Journal of Combinatorics
Volume 11, Number 2, 231–248, 2020

Zero-sum analogues of van der Waerden's theorem on arithmetic progressions

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Let r and k be positive integers with $r \mid k$. Denote by $z(k;r)$ the minimum integer such that every coloring $\chi : [1, z(k;r)] \to \{0, 1, \ldots, r-1\}$ admits a k-term arithmetic progression $a, a+d, \ldots, a+(k-1)d$ with $\sum_{j=0}^{k-1} \chi(a+jd) \equiv 0 \pmod{r}$. We investigate these numbers as well as a "mixed" monochromatic/zero-sum analogue. We also present an interesting reciprocity between the van der Waerden numbers and $z(k;r)$.

1. Introduction

Van der Waerden's theorem [16] on arithmetic progressions states that for $k, r \in \mathbb{Z}^+$, there exists a minimum integer $w(k;r)$ such that every r-coloring of $[1, w(k;r)]$ admits a monochromatic k-term arithmetic progression. The determination of these numbers is notoriously difficult; in fact, only seven such nontrivial numbers are known.

In this article we investigate some zero-sum analogues of van der Waerden's theorem.

Definition 1. Let $a_1, a_2, \ldots, a_n$ be a sequence of non-negative integers and let $r \in \mathbb{Z}^+$. We say that the sequence is *r-zero-sum* if $\sum_{i=1}^{n} a_i \equiv 0 \pmod{r}$.

The seminal result in the area of zero-sum sequences is the Erdős-Ginzberg-Ziv theorem [10], which states that any sequence of $2n-1$ integers contains an n-zero-sum subsequence of n integers. Since around 1990, research activity concerning zero-sum results has flourished, through both the lens of additive number theory and Ramsey theory. For example, the weighted Erdős-Ginzberg-Ziv theorem due to Grynkiewicz [12] allows us to multiply the integers in the Erdős-Ginzberg-Ziv theorem by weights. This result states, in particular, that if $w_1, w_2, \ldots, w_n$ is an n-zero-sum sequence and $a_1, a_2, \ldots, a_{2n-1}$ is a sequence of $2n-1$ integers, then there exists an n-term subsequence $a_{i_1}, a_{i_2}, \ldots, a_{i_n}$ and a permutation π of $\{i_1, i_2, \ldots, i_n\}$ such that $\sum_{j=1}^{n} w_j a_{\pi(i_j)} \equiv 0 \pmod{n}$. Further recent results can be found in [1], [3], and [11] among many others.

Most investigations of zero-sum sequences do not have a structure imposed on them. This is in contrast to zero-sum results on edgewise colored graphs, which have been around for many years (see, e.g., [2], [4], [7], and [9]). Some notable exceptions are found in works of Bialostocki, such as [6] and [8] where the zero-sum sequence $x_1, x_2, \ldots, x_n$ satisfies $\sum_{i=1}^{n-1} x_i < x_n$ and in [5] where $x_{i+1} - x_i \leq x_i - x_{i-1}$ for $1 \leq i \leq n-1$. These exceptions, however, do not have a rigid structure imposed on them due to the use of inequalities. In this article we investigate arithmetic progressions, thereby imposing a rigid structure on the sequences (as we note later, in [2] there is one result in this direction). In article [15] we investigate zero-sum sequences with a different rigid structure (where one term is the sum of all other terms).

The first two zero-sum analogues of van der Waerden's theorem we will investigate are given in the following definitions.

Definition 2. Let k and r be positive integers such that $r \mid k$. We denote by $z(k;r)$ the minimum integer such that every coloring $\chi : [1, z(k;r)] \to \{0, 1, \ldots, r-1\}$ admits a k-term arithmetic progression $a, a+d, \ldots, a+(k-1)d$ such that $\sum_{j=0}^{k-1} \chi(a+jd) \equiv 0 \pmod{r}$; in other word, every coloring of $[1, z(k;r)]$ with the colors $0, 1, \ldots, r-1$ (which we may refer to by $\mathbb{Z}_r$) admits a k-term r-zero-sum arithmetic progression.

Definition 3. Let k and r be positive integers such that $r \mid k$. We denote by $\bar{z}_r(k)$ the minimum integer such that every coloring $[1, \bar{z}_r(k)]$ with the colors 0 and 1 admits a k-term r-zero-sum arithmetic progression.

Implicit in the above definitions is the existences of the respective minimum numbers, both of which follow directly from the existence of $w(k;r)$. Note that we need only prove the existence of $z(k;r)$ since we easily have $\bar{z}_r(k) \leq z(k;r)$ as $\mathbb{Z}_2 \subseteq \mathbb{Z}_r$. The existence of $z(k;r)$ comes from $z(k;r) \leq w(k;r)$ as any k-term monochromatic arithmetic progression is r-zero-sum when $r \mid k$. When $r \nmid k$, coloring every integer of $\mathbb{Z}^+$ with the color 1 does not admit a k-term r-zero-sum arithmetic progression. In an interesting turn of events, we will see later in this article that the independent existence of $z(k;r)$ implies the existence of $w(k;r)$.

2. Some computation

We start with results from computer calculations. We wrote the Fortran programs `ZSAP.f` and `ZSAP2.f`, available at the author's website, for the determinations of $z(k;r)$ and $\bar{z}_r(k)$, respectively, for small values of k and r. The algorithm used in both is a standard backtrack model to exhaustively search the colorings for k-term r-zero-sum arithmetic progressions.

Based on the values in Tables 1 and 2, below, we find some patterns. For $r = 2$, we clearly have $z(k;2) = \bar{z}_2(k)$ by definition, but it appears that $z(k;2) = 2k-1$. We prove this in the next section. (It is interesting to note that $2k-1$ is the formula from the Erdős-Ginzberg-Ziv zero-sum theorem.) For $k = 3, 6, 9, 12$, we have $z(k;3) = \bar{z}_3(k) = k^2$ and we also investigate this in the next section. Along the diagonal, we see some familiar van der Waerden numbers appear which is addressed in Section 3 as well.

Table 1: Values and lower bounds for $z(k;r)$ for small k and r

$k \setminus r$	2	3	4	5
2	3	∞	∞	∞
3	∞	9	∞	∞
4	7	∞	35	∞
5	∞	∞	∞	≥ 294
6	11	36	∞	∞
7	∞	∞	∞	∞
8	15	∞	≥ 108	∞
9	∞	81	∞	∞
10	19	∞	∞	?
11	∞	∞	∞	∞
12	23	144	≥ 163	∞

Table 2: Values and lower bounds for $\bar{z}_r(k)$ for small k and r

$k \setminus r$	2	3	4	5
2	3	∞	∞	∞
3	∞	9	∞	∞
4	7	∞	35	∞
5	∞	∞	∞	178
6	11	36	∞	∞
7	∞	∞	∞	∞
8	15	∞	80	∞
9	∞	81	∞	∞
10	19	∞	∞	≥ 194
11	∞	∞	∞	∞
12	23	144	244	∞

Remark. The lower bounds for $z(8;4)$ and $z(5;5)$ were obtained within a few hours but were not improved upon after 770 hours of computation. The lower bound for $z(12;4)$ was obtained after a couple of hours of computation, with no extended time used to try to improve upon it.

