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Chains in shard intersection lattices and parabolic support posets

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For every finite Coxeter group W, we prove that the number of chains in the shard intersection lattice introduced by Reading and in the parabolic support poset introduced by Bergeron, Zabrocki and the third author, are the same. We also show that these two partial orders are related by an equality between generating series for their Möbius numbers, and provide a dimension-preserving bijection between the order complex on the parabolic support poset and the pulling triangulation of the permutahedron arising from the right weak order, analogous to the bijection defined by Reading between the order complex of the shard intersection order and the same triangulation of the permutahedron.

1. Introduction

To study finite Coxeter groups, it has appeared to be very useful to introduce several partial orders on their elements. The most usual partial orders are the Bruhat order and the weak order (or permutahedron order). They are now very classical, and have been used in plenty of works.

Introduced more recently in [Rea09, Rea11], the shard intersection order has proved to be a convenient tool to study other aspects of Coxeter groups, in particular in relation to noncrossing partitions, c-sortable elements and cluster combinatorics.

Another partial order has been introduced in [BHZ06], that will be called the parabolic support partial order on the finite Coxeter group W. It has

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the distinction of not being a lattice, having several maximal elements. This partial order is closely related in type A to the structure of some Hopf algebras on permutations; see [BHZ06] for more details.

Our aim in this note is to explain an unexpected connection between the shard intersection order and the parabolic support order. The main result is the following one.

Theorem 1.1. *Let W be a finite Coxeter group. For every subset J of simple reflections and for every integer $k \geq 1$, the number of k-chains whose maximal element has support J is the same for the shard intersection order on W and for the parabolic support order on W.*

We give two different proofs of the main theorem. The first is self-contained, and follows from a common recursion obtained in section 4. In this statement and from now on, a k-chain in a poset means a weakly increasing sequence of k elements

$$u_1 \leq u_2 \leq \cdots \leq u_k.$$

A sequence $u_1 < u_2 < \cdots < u_k$ is called a strict k-chain.

There is a general relationship in partial orders between chains and strict chains (see for example [Sta97, §3.11]). The correspondence is defined by seeing a chain as a pair (strict chain, sequence of multiplicities). This allows one to relate the enumeration of all strict chains and the enumeration of all chains.

In our context, one can restrict this correspondence to chains whose last element has a given support. This implies that the analogue of Theorem 1.1 for strict chains also holds. This has the following consequence, by summing over all subsets J.

Theorem 1.2. *Let W be a finite Coxeter group. The f-vectors of the order complexes for the shard intersection order on W and for the parabolic support order on W are equal.*

The starting point of this article was in fact a previous observation, relating some generating series of Möbius numbers for shard intersection orders and for parabolic support order. We prove a refined version of such a statement in Section 5.

An alternative approach to the main theorem is provided in Section 6. There, we give a dimension-preserving bijection between the order complex of the parabolic support poset and a certain pulling triangulation of the permutahedron. Reading, in [Rea11, Theorem 1.5], gives a dimension-preserving bijection between the order complex of the shard intersection lattice and an isomorphic triangulation of the permutahedron. This establishes the following proposition:

Proposition 1.3. *Let W be a finite Coxeter group. For every integer $k \geq 1$, the number of k-chains is the same for the shard intersection order on W and for the parabolic support order on W.*

This proposition is an obvious corollary of Theorem 1.1, but in fact, we show in Section 4 that Theorem 1.1 can also be deduced from it, by an inclusion-exclusion argument.

2. Preliminaries and first lemmas

Let (W, S) be a finite Coxeter system, where W is a Coxeter group and S is the fixed set of simple reflections. The letter n will always denote the rank of W, which is the cardinality of S. Let $\ell(w)$ denote the length of an element $w \in W$ with respect to the set S of generators. The symbol e will be the unit of W, and $w_\circ$ the unique longest element. We refer the reader who is unfamiliar with the subject to [Hum90, GP00, BB05] for general references on (finite) Coxeter groups.

For an element w in W, we will use the following (more or less standard) notations. The *set $D_R(w)$ of right descents* of w is the set of elements $s \in S$ such that $\ell(ws) < \ell(w)$. Let $S(w)$ be the *support of* w, that is, the set of elements $s \in S$ such that s appears in some (or equivalently any) reduced word for w.

