Journal of Combinatorics
Volume 11, Number 3, 457–473, 2020

Unions of 1-factors in r-graphs and overfull graphs

Ligang Jin* and Eckhard Steffen

We prove lower bounds for the fraction of edges of an r-graph which can be covered by the union of k 1-factors. The special case $r = 3$ yields some known results for cubic graphs. Furthermore, we introduce the concept of k-overfull-free r-graphs and achieve better bounds for these graphs.

Keywords and phrases: r-graphs, 1-factors, overfull graphs.

1. Introduction

We consider finite graphs G with vertex set $V(G)$ and edge set $E(G)$. Graphs do not contain loops in this paper. For $v, w \in V(G)$, the number of edges between v and w is denoted by $\mu(v, w)$ and $\mu(G) = \max\{\mu(v, w)\colon\ v, w \in V(G)\}$. $\mu(v, w)$ is called the multiplicity of vw and $\mu(G)$ the multiplicity of G. A graph is simple if $\mu(v, w) \leq 1$ for any two vertices v, w. The number of edges which are incident to vertex v is the vertex degree of v which is denoted by $d_G(v)$. The maximum vertex degree of G is $\max\{d_G(v) : v \in V(G)\}$ and it is denoted by $\Delta(G)$. Further $\delta(G)$ denotes the minimum degree of a vertex of G.

1.1. 1-factor covering

The following celebrated conjecture, often referred to as the Berge-Fulkerson conjecture, is due to Fulkerson and appears first in [5]:

Conjecture 1.1 (Berge-Fulkerson conjecture [5]). *Every bridgeless cubic graph G has six 1-factors such that each edge of G is contained in precisely two of them.*

A set of such six 1-factors in the conjecture is called a Fulkerson cover of G. It is straightforward that Berge-Fulkerson Conjecture implies the existence of five 1-factors whose union is the edge-set of the graph G. This naturally raises a seemly weaker conjecture, attributed to Berge (unpublished, see e.g. [28]).

*Grant Numbers: NSFC 11801522 and QJD 1803023.

Conjecture 1.2 (Berge conjecture)**.** *Every bridgeless cubic graph G has five 1-factors such that each edge of G is contained in at least one of them.*

A set of the five 1-factors in Berge Conjecture is called a Berge cover of G. Recently, Mazzuoccolo [12] proved that the previous two conjectures are equivalent. It is unclear whether the same equivalence holds for every single bridgeless cubic graph, in other words, does a graph having a Berge cover always have a Fulkerson cover?

Let r be a positive integer. A graph G is r-regular, if $d_G(v) = r$ for all $v \in V(G)$. Let $X \subseteq V(G)$ be a set of vertices. The subgraph of G induced by X is denoted by $G[X]$, and the set of edges with precisely one end in X by $\partial_G(X)$. An r-regular graph G is an r-graph if $|\partial_G(X)| \geq r$ for every odd set $X \subseteq V(G)$.

A cubic graph is a 3-graph if and only if it is bridgeless. Moreover, it was proved in [22] that every r-graph has a 1-factor. Hence, it is natural to consider similar questions on perfect matching covering for r-graphs as for bridgeless cubic graphs. In particular, aforementioned two conjectures were generalized to r-graphs. In 1979, Seymour [22] proposed the generalized Berge-Fulkerson conjecture:

Conjecture 1.3 (Generalized Berge-Fulkerson conjecture [22])**.** *Every r-graph has $2r$ 1-factors such that each edge is contained in precisely two of them.*

Trivially, this conjecture implies the following generalized form of Conjecture 1.2, first proposed by Mazzuoccolo [13].

Conjecture 1.4 (Generalized Berge conjecture [13])**.** *Every r-graph G has $2r - 1$ 1-factors such that each edge is contained in at least one of them.*

The value $2r - 1$ in the conjecture is best possible, that is, it can not be smaller, as shown in [13]. In the same paper, Mazzuoccolo proved the equivalence between the generalized Berge-Fulkerson conjecture and the generalized Berge conjecture, in a similar way as he did for cubic case.

