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Lecture hall P-partitions

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We introduce and study s-lecture hall P-partitions which is a generalization of s-lecture hall partitions to labeled (weighted) posets. We provide generating function identities for s-lecture hall P-partitions that generalize identities obtained by Savage and Schuster for s-lecture hall partitions, and by Stanley for P-partitions. We also prove that the corresponding (P, s)-Eulerian polynomials are real-rooted for certain pairs (P, s), and speculate on unimodality properties of these polynomials.

1. Introduction

Let $s = (s_1, \dots, s_n)$ be a sequence of positive integers. An *s-lecture hall partition* is an integer sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ satisfying $0 \le \lambda_1/s_1 \le \cdots \le \lambda_n/s_n$. These are generalizations of *lecture hall partitions*, corresponding to the case when $s = (1, 2, \dots, n)$, first studied by Bousquet-Mélou and Eriksson [3]. It has recently been made evident that s-lecture hall partitions serve as a rich model for various combinatorial structures with interesting generating functions, see [2, 3, 4, 13, 14, 19, 18, 20, 21] and the references therein.

In this paper we generalize the concept of s-lecture hall partitions to labeled posets. This constitutes a generalization of Stanley's theory of P-partitions, see [24, Ch. 3.15]. In Section 3 we derive multivariate generating function identities for s-lecture hall P-partitions, and prove a reciprocity theorem (Theorem 3.9). When P is a naturally labeled chain or an anti-chain, the generating function identities obtained produce results on s-lecture hall partitions and signed permutations, respectively (see Section 6). We also introduce and study a (P, s)-Eulerian polynomial. In Section 4 we prove that this polynomial is palindromic for sign-graded labeled posets with a specific choice of s. In Section 5 we prove that the (P, s)-Eulerian polynomial is real-rooted for certain choices of (P, s), and we also speculate on unimodality properties satisfied by these polynomials.

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2. Lecture hall *P*-partitions

In this paper a *labeled poset* is a partially ordered set on $[p] := \{1, \dots, p\}$ for some positive integer p, i.e., $P = ([p], \preceq)$, where $\preceq$ denotes the partial order. We will use the symbol $\leq$ to denote the usual total order on the integers. If P is a labeled poset, then a P-partition[1] is a map $f : [p] \to \mathbb{R}$ such that

1. if $x \prec y$, then $f(x) \leq f(y)$, and
2. if $x \prec y$ and $x > y$, then $f(x) < f(y)$.

The theory of P-partitions was developed by Stanley in his thesis and has since then been used frequently in several different combinatorial settings, see [24, 25].

Let

$$O(P) = \{f \in \mathbb{R}^p : f \text{ is a } P\text{-partition and } 0 \leq f(x) \leq 1 \text{ for all } x \in [p]\}$$

be the *order polytope* associated to P. Note that if P is naturally labeled, i.e., $x \prec y$ implies $x < y$, then $O(P)$ is a closed integral polytope. Otherwise $O(P)$ is the intersection of a finite number of open or closed half-spaces. Recall that the *Ehrhart polynomial* of an integral polytope $\mathcal{P}$ in $\mathbb{R}^p$ is defined for nonnegative integers n as

$$i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^p|,$$

where $n\mathcal{P} = \{n\mathbf{x} : \mathbf{x} \in \mathcal{P}\}$, see [24, p. 497]. For order polytopes we have the following relationship due to Stanley:

$$\sum_{n \geq 0} i(O(P), n) t^n = \frac{A_P(t)}{(1-t)^{p+1}},$$

where $A_P(t)$ is the P-Eulerian polynomial, which is the generating polynomial of the descent statistic over the set of all linear extensions of P, see [24, Ch. 3.15].

The purpose of this paper is to initiate the study of a lecture hall generalization of P-partitions. Let P be a labeled poset and let $s : [p] \to \mathbb{Z}_+ := \{1, 2, 3, \dots\}$ be an arbitrary map. We define a *lecture hall* (P, s)*-partition* to be a map $f : [p] \to \mathbb{R}$ such that

[1] What we call P-partitions are called reverse (P, ω)-partitions in [24, 25]. However the theory of (P, ω)-partitions and reverse (P, ω)-partitions are clearly equivalent.

1. if $x \prec y$, then $f(x)/s(x) \leq f(y)/s(y)$, and
2. if $x \prec y$ and $x > y$, then $f(x)/s(x) < f(y)/s(y)$.

Let

$$O(P, s) = \{f \in \mathbb{R}^p : f \text{ is a } (P, s)\text{-partition and } 0 \leq f(x)/s(x) \leq 1 \text{ for all } x \in [p]\}$$

be the *lecture hall order polytope* associated to (P, s). We also let

$$C(P, s) = \{f \in \mathbb{R}^p : f \text{ is a } (P, s)\text{-partition and } 0 \leq f(x)/s(x) \text{ for all } x \in [p]\}$$

be the *lecture hall order cone* associated to (P, s). The (P, s)-*Eulerian polynomial*, $A_{(P,s)}(t)$, is defined by

$$\sum_{n \geq 0} i(O(P, s), n)t^n = \frac{A_{(P,s)}(t)}{(1 - t)^{p+1}}.$$

3. The main generating functions

In this section we derive formulas for the main generating functions associated to lecture hall (P, s)-partitions. The outline follows Stanley's theory of P-partitions [24, Ch. 3.15]. We shall see in Section 6 that the special cases when P is naturally labeled chain or an anti-chain automatically produce results on lecture hall polytopes and signed permutations, respectively.

Let $\mathfrak{S}_p$ denote the symmetric group on $[p]$. If $\pi = \pi_1\pi_2 \cdots \pi_p \in \mathfrak{S}_p$ is a permutation written in one-line notation, we let P_π denote the labeled chain $\pi_1 \prec \pi_2 \prec \cdots \prec \pi_p$. If $P = ([p], \preceq)$ is a labeled poset, let $\mathcal{L}(P)$ denote the set

$$\mathcal{L}(P) := \{\pi \in \mathfrak{S}_p : \text{if } \pi_i \preceq \pi_j, \text{ then } i \leq j, \text{ for all } i, j \in [p]\},$$

of *linear extensions* (or the *Jordan-Hölder set*) of P. The following lemma is an immediate consequence of Stanley's decomposition of P-partitions [24, Lemma 3.15.3].

Lemma 3.1. *If P is a labeled poset and $s : [p] \to \mathbb{Z}_+$, then*

$$C(P, s) = \bigsqcup_{\pi \in \mathcal{L}(P)} C(P_\pi, s),$$

where $\bigsqcup$ denotes disjoint union.