The following notations are needed for the definition of the shard intersection order. Let $G(w)$ be the subgroup of W generated by wsw^{-1} for $s \in D_R(w)$. This is a conjugate of a standard parabolic subgroup. Let $N(w)$ be the set of (left) inversions of w, defined as $\Phi^+ \cap w(-\Phi^+)$, where Φ^+ is the set of positive roots in a root system Φ for (W, S).

We also need the standard properties of *parabolic decomposition* of elements of W. Let us recall them briefly. For a subset $I \subseteq S$, let W_I be the (standard) parabolic subgroup of W generated by the simple reflections in I. Given a subset I of S, there is a factorisation $W = W^I W_I$, where W_I is the parabolic subgroup and W^I the set of minimal length coset representatives of classes in W/W_I. Equivalently, every $w \in W$ can be uniquely written $w = w^I w_I$ where $w_I \in W_I$ and the right descents of w^I do not belong to I. Moreover, the expression $w = w^I w_I$ is reduced, meaning that $\ell(w) = \ell(w^I) + \ell(w_I)$. Following [BB05], we call the pair (w^I, w_I) *the* *parabolic components of* $w \in W$. We denote by $w_{\circ,I}$ the longest element of W_I.

Given subsets $I \subseteq J$ of S, the more general notation $W_J^I = W_J \cap W^I$ stands for the set of elements of the parabolic subgroup W_J that have no right descents in I. Then there is a unique factorisation $W_J = W_J^I W_I$, similar to the previous one and denoted in the same way; see [GP00, Chapter 3] or [BBHT92] for more details on these decompositions.

The following lemma is proved by Möbius inversion on the boolean lattice of subsets of S.

Lemma 2.1. *The number of elements of* W *with support* S *is*

$$\#\{u \in W \mid S(u) = S\} = \sum_{J \subseteq S} (-1)^{n-\#J} \#W_J. \tag{1}$$

Finally, we discuss a second lemma that is crucial for the proof of Theorem 1.1. Fix $J \subseteq S$ and $I \subseteq J$. We introduce two sets attached to the pair (I, J). To distinguish between the two orders considered in this article, we will use the symbols σ and π as markers standing for shard and for parabolic support, respectively. The first one is

$$E_\pi(I, J) = \{u \in W_J^I \mid J \setminus I \subseteq S(u)\}. \tag{2}$$

The second one is

$$E_\sigma(I, J) = \{w \in W_J \mid S(w) = J,\ I \subseteq D_R(w) \subseteq J\}. \tag{3}$$

Both these sets are subsets of W_J.

Lemma 2.2. *The map* $\psi : w \mapsto u = w^I$ *defines a bijection from* $E_\sigma(I, J)$ *to* $E_\pi(I, J)$*, with inverse map* $\varphi : u \mapsto w = uw_{\circ,I}$*.*

Proof. First, let $w \in E_\sigma(I, J)$ and decompose w in parabolic components: $w = uv$ with $u = w^I \in W_J^I$ and $v = w_I \in W_I$. Since $S(v) \subseteq I$, $S(w) = J$ and uv is a reduced expression (i.e., the concatenation of reduced words for u and for v gives a reduced word for w), we have $J \setminus I \subseteq S(u)$. Note also that, since $I \subseteq D_R(w)$, we have $v = w_{\circ,I}$. Indeed, since $\ell(ws) < \ell(w)$ for all $s \in I$, we have by definition of W^I that $\ell(vs) < \ell(v)$ for all $s \in I$. Since $v \in W_I$ this forces

$$v = w_{\circ,I}. \tag{4}$$

By uniqueness of the parabolic components, this defines a map $\psi : E_\sigma(I, J) \to E_\pi(I, J)$, defined by $w \mapsto w^I$.

Now, we show that the map $\varphi : E_\pi(I,J) \to E_\sigma(I,J)$, defined by $u \mapsto uw_{\circ,I}$, is well-defined. This will imply, using (4), that $\varphi = \psi^{-1}$ and therefore our claim.