Remark. The determination of $\bar{z}_4(12)$ took about 8 days, while the lower bound for $\bar{z}_5(10)$ was established after only a few minutes of searching, but further impovement was not achieved after several hours. All other values took less than a few hours.

3. Formulas

As previously mentioned, some interesting patterns can be seen in Tables 1 and 2. In this section we explore these patterns. We start with a formula for the $r = 2$ columns and note that it also proves the existence of $z(k; 2)$ for all even k, independent of the existence of the van der Waerden number $w(k; 2)$. This was first discovered by Alon and Caro as Proposition 4.5 in [2]. The proof here was independently discovered (and is different).

Proposition 4 (Alon and Caro [2])**.** *Let $k \in \mathbb{Z}^+$ be even. Then $z(k; 2) = \bar{z}_2(k) = 2k - 1$.*

Proof. The first equality is by definition. To show that $z(k; 2) = 2k - 1$ we will provide matching upper and lower bounds. First, it is easy to check that the 2-coloring of $[1, 2k-2]$ with all integers colored 0 except for integer k avoids k-term 2-zero-sum arithmetic progressions as any such arithmetic progression must consist of k consecutive integers and, hence, exactly one integer of color 1. Hence, $z(k; 2) \geq 2k - 1$.

We next show that $z(k; 2) \leq 2k-1$ by contradiction, assuming that there exists a coloring χ of $[1, 2k-1]$ by $\mathbb{Z}_2$ with no k-term 2-zero-sum arithmetic progression. Let $A = \{1, 3, 5, \ldots, k-1\}$ and $B = \{k+1, k+3, \ldots, 2k-1\}$. Since $A \cup B$ is a k-term arithmetic progression, we assume that the sum of the colors of the integers in $A \cup B$ is odd. Hence, one of A and B has an even number of integers of color 1, while the other has an odd number of integers of color 1. Without loss of generality, let A have an even number of integers of color 1.

Consider $S(x) = \sum_{i=x}^{k+x-1} \chi(i)$ for $x \in [1, k]$. Next note that $S(x+1) - S(x) = \chi(k+x) - \chi(x)$ for $x \in [1, k-1]$. Since we assume that $S(x) \equiv 1 \pmod 2$ for all $x \in [1, k]$ we must have $\chi(k+x) = \chi(x)$ for $x \in [1, k-1]$. This means that $[k+1, 2k-1]$ is colored in the exact same way as $[1, k-1]$. This contradicts our determination that A has an even number of integers of color 1, while B has an odd number of integers of color 1. $\square$

For the $r = 3$ column (in both Table 1 and Table 2), we can justify lower bounds that match all of the calculated numbers via Theorems 5, which we present next.

Theorem 5. *Let $k \in \mathbb{Z}^+$ with $3 \mid k$. If $k+1 \in \{p, 2p\}$, with p prime, then $\bar{z}_3(k) \geq k^2$.*

Proof. First, consider $k+1 = p$. We will show that the 2-coloring χ of $[1, k^2-1]$ defined by

$$(0^{k-1}11)^{k-1}$$

(i.e., the color pattern of $k-1$ consecutive 0s followed by two 1s, repeated $k-1$ times) avoids k-term 3-zero-sum arithmetic progressions.

Consider an arbitrary k-term arithmetic progression $a, a+d, a+2d, \dots, a+(k-1)d$. Note that $a+(k-1)d \leq k^2-1$ gives us that $d \leq k$. Since $k+1$ is prime, we have $(d, k+1) = 1$. It follows that $\{a, a+d, \dots, a+(k-1)d\}$ when reduced modulo $k+1$ is a set of k (distinct) elements of $\mathbb{Z}_{k+1}$.

Looking at our coloring, we interpret it as $\chi(x) = 1$ if $x \equiv 0$ or $k \pmod{k+1}$ and $\chi(x) = 0$ otherwise. Since our arithmetic progression hits k distinct residues modulo $k+1$ we see that $\sum_{j=0}^{k-1} \chi(a+jd) = 1$ or 2 so that it is not 3-zero-sum.

Next, consider the case when $k+1 = 2p$. Before presenting the general case, we note that case when $p = 2$, i.e., $k = 3$, holds since $\bar{z}_3(3) = 9$ (see Table 2). Hence, it is safe to assume that $p \geq 3$ for the rest of the proof.

We will show that the 2-coloring χ of $[1, k^2-1]$ defined by

$$(0^{p-2}101^{p-2}01)^{k-1}$$

avoids k-term 3-zero-sum arithmetic progressions.

Consider an arbitrary k-term arithmetic progression $a, a+d, \dots, a+(k-1)d$. Note that $a+(k-1)d \leq k^2-1$ gives us that $d \leq k$. Hence, $(d, k+1) \in \{1, 2, p\}$. We will determine $\sum_{j=0}^{k-1} \chi(a+jd) \pmod 3$ based on the value of $(d, k+1)$. We will also use the fact that $p \equiv 2 \pmod 3$ which follows from $k = 2p-1 \equiv 0 \pmod 3$.

`Case 1.` $(d, k+1) = 1$. It follows that $\{a, a+d, \dots, a+(k-1)d\}$ when reduced modulo $k+1$ is a set of k (distinct) residues of $\mathbb{Z}_{k+1}$. Hence, $\sum_{j=0}^{k-1} \chi(a+jd) = p-1$ or p. Since $p \equiv 2 \pmod 3$ we have $\sum_{j=0}^{k-1} \chi(a+jd) \equiv 1$ or $2 \pmod 3$ so that our arithmetic progression is not 3-zero-sum. ◇

`Case 2.` $(d, k+1) = 2$. By reducing all terms of $\{a, a+d, \dots, a+(k-1)d\}$ modulo $k+1$ we see that we have either $\{0, 2, 4, \dots, k-1\}$ or $\{1, 3, 5, \dots, k\}$. Looking at $0^{p-2}101^{p-2}01$ we see that the coloring of the even terms is $0^{\frac{p-3}{2}}1^{\frac{p+3}{2}}$, while the coloring of the odd terms is $0^{\frac{p+1}{2}}1^{\frac{p-3}{2}}0$.

