_s)$ where $a_1+\cdots+a_s = k-\ell$ and each $a_i \geq 1$. By setting $u_{i_1}=a_1$, $u_{i_2}=a_2$, …, $u_{i_s}=a_s$ and $u_j=0$ if $j\in[\ell]-S$, we determine the sequence u (and in turn the set T) by this choice of S and a. Note that if $u_j=0$, then u_j corresponds to a diagonal term $v_{t_j,t_{j-1}+1} = -(1+x_1t)$ and if $u_j = m$ where $m>0$, then u_j corresponds to a term on the mth subdiagonal which is $(x_m - x_{m+1})t$. Since there are $\binom{\ell}{s}$ ways to choose the set S, it follows that

$$\sum_{\{k\}\subseteq\{t_1<\cdots<t_\ell\}\subseteq[k]}\prod_{j=1}^{\ell-1} v_{t_{j+1},t_j+1} = \sum_{s=1}^{\ell}(-(1+x_1t))^{\ell-s}\binom{\ell}{s}\sum_{\substack{a_1+\cdots+a_s=k-\ell\\ 0<a_i<k}}\prod_{i=1}^{s}(x_{a_i}-x_{a_i+1})t.$$

Substituting this in (36) gives

$$\det(V^{(k)}) = (-1)^k(1+x_1t)^k + \sum_{\ell=1}^{k-1}(-1)^{k-\ell}(1+t)^{k-\ell}\sum_{s=1}^{\ell}(-(1+x_1t))^{\ell-s}\binom{\ell}{s}\sum_{\substack{a_1+\cdots+a_s=k-\ell\\ 0<a_i<k}}\prod_{i=1}^{s}(x_{a_i}-x_{a_i+1})t.$$

Hence we have the following result.

Theorem 6. *For all $k\geq 2$,*

$$\det\left(M^{(k)}\right) = -\det\left(V^{(k-1)}\right) + \sum_{i=1}^{k-1} x_i t(-(1+t))^i \det\left(V^{(k-i-1)}\right)$$

where $\det(V^{(0)}) = 1$ *and for* $k \geq 1$,

$$\det(V^{(k)}) = (-1)^k(1+x_1t)^k + \sum_{\ell=1}^{k-1}(1+t)^{k-\ell}\sum_{s=1}^{\ell}(-1)^{k-s}(1+x_1t)^{\ell-s}t^s\binom{\ell}{s}\sum_{\substack{a_1+\cdots+a_s=k-\ell\\0<a_i<k}}\prod_{i=1}^{s}(x_{a_i}-x_{a_i+1}).$$

The only factor in the formula for $\det(V^{(k)})$ which is not explicit is the term $\sum_{\substack{a_1+\cdots+a_s=k-\ell\\0<a_i<k}}\prod_{i=1}^{s}(x_{a_i}-x_{a_i+1})$. We let

$$B(m,s,x_1,\ldots,x_k) = \sum_{\substack{a_1+\cdots+a_s=m\\0<a_i<k}}\prod_{i=1}^{s}(x_{a_i}-x_{a_i+1}). \tag{37}$$

Then the function $B(m,s,x_1,\ldots,x_k)$ has the following properties:

1. $B(m,s,x_1,\ldots,x_k) = 0$ if $m < s$ or $m > s(k-1)$,
2. $B(s,s,x_1,\ldots,x_k) = (x_1-x_2)^s$ and $B(s(k-1),s,x_1,\ldots,x_k) = (x_{k-1}-x_k)^s$,
3. $B(m,1,x_1,\ldots,x_k)$ is equal to $x_m - x_{m+1}$ if $1 \leq m \leq k-1$ and is equal to 0 otherwise, and
4. $B(m,s,x_1,\ldots,x_k) = \sum_{i=1}^{k-1}(x_i-x_{i+1})B(m-i,s-1,x_1,\ldots,x_k)$ if $s \geq 2$.

We end this paper by giving a number of simple cases where we have been able to specialize our formulas to obtain explicit formulas for $\det(V^{(k)})$ and $\det(M^{(k)})$.

Case 1. Suppose that $1 \leq j \leq k$ and $x_i = p$ for $i \leq j$ and $x_i = 0$ for $i > j$.