Since $J \setminus I \subseteq S(u)$, $S(w_{\circ,I}) = I$ and the fact that the expression $uw_{\circ,I}$ is reduced, we have $S(\varphi(u)) = S(uw_{\circ,I}) = J$. Since $\ell(w_{\circ,I}s) < \ell(w_{\circ,I})$ for all $s \in I$, we have $\ell(uw_{\circ,I}s) < \ell(uw_{\circ,I})$ for all $s \in I$. Hence $I \subseteq D_R(uw_{\circ,I}) = D_R(\varphi(u))$. Therefore, $\varphi(u) \in E_\sigma(I,J)$ and the claim is proven. □

3. Shard intersection lattice and parabolic support posets

In this section, we will recall the definitions and main properties of the shard intersection order and of the parabolic support order. For proofs and details, the reader should refer to the articles [Rea09, Rea11, STW15] and [BHZ06].

Recall that, in order to distinguish between the two orders considered in this article, we will use the symbols σ and π as markers standing for shards and for parabolic support respectively.

3.1. Shard intersection order

The shard intersection order on W can be defined as follows. Let u and v in W. Then $u \leq_\sigma v$ if and only if $N(u) \subseteq N(v)$ and $G(u) \subseteq G(v)$. Recall that $N(u)$ is the set of (left) inversions of u and $G(u)$ is the subgroup defined in section 2. This definition is not the original one from [Rea09, Rea11], but comes from the reformulation explained in [STW15, §4.7].

Here are some basic properties. The unique minimal element is e, the unique maximal element is $w_\circ$. The partial order $\leq_\sigma$ is ranked, and the rank $\mathrm{rk}_\sigma(u)$ of an element u is the cardinality of the descent set $D_R(u)$.

Moreover, for every v, the interval $[e,v]$ is isomorphic to the shard intersection order for the parabolic subgroup associated to the set of right descents $D_R(v)$.

By [Rea09, Th. 4.3], the Möbius function satisfies

$$\mu_\sigma(e,v) = \sum_{J \subseteq D_R(v)} (-1)^{\#J} \#W_J. \tag{5}$$

3.2. The parabolic support order

The parabolic support partial order on W is defined as follows: $u \leq_\pi v$ if and only if $v_{S(u)} = u$.

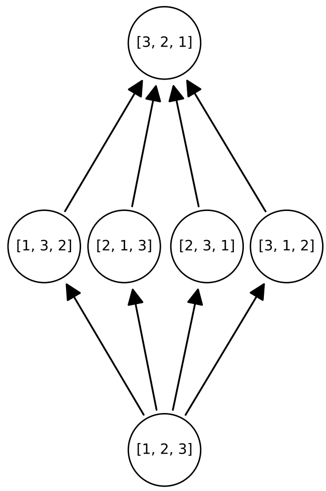

Figure 1: The shard intersection order on the symmetric group S_3.

The partial order $\leq_\pi$ is ranked and the rank $\mathrm{rk}_\pi(u)$ of an element u is the cardinality of the support $S(u)$. The unique minimal element is e, and the maximal elements are the elements of W of support S.

The Möbius function is described completely by [BHZ06, Th. 4]:

$$\mu_\pi(u,v) = \begin{cases} (-1)^{\#S(v)-\#S(u)} & \text{if } S(u) = S(v) \setminus D_R(v^{S(u)}), \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Let us now describe the principal upper ideal of an element.

Proposition 3.1. *Let u be an element with support $S(u) = I$. The principal upper ideal of u in the parabolic support poset $\leq_\pi$ is in bijection with W^I by the map that sends v to v^I.*

This is a direct consequence of the definition of the parabolic support order.

Recall the set $E_\pi(I,J)$ defined in (2).

Proposition 3.2. *The set of elements v such that $u \leq_\pi v$ and $S(v) = J$ is in bijection with $E_\pi(I,J)$.*

This follows from the previous proposition, together with the uniqueness of the decomposition $W_J = W_J^I W_I$.