The excessive index $\chi'_e(G)$ of a graph G is the minimum number of 1-factors needed to cover $E(G)$. This parameter, also called the perfect matching index in [4], was widely studied in the literature, e.g., [1, 2, 13, 15, 16, 19]. It is reasonable to consider the excessive index for r-graphs in the context that it can be arbitrary large for some family of bridgeless r-regular graphs, constructed in [16]. However, it is an open question whether there exists a constant k such that $\chi'_e(G) \leq k$ for all r-graphs G for any fixed $r \geq 3$. The result of Mazzuoccolo [13] shows that if such k exists then it is at least $2r-1$. The generalized Berge conjecture asserts that such k exists and $k = 2r - 1$.

Partial covers of r-graphs with 1-factors are of great interest, see e.g. [8, 23]. In this paper, we consider the following relaxed form of the generalized Berge conjecture: Over all r-graphs G for any fixed r, what is the maximum constant c ($c \leq 1$), such that G has $2r-1$ 1-factors whose union contains at least $c|E(G)|$ edges? Note that the generalized Berge conjecture asserts that $c = 1$. We will show that $c \geq 1 - e^{-2} \approx 0.8647$. We will also show a second lower bound for c which depends on r, but which is always greater than $1 - e^{-2}$. In fact, this second lower bound is an approximation to the following more general problem.

Given an r-graph G, let $\mathcal{M}$ be the set of distinct 1-factors in G. Fix a positive integer k. Define

$$m(r, k, G) = \max_{M_1, \ldots, M_k \in \mathcal{M}} \frac{|\bigcup_{i=1}^{k} M_i|}{|E(G)|},$$

and

$$m(r, k) = \inf_{G} m(r, k, G),$$

where the infimum is taken over all r-graphs. Clearly, $m(r, k) \leq m(r, k+1) \leq 1$. With this notation, the generalized Berge conjecture can be reformulated as follows:

Conjecture 1.5. $m(r, 2r-1) = 1$ *for every integer* r *with* $r \geq 3$.

The parameter $m(r, k)$ has primarily been studied in cubic case, i.e. $r = 3$. Berge's conjecture states that $m(3, 5) = 1$. Kaiser, Král and Norine [9] proposed a lower bound for $m(3, k)$ as

$$m(3, k) \geq 1 - \prod_{i=1}^{k} \frac{i+1}{2i+1}, \tag{1}$$

and verified it for the case $k \in \{2, 3\}$. Meanwhile, Patel [17] conjectured that $m(3, 2) = \frac{3}{5}$, $m(3, 3) = \frac{4}{5}$ and $m(3, 4) = \frac{14}{15}$. Since the example of Petersen graph, the result of Kaiser, Král and Norine confirms that $m(3, 2) = \frac{3}{5}$. But the exact values for $m(3, 3)$ and $m(3, 4)$ are still unknown. A complete proof for the lower bound in (1) was later given by Mazzuoccolo [14].

In Section 3, we obtain the following lower bound for $m(r, k)$:

$$m(r, k) \geq 1 - \prod_{i=1}^{k} \frac{(r^2 - 3r + 1)i - (r^2 - 5r + 3)}{(r^2 - 2r - 1)i - (r^2 - 4r - 1)} \tag{2}$$

Table 1: Approximate values of the two lower bounds for $m(r,k)$ presented in formulations (2) and (3) and in Theorem 3.1, shown respectively in the left and the right sides of the inequality in the table. In particular, the one for $m(r,2r-1)$ is presented in bold

	$r=3$	$r=4$	$r=5$
$m(r,2)\geq$	$0.6\geq 0.5556$	$0.45\geq 0.4375$	$0.3714\geq 0.36$
$m(r,3)\geq$	$0.7714\geq 0.7037$	$0.6\geq 0.5781$	$0.5081\geq 0.488$
$m(r,4)\geq$	$0.873\geq 0.8025$	$0.7103\geq 0.6836$	$0.6157\geq 0.5904$
$m(r,5)\geq$	$\mathbf{0.9307\geq 0.8683}$	$0.7908\geq 0.7627$	$0.7\geq 0.6723$
$m(r,6)\geq$	$0.9627\geq 0.9122$	$0.8492\geq 0.822$	$0.766\geq 0.7379$
$m(r,7)\geq$	$0.9801\geq 0.9415$	$\mathbf{0.8914\geq 0.8665}$	$0.8176\geq 0.7903$
$m(r,8)\geq$	$0.9895\geq 0.961$	$0.9219\geq 0.8999$	$0.8578\geq 0.8322$
$m(r,9)\geq$	$0.9945\geq 0.974$	$0.9439\geq 0.9249$	$\mathbf{0.8892\geq 0.8658}$