We shall denote this specialization of $B(m,s,x_1,\ldots,x_k)$, $V^{(k)}$, $M^{(k)}$, and $CNR^{(k)}(x_1,\ldots,x_{k-1},t)$ by $B_{p^j0^{k-j}}(m,s)$, $V^{(k)}_{p^j0^{k-j}}$, $M^{(k)}_{p^j0^{k-j-1}}$, and $CNR^{(k)}_{p^j0^{k-j-1}}(p,t)$, respectively. In this case, the only factor of the form $(x_{a_i}-x_{a_i+1})$ that is not zero is $(x_j - x_{j+1})$, which is equal to p so that

$$B_{p^j0^{k-j}}(m,s) = \left(\sum_{\substack{a_1+\cdots+a_s=m\\a_i=j}}\prod_{i=1}^{s}p\right) = p^s\chi(m=js).$$

It follows that

$$\sum_{s=1}^{\ell}(-1)^{k-s}(1+x_1t)^{\ell-s}t^s\binom{\ell}{s}\sum_{\substack{a_1+\cdots+a_s=k-\ell\\0<a_i<k}}\prod_{i=1}^{s}(x_{a_i}-x_{a_i+1})$$

gives a contribution if and and only if $sj=k-\ell$ and $s\le\ell$. Thus we must have $s\le k-sj$ or, equivalently, $s\le\lfloor\frac{k}{j+1}\rfloor$. Hence

$$\sum_{\ell=1}^{k-1}(1+t)^{k-\ell}\sum_{s=1}^{\ell}(-1)^{k-s}(1+x_1t)^{\ell-s}t^s\binom{\ell}{s}\sum_{\substack{a_1+\cdots+a_s=k-\ell\\0<a_i<k}}\prod_{i=1}^{s}(x_{a_i}-x_{a_i+1})$$

becomes

$$\sum_{s=1}^{\lfloor\frac{k}{j+1}\rfloor}(-1)^{k-s}(1+t)^{sj}(1+pt)^{k-s(j+1)}(pt)^s\binom{k-sj}{s}.$$

Thus

$$\det\left(V^{(k)}_{p^j0^{k-j}}\right)= (-1)^k\left((1+pt)^k+\sum_{s=1}^{\lfloor\frac{k}{j+1}\rfloor}(-1)^s(1+t)^{sj}(1+pt)^{k-s(j+1)}(pt)^s\binom{k-sj}{s}\right) \tag{38}$$

and

$$\det\left(M^{(k)}_{p^j0^{k-j-1}}\right)= -\det\left(V^{(k-1)}_{p^j0^{k-j-1}}\right)+\sum_{i=1}^{j}pt(-(1+t))^i\det\left(V^{(k-i-1)}_{p^j0^{k-i-j-1}}\right). \tag{39}$$

Here we interpret $V^{(k-i-1)}_{p^j0^{k-i-j-1}}$ as the matrix where we are setting $x_i=p$ for $i\le j$ and $x_i=0$ for $i\ge j+1$. By Theorem 3, $CNR^{(k)}_{p^j0^{k-j-1}}(p,t)=\frac{(-1)^k(1+t)^k}{\det(M^{(k)}_{p^j0^{k-j-1}})}$.

In this case $CNR^{(k)}_{p^j0^{k-j-1}}(p,t)$ is the generating function of $p^{\mathrm{rise}(w)}t^{|w|}$ over all words $w=w_1\ldots w_n\in NR([k])$ such that any rise w_iw_{i+1} must have $w_{i+1}-w_i\le j$. Thus $CNR^{(k)}_{p^{k-1}}(p,t)$ is just the generating function of

$p^{\mathrm{rise}(w)}t^{|w|}$ over all words, and $CNR^{(k)}_{p0^{k-2}}(p,t) = NR^{(k)}_{\mu}(0,p,t)$ is covered by Corollary 5.