Next, when reducing all terms of $\{a, a+d, \dots, a+(k-1)d\}$ modulo $k+1$ we see that every residue except for one of $\{0, 2, 4, \dots, k-1\}$ is congruent modulo $k+1$ to precisely two terms of the arithmetic progression. The same holds for the residue set $\{1,3,5,\ldots,k\}$.

In the situation where $\{a, a+d, \ldots, a+(k-1)d\}$ modulo $k+1$ is $\{0,2,4,\ldots,k-1\}$ we have $\sum_{j=0}^{k-1} \chi(a+jd) = 2(\frac{p+3}{2}) + \epsilon$ where $\epsilon \in \{-1,0\}$. We have $2(\frac{p+3}{2}) + \epsilon \equiv p + \epsilon \pmod 3$. Since $p \equiv 2 \pmod 3$, we see that our arithmetic progression is not 3-zero-sum in this situation.

In the situation where $\{a, a+d, \ldots, a+(k-1)d\}$ modulo $k+1$ is $\{1,3,5,\ldots,k\}$ we have $\sum_{j=0}^{k-1} \chi(a+jd) = 2(\frac{p-3}{2}) + \epsilon$ where $\epsilon \in \{-1,0\}$. We have $2(\frac{p-3}{2}) + \epsilon \equiv p + \epsilon \pmod 3$. Since $p \equiv 2 \pmod 3$, we see that our arithmetic progression is not 3-zero-sum in this situation. $\diamond$

`Case 3.` $(d, k+1) = p$. We must have $d = p$ in this situation since $d \leq k$. Looking at $0^{p-2}101^{p-2}01$ as a coloring of $[1, k+1] = [1, 2p]$ we see that $\chi(p+i) = \chi(i)+1 \pmod 2$ for $i = 1, 2, \ldots, p$. In particular, $\chi(x)+\chi(x+p) = 1$ for any $x \in [1,p]$. Given that our coloring $0^{p-2}101^{p-2}01$ is repeated $k-1$ times, we have $\chi(x) + \chi(x+p) = 1$ for any x where, for $\bar{x} \equiv x \pmod{k+1}$, we have $\bar{x} \in [1,p]$.

If $a \pmod{k+1}$ is between 1 and p, inclusive, then we have

$$\sum_{j=0}^{k-1} \chi(a+jd) = \sum_{j=0}^{k-1} \chi(a+jp) = \chi(a+(k-1)p) + \sum_{\substack{0 \leq j \leq k-3 \\ j \text{ even}}} \big(\chi(a+jp) + \chi(a+(j+1)p) \big)$$

$$= \chi(a+(k-1)p) + 1 \cdot \frac{k-1}{2}.$$

$$= \chi(a+(k-1)p) + p - 1.$$

Now, since $p \equiv 2 \pmod 3$, we see that regardless of the value of $\chi(a+(k-1)p)$ we have $\sum_{j=0}^{k-1} \chi(a+jd) \not\equiv 0 \pmod 3$ so that our arithmetic progression is not 3-zero-sum.

If $a \pmod{k+1}$ is between $p+1$ and $k+1$, inclusive, then we have

$$\sum_{j=0}^{k-1} \chi(a+jd) = \sum_{j=0}^{k-1} \chi(a+jp) = \chi(a) + \sum_{\substack{1 \leq j \leq k-2 \\ j \text{ odd}}} \big(\chi(a+jp) + \chi(a+(j+1)p) \big)$$

$$= \chi(a) + 1 \cdot \frac{k-1}{2}.$$

$$= \chi(a) + p - 1.$$

Again, since $p \equiv 2 \pmod 3$, regardless of the value of $\chi(a)$ our arithmetic progression is not 3-zero-sum. ◇

As we have exhausted all possible values of $(d, k+1)$ and shown that no k-term 3-zero-sum arithmetic progression exists under our coloring in each situation, we are done. □

Remark. Theorem 5 also gives lower bounds for $z(k;3)$ with k being a prime or twice a prime.

The next proposition explains the appearance of the van der Waerden numbers along the main diagonal of Table 2. These same numbers are lower bounds for the main diagonal of Table 1, where we see divergence occurring at $z(5;5)$.

Proposition 6. *Let $k \in \mathbb{Z}^+$. Then $w(k;2) = \bar{z}_k(k) \leq z(k;k)$,*

Proof. The equality $w(k;2) = \bar{z}_k(k)$ follows from the fact that the only way a k-term sequence of 0s and 1s can be k-zero-sum is if all terms are 0s or all terms are 1s, i.e., monochromatic. Since $\mathbb{Z}_2 \subseteq \mathbb{Z}_r$, we have $\bar{z}_r(k) \leq z(k;r)$ so we are done. □

4. A "mixed" monochromatic/zero-sum analogue

In this section we investigate an interplay between monochromatic and zero-sum arithmetic progressions. We start with the question of whether or not by avoiding certain monochromatic arithmetic progressions we can guarantee certain zero-sum arithmetic progressions. To this end, consider the following definition.

Definition 7. Let $k, \ell, r \in \mathbb{Z}^+$ with $k, \ell, r \geq 2$. Define $m(k, \ell; r)$ to be the minimum integer n such that any coloring of $[1, n]$ by $\mathbb{Z}_r$ admits either a k-term monochromatic arithmetic progression of a color other than 0 or an ℓ-term r-zero-sum arithmetic progression.

Inherent in this definition is the existence of $m(k, \ell; r)$ for all positive integers k, ℓ, and r, so we must justify this existence. The existence follows easily from van der Waerden's theorem. For the situation when $k \geq \ell$, we know that any coloring of $[1, w(k;r)]$ admits a monochromatic k-term arithmetic progression. If the color is anything but color 0, then we are done, so we assume that it is of color 0. But then we have an ℓ-term arithmetic progression of color 0, which is necessarily r-zero-sum for any r. For the situation when $k < \ell$, any coloring of $[1, w(\ell;r)]$ admits a monochromatic

ℓ-term arithmetic progression. If this color is 0, then it is r-zero-sum. Otherwise, it contains a k-term monochromatic arithmetic progression of color other than 0.

Having the existence of $m(k,\ell;r)$, we see that these "mixed" monochromatic/zero-sum numbers, in particular $m(k,k;r)$, address the non-existence of $z(k;r)$ when $r \nmid k$ (recall that the counterexample was coloring all integers with color 1).

Using the Fortran programs `MZSAP.f` and `MZSAP2.f`, available at the author's website, we have calculated the values in Tables 3 and 4.

