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Two classes of modular p-Stanley sequences

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Consider a set A with no p-term arithmetic progressions for p prime. The p-Stanley sequence of a set A is generated by greedily adding successive integers that do not create a p-term arithmetic progression. For $p > 3$ prime, we give two distinct constructions for p-Stanley sequences which have a regular structure and satisfy certain conditions in order to be modular p-Stanley sequences, a set of particularly nice sequences defined by Moy and Rolnick which always have a regular structure.

Odlyzko and Stanley conjectured that the 3-Stanley sequence generated by $\{0, n\}$ only has a regular structure if $n = 3^k$ or $n = 2 \cdot 3^k$. For $p > 3$ we find a substantially larger class of integers n such that the p-Stanley sequence generated from $\{0, n\}$ is a modular p-Stanley sequence and numerical evidence given by Moy and Rolnick suggests that these are the only n for which the p-Stanley sequence generated by $\{0, n\}$ is a modular p-Stanley sequence. Our second class is a generalization of a construction of Rolnick for $p = 3$ and is thematically similar to the analogous construction by Rolnick.

1. Introduction

For an odd prime p, a set is called *p-free* if it contains no p-term arithmetic progression. Szekeres conjectured that for p an odd prime, the maximum number of elements in a p-free subset of $\{0, 1, \ldots, n-1\}$ grows as $n^{\log_{p-1} p}$ [2]. This conjecture however has been disproved. In particular, Elkin [1] proves the best known lower bound for 3-free sets of $O(n^{1-o(1)})$ while the best proven upper bound is $O(n(\log\log n)^5/\log n)$ due to recent work of Sanders [12].

The inspiration for Szekeres's conjecture however is of interest. In particular, Szekeres's conjecture is based on the sequence constructed by starting with 0 and greedily adding each subsequent integer that does not create a p-term arithmetic progression. The sequence produced is exactly the non-negative integers that have no digit of $p-1$ in their base p expansion. In 1978, Odlyzko and Stanley generalized this construction to arbitrary sets [9].

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Definition 1.1. *Let $A := \{a_1, \ldots, a_n\}$ be a finite set of nonnegative integers that contains* 0 *with no nontrivial p-term arithmetic progressions. Furthermore take $0 = a_1 < a_2 < \cdots < a_n$ and for each integer $k \geq n$, let a_{k+1} be the least integer greater than a_k such that $\{a_1, \ldots, a_k, a_{k+1}\}$ has no p-term arithmetic progressions. The p-Stanley sequence $S_p(A)$, also written as $S_p(a_1, \ldots, a_n)$, is the sequence $a_1, \ldots, a_n, a_{n+1}, \ldots$.*

In the language of Stanley sequences the previous example is precisely $S_p(0)$. Odlyzko and Stanley noticed that for some sets A, the Stanley sequence $S_3(A)$ displays a regular pattern in terms of the ternary representations of its terms and these sequences grow as $n^{\log_2 3}$. In particular, they explicitly computed $S_3(0, 3^k)$ and $S_3(0, 2 \cdot 3^k)$ and showed that these sequences satisfy the above properties. However, for other values of m, the sequence $S_3(0, m)$ seems to grow chaotically and at the rate $n^2/\log n$. In particular, Lindhurst [5] computed $S_3(0, 4)$ for large values and observes that it appears to follow this second growth rate.

Odlyzko and Stanley provided a heuristic argument why a randomly chosen sequence should grow at the rate $n^2/\log n$ and conjectured that these two behaviors are the only possible ones. Further work on the growth of chaotic p-Stanley sequences for $p > 3$ can be found in [4]. This leads to the following conjecture, which is explicitly stated for $p = 3$ in [9].

Conjecture 1.1 (Based on [9], [4])**.** *A p-Stanley sequence $a_1, a_2, \ldots$ with p an odd prime satisfies either:*

Type 1: $a_n = \Theta(n^{\log_{(p-1)} p})$
Type 2: $a_n = \Theta\left(n^{(p-1)/(p-2)}/(\log n)^{1/(p-2)}\right)$.

To date however there has been no 3-Stanley sequence, or more generally p-Stanley sequence, that has been proven to have Type 2 growth. Despite this, there has been significant interest in studying the structure of Type 1 3-Stanley sequences ([7], [11], [10]). The most relevant class of Type 1 3-Stanley sequences stems from the work of Moy and Rolnick [7], extending work of Rolnick [10], which gave the following class of Type 1 sequences.