Let $s : [p] \to \mathbb{Z}_+$. An *s-colored permutation* is a pair $\tau = (\pi, r)$ where $\pi \in \mathfrak{S}_p$, and $r : [p] \to \mathbb{N}$ satisfies $r(\pi_i) \in \{0, 1, \dots, s(\pi_i)-1\}$ for all $1 \le i \le p$. If $P = ([p], \preceq)$ is a labeled poset, let

$$\mathcal{L}(P, s) = \{\tau : \tau = (\pi, r) \text{ where } \pi \in \mathcal{L}(P) \text{ and } \tau \text{ is an } s\text{-colored permutation}\}.$$

For $f : [p] \to \mathbb{N}$, let $q(f), r(f) : [p] \to \mathbb{N}$ be the unique functions satisfying

$$f(x) = q(f)(x) \cdot s(x) + r(f)(x), \quad \text{where } q(f)(x) \in \mathbb{N} \text{ and } 0 \le r(f)(x) < s(x),$$

for all $x \in [p]$. Let further

$$F_{(P,s)}(\mathbf{x}, \mathbf{y}) = \sum_{f \in \mathbb{N}(P,s)} \mathbf{y}^{r(f)} \mathbf{x}^{q(f)},$$

where $\mathbf{x}^r = x_1^{r(1)} x_2^{r(2)} \cdots x_p^{r(p)}$ and $\mathbb{N}(P, s) = C(P, s) \cap \mathbb{N}^p$. We say that $i \in [p-1]$ is a *descent* of $\tau = (\pi, r)$ if

$$\begin{cases} \pi_i < \pi_{i+1} \text{ and } r(\pi_i)/s(\pi_i) > r(\pi_{i+1})/s(\pi_{i+1}), \text{ or,} \\ \pi_i > \pi_{i+1} \text{ and } r(\pi_i)/s(\pi_i) \ge r(\pi_{i+1})/s(\pi_{i+1}), \end{cases}$$

Let

$$D_1(\tau) = \{i \in [p-1] : i \text{ is a descent}\}.$$

Theorem 3.2. *If P is a labeled poset and $s : [p] \to \mathbb{Z}_+$, then*

$$F_{(P,s)}(\mathbf{x}, \mathbf{y}) = \sum_{\tau = (\pi, r) \in \mathcal{L}(P,s)} \mathbf{y}^r \frac{\prod_{i \in D_1(\tau)} x_{\pi_{i+1}} \cdots x_{\pi_p}}{\prod_{i \in [p]} (1 - x_{\pi_i} \cdots x_{\pi_p})}. \tag{3.1}$$

Proof. By Lemma 3.1 we may assume that $P = P_\pi$ is a labeled chain. Let $f \in \mathbb{N}^p$, and write $f(t) = q(t)s(t) + r(t)$, where $0 \le r(t) < s(t)$ and $q(t) \in \mathbb{N}$ for all $t \in [p]$. What conditions on q and r guarantee $f \in \mathbb{N}(P, s)$? Suppose $\pi_i < \pi_{i+1}$. Then we need

$$q(\pi_i) + \frac{r(\pi_i)}{s(\pi_i)} = \frac{f(\pi_i)}{s(\pi_i)} \le \frac{f(\pi_{i+1})}{s(\pi_{i+1})} = q(\pi_{i+1}) + \frac{r(\pi_{i+1})}{s(\pi_{i+1})}. \tag{3.2}$$

If $r(\pi_i)/s(\pi_i) \leq r(\pi_{i+1})/s(\pi_{i+1})$, then (3.2) holds if and only if $q(\pi_i) \leq q(\pi_{i+1})$. If $r(\pi_i)/s(\pi_i) > r(\pi_{i+1})/s(\pi_{i+1})$, then (3.2) holds if and only if $q(\pi_i) < q(\pi_{i+1})$.

Suppose $\pi_i > \pi_{i+1}$. Then we need

$$(3.3) \qquad q(\pi_i) + \frac{r(\pi_i)}{s(\pi_i)} = \frac{f(\pi_i)}{s(\pi_i)} < \frac{f(\pi_{i+1})}{s(\pi_{i+1})} = q(\pi_{i+1}) + \frac{r(\pi_{i+1})}{s(\pi_{i+1})}.$$

If $r(\pi_i)/s(\pi_i) < r(\pi_{i+1})/s(\pi_{i+1})$, then (3.3) holds if and only if $q(\pi_i) \leq q(\pi_{i+1})$. If $r(\pi_i)/s(\pi_i) \geq r(\pi_{i+1})/s(\pi_{i+1})$, then (3.3) holds if and only if $q(\pi_i) < q(\pi_{i+1})$.

Let $\tau = (\pi, r)$, where r is fixed. Then $f = qs + r \in \mathbb{N}(P, s)$ with given (fixed) r if and only if

$$(3.4) \qquad 0 \leq q(\pi_1) \leq q(\pi_2) \leq \cdots \leq q(\pi_p),$$

where $q(\pi_i) < q(\pi_{i+1})$ if $i \in D_1(\tau)$. Hence $f = qs + r \in \mathbb{N}(P, s)$ if and only if for each $k \in [p]$:

$$q(\pi_k) = \alpha_k + |\{i \in D_1(\tau) : i < k\}|,$$

where $\alpha_k \in \mathbb{N}$ and $0 \leq \alpha_1 \leq \cdots \leq \alpha_p$. Hence

$$\sum_q \prod_{i=1}^p x_{\pi_i}^{q(\pi_i)} = \sum_{0 \leq \alpha_1 \leq \cdots \leq \alpha_p} x_{\pi_1}^{\alpha_1} \cdots x_{\pi_p}^{\alpha_p} \prod_{i \in D_1(\tau)} x_{\pi_{i+1}} \cdots x_{\pi_p}$$
$$= \frac{\displaystyle\prod_{i \in D_1(\tau)} x_{\pi_{i+1}} \cdots x_{\pi_p}}{\displaystyle\prod_{i \in [p]} (1 - x_{\pi_i} \cdots x_{\pi_p})},$$

where the first sum is over all q satisfying (3.4). The theorem follows. □

Let $\mathbb{Z}_+(P, s) = C(P, s) \cap \mathbb{Z}_+^p$ and let

$$F^+_{(P,s)}(\mathbf{x}, \mathbf{y}) = \sum_{f \in \mathbb{Z}_+(P,s)} \mathbf{y}^{r(f)} \mathbf{x}^{q(f)}.$$

Let further

$$D_2(\tau) = \begin{cases} D_1(\tau), & \text{if } r(\pi_1) \neq 0, \\ D_1(\tau) \cup \{0\}, & \text{if } r(\pi_1) = 0. \end{cases}$$

Theorem 3.3. *If P is a labeled poset and $s : [p] \to \mathbb{Z}_+$, then*

$$F^+_{(P,s)}(\mathbf{x},\mathbf{y}) = \sum_{\tau=(\pi,r)\in\mathcal{L}(P,s)} \mathbf{y}^r \frac{\prod_{i\in D_2(\tau)} x_{\pi_{i+1}}\cdots x_{\pi_p}}{\prod_{i\in[p]}(1 - x_{\pi_i}\cdots x_{\pi_p})}.$$