for any even $r\geq 4$ and any $k\geq 1$, and

$$m(r,k)\geq 1-\prod_{i=1}^{k}\frac{(r^2-2r-1)i-(r^2-4r+1)}{(r^2-r-2)i-(r^2-3r-2)} \tag{3}$$

for any odd $r\geq 3$ and any $k\geq 1$.

For instance of small r and k, the values of this lower bound are listed in Table 1.

In particular, if we take $r=3$, this lower bound coincides with the established bound in (1); if we take $k=2r-1$, it gives a partial result to the generalized Berge conjecture, and the approximate value of $m(r,2r-1)$ is shown in bold in Table 1.

Now we are going to show that the lower bounds in (2) and (3) is always better than $1-e^{-2}$. Let $f(x)$ be a function defined by

$$f(x)=\frac{(r^2-3r+1)x-(r^2-5r+3)}{(r^2-2r-1)x-(r^2-4r-1)}.$$

We can calculate that $f(1)=\frac{r-1}{r}$ and the derivative

$$f'(x)=\frac{-2}{[(r^2-2r-1)x-(r^2-4r-1)]^2}<0.$$

Hence, $f(i)\leq\frac{r-1}{r}$ for each $i\geq 1$. So when we take $k=2r-1$, the lower bound in (2) is greater than

$$1-(\frac{r-1}{r})^{2r-1}>1-e^{-2},$$

where the last inequality is given by Corollary 3.2. Similarly, when $k = 2r-1$, we can deduce that the lower bound in (3) is greater than $1 - e^{-2}$ as well.

1.2. Edge-colorings and overfull graphs

A graph G is k-overfull if $|V(G)|$ is odd, $\Delta(G) \le k$ and $\frac{|E(G)|}{\lfloor \frac{1}{2}|V(G)| \rfloor} > k$. It is easy to see that G is k-overfull if and only if $2|E(G)| > k(|V(G)| - 1)$. Furthermore, the k-deficiency of G is $k|V(G)| - 2|E(G)|$, and it is denoted by $s_k(G)$.

A k-edge-coloring of G is a mapping $c : E(G) \to \{1, \dots, k\}$ such that adjacent edges are colored differently. The chromatic index $\chi'(G)$ is the minimum number k such that G has a k-edge-coloring. Vizing [26] proved that $\Delta(G) \le \chi'(G) \le \Delta(G) + \mu(G)$, in particular if G is simple, then $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$. We say that G is class 1 if $\chi'(G) = \Delta(G)$, and it is class 2 otherwise.

Clearly, $\Delta(G)$ is a lower bound for the chromatic index of G. Overfull graphs are class 2 graphs for the trivial reason that they contain too many edges. In general we have that $\chi'(G) \ge \max_{H \subseteq G} \lceil \frac{|E(H)|}{\lfloor \frac{1}{2}|V(H)| \rfloor} \rceil$.

A graph G is critical with respect to $\chi'(G)$, if $\chi'(G - e) < \chi'(G)$ for every $e \in E(G)$. For simple graphs we have the definition of a k-critical graph which says that a critical graph H is k-critical, if $\Delta(H) = k$ and $\chi'(H) = k + 1$. Vizing [27] proved the classical result that a simple class 2 graph with maximum degree k contains a t-critical subgraph for every $t \in \{2, \dots, k\}$. These results are the motivation for the result of Section 4, which proves that a k-overfull graph contains a t-overfull subgraph for every $t \in \{2, \dots, k\}$.

A graph G is k-overfull-free, if it does not contain a k-overfull subgraph. Clearly, there are no 1-critical graphs and the 2-critical graphs are the odd circuits which are also the connected 2-overfull graphs. Hence we have: A graph is 2-overfull-free if and only if it is bipartite. We study k-overfull-free graphs in Section 4. If an r-regular graph G is class 1, then surely G is an r-graph and G is r-overfull-free. For $i \in \{1, 2\}$ let G_i be an r_i-regular graph. We say that an r-graph G is decomposable into G_1 and G_2 if $r = r_1 + r_2$, $V(G_i) = V(G)$ and $E(G) = E(G_1) \cup E(G_2)$. We will characterize some decomposable r-graphs in terms of excluded overfull subgraphs.