Definition 1.2. *Consider a set $A \subseteq \{0, \ldots, N-1\}$ with $0 \in A$ such that there is no nontrivial* 3*-term arithmetic progression mod N among the elements of A. (Trivial arithmetic progressions refer to progressions with all elements equal.) A set A is said to be* modular *if for every integer x, there exists $y \geq z$ in A such that $2y - z \equiv x \mod N$. Note that the second condition is equivalent to x, y, and z being an arithmetic progression mod N. Furthermore we say that $S_3(A)$ is a* modular *Stanley sequence if A satisfies these conditions.*

Several papers have been dedicated to understanding various properties of these modular sequences; namely the character, repeat factor, and scaling factor of these sequences. See [10], [7] for definitions of these properties and [11], [6], [8] for further work on understanding these properties. Furthermore Moy and Rolnick [7] conjecture that all 3-Stanley sequences with Type 1 growth are pseudomodular, a suitable generalization of modular sequences. In contrast, for general p-Stanley sequences, there is no such conjectured form for Type 1 sequences. However there is a natural analog of modular Stanley sequences, modular p-Stanley sequences. In particular one modifies the given definition to have no p-term arithmetic progressions and defines an analog of the second condition. This is defined more precisely in the next section.

In this paper we present two classes of modular p-Stanley sequences, one of which hints a difference between 3-Stanley sequences and p-Stanley sequences for larger primes p whereas the other appears to suggest a degree of similarity. The first demonstrates that for $p > 3$, there exists a large class of integers n for which $S_p(0, n)$ has Type 1 growth and in fact is a modular sequence. In particular for $p \geq 5$, if $2 \cdot p^{k-1} < n < p^k$ and $p^k - n$ has no $p - 1$ in its base p expansion, then $S_p(0, n)$ has Type 1 growth. This is notable as there exist $n \neq i \cdot p^k$ for $1 \leq i \leq p - 1$ such that $S_p(0, n)$ exhibits Type 1 growth, unlike the case $p = 3$ where Stanley and Odlyzko [9] conjecture that only $S_3(0, 3^k)$ and $S_3(0, 2 \cdot 3^k)$ have Type 1 growth among sequences of the form $S_3(0, n)$. Numerical evidence given by Moy and Rolnick [7] suggests that these are the only possible integer n and thus appears to give a conjectural answer to a question raised by Moy and Rolnick [7] of classifying integers n such that $S_p(0, n)$ is modular.

The second class is a generalization of Theorem 1.2 by Rolnick [10]. These constructions are notable as they are among the first explicit constructions for large classes of modular p-sequences, with the only other large class of constructions present in the literature being that of basic sequences given by Moy and Rolnick [7].

In Section 2 we provide some definitions and basic results on modular p-Stanley sequences that are used within this paper. In Section 3 we demonstrate the first class of modular p-Stanley sequences, and in Section 4 we demonstrate the second class of modular p-Stanley sequences. Section 5 contains some ideas for future work in these directions.

2. Definitions

This section provides the definitions and basic results on modular p-Stanley sequences necessary to prove our results. For further exposition, see [7].

Definition 2.1. *A set A p-covers x if there exist $x_1, x_2, \dots, x_{p-1} \in A$ such that $x_1 < x_2 < \cdots < x_{p-1} < x$ is an arithmetic progression.*

Proposition 2.1. *The p-Stanley sequence $S_p(A)$ is the unique sequence that starts with A, is p-free, and p-covers all $x \notin S_p(A)$ with $x > \max(A)$.*

Proof. Since $x > \max(A)$ there are two cases. If x is in $S_p(A)$, its addition to the sequence preserves that the sequence is p-free. If x is not in $S_p(A)$, it follows that the addition of x would have created a p-term arithmetic progression with largest term x and with the remaining terms in $S_p(A)$. □

Definition 2.2. *A set $A \subseteq \{0, 1, \dots, N-1\}$ is said to p-cover x mod N if there exist $x_1, x_2, \dots, x_{p-1} \in A$ such that $x_1 < x_2 < \cdots < x_{p-1}$ and x form an arithmetic progression mod N. Restricting $0 \le x < N$ and given the size restrictions for A this is equivalent to $x_1 < x_2 < \cdots < x_{p-1} < x$ or $x_1 < x_2 < \cdots < x_{p-1} < x + N$ forming an arithmetic progression.*

Definition 2.3. *A set $A \subseteq \{0, 1, \dots, N-1\}$ is a modular p-free set mod N if A contains 0, is p-free mod N, and p-covers all x with $0 \le x < N$ and $x \notin A$. A p-Stanley sequence is a modular p-Stanley sequence if it has the form $S_p(A)$ for a modular p-free set A.*

We will refer to "p-covering" and "modular p-free" simply as "covering" and "modular" when p is obvious. We write $A+B$ for $\{a+b \mid a \in A, b \in B\}$ and $c \cdot A$ for $\{c \cdot a \mid a \in A\}$. The following is the main theorem on modular p-Stanley sequences proved in [7]. It implies that a modular Stanley sequence grows asymptotically as $S_p(0)$.