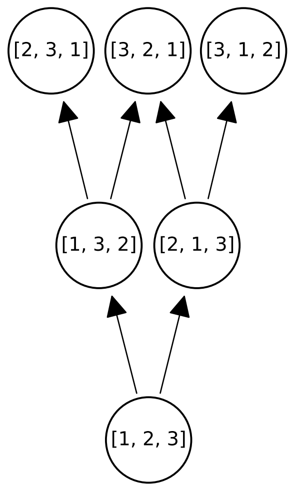

Figure 2: The parabolic support order on the symmetric group S_3.

4. Counting chains and proof of Theorem 1.1

In this section, we obtain recursions (on the length k) for the numbers of k-chains for both shard intersection order and parabolic support order.

It turns out that Lemma 2.2 allows us to identify these recursions. The initial conditions for $k = 1$ also match, as they amount to counting elements with given support in W. This implies Theorem 1.1.

4.1. Counting chains in shards

Let $\mathrm{Ch}^{\sigma}_{J,k}$ be the set of k-chains in the shard intersection order such that the top element of the chain has support J. Recall the set $E_\sigma(I, J)$ defined in (3).

Proposition 4.1. *For $k \geq 2$, the numbers of k-chains in the shard intersection order satisfy*

$$\# \mathrm{Ch}^{\sigma}_{J,k} = \sum_{I \subseteq J} \# E_\sigma(I, J) \, \# \mathrm{Ch}^{\sigma}_{I,k-1} \, . \tag{7}$$

Proof. The cardinality of $\mathrm{Ch}^{\sigma}_{J,k}$ is the sum

$$\sum_{\{x | S(x) = J\}} \# \mathrm{Ch}^{\sigma}_{x,k},$$

where $\mathrm{Ch}^{\sigma}_{x,k}$ is the set of k-chains ending with x.

Now k-chains with top element x are in bijection with k-chains in the shard intersection order for a parabolic subgroup of type $D_R(x)$. Their number only depends on the set $D_R(x)$ and is the sum

(8)
$$\sum_{I \subseteq D_R(x)} \# \,\mathrm{Ch}^{\sigma}_{I,k-1} \,.$$

Therefore one gets

$$\# \,\mathrm{Ch}^{\sigma}_{J,k} = \sum_{\{x | S(x)=J\}} \sum_{I \subseteq D_R(x)} \# \,\mathrm{Ch}^{\sigma}_{I,k-1} \,.$$

Exchanging the summations, one obtains

$$\sum_{I \subseteq J} \left(\sum_{\{x | S(x)=J,\, I \subseteq D_R(x)\}} 1 \right) \# \,\mathrm{Ch}^{\sigma}_{I,k-1} \,.$$

The inner sum is exactly the cardinality of $E_\sigma(I,J)$ as defined in (3). □

4.2. Counting chains in parabolic support posets

Let $\mathrm{Ch}^{\pi}_{J,k}$ be the set of k-chains in the parabolic support order such that the top element of the chain has support J. Recall the set $E_\pi(I,J)$ defined in (2).

Proposition 4.2. *For $k \geq 2$, these numbers of k-chains in the parabolic support order satisfy*

(9)
$$\# \,\mathrm{Ch}^{\pi}_{J,k} = \sum_{I \subseteq J} \# E_\pi(I,J) \,\# \,\mathrm{Ch}^{\pi}_{I,k-1} \,.$$

Proof. There is a bijection between k-chains and triples (element u, $(k-1)$-chain with top element u, element v greater than or equal to u).

By proposition 3.2, given an element u of support I, the number of possible v with $u \leq_\pi v$ and $S(v) = J$ is the cardinality of $E_\pi(I,J)$.

Therefore the number of k-chains with last element of support J and next-to-last element u only depends on the support I of u. The formula follows by gathering k-chains according to the support of their next-to-last element u. □

4.3. Equivalence of Theorem 1.1 and Proposition 1.3

It is clear that Proposition 1.3 is a weakening of Theorem 1.1. We wish to show, conversely, that Theorem 1.1 can be deduced from Proposition 1.3. In section 6, we will then give an independent proof of Proposition 1.3, which together with the current argument constitutes a separate proof of Theorem 1.1.

Suppose that we know Proposition 1.3 holds for any finite Coxeter group. Write $\#\operatorname{Ch}^{\sigma}_{k}(W)$ for the number of k-chains in W with respect to $\leq_\sigma$.