2. The perfect matching polytope

Let G be a graph and w be a vector of $\mathbb{R}^{E(G)}$. The entry of w corresponding to an edge e is denoted by $w(e)$, and for $A \subseteq E$, we define $w(A) = \sum_{e \in A} w(e)$. The vector w is a *fractional 1-factor* if it satisfies

(i) $0 \leq w(e) \leq 1$ for every $e \in E(G)$, and
(ii) $w(\partial(\{v\})) = 1$ for every $v \in V(G)$, and
(iii) $w(\partial(S)) \geq 1$ for every $S \subseteq V(G)$ with odd cardinality.

Let $F(G)$ denote the set of all fractional 1-factors of a graph G. If M is a 1-factor, then its characteristic vector χ^M is contained in $F(G)$. Furthermore, if $w_1, \ldots, w_n \in F(G)$, then any convex combination $\sum_{i=1}^{n} \alpha_i w_i$ (where $\alpha_1, \ldots, \alpha_n$ are nonnegative real numbers summing up to 1) also belongs to $F(G)$. It follows that $F(G)$ contains the convex hull of all the vectors χ^M where M is a 1-factor of G. The following theorem by Edmonds asserts that the converse inclusion also holds:

Theorem 2.1 (Perfect Matching Polytope Theorem [3])**.** *For any graph G, the set $F(G)$ coincides with the convex hull of the characteristic vectors of all 1-factors of G.*

Towards the generalized Berge-Fulkerson conjecture, Seymour [22] gave an alternative proof of the following theorem, which is a corollary of Edmonds's matching polytope theorem (see [22] for the details between these two theorems).

Theorem 2.2 ([22])**.** *For any r-graph G, there is a positive integer p such that G has rp 1-factors and each edge is contained in precisely p of them.*

We will use this theorem to deduce our first lower bound in the next section. Moreover, the following property on fractional 1-factors will play a crucial role in the proof for our second lower bound.

Lemma 2.3 ([9])**.** *Let w be a fractional 1-factor of a graph G and $c \in \mathbb{R}^{E(G)}$. Then G has a 1-factor M such that $c \cdot \chi^M \geq c \cdot w$, where $\cdot$ denotes the scalar product, and $|M \cap C| = 1$ for each edge-cut C of odd cardinality and with $w(C) = 1$.*

The proof of this lemma was given in [9], where Theorem 2.1 is the main tool for the proof.

3. Lower bounds for $m(r, k)$

We are going to deduce a lower bound for the parameter $m(r, k)$ by using Theorem 2.2 only.

Theorem 3.1. *$m(r, k) \geq 1 - (\frac{r-1}{r})^k$ for every positive integers r and k with $r \geq 3$.*

Proof. (induction on k.) Since every r-graph has a 1-factor, which covers a fraction $\frac{1}{r}$ of the edges, the proof is trivial for $k = 1$. We proceed to the induction step. Let G be any r-graph and $E = E(G)$. By the induction hypothesis, G has $k - 1$ many 1-factors $M_1, \ldots, M_{k-1}$ such that

$$\frac{|\bigcup_{i=1}^{k-1} M_i|}{|E|} \geq 1 - (\frac{r-1}{r})^{k-1}. \tag{4}$$

Moreover, by Theorem 2.2, there exists a positive integer p such that G has rp 1-factors $F_1, \ldots, F_{rp}$ and each edge is contained in precisely p of them. It follows that for every $X \subseteq E$, graph G has a 1-factor F among $F_1, \ldots, F_{rp}$ such that $|F \cap X| \geq \frac{|X|}{r}$. In particular, let $X = E \backslash \bigcup_{i=1}^{k-1} M_i$ and consequently, take $M_k = F$. Thus,

$$|M_k \cap (E \setminus \bigcup_{i=1}^{k-1} M_i)| \geq \frac{|E \setminus \bigcup_{i=1}^{k-1} M_i|}{r}, \tag{5}$$

that is,

$$\frac{|\bigcup_{i=1}^{k} M_i| - |\bigcup_{i=1}^{k-1} M_i|}{|E|} \geq \frac{1}{r}(1 - \frac{|\bigcup_{i=1}^{k-1} M_i|}{|E|}). \tag{6}$$