Theorem 2.1 (Theorem 6.5 in [7])**.** *If A is a modular p-free set mod N, then $S_p(A) = A + N \cdot S_p(0)$. Note that $S_p(0)$ consists of all nonnegative integers with no $p-1$ in their base p expansions.*

Corollary 2.1 (Corollary 6.6 in [7])**.** *Any modular p-Stanley sequence exhibits Type 1 growth.*

3. First class of p-Stanley sequences

We use the notation $t_i(x)$ to refer to the digit corresponding to p^i in the base p expansion of x. We initially define a pair of sets which are critical for this section.

Definition 3.1. *Let A_p^k be the set of positive integers n such that $2 \cdot p^{k-1} < n \le p^k$ with $p^k - n \in S_p(0)$. This is equivalent to $t_i(p^k - n) \ne p - 1$ for all i and additionally $t_{k-1}(p^k - n) \ne p - 2$. Let $A_p = \bigcup_{k=0}^{\infty} A_p^k$.*

For example the set A_5 begins $\{1, 3, 4, 5, 12, 13, 14, 15, 17, 18\ldots\}$.

Notation 3.1. *Let* $S_p^k = \{x \mid x \in S_p(0), x < p^k\}$. *Note by Lemma 6.4 in [7],* S_p^k *is* p*-free mod* p^k *and covers* $\{0, 1, \ldots, p^k - 1\} \setminus S_p^k$.

In a manner closely related to the proof of Lemma 6.4 in [7], we define a key procedure for the proof of Theorem 3.4.

Definition 3.2. *For* $0 \le x < p^k$ *define the canonical covering of* x *to be the sequence* $x_1, x_1, \ldots, x_{p-1}$ *where* $x_j = \sum_i t_i^{(j)} p^i$ *and* $t_i^{(j)} = t_i(x)$ *if* $t_i(x) \neq p-1$ *and* $t_i^{(j)} = j - 1$ *if* $t_i(x) = p - 1$.

Note that the canonical covering is contained in S_p^k and, as suggested by its name, p-covers x. Using these definitions it possible to prove our first result on modular p-Stanley sequences.

Theorem 3.1. *For* $p > 3$ *a prime and* $n \in A_p$, $S_p(0, n)$ *is a modular* p*-Stanley sequence.*

Proof. Suppose that k is such that $p^{k-2} < n \le p^{k-1}$, and let $A = \{0\} \cup (n + S_p^k) \setminus \{p^{k-1}(p-1)\}$. Note that $\max(A) < p^k$. Therefore it suffices to demonstrate $S_p(0, n) = S_p(A)$ and that A is modular mod p^k.

To demonstrate that $S_p(0, n) = S_p(A)$, it suffices by Proposition 2.1 to prove that A is p-free and covers all $n < x < p^k$ with $x \notin A$. To demonstrate that A is modular mod p^k, it suffices to prove that A is p-free mod p^k and covers all $0 \le x < p^k$ mod p^k with $x \notin A$. Thus it is sufficient to show the slightly stronger statement that A is p-free mod p^k and covers all $n < x < p^k + n$ with $x \notin A$ and $x \neq p^k$. Let $A' = -n + A = \{-n\} \cup S_p^k \setminus \{p^{k-1}(p-1) - n\}$. We demonstrate that A' has no arithmetic progressions mod p^k which will give us the first of our two desired results.

Since S_p^k is p-free mod p^k, any arithmetic progression in A' must contain $-n$. Suppose there is an arithmetic progression $\{a_i\}$ mod p^k and define $b_i \equiv a_i \mod p^{k-1}$ with $0 \le b_i < p^{k-1}$. It follows that $\{b_i\}$ is an arithmetic progression mod p^{k-1}. By the definition of A_p, we know that $p^{k-1} - n \in S_p^k$, so the progression $\{b_i\}$ is in fact an arithmetic progression mod p^{k-1} in S_p^{k-1}. Thus the progression $\{b_i\}$ must be the constant arithmetic progression. It follows that $a_0 \equiv a_1 \equiv \cdots \equiv a_{p-1} \equiv -n \pmod{p^{k-1}}$ and therefore the only possible arithmetic progression mod p^k in A' is $i \cdot p^{k-1} - n$ for $0 \le i < p$. However, since $(p-1)p^{k-1} - n \notin A'$, it follows that A' is p-free mod p^k.