Table 3: Values and lower bounds for $m(k,\ell;r)$ for small k,ℓ, and r

k	r	$\ell=2$	$\ell=3$	$\ell=4$	$\ell=5$
2	2	3	6	7	10
	3	3	7	7	15
	4	4	8	12	20
3	2	3	7	7	15
	3	7	9	16	25
	4	7	21	28	47
4	2	3	8	7	20
	3	7	9	18	33
	4	11	53	35	≥ 97
5	2	3	8	7	21
	3	10	9	21	33
	4	15	219	35	≥ 103

Remark. All exact values were achieved within a few hours of computation time. The lower bound for $m(4,5;4)$ was attained quite quickly but not improved upon after 370 hours of computation. The lower bound for $m(5,5;4)$ was reached after about 15 minutes, but was not improved after several hours of computation.

Examining Table 3, patterns do not pop out as they did in Tables 1 and 2. There seems to be different behavior for a given k depending on, perhaps, the value of the $\gcd(\ell,r)$ (see, e.g., the rows for $k=4,5$). We do see that for $k=2,3$, and 4 we have $m(k,k;k)=w(k;2)$, and for $\ell=3,4$, and 5 we have $m(3,\ell)=\ell^2$. However, further calculation shows that $m(3,6)=33\neq 6^2$. We can, however, provide formulas for the first three rows.

Theorem 8. *Let $\ell \geq 2$ be an integer. Then*

$$m(2,\ell;2)=\begin{cases} 2\ell-1 & \text{if } 2 \mid \ell \\ 2\ell & \text{if } 2 \nmid \ell \end{cases};$$

$$m(2,\ell;3) = \begin{cases} 2\ell-1 & \text{if } \ell \equiv 0,2,4 \pmod 6 \\ 3\ell-2 & \text{if } \ell \equiv 3 \pmod 6 \\ 3\ell & \text{if } \ell \equiv 1,5 \pmod 6 \end{cases};$$

and

$$m(2,\ell;4) = \begin{cases} 3\ell & \text{if } \ell \equiv 0,2,3,4 \pmod 6 \\ 4\ell & \text{if } \ell \equiv 1,5 \pmod 6 \end{cases}.$$

Proof. $\boldsymbol{m(2,\ell;2)}$. We first consider $m(2,\ell;2)$. If ℓ is even, consider the coloring $0^{\ell-1}10^{\ell-2}$; if ℓ is odd, consider the coloring $0^{\ell-1}10^{\ell-1}$. It is routine to check that these do not admit 2-term arithmetic progressions of color 1 or ℓ-term 2-zero-sum arithmetic progressions. Now, with ℓ even, let $n = 2\ell - 1$ and assume, for a contradiction, that there exists a 2-coloring of $[1,n]$ that avoids the requisite arithmetic progressions. Clearly, we may have at most one integer of color 1. Furthermore, we can have at most $\ell - 1$ consecutive integers of color 0. Hence, the only way to have a valid 2-coloring of $[1,n]$ is with $0^{\ell-1}10^{\ell-1}$. But then $1,3,5,\ldots,2\ell-1$ is an ℓ-term 2-zero-sum arithmetic progression, a contradiction. The case when ℓ is odd is easier since, by the same reasoning, we can only have $2\ell-1$ integers colored (by $0^{\ell-1}10^{\ell-1}$) and avoid the relevant arithmetic progression. Hence, every 2-coloring of $[1,2\ell]$ admits either a 2-term arithmetic progressions of color 1 or an ℓ-term 2-zero-sum arithmetic progressions.

$\boldsymbol{m(2,\ell;3)}$. Next, consider $m(2,\ell;3)$ and let $\ell \equiv 0,2,4 \pmod 6$ so that ℓ is even. Again, it is easy to see that $0^{\ell-1}10^{\ell-2}$ does not admit 2-term arithmetic progressions of color 1 or 2, or ℓ-term 2-zero-sum arithmetic progressions. Assume, for a contradiction, that $\chi : [1,2\ell-1] \to \{0,1,2\}$ does not admit the relevant arithmetic progressions. Then at most one integer has color 1 and at most one integer has color 2. Hence, if we use both colors 1 and 2 then our coloring has form $0^s10^t20^u$ or its reverse. The following argument works for either form, so we will assume we have $0^s10^t20^u$. We know that $s,t,u \le \ell-1$, subject to $s+t+u = 2\ell-3$. However, if $t \le \ell-2$ then $0^x10^t20^y$ with $x+y+t = \ell-2$ is an ℓ-term 3-zero-sum arithmetic progression. Since $s+t+u = 2\ell-3$ while 10^t2 is at most ℓ terms, we see that $s+u \ge \ell-1$, which gives us that for some $x,y \ge 0$ we indeed have the existence of $0^x10^t20^y$ with $x+y+t = \ell-2$ contained in χ, a contradiction. If χ does not use both colors 1 and 2, then our coloring has form 0^s10^t or 0^s20^t. We will assume the former (the argument is the same for the latter). We must have $s+t = 2\ell-2$ and, hence, $s = t = \ell-1$. But then $1,3,5,\ldots,2\ell-1$ is an ℓ-term 3-zero-sum arithmetic progression, a contradiction. Hence, $m(2,\ell;3) = 2\ell-1$ for ℓ even.

We next look at $m(2,\ell;3)$ for $\ell \equiv 3 \pmod 6$. For a lower bound, consider the coloring of $[1,3\ell-3]$ given by $0^{\ell-1}10^{\ell-1}20^{\ell-3}$. Clearly we have no 2-term arithmetic progression of color 1 or 2 and we do not have ℓ consecutive integers that are 3-zero-sum. Hence, the only possible ℓ-term 3-zero-sum arithmetic progression must have common gap 2. Since $\ell \equiv 3 \pmod 6$ we see that ℓ is odd. This means that any arithmetic progression with common gap 2 cannot contain both the color 1 and color 2. Let $a, a+2, \ldots, a+2(\ell-1)$ be any arbitrary arithmetic progresion with common gap 2. In order to have $a + 2(\ell - 1) \leq 3(\ell - 1)$ we must have $a \leq \ell - 1$. This means that one of the terms must have either color 1 or color 2, but that both colors 1 and 2 cannot occur in the progression. Hence, $\sum_{i=0}^{\ell-1} \chi(a + 2i) = 1$ or 2 and is not 3-zero-sum. We conclude that $m(2, \ell; 3) \geq 3\ell - 2$.