Because there are natural inclusions of posets $(W_I, \leq_\sigma) \subseteq (W_J, \leq_\sigma)$ for subsets $I \subseteq J$, one can show that

$$\#\operatorname{Ch}^{\sigma}_{J,k}(W) = \#\left(\operatorname{Ch}^{\sigma}_{k}(W_J) \setminus \bigcup_{K \subset J} \operatorname{Ch}^{\sigma}_{k}(W_K)\right).$$

By inclusion-exclusion, this can be expressed as:

$$\#\operatorname{Ch}^{\sigma}_{J,k}(W) = \sum_{K \subseteq J} (-1)^{|J|-|K|} \#\operatorname{Ch}^{\sigma}_{k}(W_K).$$

Similarly, because there are also natural inclusions $(W_I, \leq_\pi) \subseteq (W_J, \leq_\pi)$ for subsets $I \subseteq J$, one can show that

$$\#\operatorname{Ch}^{\pi}_{J,k}(W) = \sum_{K \subseteq J} (-1)^{|J|-|K|} \#\operatorname{Ch}^{\pi}_{k}(W_K).$$

The right-hand sides of these two equations are equal by Proposition 1.3, and this shows that Proposition 1.3 implies Theorem 1.1.

5. An equality of characteristic polynomials

In this section, we will compare two generating series, the first one built from Möbius numbers from the bottom in the shard intersection order, the second one from Möbius numbers to the top elements in the parabolic support order. They turn out to be equal, up to sign.

5.1. Characteristic polynomials for shards

Let us consider the following polynomial in one variable:

$$\chi_\sigma(q) = \sum_{v} \mu_\sigma(e, v) q^{n-\operatorname{rk}_\sigma(v)}. \tag{10}$$

This is exactly the characteristic polynomial of the shard intersection poset, as defined usually for graded posets with a unique minimum. We will instead compute the more refined generating series

$$X_\sigma = \sum_v \mu_\sigma(e, v) Z_{D_R(v)}, \tag{11}$$

with variables Z indexed by subsets of S. This reduces to $\chi_\sigma(q)$ by sending Z_I to $q^{n-\#I}$.

Using the known value (5) for the Möbius function in the shard intersection order, one finds

$$X_\sigma = \sum_v \sum_{I \subseteq D_R(v)} (-1)^{\#I} \# W_I Z_{D_R(v)}, \tag{12}$$

Introducing J to represent $D_R(v)$ and exchanging the summations, this becomes

$$X_\sigma = \sum_{I \subseteq J} (-1)^{\#I} \# W_I Z_J \sum_{\{v \mid D_R(v) = J\}} 1. \tag{13}$$

5.2. Characteristic polynomials for parabolic support posets

Let us consider the following polynomial

$$\chi_\pi(q) = \sum_{\substack{u \leq_\pi v \\ \mathrm{rk}(v) = n}} \mu_\pi(u, v) q^{n - \mathrm{rk}_\pi(u)}. \tag{14}$$

This is something like a characteristic polynomial of the opposite of the parabolic support poset, except that there are several maximal elements, so there is an additional summation over them.

In fact, let us instead compute the more refined generating series

$$X_\pi = \sum_{\substack{u \leq_\pi v \\ \mathrm{rk}(v) = n}} \mu_\pi(u, v) Z_{S(u)}, \tag{15}$$

with variables Z indexed by subsets of S.

By the known value of the Möbius function (6) in the parabolic support order, one obtains

$$\sum_{\{u \leq_\pi v \mid S(v) = S,\, S(u) = S(v) \setminus D_R(v^{S(u)})\}} (-1)^{\#S(v) - \#S(u)} Z_{S(u)}. \tag{16}$$

Let us use J to denote $S(u)$. Then for v fixed, u is determined from J by the relation $u = v_J$, by definition of the parabolic support partial order. One can therefore replace the summation over u and v by a summation over J, v^J and v_J. One obtains

$$\sum_J (-1)^{n-\#J} Z_J \sum_{\{v^J \in W^J, v_J \in W_J | S(v_J)=J,\, J=S\setminus D_R(v^J)\}} 1, \tag{17}$$

where the condition that $S(v^J v_J) = S$ has been removed because it is implied by the other conditions. The inner sum can be factorised into the product of

$$\sum_{\{v_J \in W_J | S(v_J)=J\}} 1 = (-1)^{\#J} \sum_{I \subseteq J} (-1)^{\#I} \#W_I, \tag{18}$$

and

$$\sum_{\{v^J \in W^J | D_R(v^J)=S\setminus J\}} 1 = \sum_{\{x | D_R(x)=S\setminus J\}} 1, \tag{19}$$

where we used the definition of W^J.