To prove the second result we demonstrate that A' covers $0 < x < p^k$ with $x \notin A'$ and $x \neq p^k - n$. If $x = p^{k-1}(p-1) - n$, then x is covered by $\{ip^{k-1} - n\}$ for $0 \le i < p - 1$. Otherwise, $x \notin S_p^k$. Since x is covered by its canonical covering in S_p^k, the only cases we have to consider are those in which the canonical covering of x contains $p^{k-1}(p-1)-n$.

Let $m = p^{k-1}(p-1)-n$, since $n \in A_p$, we know that $t_{k-1}(m) = p-2$, $t_{k-2}(m) < p-2$, and $t_i(m) \neq p-1$ for all i. Any $0 < x < p^k$ whose canonical covering contains m can be written in the form

$$x_S = \sum_{\substack{i=0\\ i\notin S}}^{k-1} t_i(m)p^i + \sum_{i\in S}(p-1)p^i,$$

where $S \subseteq \{0, 1, \ldots, k-1\}$ is a set of digits such that $t_i(m)$ is the same for all $i \in S$. We earlier assumed that $x \neq m$ and $x \neq p^k - n = p^{k-1} + m$. This implies that $S \neq \emptyset, \{k-1\}$.

For the remainder of the proof fix an integer a and an $S \subseteq \{0, 1, \ldots, k-1\}$ such that $a = t_i(m)$ for all $i \in S$ and $S \neq \emptyset, \{k-1\}$. Let j be $\max(S\backslash\{k-1\})$ and let $b = t_{j+1}(m)$.

We know that $t_{k-1}(m) = p-2$ and $t_{k-2}(m) < p-2$, which implies that $\{k-2, k-1\} \not\subseteq S$. Thus this implies that if $j = k-2$, then $k-1 \notin S$.

We know that $0 \leq a, b < p-1$, and we now consider four cases.

Case 1: $a = 0$.

Let $\Delta = \sum_{i\in S} p^i$. Then $\{p^{k-1}(p-1) - n + i\cdot\Delta\}$ for $0 \leq i < p-1$ is the canonical covering of x_S as we are preserving all digits not equals to $p-1$ in x_S and using $\{0, \ldots, p-2\}$ where x_S has a digit $p-1$. However $\{i \cdot p^{k-1} - n + i\cdot\Delta\}$ for $0 \leq i < p-1$ also covers x_S.

We need to check that all of these terms are in A'. Since $p^{k-1}(p-1) - n + i\Delta \in S_p^k$ with first digit $p-2$, then $i\cdot p^{k-1} - n + i\cdot\Delta$ is identical except the first digit ranges from 0 through $p-2$ for $0 < i < p-1$ while for $i = 0$ it follows as $i\cdot p^{k-1} - n + i\cdot\Delta = -n \in A'$.

Case 2: $0 < a < p-1$ and $0 \leq b < (p-3)/2$.

Let $j' > j$ be the smallest integer such that $t_{j'}(m) \geq (p-1)/2$. Note j' exists since $t_{k-1}(m) = p-2 \geq (p-1)/2$. In this case take

$$\begin{aligned}\Delta &= \sum_{i=j}^{j'-1} p^i(p-1)/2 + \sum_{i\in S\backslash\{j,j+1,\ldots,j'\}} p^i,\\ &= (p^{j'} - p^j)/2 + \sum_{i\in S\backslash\{j,j+1,\ldots,j'\}} p^i\end{aligned}$$

and consider the arithmetic progression $\{x_S - i\cdot\Delta\}$ for $0 < i \leq p-1$. We claim this set is contained in A'.

We can compute the digits of each of these numbers. Write the digit expansion of $x_S - i \cdot \Delta$ as $x_S - i \cdot \Delta = \sum_l t_l^{(i)} p^l$. For $l \notin \{j, j+1, \dots, j'\}$, then $t_l^{(i)}$ matches the canonical covering. In particular, $t_l^{(i)} = t_l(m)$ if $i \notin S$ and otherwise $t_l^{(i)} = p - 1 - i$.

Using explicit computation it is possible to determine the remaining digits. First note that $t_{j'}^{(i)} = t_{j'} - \lceil i/2 \rceil$. For $j+1 < l < j'$, we have $t_l^{(i)} = t_l(m)$ for i even and $t_l^{(i)} = t_l(m) + (p-1)/2$ for i odd. Furthermore, $t_{j+1}^{(i)} = t_l(m) + 1$ for $i > 0$ even and $t_{j+1}^{(i)} = t_{j+1}(m) + 1 + (p-1)/2$ for i odd. Finally, $t_j^{(i)} = i/2 - 1$ for $i > 0$ even and $t_j^{(i)} = (p-1)/2 + (i-1)/2$ for i odd.