Here are a few examples of the formulas for $CNR^{(k)}_{p^j 0^{k-j-1}}(p,t)$ for small values of k and j.

$$CNR^{(4)}_{p^2 0}(p,t) = \frac{(1+t)^4}{1 - pt^2(5+2t) - p^2t^3(4+t) - p^3t^4}$$
$$CNR^{(4)}_{p^3}(p,t) = \frac{(1+t)^4}{1 - pt^2(6+4t+t^2) - p^2t^3(4+t) - p^3t^4}$$
$$CNR^{(5)}_{p^2 0^2}(p,t) = \frac{(1+t)^5}{1 - pt^2(7+3t) - p^2t^3(8+t-t^2) - p^3t^4(5+t) - p^4t^5}$$
$$CNR^{(5)}_{p^3 0}(p,t) = \frac{(1+t)^5}{1 - pt^2(9+7t+2t^2) - p^2t^3(10+5t+t^2) - p^3t^4(5+t) - p^4t^5}$$
$$CNR^{(5)}_{p^4}(p,t) = \frac{(1+t)^5}{1 - pt^2(10+10t+5t^2+t^3) - 3p^2t^3(10+5t+t^2) - p^3t^4(5+t) - p^4t^5}$$

Case 2. Suppose that $2 \le j \le k$ and $x_i = 0$ for $i < j$ and $x_i = p$ for $i \ge j$.

We shall denote this specialization of $B(m,s,x_1,\ldots,x_k)$, $V^{(k)}$, $M^{(k)}$, and $CNR^{(k)}(x_1,\ldots,x_{k-1},t)$ by $B_{0^{j-1}p^{k-j+1}}(m,s)$, $V^{(k)}_{0^{j-1}p^{k-j+1}}$, $M^{(k)}_{0^{j-1}p^{k-j}}$, and $CNR^{(k)}_{0^{j-1}p^{k-j}}(p,t)$, respectively. In this case, the only factor of the form $(x_{a_i} - x_{a_i+1})$ that is not zero is $(x_{j-1} - x_j)$, which is equal to $-p$ so that

$$B_{0^{j-1}p^{k-j+1}}(m,s) = \left(\sum_{\substack{a_1+\cdots+a_s=m \\ a_i=j-1}} \prod_{i=1}^{s}(-p) \right) = (-p)^s \chi(m = (j-1)s).$$

It follows that

$$\sum_{s=1}^{\ell} (-1)^{k-s}(1+x_1t)^{\ell-s}t^s \binom{\ell}{s} \sum_{\substack{a_1+\cdots+a_s=k-\ell \\ 0<a_i<k}} \prod_{i=1}^{s}(x_{a_i} - x_{a_i+1})$$

gives a contribution if and and only if $s(j-1) = k-\ell$ and $s \le \ell$. Thus we must have $s \le k - s(j-1)$ or, equivalently, $s \le \lfloor k/j \rfloor$. Hence

$$\sum_{\ell=1}^{k-1}(1+t)^{k-\ell} \sum_{s=1}^{\ell}(-1)^{k-s}(1+x_1t)^{\ell-s}t^s \binom{\ell}{s} \sum_{\substack{a_1+\cdots+a_s=k-\ell \\ 0<a_i<k}} \prod_{i=1}^{s}(x_{a_i} - x_{a_i+1})$$

becomes

$$\sum_{s=1}^{\lfloor k/j \rfloor} (-1)^k (1+t)^{s(j-1)} \binom{k-s(j-1)}{s} (pt)^s.$$

Thus

$$\det\left(V^{(k)}_{0^{j-1}p^{k-j+1}}\right) = (-1)^k \left(1 + \sum_{s=1}^{\lfloor k/j \rfloor} (1+t)^{s(j-1)} \binom{k-s(j-1)}{s} (pt)^s\right) \tag{40}$$

and

$$\det\left(M^{(k)}_{0^{j-1}p^{k-j}}\right) = -\det\left(V^{(k-1)}_{0^{j-1}p^{k-j}}\right) + \sum_{i=j}^{k-1} pt(-(1+t))^i \det\left(V^{(k-i-1)}_{0^{j-1}p^{k-i-j}}\right). \tag{41}$$

Here we interpret $V^{(k-i-1)}_{0^{j-1}p^{k-i-j}}$ as the matrix where we are setting $x_i = 0$ for $i < j$ and $x_i = p$ for $i \geq j$.