Proof. Consider (P', s') where P' is obtained from P by adjoining a least element $\hat{0}$ labeled $p+1$, and $s' : [p+1] \to \mathbb{Z}_+$ is such that s' restricted to $[p]$ agrees with s. Let also $s'(p+1) > \max\{s(t) : t \in [p]\}$. Then $f \in \mathbb{N}(P', s')$ if and only if $f|_{[p]} \in \mathbb{N}(P, s)$ and

$$0 \le \frac{f(p+1)}{s'(p+1)} < \frac{f(x)}{s(x)}, \quad \text{for all } x \in [p].$$

Thus $F^+_{(P,s)}(\mathbf{x},\mathbf{y})$ is obtained from $F_{(P',s')}(\mathbf{x},\mathbf{y})$ when we restrict to all $f \in \mathbb{N}(P', s')$ with $f(p+1) = 1$, i.e., $q(p+1) = 0$ and $r(p+1) = 1$, and then shift the indices. Hence $i = 0$ is a descent in $((p+1)\pi_1\pi_2\cdots\pi_p, r)$ if and only if $r(\pi_1) = 0$, and the proof follows. □

For $f : [p] \to \mathbb{Z}_+$, let $q'(f), r'(f) : [p] \to \mathbb{N}$ be the unique functions satisfying

$$f(x) = q'(f)(x)\cdot s(x) + r'(f)(x), \quad \text{where } q'(f)(x) \in \mathbb{N} \text{ and } 0 < r'(f)(x) \le s(x),$$

for all $x \in [p]$. Let further

$$G_{(P,s)}(\mathbf{x},\mathbf{y}) = \sum_{f\in\mathbb{Z}_+(P,s)} \mathbf{y}^{r'(f)}\mathbf{x}^{q'(f)}.$$

Let $D_3(\tau)$ be the set of all $i \in [p-1]$ for which

$$\pi_i < \pi_{i+1} \text{ and } (r(\pi_i)+1)/s(\pi_i) > (r(\pi_{i+1})+1)/s(\pi_{i+1}), \text{ or,}$$
$$\pi_i > \pi_{i+1} \text{ and } (r(\pi_i)+1)/s(\pi_i) \ge (r(\pi_{i+1})+1)/s(\pi_{i+1}).$$

Theorem 3.4. *If P is a labeled poset and $s : [p] \to \mathbb{Z}_+$, then*

$$G_{(P,s)}(\mathbf{x},\mathbf{y}) = \sum_{\tau=(\pi,r)\in\mathcal{L}(P,s)} \mathbf{y}^{r+\mathbf{1}} \frac{\prod_{i\in D_3(\tau)} x_{\pi_{i+1}}\cdots x_{\pi_p}}{\prod_{i\in[p]}(1 - x_{\pi_i}\cdots x_{\pi_p})},$$

where $\mathbf{1} = (1, 1, \ldots, 1)$ *is the all ones vector.*

Proof. The proof is almost identical to that of Theorem 3.2, and is therefore omitted. □

For $n \in \mathbb{N}$, let

$$\mathbb{N}_{\leq n}(P, s) = \{f \in \mathbb{N}(P, s) : f(x)/s(x) \leq n \text{ for all } x \in [p]\},$$

and let

$$F_{(P,s)}(\mathbf{x}, \mathbf{y}; n) = \sum_{f \in \mathbb{N}_{\leq n}(P,s)} \mathbf{y}^{r(f)} \mathbf{x}^{q(f)}.$$

The polynomials $F^+_{(P,s)}(\mathbf{x}, \mathbf{y}; n)$ and $G_{(P,s)}(\mathbf{x}, \mathbf{y}; n)$ are defined analogously over $\{f \in \mathbb{Z}_+(P, s) : f(x)/s(x) \leq n \text{ for all } x \in [p]\}$. Let also

$$\mathbb{N}_{<n}(P, s) = \{f \in \mathbb{N}(P, s) : f(x)/s(x) < n \text{ for all } x \in [p]\},$$

and

$$F'_{(P,s)}(\mathbf{x}, \mathbf{y}; n) = \sum_{f \in \mathbb{N}_{<n}(P,s)} \mathbf{y}^{r(f)} \mathbf{x}^{q(f)}.$$

For $\tau = (\pi, r) \in \mathcal{L}(P, s)$, define

$$D(\tau) = \begin{cases} D_1(\tau), & \text{if } r(\pi_p) = 0, \\ D_1(\tau) \cup \{p\}, & \text{if } r(\pi_p) > 0, \end{cases}$$

and

$$D_4(\tau) = \begin{cases} D_2(\tau), & \text{if } r(\pi_p) = 0, \\ D_2(\tau) \cup \{p\}, & \text{if } r(\pi_p) > 0. \end{cases}$$

Proposition 3.5. *If P is a labeled poset and $s : [p] \to \mathbb{Z}_+$, then*

(3.5)

$$\sum_{n \geq 0} F_{(P,s)}(\mathbf{x}, \mathbf{y}; n) t^n = \sum_{\tau = (\pi,r) \in \mathcal{L}(P,s)} \mathbf{y}^r \frac{\prod_{i \in D(\tau)} x_{\pi_{i+1}} \cdots x_{\pi_p}}{\prod_{i \in [p]} (1 - x_{\pi_i} \cdots x_{\pi_p} t)} \frac{t^{|D(\tau)|}}{1 - t},$$

(3.6)

$$\sum_{n \geq 0} F'_{(P,s)}(\mathbf{x}, \mathbf{y}; n) t^n = \sum_{\tau = (\pi,r) \in \mathcal{L}(P,s)} \mathbf{y}^r \frac{\prod_{i \in D_1(\tau)} x_{\pi_{i+1}} \cdots x_{\pi_p}}{\prod_{i \in [p]} (1 - x_{\pi_i} \cdots x_{\pi_p} t)} \frac{t^{|D_1(\tau)|+1}}{1 - t},$$

(3.7)

$$\sum_{n\geq 0} F^{+}_{(P,s)}(\mathbf{x},\mathbf{y};n)t^n = \sum_{\tau=(\pi,r)\in\mathcal{L}(P,s)} \mathbf{y}^r \frac{\prod_{i\in D_4(\tau)} x_{\pi_{i+1}}\cdots x_{\pi_p}}{\prod_{i\in[p]}(1-x_{\pi_i}\cdots x_{\pi_p}t)} \frac{t^{|D_4(\tau)|}}{1-t},$$

(3.8)

$$\sum_{n\geq 0} G_{(P,s)}(\mathbf{x},\mathbf{y};n)t^n = \sum_{\tau=(\pi,r)\in\mathcal{L}(P,s)} \mathbf{y}^{r+\mathbf{1}} \frac{\prod_{i\in D_3(\tau)} x_{\pi_{i+1}}\cdots x_{\pi_p}}{\prod_{i\in[p]}(1-x_{\pi_i}\cdots x_{\pi_p}t)} \frac{t^{|D_3(\tau)|+1}}{1-t}.$$

Proof. For (3.5) consider (P', s') where P' is obtained from P by adjoining a greatest element $\hat{1}$ labeled $p+1$, and $s' : [p+1] \to \mathbb{Z}_+$ restricted to $[p]$ agrees with s, while $s'(p+1) = 1$. If we set $x_{p+1} = t$, then

$$\sum_{n\geq 0} F_{(P,s)}(\mathbf{x},\mathbf{y};n)t^n = F_{(P',s')},$$

and

$$\mathcal{L}(P',s') = \{(\pi_1\cdots\pi_p(p+1), r') : (\pi_1\cdots\pi_p, r'|_P) \in \mathcal{L}(P,s) \text{ and } r'(p+1) = 0\}.$$

The identity (3.5) follows by noting that $i = p$ is a descent of $(\pi_1\cdots\pi_p(p+1), r')$ if and only if $r(\pi_p)/s(\pi_p) > r'(p+1)/s'(p+1) = 0$.