To show that $m(2, \ell; 3) \leq 3\ell - 2$ for $\ell \equiv 3 \pmod 6$, assume that $\chi : [1, 3\ell - 2] \to \{0, 1, 2\}$ is a coloring with no 2-term arithmetic progression of color 1 or 2 and no ℓ-term 3-zero-sum arithmetic progression. We easily see that χ must use all 3 colors for otherwise we cannot have more than $2\ell - 1$ integers of colors only 0 and 1 (or 0 and 2) since we are allowed only one integer of a non-zero color and we cannot have ℓ consecutive integers of color 0. Thus, we see that χ has form $0^s10^t20^u$ (or its reverse) with $s+t+u = 3\ell-4$ and $s, t, u \leq \ell - 1$. As argued previously, we must have t be even, and hence $t = \ell - 1$ so that we have $0^s10^{\ell-1}20^t$ with $s + t = 2\ell - 3$. Hence, one of s and t must be $\ell - 1$ and the other must be $\ell - 2$. We assume $s = \ell - 1$ (the case $s = \ell - 2$ is very similar). Now that we have $0^{\ell-1}10^{\ell-1}20^{\ell-2}$, consider $1, 4, 7, \ldots, 3\ell-2$ (note that we are using the fact that $\ell \equiv 3 \pmod 6$ to end our arithmetic progression at $3\ell - 2$). Notice that this arithmetic progression consists of integers congruent to $1 \pmod 3$ while the colors 1 and 2 are on integers congruent to $0 \pmod \ell$. Since $\ell \equiv 0 \pmod 3$, the arithmetic progression $1, 4, 7, \ldots, 3\ell - 2$ is monochromatic of color 0, and hence is an ℓ-term 3-zero-sum arithmetic progression, a contradiction. Thus, we can conclude that $m(2, \ell; 3) = 3\ell - 2$ for $\ell \equiv 3 \pmod 6$.

Lastly, for the $r = 3$ case, we consider $m(2, \ell; 3)$ for $\ell \equiv 1, 5 \pmod 6$. For the lower bound, consider the coloring of $[1, 3\ell-1]$ given by $0^{\ell-1}10^{\ell-1}20^{\ell-1}$. As argued above, the only necessary arithmetic progressions to checkare ℓ-term ones with common gap 3. The possibilities are $1, 4, 7, \ldots, 3\ell - 2$ and $2, 5, 8, \ldots, 3\ell-1$, i.e., the integers congruent to 1 modulo 3 and the integers congruent to 2 modulo 3, respectively. Since we know that $\ell \equiv 1, 5 \pmod 6$ we have that one of ℓ and 2ℓ is congruent to 1 modulo 3, while the other is congruent to 2 modulo 3. Regardless of which is which, we see that neither of these arithmetic progressions are 3-zero-sum, thereby proving that $m(2, \ell; 3) \geq 3\ell$ for $\ell \equiv 1, 5 \pmod 6$.

The upper bound is easy in this case since any 3-coloring of $[1, 3\ell]$ using only one 1 and one 2 must have $3\ell - 2$ integers of color 0, without ℓ consecutive integers of color 0. This is not possible. Hence, we can conclude that $m(2, \ell; 3) = 3\ell$ for $\ell \equiv 1, 5 \pmod 6$.

$\boldsymbol{m(2,\ell;4)}$. We now move onto $m(2,\ell;4)$. We will start with the lower bounds by giving colorings that avoid the relevant arithmetic progressions.

For $\ell \equiv 0,2 \pmod 6$, we may assume $\ell \geq 6$ since we have calculated the value of the function when$\ell = 2$. Consider the coloring of $[1, 3\ell-1]$ given by $0^{\ell-1}10^{\ell-2}20^{\ell-5}30^4$. In this situation, $\{\ell, 2\ell-1, 3\ell-5\}$, which are the non-zero colored integers, forms a complete residue system modulo 3. Hence, any possible ℓ-term 4-zero-sum arithmetic progression $a, a+d, \dots, a+(\ell-1)d$ cannot have $d=3$. Clearly we do not have such a progression with $d=1$; so, $d=2$ is the only possibility to check. Since we require $a+(\ell-1)d \leq 3\ell-1$ we see that $a \leq \ell+1$. If $a \leq \ell-1$, our progression contains exactly one of ℓ and $2\ell-1$, so that it cannot be 4-zero-sum. If $a=\ell$, then in order for our progression to be 4-zero-sum, $3\ell-5$ (which has color 3) must be part of the progression. This means that $a+2j = 3\ell-5$ for some $j \in \{1,2,\dots,\ell-1\}$. Since $a=\ell$ this means $2j = 2\ell-5$, which is not possible. Lastly, if $a=\ell+1$, then our progression has sum congruent to 1 modulo 4 as it contains both $2\ell-1$ and $3\ell-5$.

For $\ell \equiv 3 \pmod 6$, we may assume $\ell \geq 9$ since we have calculated the value of the function when $\ell = 3$. Consider the coloring of $[1, 3\ell-1]$ given by $0^{\ell-1}10^{\ell-3}20^{\ell-6}30^6$. The non-zero colored elements here are $\ell, 2\ell-2$, and $3\ell-7$ and these form a complete residue system modulo 3. Hence, the only possible ℓ-term 4-zero-sum arithmetic progression $a, a+d, \dots, a+(\ell-1)d$ has $d=2$. We must still have $a \leq \ell+1$. If $a \leq \ell$ we use the facts that ℓ is odd while $2\ell-2$ and $3\ell-7$ are even to see that our progression cannot be 4-zero-sum as every such progression contains at least one of these elements, but cannot contain both ℓ and $3\ell-7$. If $a = \ell+1$, then the arithmetic progression contains both $2\ell-2$ and $3\ell-7$ and is not 4-zero-sum.

For $\ell \equiv 4 \pmod 6$, we consider the coloring of $[1, 3\ell-1]$ given by $0^{\ell-1}10^{\ell-2}20^{\ell-1}3$. The non-zero colored elements here are ℓ, $2\ell-1$, and $3\ell-1$, the first two being congruent to 1 modulo 3 and the last congruent to 2 modulo 3. Letting $a, a+d, \dots, a+(\ell-1)d$ be an arbitrary ℓ-term arithmetic progression, consider $d=3$. If our progression consists of integers congruent to 1 modulo 3 then its sum of colors is $1+2 \equiv 3 \pmod 4$; if it consists of integers congruent to 2 modulo 3 then its sum of colors is $3 \pmod 4$; if it consists of integers congruent to 0 modulo 3, then $a+(\ell-1)d \geq 3+3(\ell-1) = 3\ell > 3\ell-1$. This leaves $d=2$ as the only possibility. As above, we have $a \leq \ell+1$. We also have than ℓ is even while $2\ell-1$ and $3\ell-1$ are odd. Hence, for $a \leq \ell$ our progression contains ℓ but not $3\ell-1$ or it contains $2\ell-1$. In all situations, our progression is not 4-zero-sum. If $a = \ell+1$, the progression has color sum $2+3 \equiv 1 \pmod 4$ and, again, the progression is not 4-zero-sum.