At the end, one finds

$$X_\pi = (-1)^n \sum_{I \subseteq J} (-1)^{\#I} \#W_I Z_J \sum_{\{x | D_R(x)=S\setminus J\}} 1. \tag{20}$$

Note that by using the involution $v \leftrightarrow w_\circ v$, the inner sum is the same as the sum over x such that $D_R(x) = J$. This proves the following result.

Proposition 5.1. *For every finite Coxeter group W, there is an equality $X_\sigma = (-1)^n X_\pi$.*

6. A bijection between strict chains in the parabolic support poset and the faces of a pulling triangulation of the permutahedron

As before, (W, S) is a finite Coxeter system. In [Rea11, Theorem 1.5], the author exhibits a bijection between strict k-chains in the shard intersection lattice and $(k-1)$-faces (faces of dimension $k-1$) of a pulling triangulation of the W-permutahedron. We exhibit here a bijection between strict k-chains in the parabolic support poset and $(k-1)$-faces of an isomorphic pulling triangulation of the W-permutahedron.

Recall that the W-permutahedron $\mathrm{Perm}(W)$ is the convex hull of the W-orbit of a generic point in the space on which W acts as a finite reflection group. The faces of $\mathrm{Perm}(W)$ are naturally indexed by the cosets W/W_I for all $I \subseteq S$: the face F_{wW_I} is of dimension $\#I$ for any $w \in W$ and can be identified with $\mathrm{Perm}(W_I)$; see for instance [Hoh12]. Moreover, the set of vertices in a face of $\mathrm{Perm}(W)$ is an interval in the right weak order. The *right weak order* $\leq_R$ on W is defined by $u \leq_R v$ if a reduced word for $u \in W$ is a prefix of a reduced word for $v \in W$, i.e., $\ell(u^{-1}v) = \ell(v) - \ell(u)$; see [BB05, §3]. Then the set of vertices of the face F_{wW_I} is the set $wW_I = [w^I, w^I w_{\circ,I}]_R$.

For Q a polytope, with a fixed total order on its vertices, the corresponding *pulling triangulation* of Q is defined as follows (see [Lee97]). Let v be the initial (minimum) vertex. Inductively, determine the pulling triangulation of each facet which does not include vertex v (with respect to the total order on its vertices defined by restriction). Then, cone each of the simplices from these triangulations over the vertex v. Remark that it is not actually necessary to start with a total order on the vertices: it is sufficient if there is a unique minimum vertex for each face of the polytope. We can therefore define $\Delta(W)$ to be the pulling triangulation of $\mathrm{Perm}(W)$ with respect to right weak order.

Remark 6.1. Reading in [Rea11, Theorem 1.5] gave a bijection from the faces of the order complex of the shard intersection lattice to the pulling triangulation defined with respect to the reverse of weak order. These two triangulations are equivalent under the linear transformation defined by multiplication on the left by $w_\circ$.

We first define our map for strict k-chains containing e. To $e = u_1 <_\pi u_2 <_\pi \cdots <_\pi u_k$ in the parabolic support poset, $k \in \mathbb{N}^*$, we associate the following subset of W:

$$\varphi(u_1 <_\pi u_2 <_\pi \cdots <_\pi u_k) = \{w_j \mid 1 \leq j \leq k\}, \tag{21}$$

where $w_j = (u_k)^{S(u_j)} \in W^{S(u_j)}$, or equivalently, $w_j = u_k u_j^{-1}$. Observe that $w_1 = u_k$ is the maximal element of the input chain, while $w_k = u_k^{S(u_k)} = e$.