Now we check that all of these terms are in A'. The jth digit cycles through each value when $0 \le i \le p-1$, and since it equals $p-1$ when $i = 0$, it never equals $p-1$ in the range $0 < i \le p-1$ that we are using to cover x_S. Since $t_{j'}(m) \ge (p-1)/2$, $t_{j'}^{(i)}$ never goes below 0, and $t_{j'}^{(i)} < t_{j'}(m)$. Therefore we have $t_{j'}^{(i)} < p-1$ for $i > 0$. Furthermore since $t_l(m) < (p-1)/2$ for $j < l < j'$, neither of the two values that this digit takes is $p-1$. Furthermore the $(j+1)$st digit only takes on 3 values, none of which is $p-1$ since $t_{j+1}(m) = b < (p-3)/2$. Finally, $t_{j+1}^{(i)} \neq t_{j+1}(m)$ for $i > 0$. Since $t_{j+1}(m)$ never takes on its original value again, none of the terms in this sequence are m.

Case 3: $0 < a < p-1$ and $(p-3)/2 \le b < p-1$ and $(a, b, p) \neq (2, 1, 5)$.

We claim we can find $1 \le d \le b+1$ such that $d \not\mid p - a - 1$ given the conditions in this case. If $p > 5$ it is not hard to check[1] that $\mathrm{lcm}(1, 2, \dots, (p-1)/2) \ge p-1$, so a number in this range must not divide $p - a - 1 < p - 1$. If $p = 5$, we can use $d = 2$ unless $a = 2$ (and therefore $p - a - 1 = 2$). Furthermore if $p = 5$, $a = 2$, $b \ge 2$, we can use $d = 3$.

Let

$$\Delta = d \cdot p^j + \sum_{i \in S \setminus \{j\}} p^i.$$

We claim that the arithmetic progression $\{x_S - i \cdot \Delta\}$ for $0 < i \le p-1$ is contained in A'.

None of the digits of $x_S - i \cdot \Delta$ is equal to $p-1$ except for possibly the jth and $(j+1)$st digits. The jth digit decreases by $d \pmod p$ so it only takes on the value $p-1$ when $i = 0$. Moreover, subtracting Δ, the jth digit

[1]Let $\prod_i p_i^{e_i}$ be the prime factorization of $p-1$. If $p-1$ is not a prime power, then $p_i^{e_i} \in \{1, \dots, (p-1)/2\}$ for all i. Otherwise, since p is odd, we can write $p - 1 = 2^k$. Then since $k > 2$, 2^{k-1} and 3 are elements in $\{1, 2, \dots, (p-1)/2\}$ and thus the least common multiple is at least $3 \cdot 2^{k-1} \ge 2^k = p - 1$.

forces the $(j+1)$st to decrement exactly $d-1$ times (due to a "borrow"). Since $p-1 > b \geq (p-3)/2 \geq d-1$, the $(j+1)$st digit never takes on the value $p-1$ and never itself "borrows" from the $(j+2)$nd digit.

Thus it suffices to check that no term is equal to $p^{k-1}(p-1)-n$. This must occur before the $(j+1)$st digit has changed its value from $t_{j+1}(m)$. In this range, the jth digit has value $t_j(x_S) - i \cdot d = (p-1) - i \cdot d$. However if $(p-1) - i \cdot d = a$, then $d \mid p-a-1$, a contradiction. Thus this arithmetic progression is contained in A', as desired.

Case 4: $a = 2$, $b = 1$, and $p = 5$.

This special case is similar to Case 2. Note that for $j < j' < k$, it is not the case that $j' \in S$. In particular the only possibility is $j' = k-1$, but $\{j, k-1\} \subseteq S$ implies that $t_j(m) = t_{k-1}(m)$ and $t_j(m) = a = 2$ whereas $t_{k-1}(m) = p-2 = 3$. Furthermore note that $j+1 \neq k-1$ since $t_{k-1}(m) = 3 \neq 1 = t_{j+1}(m)$. Now if $t_{j+2}(m) \geq 1$, letting

$$\Delta = 5^{j+1} + 3 \cdot 5^j + \sum_{i \in S \setminus \{j\}} 5^i,$$

it is easy to check that $\{x_S - i \cdot \Delta\}$ for $0 < i \leq 4$ is in A'.