The other identities follows similarly. For example (3.6) follows by considering (P', s') where P' is obtained from P by adjoining a greatest element $\hat{1}$ labeled 0 (and then relabel so that P' has ground set $[p+1]$). For (3.8) consider again (P', s'), where P' is obtained from P by adjoining a greatest element $\hat{1}$ labeled $p+1$, and s' is defined as for the case of (3.5). Note that since $r'(p+1) = 1$ we have $q'(p+1) = n-1$ if $f(p+1) = n$. This explains the shift by one in the exponent on the right hand side of (3.8), i.e., $|D_3(\tau)|+1$. □

If q is a variable, let $[0]_q := 0$ and $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$ for $n \geq 1$. For the special case of (3.5) when P is an anti-chain we acquire the following corollary, which is a generalization of [1, Theorem 5.23].

Corollary 3.6. *If P is an anti-chain and $s : [p] \to \mathbb{Z}_+$, then*

$$\sum_{n\geq 0}\prod_{i=1}^{p}\left(x_i^n + [n]_{x_i}[s(i)]_{y_i}\right)t^n = \sum_{\tau=(\pi,r)\in\mathcal{L}(P,s)} \mathbf{y}^r \frac{\prod_{i\in D(\tau)} x_{\pi_{i+1}}\cdots x_{\pi_p}}{\prod_{i\in[p]}(1 - x_{\pi_i}\cdots x_{\pi_p}t)} \frac{t^{|D(\tau)|}}{1-t}.$$

Proof. Let P be an anti-chain and let $s : [p] \to \mathbb{Z}_+$. Consider $f \in \mathbb{N}_{\leq n}(P, s)$. Since P is an anti-chain, $f(i)$ and $f(j)$ are independent for all $1 \leq i < j \leq p$, and the only restriction is $0 \leq f(i) \leq ns(i)$ for all $1 \leq i \leq p$. We write $f(i) = s(i)q(i) + r(i)$, where $0 \leq r(i) < s(i)$. Then $f \in \mathbb{N}_{\leq n}(P, s)$ if and only if either $q(i) = n$ and $r(i) = 0$, or $0 \leq q(i) \leq n - 1$ and $0 \leq r(i) \leq s(i) - 1$. Hence

$$\begin{aligned}\sum_{f\in\mathbb{N}_{\leq n}(P,s)} \mathbf{y}^{r(f)}\mathbf{x}^{q(f)} &= \prod_{i=1}^{p}\left(x_i^0[s(i)]_{y_i} + \cdots + x_i^{n-1}[s(i)]_{y_i} + x_i^n\right) \\ &= \prod_{i=1}^{p}\left(x_i^n + [n]_{x_i}[s(i)]_{y_i}\right).\end{aligned}$$

The corollary now follows from (3.5). □

Note that the special case of (3.5) when P is a naturally labeled chain gives an analogue (by an appropriate change of variables) to one of the main results in [20], see Theorem 5 therein. From (3.5) we also get an interpretation of the Eulerian polynomial $A_{(P,s)}(t)$. For $\tau \in \mathcal{L}(P, s)$, let $\operatorname{des}_s(\tau) = |D(\tau)|$.

Corollary 3.7. *If P is a labeled poset and $s : [p] \to \mathbb{Z}_+$, then*

$$A_{(P,s)}(t) = \sum_{\tau\in\mathcal{L}(P,s)} t^{\operatorname{des}_s(\tau)}.$$

The next corollary follows from Proposition 3.5 by setting the x- and y-variables to 1.

Corollary 3.8. *If P is a labeled poset and $s : [p] \to \mathbb{Z}_+$, then*

$$\sum_{\tau\in\mathcal{L}(P,s)} t^{|D_4(\tau)|} = \sum_{\tau\in\mathcal{L}(P,s)} t^{|D_3(\tau)|+1},$$

and if $s(x) = 1$ *for all minimal elements* x *in* P*, then*

$$A_{(P,s)}(t) = \sum_{\tau \in \mathcal{L}(P,s)} t^{|D(\tau)|} = \sum_{\tau \in \mathcal{L}(P,s)} t^{|D_3(\tau)|}. \tag{3.9}$$

Let $P = ([p], \preceq)$ be a labeled poset. For $i \in [p]$, let $i^* = p + 1 - i$, and let (P^*, s^*) be defined by $P^* = ([p], \preceq^*)$ with

$$i \preceq j \text{ in } P \quad \text{if and only if} \quad i^* \preceq^* j^* \text{ in } P^*, \quad \text{for all } i, j \in [p],$$

and $s^*(i^*) = s(i)$ for all $i \in [p]$. The poset P^* is called the *dual* of P.

Theorem 3.9 (Reciprocity theorem)**.** *If* P *is a labeled poset and* $s : [p] \to \mathbb{Z}_+$*, then*

$$G_{(P^*,s^*)}(\mathbf{x}^*, \mathbf{y}^*) = (-1)^p \frac{y_1^{s(1)} \cdots y_p^{s(p)}}{x_1 \cdots x_p} F_{(P,s)}(\mathbf{x}^{-1}, \mathbf{y}^{-1}),$$

where $\mathbf{x}^* = (x_p, x_{p-1}, \dots, x_1)$ *and* $\mathbf{x}^{-1} = (x_1^{-1}, \dots, x_p^{-1})$.