It follows that

$$\frac{|\bigcup_{i=1}^{k} M_i|}{|E|} \geq (1 - \frac{1}{r})\frac{|\bigcup_{i=1}^{k-1} M_i|}{|E|} + \frac{1}{r} \geq 1 - (\frac{r-1}{r})^k \tag{7}$$

where the last inequality follows by using the inequality (4). Therefore, $m(r, k, G) \geq 1 - (\frac{r-1}{r})^k$ and by the choice of G, we have $m(r, k) \geq 1 - (\frac{r-1}{r})^k$. □

In particular, if we take $k = 2r - 1$, we can further deduce from this theorem a constant lower bound for $m(r, 2r - 1)$.

Corollary 3.2. *For every integer $r \geq 3$, we have $m(r, 2r - 1) \geq 1 - e^{-2} \approx 0.8647$.*

Proof. Let $f(r)$ denote the function $1 - (\frac{r-1}{r})^{2r-1}$. It is easy to see that $f(r)$ is strictly monotonic decreasing with respect to r. Moreover, $\lim_{r \to +\infty} f(r) = 1 - e^{-2}$. It follows with Theorem 3.1 that $m(r, 2r - 1) \geq f(r) \geq 1 - e^{-2}$. □

We now prove the following theorem, which will be used to deduce a second lower bound for $m(r,k)$. An i-cut of a graph G is an edge cut of G of cardinality i. The proof of the theorem is conducted by induction. In the induction step, we apply Lemma 2.3 to a well-chosen fractional 1-factor, whose existence can be guaranteed by both inclusions of the induction hypothesis, one on the union of 1-factors and the other on i-cuts. The resulting 1-factor with its properties described in Lemma 2.3 and the 1-factors given by the induction hypothesis together complete the proof.

Theorem 3.3. *Let G be an r-graph, and $V = V(G)$ and $E = E(G)$.*

(a) If r is even and $r \geq 4$, then for any positive integer k, graph G has k 1-factors $M_1, \dots, M_k$ such that

$$\frac{|\bigcup_{i=1}^{k} M_i|}{|E|} \geq 1 - \prod_{i=1}^{k} \frac{(r^2-3r+1)i-(r^2-5r+3)}{(r^2-2r-1)i-(r^2-4r-1)}$$

and $\sum_{i=1}^{k} \chi^{M_i}(C) \leq (r-1)k+2$ for each $(r+1)$-cut C.

(b) If r is odd and $r \geq 3$, then for any positive integer k, graph G has k 1-factors $M_1, \dots, M_k$ such that

$$\frac{|\bigcup_{i=1}^{k} M_i|}{|E|} \geq 1 - \prod_{i=1}^{k} \frac{(r^2-2r-1)i-(r^2-4r+1)}{(r^2-r-2)i-(r^2-3r-2)},$$

$\sum_{i=1}^{k} \chi^{M_i}(C) = k$ for each r-cut C and $\sum_{i=1}^{k} \chi^{M_i}(D) \leq rk+2$ for each $(r+2)$-cut D.

Proof. (induction on k).

Statement (a). The statement holds for $k=1$, since the required M_1 can be an arbitrary 1-factor of G. Assume that $k \geq 2$. By the induction hypothesis, G has $k-1$ many 1-factors $M_1, \dots, M_{k-1}$ such that

$$\frac{|\bigcup_{i=1}^{k-1} M_i|}{|E|} \geq 1 - \prod_{i=1}^{k-1} \frac{(r^2-3r+1)i-(r^2-5r+3)}{(r^2-2r-1)i-(r^2-4r-1)}$$

and

$$\sum_{i=1}^{k-1} \chi^{M_i}(C) \leq (r-1)(k-1)+2 \tag{8}$$

for each $(r+1)$-cut C.