For $\ell \equiv 1,5 \pmod 6$, we consider the coloring of $[1,4\ell-1]$ given by $0^{\ell-1}10^{\ell-1}20^{\ell-1}30^{\ell-1}$. The non-zero colored elements $\ell, 2\ell$, and 3ℓ form a complete residue system modulo 3. Let $a, a+d, \dots, a+(\ell-1)d$ be an arbitrary ℓ-term arithmetic progression. If $d=3$ we have $a \leq \ell+2$. If $a \leq \ell$, then our progression contains exactly one of ℓ, 2ℓ, and 3ℓ and is not 4-zero-sum. If $a=\ell+1$, then $a \equiv 2 \pmod 3$ if $\ell \equiv 1 \pmod 6$ and $a \equiv 0 \pmod 3$ if $\ell \equiv 5 \pmod 6$. In the former case, our progression contains 2ℓ; in the latter case, our progression contains 3ℓ. In either case, we see that the arithmetic progression is not 4-zero-sum. If $a=\ell+2$ then we cannot have $\ell \equiv 1 \pmod 6$ since then $a \equiv 0 \pmod 3$, which tells us that 3ℓ is part of the progression so that the progression cannot be 4-zero-sum. Hence, we have $a \equiv 1 \pmod 3$ since we have $\ell \equiv 5 \pmod 6$. But then $2\ell \equiv 1 \pmod 3$ so our progression contains 2ℓ, giving us that our progression cannot be 4-zero-sum. Noting that ℓ and 3ℓ are odd, while 2ℓ is even, consider $d=2$. Note that, in this situation, ℓ and 3ℓ cannot both be members of our progression. We must have $a \leq 2\ell+1$ so that our progression clearly contains at least one of $\ell, 2\ell, 3\ell$ and not both ℓ and 3ℓ, yielding that the arithmetic progression is not 4-zero-sum. Hence, if $d=2$ our progression is not 4-zero-sum. What remains is the case $d=4$. Here we must have $a \leq 3$. If $a=1$, or $a=3$, then our progression consists of integers congruent to 1 modulo 4, respectively, 3 modulo 4. We next note that one of ℓ and 3ℓ is congruent to 1 modulo 4 while the other is congruent to 3 modulo 4. Hence, our progression cannot be 4-zero-sum. If $a=2$, then 2ℓ is a member of the arithmetic progression so that it is not 4-zero-sum.

We now move onto the upper bounds for $m(2,\ell;4)$.

The cases $\ell \equiv 1,5 \pmod 6$ are easy so we will do them first. Assume, for a contradiction, that there exists a coloring of $[1,4\ell]$ by $\mathbb{Z}_4$ that does not admit two terms of the same non-zero color or an ℓ-term 4-zero-sum arithmetic progression. Since any such coloring of $[1,4\ell]$ uses at most one of each non-zero color, it must have at least $4\ell-3$ integers of color 0. This gives us at least ℓ consecutive integers of color 0, a contradiction since these consecutive integers form an ℓ-term 4-zero-sum arithmetic progression.

For the cases $\ell \equiv 0,2,3,4 \pmod 6$, we will consider the coloring forms of $[1,3\ell]$ given by: (i) $0^s10^t20^u30^v$; (ii) $0^s10^t30^u20^v$; and (iii) $0^s20^t10^u30^v$, and leave the reverse colorings' argument details to the reader (which follow by application of the involution of $[1,3\ell]$ given by $i \mapsto 3\ell+1-i$). We assume, for a contradiction, that each coloring avoids the requisite progressions.

For any of the colorings we have: $s,t,u,v \leq \ell-1$ and $s+t+u+v=3\ell-2$. Furthermore, in order to avoid the ℓ-term 4-zero-sum arithmetic progression given by all integers congruent to i modulo 3 for some i, we see that all non-zero colored integers must form a complete residue system modulo 3.

We start with coloring (i): $0^s10^t20^u30^v$. We know that $s+t+u \geq 2\ell-2$ so that $0^s10^t20^u$ contains the coloring of $[1, 2\ell]$. In order for $1, 3, 5, \ldots, 2\ell-1$ and $2, 4, \ldots, 2\ell$ to avoid being 4-zero-sum, the parity of the integers colored 1 and 2 must be different. Similarly, by considering $0^t20^u30^v$ we can deduce that the parity of the integers colored 2 and 3 must be different. Hence, the integers colored 1 and 3 have the same parity. The integers colored 1 and 3 are $s+1$ and $s+t+u+3$. If $t+u < 2\ell-3$ then $s+1, s+3, \ldots, s+t+u+3, \ldots, s+2\ell-1$ is an ℓ-term 4-zero-sum arithmetic progression. We can conclude that $t+u \geq 2\ell-3$. Since $t, u \leq \ell-1$, we may have (a) $t = u = \ell-1$; (b) $t = \ell-1$ and $u = \ell-2$; or (c) $t = \ell-2$ and $u = \ell-1$.

If we have (a), then our non-zero integers are $s+1, s+\ell+1$, and $s+2\ell+1$. Since these must form a complete residue system modulo 3, we cannot have $\ell \equiv 0, 3 \pmod 6$. Since $s+1$ and $s+\ell+1$ must have different parities, we cannot have $\ell \equiv 2, 4 \pmod 6$. We conclude that (a) may not occur.

If (b) holds, then $t+1 = \ell$. Since the non-zero integers must have different values modulo 3, we cannot have $t+1 \equiv 0 \pmod 3$. Hence, $\ell \not\equiv 0, 3 \pmod 6$. Similarly, we cannot have $u+1 \equiv 0 \pmod 3$. Since $u = \ell-2$ we have $u+1 \equiv \ell-1$ so that $\ell \not\equiv 4 \pmod 6$. Lastly, in order for the integers colored 1 and 3 to be different modulo 3, we cannot have $t+u+2 \equiv 0 \pmod 3$. Since $t+u = 2\ell-3$, we cannot have $\ell \equiv 2 \pmod 6$.

If (c) holds, essentially the same argument as that for (b) can be employed.

Next, we consider the coloring (ii): $0^s10^t30^u20^v$. We must have $t = \ell-1$ for otherwise $0^s10^t30^u$ contains ℓ consecutive terms, including the integers colored 1 and 3; that is, an ℓ-term 4-zero-sum arithmetic progression. Further, by considering $1, 3, 5, \ldots, 2\ell-1$ and $2, 4, \ldots, 2\ell$ we see that the integers colored 1 and 3 cannot have the same parity. This means that ℓ must be odd so that $\ell \not\equiv 0, 2, 4 \pmod 6$. However, if $\ell \equiv 3 \pmod 6$ we see that these two integers are equivalent modulo 3, which is also not allowed.