Proof. For $\tau = (\pi, r) \in \mathcal{L}(P, s)$, let $\tau^* = (\pi_1^* \pi_2^* \cdots \pi_p^*, r^*)$ where $r^*(i^*) = s(i) - 1 - r(i)$ for all $i \in [p]$. Clearly the map $\tau \mapsto \tau^*$ is a bijection between $\mathcal{L}(P, s)$ and $\mathcal{L}(P^*, s^*)$. Moreover if $i \in [p-1]$, then $i \in D_3(\tau)$ if and only if

$$\begin{cases} \pi_i < \pi_{i+1} \text{ and } (r(\pi_i) + 1)/s(\pi_i) > (r(\pi_{i+1}) + 1)/s(\pi_{i+1}), \text{ or,} \\ \pi_i > \pi_{i+1} \text{ and } (r(\pi_i) + 1)/s(\pi_i) \geq (r(\pi_{i+1}) + 1)/s(\pi_{i+1}), \end{cases}$$

if and only if

$$\begin{cases} \pi_i^* > \pi_{i+1}^* \text{ and } r^*(\pi_i^*)/s^*(\pi_i^*) < r^*(\pi_{i+1}^*)/s^*(\pi_{i+1}^*), \text{ or,} \\ \pi_i^* < \pi_{i+1}^* \text{ and } r^*(\pi_i^*)/s^*(\pi_i^*) \leq r^*(\pi_{i+1}^*)/s^*(\pi_{i+1}^*) \end{cases}$$

if and only if $i \in [p-1] \setminus D_1(\tau^*)$. Thus

$$D_3(\tau) = [p-1] \setminus D_1(\tau^*) \quad \text{and} \quad D_1(\tau) = [p-1] \setminus D_3(\tau^*), \tag{3.10}$$

for all $\tau \in \mathcal{L}(P, s)$. Now

$$F_{(P,s)}(\mathbf{x}, \mathbf{y}) = \sum_{\tau \in \mathcal{L}(P,s)} \mathbf{y}^r \frac{\prod_{i \in D_1(\tau)} x_{\pi_{i+1}} \cdots x_{\pi_p}}{\prod_{i \in [p]} (1 - x_{\pi_i} \cdots x_{\pi_p})}$$

$$= \sum_{\tau \in \mathcal{L}(P,s)} \mathbf{y}^r \frac{\prod_{i \in [p-1] \setminus D_3(\tau^*)} x_{\pi_{i+1}} \cdots x_{\pi_p}}{\prod_{i \in [p]} (1 - x_{\pi_i} \cdots x_{\pi_p})}$$

$$= \sum_{\tau \in \mathcal{L}(P,s)} \frac{\mathbf{y}^s (\mathbf{y}^*)^{-(r^*+\mathbf{1})}}{x_1 \cdots x_p} \frac{\prod_{i \in D_3(\tau^*)} x_{\pi_{i+1}}^{-1} \cdots x_{\pi_p}^{-1}}{\prod_{i \in [p]} (1 - x_{\pi_i} \cdots x_{\pi_p})} \prod_{i \in [p]} x_{\pi_i} \cdots x_{\pi_p}$$

$$= (-1)^p \frac{y_1^{s(1)} \cdots y_p^{s(p)}}{x_1 \cdots x_p} \sum_{\tau \in \mathcal{L}(P,s)} (\mathbf{y}^*)^{-(r^*+\mathbf{1})} \frac{\prod_{i \in D_3(\tau^*)} x_{\pi_{i+1}}^{-1} \cdots x_{\pi_p}^{-1}}{\prod_{i \in [p]} (1 - x_{\pi_i}^{-1} \cdots x_{\pi_p}^{-1})}$$

$$= (-1)^p \frac{y_1^{s(1)} \cdots y_p^{s(p)}}{x_1 \cdots x_p} G_{(P^*,s^*)}((\mathbf{x}^*)^{-1}, (\mathbf{y}^*)^{-1}),$$

from which the theorem follows. □

Remark 3.1. Theorem 3.9 generalizes the reciprocity theorem in [4] which follows as the special case when P is a naturally labeled chain.

4. Sign-ranked posets

Let $P = \{1 \prec 2 \prec \cdots \prec p\}$ be a naturally labeled chain, and let $s(i) = i$ for all $i \in [p]$. Savage and Schuster [20, Lemma 1] proved that $A_{(P,s)}(t)$ is equal to the Eulerian polynomial

$$A_p(t) = \sum_{\pi \in \mathfrak{S}_p} t^{\mathrm{des}(\pi)},$$

where $\mathrm{des}(\pi) = |\{i \in [p] : \pi_i > \pi_{i+1}\}$. Recall that a polynomial $g(t)$ is *palindromic* if $t^N g(1/t) = g(t)$ for some integer N. It is well known that $A_p(t)$ is palindromic (in fact $t^{p-1}A_p(1/t) = A_p(t)$). The same is known to be true for the P-Eulerian polynomial of any naturally labeled graded poset, see [24, Corollary 3.15.18], and more generally for P-Eulerian polynomials of so called sign-graded labeled posets [10, Corollary 2.4]. We shall here generalize these results to (P, s)-Eulerian polynomials.

Recall that a pair of elements elements (x, y) taken from a labeled poset P is a *covering relation* if $x \prec y$ and $x \prec z \prec y$ for no $z \in P$. Let $\mathcal{E}(P)$ denote the set of covering relations of P. If P is a labeled poset define a function $\epsilon : \mathcal{E}(P) \to \{-1, 1\}$ by

$$\epsilon(x, y) = \begin{cases} 1, & \text{if } x < y, \text{ and} \\ -1, & \text{if } x > y. \end{cases}$$

Sign-graded (labeled) posets, introduced in [10], generalize graded naturally labeled posets. A labeled poset P is *sign-graded* of *rank* r, if

$$\sum_{i=1}^{k} \epsilon(x_{i-1}, x_i) = r$$

for each maximal chain $x_0 \prec x_1 \prec \cdots \prec x_k$ in P. A sign-graded poset is equipped with a well-defined *rank-function*, $\rho : P \to \mathbb{Z}$, defined by

$$\rho(x) = \sum_{i=1}^{k} \epsilon(x_{i-1}, x_i),$$

where $x_0 \prec x_1 \prec \cdots \prec x_k = x$ is any unrefinable chain, x_0 is a minimal element and $x_k = x$. Hence a naturally labeled poset is sign-graded if and only if it is graded. A labeled poset P is *sign-ranked* if for each maximal element $x \in P$, the subposet $\{y \in P : y \preceq x\}$ is sign-graded. Note that each sign-ranked poset has a well-defined rank function $\rho : P \to \mathbb{Z}$. Thus a naturally labeled poset is sign-ranked if and only if it is ranked.

Theorem 4.1. *Let P be a sign-ranked labeled poset and suppose its rank function attains non-negative values only. Let $s(x) = \rho(x) + 1$ for each $x \in [p]$, and define $u : \mathbb{N}(P, s) \to \mathbb{Z}^p$ by $u(f)(x^*) = f(x) + \rho(x)$. Then $u : \mathbb{N}_{\leq n}(P, s) \to \mathbb{N}_{<n+1}(P^*, s^*)$ is a bijection for each $n \in \mathbb{N}$.*

Proof. We first prove $u : \mathbb{N}(P, s) \to \mathbb{N}(P^*, s^*)$. Note that f is a (P, s)-partition if and only if

1. if $(x, y) \in \mathcal{E}(P)$, then $f(x)/s(x) \leq f(y)/s(y)$, and
2. if $(x, y) \in \mathcal{E}(P)$ and $\epsilon(x, y) = -1$, then $f(x)/s(x) < f(y)/s(y)$.

Hence it suffices to consider covering relations when proving that $u : \mathbb{N}(P, s) \to \mathbb{N}(P^*, s^*)$.