In this case $CNR^{(k)}_{0^{j-1}p^{k-j}}(p,t) = \frac{(-1)^k(1+t)^k}{\det(M^{(k)}_{0^{j-1}p^{k-j}})}$ is the generating function of $p^{\mathrm{rise}(w)}t^{|w|}$ over all words $w = w_1 \ldots w_n \in NR[k]$ such that any rise $w_i w_{i+1}$ must have $w_{i+1} - w_i \geq j$.

One case is very simple, namely, we claim that

$$CNR^{(k)}_{0^{k-2}p}(p,t) = \frac{(1+t)^k}{1 - pt^2(1+t)^{k-2}}.$$

In fact, there is a simple combinatorial proof of this fact. That is, in this case, the only rises that are allowed are $1k$. Now if w has ℓ occurrences of $1k$ and no other rises, then $w = w_0 1k w_1 1k \ldots 1k w_{\ell-1} 1k w_\ell$ where w_0 is a decreasing word over $\{2, \ldots, k\}$, w_i is a decreasing word over $\{2, \ldots, k-1\}$ for $i \in \{1, \ldots, \ell-1\}$, and w_ℓ is a decreasing word over $\{1, \ldots, k-1\}$. Thus, w_0 gives rise to a factor of $(1+t)^{k-1}$, each w_i for $i \in \{1, \ldots, \ell-1\}$ gives rise to a factor of $(1+t)^{k-2}$, and w_ℓ gives rise to a factor of $(1+t)^{k-1}$. Hence the generating function for $p^{(k-1)\mathrm{rise}(w)}t^{|w|}$ over all words w such that $(k-1)\mathrm{rise}(w) = \ell$ is

$$((1+t)^{k-1})^2((1+t)^{k-2})^{\ell-1}(pt^2)^\ell = (1+t)^k(pt^2(1+t)^{k-2})^\ell.$$

Also, the generating function for for $p^{(k-1)\text{rise}(w)}t^{|w|}$ over all words w such that $(k-1)\text{rise}(w)=0$ is clearly $(1+t)^k$ since they are just all decreasing words. It follows that

$$\begin{aligned} CNR^{(k)}_{0^{k-2}p}(p,t) &= (1+t)^k + \sum_{\ell\geq 1}(1+t)^k\left(pt^2(1+t)^{k-2}\right)^{\ell} \\ &= \frac{(1+t)^k}{1-pt^2(1+t)^{k-2}}. \end{aligned}$$

Here are some examples of $CNR^{(k)}_{0^{j-1}p^{k-j}}(p,t)$ for small values of k and j.

$$\begin{aligned} CNR^{(4)}_{0p^2}(p,t) &= \frac{(1+t)^4}{1-pt^2(3+4t+t^2)} \\ CNR^{(5)}_{0p^3}(p,t) &= \frac{(1+t)^5}{1-pt^2(6+10t+5t^2+t^3)-p^2t^3(1+t)^2} \\ CNR^{(5)}_{0^2p^2}(p,t) &= \frac{(1+t)^5}{1-pt^2(1+t)^2(3+t)} \\ CNR^{(6)}_{0p^4}(p,t) &= \frac{(1+t)^6}{1-pt^2(10+20t+15t^2+t^4)-p^2t^3(1+t)^2(4+t)} \\ CNR^{(6)}_{0^2p^3}(p,t) &= \frac{(1+t)^6}{1-pt^2(1+t)^2(6+4t+t^2)} \\ CNR^{(6)}_{0^3p^2}(p,t) &= \frac{(1+t)^6}{1-pt^2(1+t)^3(3+t)} \end{aligned}$$

Case 3. Suppose that for $2\leq j\leq k-1$, we set $x_j=p$ and $x_i=0$ for all $i\neq j$.

We shall denote this specialization of $B(m,s,x_1,\ldots,x_k)$, $V^{(k)}$, $M^{(k)}$, and $CNR^{(k)}(x_1,\ldots,x_k,t)$ by $B_{0^{j-1}p0^{k-j}}(m,s)$, $V^{(k)}_{0^{j-1}p0^{k-j}}$, $M^{(k)}_{0^{j-1}p0^{k-j-1}}$, and $CNR^{(k)}_{0^{j-1}p0^{k-j-1}}(p,t)$, respectively.