Otherwise, $t_{j+2}(m) = 0$. Let $j' > j+2$ be the smallest integer such that $t_{j'}(m) \geq 2$. This exists for the same reason as in Case 2. Now let

$$\begin{aligned}\Delta &= \left(\sum_{i=j+2}^{j'-1} 2 \cdot 5^i\right) + 5^{j+1} + 3 \cdot 5^j + \sum_{i \in S \setminus \{j, j+1, \ldots, j'\}} 5^i, \\ &= (5^{j'} - 5^{j+2})/2 + 5^{j+1} + 3 \cdot 5^j + \sum_{i \in S \setminus \{j, j+1, \ldots, j'\}} 5^i.\end{aligned}$$

We cover x_S by $\{x_S - i \cdot \Delta\}$ for $0 < i \leq 4$. By exactly the same reasoning as in Case 2, this covering is in A'. □

We conjecture, but cannot currently prove, that these are the only integers n such that $S_5(0, n)$ exhibits Type 1 growth. Computational evidence provided by Moy and Rolnick [7] suggests that the integers less than 100 such that $S_5(0; n)$ are well-behaved and in particular modular are as follows:

$$\begin{aligned}&1, 3, 4, 5, 12, 13, 14, 15, 17, 18, 19, 20, 22, 23, 24, 25, 37, 39, 40, 42, 43, 44, 45, \\ &\quad 47, 57, 58, 59, 60, 62, 63, 64, 65, 67, 68, 69, 70, 72, 73, 74, 75, 82, 83, 84, 85, \\ &\quad 87, 88, 89, 90, 92, 93, 94, 95, 97, 98, 99.\end{aligned}$$

See Problem 6.7 in [7] for more detail. This matches exactly the integers which Theorem 3.4 would suggest, giving some support for this conjecture.

4. Second construction of p-Stanley sequences

This section presents a generalization of Theorem 1.2 given by Rolnick [11] with a proof that is similar in spirit to that of Theorem 1.2. For this section, fix an odd prime p, and recall that $t_i(x)$ refers to the ith digit of x in base p.

Definition 4.1. *We say a (positive) integer x* dominates *an integer y if $t_i(x) \geq t_i(y)$ for all integers i.*

Note that the set S_p^k defined in Section 3 is exactly the set of integers dominated by $\sum_{i=0}^{k-1}(p-2)p^i$.

Theorem 4.1. *Let $T \subseteq S_p^k$ be a nonempty set that is downward-closed under the domination ordering. Namely if $x \in T$ and y is dominated by x, then $y \in T$. Then $S_p(T \cup \{p^k\})$ and $S_p(T \cup \{(p-1)p^k\})$ are modular p-Stanley sequences.*

Note that for $p = 3$ this is Theorem 1.2 in Rolnick [10].

Proof. In both cases, we give an explicit description of the Stanley p-sequences and prove that this is the correct sequence.

We claim that $x \in S_p(T \cup \{p^k\})$ if and only if the following three conditions hold

- $t_i(x) \neq p-1$ for $i \neq k$,
- $t_k(x) = 0$ implies that $\sum_{i=0}^{k-1} t_i(x)p^i \in T$,
- $t_k(x) = p-1$ implies that $\sum_{i=0}^{k-1} t_i(x)p^i \notin T$.

For convenience let L be the set of integers satisfying the above relations. Note that $L \cap \{0, 1, \dots, p^k\} = T \cup \{p^k\}$. It suffices by Proposition 2.2 to demonstrate that L does not contain any p-term arithmetic progressions and that every integer not in L and greater than p^k is covered by a p-term arithmetic progression in L.

To show that L is p-free we proceed by contradiction. Suppose that $x_1 < \cdots < x_p$ form an arithmetic progression. Let i be the smallest integer such that $t_i(x_1), \dots, t_i(x_p)$ are not all equal. Since p is prime and the first i digits of $x_1, \dots, x_p$ are the same, this implies that $\{t_i(x_1), \dots, t_i(x_p)\} = \{0, \dots, p-1\}$. Since $t_i(x) \neq p-1$ for $i \neq k$, we conclude that $i = k$.

Now there are some j, j' such that $t_k(x_j) = 0$ and $t_k(x_{j'}) = p-1$. By the definition of L, this implies that $\sum_{i=0}^{k-1} t_i(x_j)p^i \in T$ and $\sum_{i=0}^{k-1} t_i(x_{j'})p^i \notin T$. However, since $t_i(x_j) = t_i(x_{j'})$ for $i < k$, this is a contradiction.