Let $f \in \mathbb{N}(P, s)$. Suppose y covers x and $\epsilon(x, y) = 1$. Then $f(x)/s(x) \leq f(y)/s(y)$ and $s(x) < s(y)$, and thus

$$\frac{u(f)(x^*)}{s^*(x^*)} = \frac{f(x) + s(x) - 1}{s(x)} \leq \frac{f(y)}{s(y)} + 1 - \frac{1}{s(x)} < \frac{f(y)}{s(y)} + 1 - \frac{1}{s(y)} = \frac{u(f)(y^*)}{s^*(y^*)},$$

as desired.

Suppose y covers x and $\epsilon(x,y) = -1$. Then $f(x)/s(x) < f(y)/s(y)$ and $s(x) = s(y) + 1$ so that

$$\frac{u(f)(y^*)}{s^*(y^*)} - \frac{u(f)(x^*)}{s^*(x^*)} = \frac{f(y)}{s(y)} - \frac{f(x)}{s(y)+1} - \left(\frac{1}{s(y)} - \frac{1}{s(y)+1}\right).$$

We want to prove that the quantity on either side of the equality above is nonnegative. By assumption

$$\frac{f(y)}{s(y)} - \frac{f(x)}{s(y)+1} = \frac{(s(y)+1)f(y) - s(y)f(x)}{s(y)(s(y)+1)} > 0.$$

Hence $(s(y)+1)f(y) - s(y)f(x)$ is a positive integer, so that

$$\frac{f(y)}{s(y)} - \frac{f(x)}{s(y)+1} \geq \frac{1}{s(y)(s(y)+1)},$$

as desired. Note that $u(f)$ is nonnegative since it is increasing and $u(f)(x^*) = f(x)$ when x^* is a minimal element in P^*. Hence $u(f) \in \mathbb{N}(P^*, s^*)$.

Let $\eta : \mathbb{N}(P^*, s^*) \to \mathbb{Z}^P$ be defined by $\eta(g)(x) = g(x^*) - \rho(x) = g(x^*) + \rho^*(x^*)$, where ρ^* is the rank function of P^*. Clearly $\eta : \mathbb{N}(P^*, s^*) \to \mathbb{N}(P, s)$ by the exact same arguments as above. Thus $u^{-1} = \eta$ and $u : \mathbb{N}(P, s) \to \mathbb{N}(P^*, s^*)$ is a bijection.

Now $u(f)(x^*)/s^*(x^*) = f(x)/s(x) + (s(x) - 1)/s(x) < n + 1$ if $f \in \mathbb{N}_{\leq n}(P, s)$ and $x \in P$, so that $u : \mathbb{N}_{\leq n}(P, s) \to \mathbb{N}_{<n+1}(P^*, s^*)$ for each $n \in \mathbb{N}$.

On the other hand if $g \in \mathbb{N}_{<n+1}(P^*, s^*)$, then $g(x^*) = q(x^*)(\rho(x) + 1) + r(x^*)$ where $0 \leq q(x^*) \leq n$ and $0 \leq r(x^*) \leq \rho(x)$. Hence

$$\frac{\eta(g)(x)}{s(x)} = \frac{g(x^*)}{\rho(x)+1} - \frac{\rho(x)}{\rho(x)+1} \leq n + \frac{r(x^*)}{\rho(x)+1} - \frac{\rho(x)}{\rho(x)+1} \leq n.$$

Thus $\eta : \mathbb{N}_{<n+1}(P^*, s^*) \to \mathbb{N}_{\leq n}(P, s)$ which proves the theorem. □

Theorem 4.2. *If P is a sign-ranked labeled poset with nonnegative rank function ρ and $s = \rho + 1$, then*

$$A_{(P,s)}(t) = t^{p-1} A_{(P,s)}(t^{-1})$$

and

$$(-1)^p i(O(P, s), -t) = i(O(P, s), t - 2).$$

Proof. By (3.5), (3.6) and Theorem 4.1

$$A_{(P,s)}(t) = \sum_{\tau \in \mathcal{L}(P,s)} t^{|D(\tau)|} = \sum_{\tau^* \in \mathcal{L}(P^*,s^*)} t^{|D_1(\tau^*)|}.$$

The first part of the theorem now follows from (3.9) and (3.10). The second part follows from e.g., [24, Lemma 3.15.11]. ☐

5. Real-rootedness and unimodality

The Neggers-Stanley conjecture asserted that for each labeled poset P, the Eulerian polynomial $A_P(t)$ is real-rooted. Although the conjecture is refuted in its full generality [9, 26], it is known to hold for certain classes of posets [6, 27]. Moreover, when P is sign-graded, then the coefficients of $A_P(t)$ form a unimodal sequence [10, 16]. It is natural to ask for which pairs (P, s)

(a) is $A_{(P,s)}(t)$ real-rooted?
(b) do the coefficients of $A_{(P,s)}(t)$ form a unimodal sequence?

We first address (a). Suppose $P = ([p], \preceq_P)$, $Q = ([q], \preceq_Q)$ and $R = ([p + q], \preceq_R)$ are labeled posets such that $[p + q]$ is the disjoint union of the two sets $\{u_1 < u_2 < \cdots < u_p\}$ and $\{v_1 < v_2 < \cdots < v_q\}$, and $x \preceq_R y$ if and only if either

- $x = u_i$ and $y = u_j$ for some $i, j \in [p]$ with $i \preceq_P j$, or
- $x = v_i$ and $y = v_j$ for some $i, j \in [q]$ with $i \preceq_Q j$.

We say that R is a *disjoint union* of P and Q and write $R = P \sqcup Q$. Moreover if $s_P : [p] \to \mathbb{Z}_+$ and $s_Q : [q] \to \mathbb{Z}_+$, then we define $s_{P\sqcup Q} : [p + q] \to \mathbb{Z}_+$ as the unique function satisfying $s_{P\sqcup Q}(u_i) = s_P(i)$ and $s_{P\sqcup Q}(v_j) = s_Q(j)$.

Proposition 5.1. *If the polynomials $A_{(P,s_P)}(t)$ and $A_{(Q,s_Q)}(t)$ are real-rooted, then so is the polynomial $A_{(P\sqcup Q,s_P\sqcup s_Q)}(t)$.*

Proof. Clearly

$$i((P \sqcup Q, s_P \sqcup s_Q), t) = i(O(P, s_P), t) \cdot i(O(Q, s_Q), t),$$

so the proposition follows from [28, Theorem 0.1]. ☐

It was proved in [22] that if $P = \{1 \prec 2 \prec \cdots \prec p\}$ and $s : [p] \to \mathbb{Z}_+$ is arbitrary, then $A_{(P,s)}(t)$ is real-rooted. In Theorem 5.2 below we generalize this result to ordinal sums of anti-chains. If $P = (X, \preceq_P)$ and $Q = (Y, \preceq_Q)$ are posets on disjoint ground sets, then the *ordinal sum*, $P \oplus Q = (X \cup Y, \preceq)$, is the poset with relations

1. $x_1 \prec x_2$, for all $x_1, x_2 \in X$ with $x_1 \prec_P x_2$,
2. $y_1 \prec y_2$, for all $y_1, y_2 \in X$ with $y_1 \prec_Q y_2$, and
3. $x \prec y$ for all $x \in X$ and $y \in Y$.