For $e \in E$, let $n(e)$ denote the number of 1-factors among $M_1, \ldots, M_{k-1}$ which contain e, and define

$$w_k(e) = \frac{(r-2)k - (r-4) - n(e)}{(r^2-2r-1)k - (r^2-4r-1)}.$$

We claim that w_k is a fractional 1-factor of G, that is, $w_k \in F(G)$. Since $k \geq 2, r \geq 4$ and $0 \leq n(e) \leq k-1$, we can deduce that $\frac{1}{r+3} < w_k(e) < 1$. Moreover, note that for every $X \subseteq E$, the equality $\sum_{e \in X} n(e) = \sum_{i=1}^{k-1} \chi^{M_i}(X)$ always holds and so

$$w_k(X) = \sum_{e \in X} w_k(e) = \frac{[(r-2)k-(r-4)]|X| - \sum_{i=1}^{k-1} \chi^{M_i}(X)}{(r^2-2r-1)k-(r^2-4r-1)}. \tag{9}$$

Thus for $v \in V$, since $\sum_{i=1}^{k-1} \chi^{M_i}(\partial(\{v\})) = k-1$, we have $w_k(\partial(\{v\})) = \frac{[(r-2)k-(r-4)]r-(k-1)}{(r^2-2r-1)k-(r^2-4r-1)} = 1$. Finally, let $S \subseteq V$ with odd cardinality. Since G is an r-graph, we have $|\partial(S)| \geq r$. On the other hand, by recalling that $w_k(e) > \frac{1}{r+3}$ for each edge e, we have $w_k(\partial(S)) > 1$ provided by $|\partial(S)| \geq r+3$. Hence, we may next assume that $|\partial(S)| = r+1$ by parity. Since in this case S is a $(r+1)$-cut, the formula (8) implies $\sum_{i=1}^{k-1} \chi^{M_i}(\partial(S)) \leq (r-1)(k-1)+2$, and thus with the help of the formula (9), we deduce $w_k(\partial(S)) \geq \frac{[(r-2)k-(r-4)](r+1)-[(r-1)(k-1)+2]}{(r^2-2r-1)k-(r^2-4r-1)} = 1$. This completes the proof of the claim.

By Lemma 2.3, the graph G has a 1-factor M_k such that

$$(1-\chi^{\bigcup_{i=1}^{k-1} M_i}) \cdot \chi^{M_k} \geq (1-\chi^{\bigcup_{i=1}^{k-1} M_i}) \cdot w_k.$$

Since the left side is just $|\bigcup_{i=1}^{k} M_i| - |\bigcup_{i=1}^{k-1} M_i|$ and the right side equals to $\frac{(r-2)k-(r-4)}{(r^2-2r-1)k-(r^2-4r-1)}(|E| - |\bigcup_{i=1}^{k-1} M_i|)$, it follows that

$$\begin{aligned}|\bigcup_{i=1}^{k} M_i| \geq \frac{(r^2-3r+1)k-(r^2-5r+3)}{(r^2-2r-1)k-(r^2-4r-1)} |\bigcup_{i=1}^{k-1} M_i| \\ + \frac{(r-2)k-(r-4)}{(r^2-2r-1)k-(r^2-4r-1)}|E|,\end{aligned}$$

which leads to

$$\frac{|\bigcup_{i=1}^{k} M_i|}{|E|} \geq 1 - \prod_{i=1}^{k} \frac{(r^2-3r+1)i-(r^2-5r+3)}{(r^2-2r-1)i-(r^2-4r-1)},$$

as desired.

Moreover, let C be an edge cut with cardinality $r+1$. Clearly, $\chi^{M_k}(C) \leq r+1$. Thus, if $\sum_{i=1}^{k-1}\chi^{M_i}(C) \leq (r-1)(k-1)$ then $\sum_{i=1}^{k}\chi^{M_i}(C) \leq (r-1)k+2$, as desired. By the formula (8) and by parity, we may next assume that $\sum_{i=1}^{k-1}\chi^{M_i}(C) = (r-1)(k-1)+2$. In this case, we calculate from the formula (9) that $w_k(C)=1$. Thus $\chi^{M_k}(C)=1$ by Lemma 2.3, which yields $\sum_{i=1}^{k}\chi^{M_i}(C) = (r-1)k-r+4 < (r-1)k+2$, as desired. This completes the proof of Statement (a).