It remains to show that every integer $x > p^k$ is covered by a p-term arithmetic progression. In order to do so we explicitly construct a p-term arithmetic progression $x_1 \leq x_2 \leq \cdots \leq x_{p-1} \leq x$ with the x_i in L. If we have equality anywhere in this chain then x in L; otherwise $x_1 < x_2 < \cdots < x_{p-1} < x$ as desired. For $0 \leq i \leq k-1$ if $t_i(x) = \ell < p-1$, then set $t_i(x_j) = \ell$ for $1 \leq j \leq p-1$. If instead $t_i(x) = p-1$, set $t_i(x_j) = j-1$ for $1 \leq j \leq p-1$. Note that this is exactly the canonical covering from earlier. Now we subdivide into several possible cases.

Case 1: $t_k(x) \neq 0,\ p-1$

Set $t_k(x_i) = \ell$. For the remaining digits, use the canonical covering as before.

Case 2: $t_k(x) = p-1$

We have two cases. If the last k digits of x_1 are in T, then set $t_k(x_j) = j-1$. Otherwise set $t_k(x_j) = p-1$. In either case, use the canonical covering for the remaining digits.

Case 3: $t_k(x) = 0$

If the last k digits of x_{p-1} are in T, set $t_k(x_j) = 0$ and use the canonical covering for the remaining digits. Otherwise, set $t_k(x_j) = j$ and perform the canonical covering for $x - p^{k+1}$ for the remaining higher digits. (Note that since $x > p^k$ and $t_k(x) = 0$ it follows that $x \geq p^{k+1}$.)

It is routine to verify in each case that the x_j constructed are in L, completing the proof that $S_p(T \cup \{p^k\}) = L$. To show that this is a modular Stanley sequence, let $L^* = \{x \mid x \in L, x < p^{k+1}\}$. We claim that L^* is a modular set. The proof of this fact is nearly identical to the above analysis. Consider just the digits $t_i(x)$ for $0 \leq i \leq k$.

Next we prove that $S(T \cup \{(p-1)p^k\})$ is a modular p-Stanley sequence. This proof is similar to the above argument though slightly more involved. We claim that $x \in S(T \cup \{(p-1)p^k\})$ if and only if the following four conditions hold

- $t_i(x) \neq p-1$ for $i \neq k, k+1$,
- $t_k(x) \neq p-2$,
- $t_{k+1}(x) = 0$ implies that $t_k(x) = 0$ and $\sum_{i=0}^{k-1} t_i(x)p^i \in T$ or $t_k(x) = p-1$,
- $t_{k+1}(x) = p-1$ implies that $t_k(x) \neq p-2, p-1$, and if $t_k(x) = 0$, then $\sum_{i=0}^{k-1} t_i(x)p^i \notin T$.

Again let L be the set defined by these four conditions. We show that L is p-free and p-covers the part of its complement greater than $(p-1)p^k$.

For the sake of contradiction, suppose that $x_1 < x_2 < \cdots < x_p$ form an arithmetic progression with $x_i \in L$. Using the same idea as above we see that $t_i(x_1) = \cdots = t_i(x_p)$ for $0 \leq i \leq k-1$. Since $t_k(x) \neq p-2$, it follows that $t_k(x_1) = \cdots = t_k(x_p)$. Now if $\{t_{k+1}(x_1), \ldots, t_{k+1}(x_p)\} = \{0, \ldots, p-1\}$, then there exist j, j' such that $t_{k+1}(x_j) = 0$ and $t_{k+1}(x_{j'}) = p-1$. Then we see that $\sum_{i=0}^{p-1} t_i(x_j)p^i \in T$ and $\sum_{i=0}^{p-1} t_i(x_{j'})p^i \notin T$. Thus we conclude that $t_{k+1}(x_1) = \cdots = t_{k+1}(x_p)$, and by the same reasoning we see that $x_1 = \ldots = x_p$, a contradiction.

It remains to show that every integer $x > (p-1)p^k$ is covered by a p-term arithmetic progression. In order to do so, we explicitly construct a p-term arithmetic progression, $x_1 \le x_2 \le \cdots \le x_{p-1} \le x$ with $x_i \in L$. If we have equality anywhere in this chain then $x \in L$. Otherwise, $x_1 < x_2 < \cdots < x_{p-1}$ as desired. For $0 \le i \le k-1$, if $t_i(x) = \ell < p-1$, then set $t_i(x_j) = \ell$ for $1 \le j \le p-1$. Otherwise $t_i(x) = p-1$, and we set $t_i(x_j) = j-1$ for $1 \le j \le p-1$. We will define this procedure as earlier to be the canonical covering. Now we subdivide into several possible cases and note that several of these cases degenerate when $p = 3$.

Case 1:$t_{k+1}(x) = 1, \ldots, p-2$ and $t_k(x) \neq p-2$

Set $t_{k+1}(x) = t_{k+1}(x_j)$ and $t_k(x) = t_k(x_j)$ for $1 \le j \le p-1$. For the remaining digits, use the canonical covering.