The coloring (iii) can be analyzed by essentially the same argument as that given for (ii).

Having provided matching upper and lower bounds for this last formula, the proof is complete. □

We can also provide a formula for the first column of Table 3. The fact that $m(k, 2; 2) = 3$ for all k is trivial.

Theorem 9. *Let $k \geq 2$ be an integer. Then*

$$m(k,2;3) = \begin{cases} 2k-1 & \text{if } 2 \mid k \\ 2k & \text{if } 2 \nmid k \end{cases}$$

and

$$m(k,2;4) = \begin{cases} 3k-2 & \text{if } k \equiv 0,2,3 \pmod{6} \\ 3k-1 & \text{if } k \equiv 4 \pmod{6} \\ 3k & \text{if } k \equiv 1,5 \pmod{6}. \end{cases}$$

Proof. We first prove the formula for $m(k,2;3)$. For the lower bounds, consider the colorings $1^{k-1}01^{k-2}$ for k odd and $1^{k-1}01^{k-1}$ for k even. For the upper bounds, to avoid 2-term 3-zero-sum arithmetic progressions we may only use the color 0 once and we cannot have both colors 1 and 2. Let c be 1 or 2. We cannot have k consecutive integers of color c, so our coloring has form c^s0c^t with $s,t \leq k-1$. If k is odd, then we are done since our coloring has maximum length $2k-1$. If k is even, we cannot have $s = t = k-1$ for otherwise the k-term arithmetic progressions $1,3,5,\ldots,2k-1$ is monochromatic of color c. This completes the proof of the formula for $m(k,2;3)$.

We now consider $m(k,2;4)$. For the lower bounds, it is left to the reader to check that the following colorings avoid the requisite arithmetic progressions: $1^{k-2}01^{k-1}21^{k-2}$ for $k \equiv 0,2,3 \pmod{6}$; $1^{k-1}01^{k-1}21^{k-2}$ for $k \equiv 4 \pmod{6}$; and $1^{k-1}01^{k-1}21^{k-1}$ for $k \equiv 1,5 \pmod{6}$

To finish the proof, we now justify upper bounds for $m(k,2;4)$. We assume, for a contradiction, that in each case we have a coloring that avoids the relevant arithmetic progressions. We can have at most one integer of each of color 0 and 2. Further, we cannot have both colors 1 and 3. Let c be either 1 or 3. Hence, we can conclude that any coloring that avoids k-term monochromatic arithmetic progressions of a non-zero color and 2-term 3-zero-sum arithmetic progressions has form $c^s0c^t2c^u$ or its reverse (the situation where we use only one of the colors 0 and 2 is easily dismissed as a possibility since we cannot have length longer than $2k$). We will only consider this coloring and leave the reverse coloring's analysis to the reader.

If $k \equiv 1,5 \pmod{6}$, the argument is essentially identical to the one for $m(2,\ell;3)$ in the proof of Theorem 8 by replacing ℓ with k, changing the color 0 to c, and changing the color 1 to 0.

If $k \equiv 4 \pmod{6}$, in order for $c^s0c^t2c^u$ with $s,t,u \leq k-1$ to have length $3k-1$, we must have $s = t = u = k-1$. Hence, our coloring is $c^{k-1}0c^{k-1}2c^{k-1}$. By considering the integers congruent to 2 modulo 3, we see that the integers of color 0 and 2 are congruent to 0 modulo k, and hence are both congruent to 1 modulo 3. Hence, $2, 5, 7, \ldots, 3k-1$ is a k-term monochromatic arithmetic progression of color $c \neq 0$, a contradiction, thereby finishing this case.

If $k \equiv 0, 2, 3 \pmod 6$, in order for $c^s 0 c^t 2 c^u$ with $s, t, u \le k-1$ to have length $3k-2$, we must have one of s, t, u equal to $k-2$ and the other two equal to $k-1$. If $s = t = k-1$ and $u = k-2$, then the integers congruent to 1 modulo 3 form a monochromatic k-term arithmetic progression of color c since the integers of color 0 and 2 are congruent to either 0 or 2 modulo 3. If $s = k-2$ and $t = u = k-1$, first consider $k \equiv 0, 2 \pmod 6$ so that k is even. Then $k, k+2, k+4, \ldots, 3k-2$ is a k-term monochromatic arithmetic progression of color c. Next, consider $k \equiv 3 \pmod 6$. In this situation, the integers of color 0 and 2 are both congruent to 2 modulo 3. Hence, the integers congruent to 1 modulo 3 form a monochromatic k-term arithmetic progression of color c. Lastly, consider $s = u = k-1$ and $t = k-2$. Then the integers of color 0 and 2 are k and $2k-1$. Since $k \equiv 3 \pmod 6$ we see that both of these integers are congruent to either 0 or 2 modulo 3. Hence, the integers congruent to 1 modulo 3 form a monochromatic k-term arithmetic progression of color c. □

One final piece we can take from Table 3 concerns $m(k, k; k)$. We have $m(k, k; k) \ge w(k; 2)$. This holds, since, by definition, there exists a 2-coloring $\chi : [1, w(k; 2) - 1] \to \{0, 1\}$ that does not admit a monochromatic k-term arithmetic progression. Necessarily, we do not have a k-term k-zero-sum arithmetic progression since such a progression must be monochromatic.

As was done when considering the 2-color restriction $\bar{z}_r(k)$ of $z(k; r)$, we investigate what happens when we restrict the number of colors to two for these mixed numbers. This will hopefully allow us to bound $m(k, \ell; r)$.

Definition 10. Let $k, \ell, r \in \mathbb{Z}^+$ with $k, \ell, r \ge 2$. Define $\bar{m}_r(k, \ell)$ to be the minimum integer n such that any coloring of $[1, n]$ by $\mathbb{Z}_2$ admits either a k-term monochromatic arithmetic progression of color 1 or an ℓ-term r-zero-sum arithmetic progression.

The existence of $\bar{m}_r(k, \ell)$ is clear since $\bar{m}_r(k, \ell) \le m(k, \ell; r)$ holds because $\mathbb{Z}_2 \subseteq \mathbb{Z}_r$.

As we can see, the values in Table 4 are not as irregular as in Table 3. However, as explained in the observations below, attempting to find a formula or constructive lower bound other than the apparent $(2k-1)/2k$ and $(2\ell-1)/2\ell$ formula occurring in the first column and first few rows (which we leave to the reader to investigate), does not seem hopeful.