For a strict k-chain not containing e, we add in e, and apply the above map on strict $k+1$-chains containing e, and remove e from the resulting set.

Theorem 6.2. *The map φ is a dimension-preserving bijection from the order complex of the parabolic support poset to $\Delta(W)$.*

Notice that the above theorem, together with [Rea11, Theorem 1.5], proves Proposition 1.3. As we showed in Section 4, Theorem 1.1 follows. This provides a second (not self-contained) proof of the main theorem.

Beware that the bijection φ is, however, not an isomorphism of simplicial complexes, as it does not preserve the incidence relation of simplices.

By its very construction, φ is given essentially by applying the same map twice, one time mapping chains containing e to simplices containing e, and another time mapping chains not containing e to simplices not containing e. Adding or removing e is a bijective process that allows one to go from one case to the other. Therefore, proving the bijectivity of φ can be done by looking only at the case where e is present.

Before proving the theorem, we give some properties of the map φ.

Lemma 6.3. *If* $(e = u_1 <_\pi \cdots <_\pi u_k)$ *is a k-chain in the parabolic support poset starting with e, and* $w_i = u_k u_i^{-1}$*, then* $w_1 >_R w_2 >_R \cdots >_R w_k$.

Proof. By [BHZ06, Proposition 6(3)], since $u_1 <_\pi \cdots <_\pi u_k$, we have that $u_1^{-1} <_R \cdots <_R u_k^{-1}$. Since $u_k = w_i u_i$ is a reduced factorization for each i, it follows that $w_1 >_R \cdots >_R w_k$. □

Note that this lemma implies that although the image of φ is, by definition, just a set of elements in W, we can determine the numbering of the elements by looking at their relative order with respect to $<_R$.

Lemma 6.4. *The map φ is injective.*

Proof. It suffices to consider φ applied to strict k-chains containing e. Suppose $u_1 = e$ and $\varphi(u_1 <_\pi \cdots <_\pi u_k) = \{w_i \mid 1 \leq i \leq k\}$. As discussed above, from $\{w_i \mid 1 \leq i \leq k\}$, we can reconstruct the numbering of the elements. In particular, this allows us to determine $u_k = w_1$, and then we can reconstruct $u_i = w_i^{-1} u_k$. Since we can reconstruct the u_i on the basis of their image under φ, it must be that φ is injective. □

Lemma 6.5. *Let* $e = u_1 <_\pi u_2 <_\pi \cdots <_\pi u_k$ *be a strict k-chain in the parabolic support poset, and* $\varphi(u_1 <_\pi u_2 <_\pi \cdots <_\pi u_k) = \{w_j \mid 1 \leq j \leq k\}$ *as in* (21)*. Then:*

1. *For* $i \leq k-1$*, we have* $w_i \in w_{k-1} W_{S(u_{k-1})}$.
2. $\varphi(u_1 <_\pi u_2 <_\pi \cdots <_\pi u_k) \setminus \{e\}$ *is a subset of the vertices of the face* $F_{w_{k-1} W_{S(u_{k-1})}}$ *of* Perm(W) *for which* w_{k-1} *is the smallest vertex.*

Proof. Let $i \leq k-1$. Observe that $w_i = w_{k-1} u_{k-1} u_i^{-1}$. Since $w_i \geq_R w_{k-1}$, we have that $\ell(w_i) = \ell(w_{k-1}) + \ell(u_{k-1} u_i^{-1})$. Since $w_{k-1} \in W^{S(u_{k-1})}$, while $u_{k-1} u_i^{-1} \in W_{S(u_{k-1})}$, it follows that the parabolic factorization of w_i with respect to the parabolic subgroup $W_{S(u_{k-1})}$ is $(w_{k-1}, u_{k-1} u_i^{-1})$. This establishes the first point. The second point is just a rephrasing of the first point, using the description of the faces of Perm(W) given at the beginning of this section. □

We also need the following standard facts on pulling triangulations.