Case 2: Either $t_{k+1}(x) = p-1$ and $t_k(x) = 1, \ldots, p-3$ or $t_{k+1}(x) = 0$ and $t_k(x) = p-1$

Set $t_{k+1}(x) = t_{k+1}(x_j)$ and $t_k(x) = t_k(x_j)$ for $1 \le j \le p-1$. For the remaining digits, use the canonical covering as before.

Case 3: $t_{k+1}(x) = p-1$ and $t_k(x) = p-1$

Set $t_{k+1}(x_j) = j-1$ and $t_k(x_j) = j-1$ for $1 \le j \le p-1$. For the remaining digits, use the canonical covering as before.

Case 4: $t_{k+1}(x) = 1, \ldots, p-1$ and $t_k(x) = p-2$

Set $t_{k+1}(x_j) = t_{k+1}(x)$ and $t_{k+1}(x_j) = j-2$ for $2 \le j \le p-1$ while $t_{k+1}(x_1) = t_{k+1}(x) - 1$ and $t_k(x_1) = p-1$. For the remaining digits, use the canonical covering as before.

Case 5: $t_{k+1}(x) = 0$ and $t_k(x) = 1, \ldots, p-3$

Set $t_{k+1}(x_j) = j$ and $t_k(x_j) = t_k(x)$ for $1 \le j \le p-1$. For the remaining digits, use the canonical covering $x - p^{k+2}$.

Case 6: $t_{k+1}(x) = 0$ and $t_k(x) = p-2$

Set $t_{k+1}(x_j) = j$ and $t_k(x_j) = j-2$ for $2 \le j \le p-1$. Also put $t_{k+1}(x_1) = 0$ and $t_k(x_j) = p-1$. For the remaining digits, use the canonical covering $x - p^{k+2}$.

Case 7: $t_{k+1}(x) = t_k(x) = 0$

Consider x_{p-1} before setting $t_{k+1}(x_{p-1})$ and $t_k(x_{p-1})$. If $x_{p-1} \in L$, then set $t_{k+1}(x_j) = t_k(x_j) = 0$ for $1 \le j \le p-1$ and for the remaining digits, use the canonical covering x. Otherwise, set $t_{k+1}(x_j) = j$ and $t_k(x_j) = 0$ for

$1 \leq j \leq p-1$ and use the canonical covering $x - p^{k+2}$ for the remaining digits.

Case 8: $t_{k+1}(x) = p-1$ and $t_k(x) = 0$

Consider x_{p-1} before setting $t_{k+1}(x_{p-1})$ and $t_k(x_{p-1})$. If $x_{p-1} \in L$, then set $t_{k+1}(x_j) = j-1$ and $t_k(x_j) = 0$ for $1 \leq j \leq p-1$ and for the remaining digits, use the canonical covering x. Otherwise, set $t_{k+1}(x_j) = p-1$ and $t_k(x_j) = 0$ for $1 \leq j \leq p-1$ and for the remaining digits, use the canonical covering x.

In each case it is routine to verify that the x_j constructed are in L and form an arithmetic progression with x being the largest term. Finally, to show that this sequence is modular, let $L^* = \{x \mid x \in L, x < p^{k+2}\}$. We claim that L^* is a modular set. Demonstrating that L^* is modular is nearly identical to above analysis considering $t_i(x)$ for $0 \leq i \leq k+1$ and is omitted. □

5. Conclusions

The two constructions in this paper are among the first classes of large modular p-Stanley sequences. These constructions raise several natural questions. The first follows naturally from the computational evidence in Section 3 and conjecturally answers a question of Moy and Rolnick [7] regarding which sets $\{0, n\}$ generate modular p-Stanley sequences.

Conjecture 5.1. *The sequence $S_p(0, n)$ is a modular p-Stanley sequence if and only if $n \in A_p$.*

The next question deals with p-Stanley sequences generated in manners similar to that the second construction.

Question 5.1. *Consider a set $S \subseteq \{1, \ldots, p^k - 1\}$ and $1 \leq i \leq p-2$. Under what conditions is $S_p(S \cup \{0, p^k, \ldots, i \cdot p^k\})$ a modular p-Stanley sequence?*

Finally, we end on another construction of p-Stanley sequences that appears to hold for small integers x but for which an explicit characterization appears difficult. This is the natural analog of Lemma 3.5 in Rolnick [10] and appears to suggest a further connection between the domination order and p-Stanley sequences.

Conjecture 5.2. *Consider an integer x with no $p-1$ in its base p expansion. If T is the set of all integers dominated by x, then $S_p(T)$ is a modular p-Stanley sequence.*

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