Then it is easy to see that

$$\sum_{\substack{a_1+\cdots+a_s=m\\ 0<a_i<k}}\prod_{i=1}^{s}(x_{a_i}-x_{a_i+1})$$

equals 0 unless $a_i\in\{j-1,j\}$ for all i and $s(j-1)\leq m\leq sj$, or, equivalently, $\lceil m/j\rceil\leq s\leq\lfloor m/(j-1)\rfloor$. If $s(j-1)\leq m\leq sj$, then we must have $m-s(j-1)$

of the a_i's equal to j and we can choose such i's in $\binom{s}{m-s(j-1)}$ ways. It follows that in this case,

$$\begin{aligned} B_{0^{j-1}p0^{k-j}}(m,s) &= \sum_{\substack{a_1+\cdots+a_s=m \\ a_i\in\{j-1,j\}}} \prod_{i=1}^{s}(p\chi(a_i=j)-p\chi(a_i=j-1)) \\ &= (-1)^{sj-m}\binom{s}{m-s(j-1)}p^s \end{aligned}$$

if $s(j-1)\le m\le sj$ and is equal to 0 otherwise. Thus

$$\det\left(V^{(k)}_{0^{j-1}p0^{k-j}}\right) =$$
$$(-1)^k +$$
$$(-1)^k\sum_{\ell=1}^{k-1}(1+t)^{k-\ell}\sum_{s=\lceil (k-\ell)/j\rceil}^{\min(\ell,\lfloor (k-\ell)/(j-1)\rfloor)}(-1)^{k-s}(tp)^s\binom{\ell}{s}(-1)^{sj-(k-\ell)}\binom{s}{k-\ell-s(j-1)} =$$
$$(-1)^k + \sum_{\ell=1}^{k-1}(1+t)^{k-\ell}\sum_{s=\lceil (k-\ell)/j\rceil}^{\min(\ell,\lfloor (k-\ell)/(j-1)\rfloor)}(-1)^{\ell-s(j-1)}\binom{\ell}{s}\binom{s}{k-\ell-s(j-1)}(tp)^s$$

and

$$\det\left(M^{(k)}_{0^{j-1}p0^{k-j-1}}\right) = -\det\left(V^{(k-1)}_{0^{j-1}p0^{k-j-1}}\right) + (-1)^j pt(1+t)^j \det\left(V^{(k-j-1)}_{0^{j-1}p0^{k-2j-1}}\right). \tag{42}$$

Here we interpret $V^{(k-j-1)}_{0^{j-1}p0^{k-2j-1}}$ as the matrix where we are setting $x_j=p$ and $x_i=0$ for all $i\neq j$.

In this case

$$CNR^{(k)}_{0^{j-1}p0^{k-j-1}}(p,t) = \frac{(-1)^k(1+t)^k}{\det\left(M^{(k)}_{0^{j-1}p0^{k-j-1}}\right)}$$

is the generating function of $p^{\mathrm{rise}(w)}t^{|w|}$ over all words $w=w_1\ldots w_n\in NR[k]$ such that any rise w_iw_{i+1} must have $w_{i+1}-w_i=j$. Here are some values of $CNR^{(k)}_{0^{j-1}p0^{k-j-1}}(p,t)$ for small k and j.

$$CNR^{(4)}_{0p0}(p,t) = \frac{(1+t)^4}{1-pt^2(2+2t)}$$
$$CNR^{(5)}_{0p0^2}(p,t) = \frac{(1+t)^5}{1-pt^2(3+3t)-p^2t^3(1+t)^2}$$

$$CNR^{(5)}_{0^2p0}(p,t) = \frac{(1+t)^5}{1-2pt^2(1+t)^2}$$
$$CNR^{(6)}_{0p0^3}(p,t) = \frac{(1+t)^6}{1-pt^2(4+4t)-p^2t^3(2-t)(1+t)^2}$$
$$CNR^{(6)}_{0^2p0^2}(p,t) = \frac{(1+t)^6}{1-3pt^2(1+t)^2}$$
$$CNR^{(6)}_{0^3p0}(p,t) = \frac{(1+t)^6}{1-2pt^2(1+t)^3}$$

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