Table 4: Values for $\bar{m}_r(k,\ell)$ for small k, ℓ, and r

k	r	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$
2	2	3	6	7	10
	3	3	6	7	10
	4	3	6	7	10
3	2	3	7	7	15
	3	6	9	14	21
	4	6	9	18	22
4	2	3	8	7	20
	3	7	9	16	23
	4	7	18	35	33
5	2	3	8	7	21
	3	10	9	18	26
	4	10	22	35	37

Observations. Via essentially the same argument as that presented for the proof of Proposition 6, we have $\bar{m}_k(k,k) = w(k;2)$. We also have relationships with the classical van der Waerden numbers in at least two other ways. First, we have $\bar{m}_t(k,3) = w(k,3)$ for all $t \geq 4$, where $w(k,3)$ is the minimum integer such that any 2-coloring of $[1, w(k,3)]$ admits either a monochromatic k-term arithmetic progression of the first color or a monochromatic 3-term arithmetic progression of the second color. Hence, a result due to Li and Shu [14] gives us that $\bar{m}_t(k,3) > \left(\frac{8}{729}\right)\frac{k^2}{\log^2 k}$ for sufficiently large k when $t \geq 4$. Generalizing this, we see that $\bar{m}_t(k,\ell) = w(k,\ell)$ for all $t \geq \ell + 1$. Second, we have $\bar{m}_\ell(k,\ell) = \bar{m}_\ell(\ell,\ell) = w(\ell;2)$ for all $k \geq \ell$. This holds since the first ℓ terms of a k-term arithmetic progression of color 1 form an ℓ-term ℓ-zero-sum arithmetic progression provided $k \geq \ell$.

Given that many instances of $\bar{m}_r(k,\ell)$ are equal to certain classical van der Waerden numbers, attempting to find a formula for these does not seem to be a good use of time.

5. Conclusion and open questions

If we had a proof of the existence of $z(k;k)$ (resp., $m(k,k;k)$) that did not rely on the existence of $w(k;k)$, we would have a proof of the existence of $w(k;2)$ by Proposition 6 (resp., the observation above). It is an elementary exercise (see [13]) to deduce the existence of $w(k;r)$ from $w(k;2)$ for any $r \in \mathbb{Z}^+$. Hence, the independent existence of $z(k;r)$ (or $m(k,k;k)$) implies the existence of $w(k;r)$. We can state this (for $z(k;r)$) in the following manner.

Theorem 11. *Under the condition that $r \mid k$ the following holds: $z(k;r)$ exists for all r and k if and only if $w(k;r)$ exists for all r and k.*

Unfortunately, all attempts by this author to prove the existence of $z(k;r)$ independently from the existence of $w(k;r)$ and its proofs have been unsuccessful.

We end with some open questions and problems.

- Q1. Is it true that $z(k;3) = \bar{z}_3(k)$?
- Q2. Prove or disprove: $\bar{z}_3(k) = k^2$.
- Q3. Prove the existence of $z(k;r)$ and/or $m(k,\ell;r)$ independently from van der Waerden's theorem and its proofs.
- Q4. One useful extension of van der Waerden's theorem is that we can also guarantee that the common gap in the arithmetic progression has the same color as the arithmetic progression. Along these lines, when $r \mid k$, investigate the minimum integer $\bar{z}_r^*(k)$ such that every coloring $\chi : [1, \bar{z}_r^*(k)] \to \{0,1\}$ admits a $(k-1)$-term arithmetic progression $a, a+d, a+2d, \ldots, a+(k-2)d$ such that $\chi(d) + \sum_{i=0}^{k-2} \chi(a+id) \equiv 0 \pmod{r}$. The same can be investigated via an appropriate analogue of $m(k,\ell;r)$.

Acknowledgements

The author thanks Yair Caro for helpful communication relating to this paper as well as the anonymous referee whose detailed comments bettered the presentation.

References

[1] S. D. Adhikari and E. Mazumdar, The polynomial method in the study of zero-sum theorems, *Int. J. Number Theory* **11** (2015), 1451–1461. MR3376220

[2] N. Alon and Y. Caro, On three zero-sum Ramsey-type problems, *J. Graph Theory* **17** (1993), 177–192. MR1217393

[3] P. Balister, Y. Caro, C. Rousseau, and R. Yuster, Zero-sum square matrices, *European J. Combin.* **23** (2002), 489–497. MR1931933

[4] A. Bialostocki, Zero sum trees: a survey of results and open problems, in *Finite and Infinite Combinatorics in Sets and Logic*, NATO ASI Series **411** (Series C: Math. and Physical Sci.), Springer, Dordrecht, 1993. MR1261194

[5] A. Bialostocki, G. Bialostocki, Y. Caro, and R. Yuster, Zero-sum ascending waves, *J. Combin. Math. Combin. Comput.* **32** (2000), 103–114. MR1748896

[6] A. Bialostocki, G. Bialostocki, and D. Schaal, A zero-sum theorem, *J. Combin. Theory Ser. A* **101** (2003), 147–152. MR1953285

[7] A. Bialostocki and P. Dierker, On the Erdős-Ginzburg-Ziv theorem and the Ramsey numbers for stars and matchings, *Discrete Math.* **110** (1992), 1–8. MR1197440

[8] A. Bialostocki, R. Sabar, and D. Schaal, On a zero-sum generalization of a variation of Schur's equation, *Graphs Combin.* **24** (2008), 511–518. MR2461590

[9] Y. Caro, Zero-sum problems – a survey, *Discrete Math.* **152** (1996), 93–113. MR1388634

[10] P. Erdős, A. Ginzberg, and A. Ziv, Theorem in additive number theory, *Bulletin Research Council Israel* **10F** (1961–2), 41–43. MR3618568

[11] W. Gao and A. Geroldinger, Zero-sum problems in finite abelian groups: a survey, *Expo. Math.* **24** (2006), 337–369. MR2313123

[12] D. Grynkiewicz, A weighted Erdős-Ginzberg-Ziv theorem, *Combinatorica* **26** (2006), 445–453. MR2260848

[13] B. Landman and A. Robertson, *Ramsey Theory on the Integers*, second edition, American Math. Society, 2014. MR3243507

[14] Y. Li and J. Shu, A lower bound for off-diagonal van der Waerden numbers, *Adv. Appl. Math.* **44** (2010), 243–247. MR2593310

[15] A. Robertson, Zero-sum generalized Schur numbers, *J. Comb. Num. Theory*, to appear.

[16] B. L. Van der Waerden, Beweis einer baudetschen Vermutung, *Nieuw Archief voor Wiskunde* **15** (1927), 212–216.

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Received February 13, 2018