Let f and g be two real-rooted polynomials in $\mathbb{R}[t]$ with positive leading coefficients. Let further $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_m$ be the zeros of f and g, respectively. If

$$\cdots \leq \alpha_2 \leq \beta_2 \leq \alpha_1 \leq \beta_1$$

we say that f is an *interleaver* of g and we write $f \ll g$. We also let $f \ll 0$ and $0 \ll f$. We call a sequence $F_n = (f_i)_{i=1}^n$ of real-rooted polynomials *interlacing* if $f_i \ll f_j$ for all $1 \leq i < j \leq n$. We denote by $\mathcal{F}_n$ the family of all interlacing sequences $(f_i)_{i=1}^n$ of polynomials and we let $\mathcal{F}_n^+$ be the family of $(f_i)_{i=1}^n \in \mathcal{F}_n$ such that f_i has nonnegative coefficients for all $1 \leq i \leq n$.

To avoid unnecessary technicalities we here redefine a labeled poset to be a poset $P = (S, \preceq)$, where S is any set of positive integers. Thus $\mathcal{L}(P)$ is now the set of rearrangements of S that are also linear extensions of P.

Equip $X(P, s) := \{(k, x) : x \in P \text{ and } 0 \leq k < s(x)\}$ with a total order defined by $(k, x) < (\ell, y)$ if $k/s(x) < \ell/s(y)$, or $k/s(x) = \ell/s(y)$ and $x < y$. For $\gamma \in X(P, s)$, let

$$A^{\gamma}_{(P,s)}(t) = \sum_{\substack{\tau=(\pi,r)\in\mathcal{L}(P,s)\\ (r(\pi_1),\pi_1)=\gamma}} t^{\mathrm{des}_s(\tau)}.$$

Theorem 5.2. *Suppose $P = A_{p_1} \oplus \cdots \oplus A_{p_m}$ is an ordinal sum of anti-chains, and let $s : P \to \mathbb{Z}_+$ be a function which is constant on A_{p_i} for $1 \leq i \leq m$. Then $\{A^{\gamma}_{(P,s)}(t)\}_{\gamma \in X}$, where $X = X(P, s)$, is an interlacing sequence of polynomials.*

In particular $A_{(P,s)}(t)$ and $A^{\gamma}_{(P,s)}(t)$ are real-rooted for all $\gamma \in X$.

Proof. The proof is by induction over m. Suppose $m = 1$, $p_1 = n$, A_n is the anti-chain on $[n]$, and $s(A_n) = \{s\}$. We prove the case $m = 1$ by induction over n. If $n = 1$ we get the sequence $1, t, t, \ldots, t$ which is interlacing. Otherwise if $\gamma = (k, \pi_1)$, then

$$A^{\gamma}_{(A_n,s)}(t) = \sum_{\kappa<\gamma} tA^{\kappa}_{(A_{n-1},s')}(t) + \sum_{\kappa\geq\gamma} A^{\kappa}_{(A_{n-1},s')}(t),$$

where s' is s restricted to A_{n-1}. This recursion preserves the interlacing property, see [22, Theorem 2.3] and [11], which proves the case $m = 1$ by induction.

Suppose $m > 1$. The proof for m is again by induction over $p_1 = n$. If $p_1 = 1$, then

$$A^{\gamma}_{(P,s)}(t) = \sum_{\kappa<\gamma} tA^{\kappa}_{(P',s')}(t) + \sum_{\kappa>\gamma} A^{\kappa}_{(P',s')}(t),$$

Where $P' = A_2 \oplus \cdots \oplus A_m$, and where s' is the restrictions to P'. Hence the case $p_1 = 1$ follows by induction (over m) since this recursion preserves the interlacing property, see [22, Theorem 2.3].

The case $m > 1$ and $p_1 > 1$ follows by induction over p_1 just as for the case $m = 1$, $n > 1$.

Hence $\{A^{\gamma}_{(P,s)}(t)\}_{\gamma}$ is an interlacing sequence, and thus

$$A_{(P,s)}(t) = \sum_{\gamma} A^{\gamma}_{(P,s)}(t),$$

is real-rooted by e.g., [22, Theorem 2.3]. □

Next we address (b). A palindromic polynomial $g(t) = a_0 + a_1 t + \cdots + a_n t^n$ may be written uniquely as

$$g(t) = \sum_{k=0}^{\lfloor d/2 \rfloor} \gamma_k(g) t^k (1+t)^{d-2k},$$

where $\{\gamma_k(g)\}_{k=0}^{\lfloor d/2 \rfloor}$ are real numbers. If $\gamma_k(g) \geq 0$ for all k, then we say that $g(t)$ is γ-*positive*, see [11]. Note that if $g(t)$ is γ-positive, then $\{a_i\}_{i=0}^{n}$ is a *unimodal sequence*, i.e., there is an index m such that $a_0 \leq \cdots \leq a_m \geq a_{m+1} \geq \cdots \geq a_n$.

Conjecture 5.3. *Suppose P is a sign-ranked labeled poset with nonnegative rank function ρ and $s = \rho + 1$, then $A_{(P,s)}(t)$ is γ-positive.*

Remark 5.1. Let P be a sign-ranked labeled poset with a rank function ρ with values only in $\{0, 1\}$, and let $s = \rho + 1$. Following the proof of [10, Theorem 4.2], with the use of Theorem 5.2, it follows that Conjecture 5.3 holds for (P, s). We omit the technical details in recalling the proof here.

If P is a naturally labeled ranked poset and $s = \rho + 1$, then $O(P, s)$ is a closed integral polytope and $A_{(P,s)}(t)$ is the so called h^*-*polynomial* of $O(P, s)$. If the following conjecture is true, then the coefficients of $A_{(P,s)}(t)$ form a unimodal sequence by a powerful theorem of Bruns and Römer [8, Theorem 1].

Conjecture 5.4. *Suppose P is a naturally labeled ranked poset, and let $s = \rho + 1$. Then $O(P, s)$ (or some related polytope with the same Ehrhart polynomial) has a regular and unimodular triangulation.*

Remark 5.2. Evidence for Conjectures 5.3 and 5.4 is provided by [23] where it is proved that the coefficients of $A_{(P,s)}(t)$ form unimodal sequence whenever P is a naturally labeled ranked poset with a least element, and $s = \rho + 1$.