Lemma 6.6. *Let Q be a convex polytope with a fixed order on its vertices.*

1. *The pulling triangulation of Q restricts to the pulling triangulation of each face of Q.*
2. *Each face of the pulling triangulation is contained in a smallest face of Q, and contains the minimal vertex of that face (with respect to the order on the vertices).*

Proof. The first point is established by induction on the dimension of Q. Every face of the pulling triangulation is contained in a smallest face R of Q because the faces of Q form a lattice under the inclusion order. By the first point, the pulling triangulation of Q restricts to the pulling triangulation of R. By the definition of the pulling triangulation of R, the faces of it which do not include the minimal vertex of R are each contained in a facet of R, and therefore do not have R as the smallest face in which they are contained. □

Proof of Theorem 6.2. The map φ is injective by Lemma 6.4.

We now show that the image of a strict k-chain $u_1 <_\pi \cdots <_\pi u_k$ is a k-face of $\Delta(W)$. The proof is by induction on n. The base case is trivial. Also, it suffices to consider the case that $u_1 = e$, and $k \geq 2$.

For $1 \leq i \leq k-1$, define $w'_i = u_{k-1}u_i^{-1}$. Now $\varphi(u_1 <_\pi \cdots <_\pi u_{k-1}) = \{w'_i \mid 1 \leq i \leq k-1\}$. The chain $u_1 <_\pi \cdots <_\pi u_{k-1}$ lies in $W_{S(u_{k-1})}$ whose rank is less than n, so by induction $\{w'_i \mid 1 \leq i \leq k-1\}$ is a face of the pulling triangulation of $\mathrm{Perm}(W_{S(u_{k-1})})$.

For $1 \leq i \leq k$, define $w_i = u_k u_i^{-1}$. Note that for $1 \leq i \leq k-1$, we have $w_i = u_k u_{k-1}^{-1} w'_i$, which can be rewritten as $w_i = w_{k-1} w'_i$. Left-multiplication by w_{k-1} is an order-preserving bijection from $W_{S(u_{k-1})}$ to $w_{k-1}W_{S(u_{k-1})}$. This map takes the pulling triangulation of $\mathrm{Perm}(W_{S(u_{k-1})})$ to the pulling triangulation of the face $F_{w_{k-1}W_{S(u_{k-1})}}$ of $\mathrm{Perm}(W)$.

Now,

$$\varphi(u_1 <_\pi \cdots <_\pi u_k) = \{w_i \mid 1 \leq i \leq k\} = \{e\} \cup w_{k-1}\{w'_i \mid 1 \leq i \leq k-1\}.$$

This set forms the vertices of a simplex with one vertex at e, and the others forming a face of the pulling triangulation of $F_{w_{k-1}W_{S(u_{k-1})}}$. Since $w_{k-1} >_R w_k = e$, the face $F_{w_{k-1}W_{S(u_{k-1})}}$ of $\mathrm{Perm}(W)$ does not contain e. Thus, $\{w_i \mid 1 \leq i \leq k\}$ is a k-face of the pulling triangulation of $\mathrm{Perm}(W)$.

Conversely, suppose we have a k-face H in $\Delta(W)$. We may assume $e \in H$. Write $G = H \setminus \{e\}$. G is also a face of $\Delta(W)$. By Lemma 6.6(2), G lies in a well-defined smallest face of $\mathrm{Perm}(W)$. That face can be uniquely written as wW_J with $J \subset S$ and $w \in W^J$. By Lemma 6.6(2), G contains the smallest vertex of this face, which is w. Now $w^{-1}G$ defines a $(k-1)$-face of $\Delta(W_J)$ containing e. By induction, it is the image under φ of a chain $e = u_1 <_\pi \cdots <_\pi u_{k-1}$. Since $w^{-1}G$ is not contained in W_I for any $I \subset J$, $\operatorname{supp} u_{k-1} = J$.

Define $u_k = wu_{k-1}$. Observe that $(u_k)_{S(u_{k-1})} = (u_k)_J = u_{k-1}$. Thus $u_{k-1} <_\pi u_k$.

We therefore have constructed a chain $e = u_1 <_\pi \cdots <_\pi u_k$ in the parabolic support poset. We now claim that $\varphi(u_1 <_\pi \cdots <_\pi u_k) = H$. The necessary calculation follows exactly the same logic as the proof that the image of a k-chain is a k-face of $\Delta(W)$. □

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