6. Applications

In this section we derive some applications of the generating function identities obtained in Section 3. If $\alpha = (\alpha_1, \dots, \alpha_p)$ is a sequence, let $|\alpha| = \alpha_1 + \cdots + \alpha_p$. For $\tau = (\pi, r) \in \mathcal{L}(P, s)$, let

$$\text{comaj}(\tau) = \sum_{i \in D(\tau)} p - i, \text{ and}$$
$$\text{lhp}(\tau) = |r| + \sum_{i \in D(\tau)} s(\pi_{i+1}) + \cdots + s(\pi_p)$$

Theorem 6.1. *If P is a labeled poset and $s : [p] \to \mathbb{Z}_+$, then*

$$\sum_{n \geq 0} \left(\sum_{f \in \mathbb{N}_{\leq n}(P,s)} q^{|r(f)|} u^{|q(f)|} \right) t^n = \frac{\sum_{\tau \in \mathcal{L}(P,s)} q^{|r|} u^{\text{comaj}(\tau)} t^{\text{des}_s(\tau)}}{\prod_{i=0}^{p} (1 - u^i t)}. \tag{6.1}$$

Proof. Set $x_i = u$ and $y_i = q$ for all $1 \leq i \leq p$ in (3.5). Then

$$\sum_{\tau \in \mathcal{L}(P,s)} \mathbf{y}^r \frac{\prod_{i \in D(\tau)} x_{\pi_{i+1}} \cdots x_{\pi_p}}{\prod_{i \in [p]} (1 - x_{\pi_i} \cdots x_{\pi_p} t)} \frac{t^{|D(\tau)|}}{1 - t} = \sum_{\tau \in \mathcal{L}(P,s)} \frac{q^{|r|} u^{\text{comaj}(\tau)} t^{\text{des}_s(\tau)}}{\prod_{i \in [p]} (1 - tu^{p+1-i})(1 - t)}$$
$$= \frac{\sum_{\tau \in \mathcal{L}(P,s)} q^{|r|} u^{\text{comaj}(\tau)} t^{\text{des}_s(\tau)}}{\prod_{i \in [p]} (1 - tu^i)(1 - t)}.$$

The theorem follows. ∎

Theorem 6.2. *If P is a labeled poset and $s : [p] \to \mathbb{Z}_+$, then*

$$\sum_{n\geq 0}\left(\sum_{f\in\mathbb{N}_{\leq n}(P,s)} q^{|f|}\right)t^n = \sum_{\tau\in\mathcal{L}(P,s)} \frac{q^{\mathrm{lhp}(\tau)}t^{\mathrm{des}_s(\tau)}}{\prod_{i\in[p]}\left(1-tq^{\sum_{j=i}^{p} s(\pi_j)}\right)(1-t)}. \tag{6.2}$$

Proof. Set $x_i = q^{s(i)}$ and $y_i = q$ for all $1 \leq i \leq p$ in (3.5). □

Corollary 6.3. *If P is an anti-chain and $s : [p] \to \mathbb{Z}_+$, then*

$$\sum_{n\geq 0}\prod_{i=1}^{p}\left(u^n + [n]_u[s(i)]_q\right)t^n = \frac{\sum_{\tau\in\mathcal{L}(P,s)} q^{|r|}u^{\mathrm{comaj}(\tau)}t^{\mathrm{des}_s(\tau)}}{\prod_{i=0}^{p}(1-u^i t)}. \tag{6.3}$$

Proof. The corollary follows from Theorem 6.1 and Corollary 3.6. □

The *wreath product* of $\mathfrak{S}_p$ with a cyclic group of order k has elements

$$\mathbb{Z}_k \wr \mathfrak{S}_p = \{(\pi, r) : \pi \in \mathfrak{S}_p \text{ and } r : [p] \to \mathbb{Z}_k\}.$$

The elements of $\mathbb{Z}_k \wr \mathfrak{S}_p$ are often thought of as r-colored permutations. We may identify $\mathbb{Z}_k \wr \mathfrak{S}_p$ with $\mathcal{L}(P, s)$ where P is an anti-chain on $[p]$ and $s(i) = k$ for all $k \in [p]$. For $\tau = (\pi, r) \in \mathbb{Z}_k \wr \mathfrak{S}_p$ define

$$\mathrm{fmaj}(\tau) = |r| + k \cdot \mathrm{comaj}(\tau).$$

Note that $\mathrm{lhp}(\tau)$ agrees with $\mathrm{fmaj}(\tau)$ when $s = (k, k, \ldots, k)$.

Below we derive a Carlitz formula for $\mathbb{Z}_k \wr \mathfrak{S}_p$ first proved by Chow and Mansour in [12].

Corollary 6.4. *For positive integers p and k,*

$$\sum_{n\geq 0}[kn+1]_q^p t^n = \frac{\sum_{\tau\in\mathbb{Z}_k\wr\mathfrak{S}_p} t^{\mathrm{des}_s(\tau)}q^{\mathrm{fmaj}(\tau)}}{\prod_{i=0}^{p}\left(1-tq^{ki}\right)}. \tag{6.4}$$

Proof. Let $s = (k, k, \ldots, k)$ and set $u = q^k$ in (6.3). Then

$$\begin{aligned}\prod_{i=1}^{p} (u^n + [n]_u [s(i)]_q) &= \left(q^{nk} + [n]_{q^k}[k]_q\right)^p \\ &= \left(q^{nk} + \frac{q^{kn}-1}{q^k-1}\frac{q^k-1}{q-1}\right)^p \\ &= [nk+1]_q^p.\end{aligned}$$

The right hand side follows since $s(i) = k$ for all $1 \le i \le p$, and thus we sum over all $\tau \in \mathbb{Z}_k \wr \mathfrak{S}_p$. □

Remark 6.1. The definition of fmaj above differs from the definition of the flag major index fmaj_r in [12]. By the change in variables $q \to q^{-1}$ and $t \to tq^{kp}$ and by noting that $[kn+1]_q^p t^n$ is invariant under this change of variables we find that the two flag major indices have the same distribution.

Corollary 6.5. *For positive integers p and k,*

$$\sum_{n\ge 0} \prod_{i=1}^{p} (1 + n[k]_{q_i}) t^n = \frac{\sum_{\tau \in \mathbb{Z}_k \wr \mathfrak{S}_p} q_1^{r_1} q_2^{r_2} \cdots q_p^{r_p} t^{\mathrm{des}_s(\tau)}}{(1-t)^{p+1}}.$$

Proof. Let $s = (k, k, \ldots, k)$ and set $x_i = 1$ for all $1 \le i \le p$ in the equation displayed in Corollary 3.6. □

Remark 6.2. Note that when $q_i \ge 0$ for all $1 \le i \le p$, the polynomial

$$n \mapsto \prod_{i=1}^{p} (1 + n[k]_{q_i})$$

has all its zeros in the interval $[-1, 0]$. By an application of [28, Theorem 0.1] it follows that the polynomial

$$\sum_{\tau \in \mathbb{Z}_k \wr \mathfrak{S}_p} q_1^{r_1} q_2^{r_2} \cdots q_p^{r_p} t^{\mathrm{des}_s(\tau)}$$

is real-rooted in t. This generalizes [7, Theorem 6.4], where the case $k = 2$ was obtained.

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