Statement (b). We follow a similar way to prove this statement as we did for Statement (a). Let w_1 be a vector of $\mathbb{R}^E$ defined by $w_1(e)=\frac{1}{r}$ for $e \in E$. Clearly, $w_1 \in F(G)$. By Lemma 2.3, G has a 1-factor M_1 such that $\chi^{M_1}(C)=1$ for each edge cut C with odd cardinality and with $w_1(C)=1$, that is, for each r-cut C. Therefore, the statement is true for $k=1$.

Assume $k \geq 2$. By the induction hypothesis, G has $k-1$ many 1-factors $M_1,\dots,M_{k-1}$ such that

$$\frac{|\bigcup_{i=1}^{k-1} M_i|}{|E|} \geq 1 - \prod_{i=1}^{k-1} \frac{(r^2-2r-1)i-(r^2-4r+1)}{(r^2-r-2)i-(r^2-3r-2)},$$

and for each r-cut C

$$\sum_{i=1}^{k-1}\chi^{M_i}(C) = k-1, \tag{10}$$

and for each $(r+2)$-cut D

$$\sum_{i=1}^{k-1}\chi^{M_i}(D) \leq r(k-1)+2. \tag{11}$$

For $e \in E$, let $n(e)$ denote the number of 1-factors among $M_1,\dots,M_{k-1}$ that contains e, and define

$$w_k(e) = \frac{(r-1)k-(r-3)-2n(e)}{(r^2-r-2)k-(r^2-3r-2)}.$$

We claim that $w_k \in F(G)$. Since $k \geq 2, r \geq 3$ and $0 \leq n(e) \leq k-1$, we can deduce that $0 < \frac{1}{r+4} < w_k(e) < 1$. Moreover, note that for every

$X \subseteq E$, the equality $\sum_{e\in X} n(e) = \sum_{i=1}^{k-1} \chi^{M_i}(X)$ always holds and so

$$w_k(X) = \frac{[(r-1)k-(r-3)]|X| - 2\sum_{i=1}^{k-1} \chi^{M_i}(X)}{(r^2-r-2)k-(r^2-3r-2)}. \tag{12}$$

Thus for $v \in V$, since $\sum_{i=1}^{k-1} \chi^{M_i}(\partial(\{v\})) = k-1$, we have $w_k(\partial(\{v\})) = \frac{[(r-1)k-(r-3)]r-2(k-1)}{(r^2-r-2)k-(r^2-3r-2)} = 1$. Finally, let $S \subseteq V$ with odd cardinality. Since G is an r-graph, $|\partial(S)| \geq r$. On the other hand, by recalling that $w_k(e) > \frac{1}{r+4}$ for each edge e, we have $w_k(\partial(S)) > 1$ provided by $|\partial(S)| \geq r+4$. Hence, we may next assume that either $|\partial(S)| = r$ or $|\partial(S)| = r+2$ by parity. In the former case, the formula (10) implies $\sum_{i=1}^{k-1} \chi^{M_i}(\partial(S)) = k-1$, and thus we can calculate from the formula (12) that $w_k(\partial(S)) = 1$. In the latter case, the formula (11) implies $\sum_{i=1}^{k-1} \chi^{M_i}(\partial(S)) \leq r(k-1)+2$ and similarly, we get $w_k(\partial(S)) \geq \frac{[(r-1)k-(r-3)](r+2)-2[r(k-1)+2]}{(r^2-r-2)k-(r^2-3r-2)} = 1$. This proves the claim.

By Lemma 2.3, the graph G has a 1-factor M_k such that

$$(1-\chi^{\bigcup_{i=1}^{k-1} M_i}) \cdot \chi^{M_k} \geq (1-\chi^{\bigcup_{i=1}^{k-1} M_i}) \cdot w_k.$$

Since the left side is just $|\bigcup_{i=1}^{k} M_i| - |\bigcup_{i=1}^{k-1} M_i|$ and the right side equals to $\frac{(r-1)k-(r-3)}{(r^2-r-2)k-(r^2-3r-2)}(|E| - |\bigcup_{i=1}^{k-1} M_i|)$, it follows that

$$|\bigcup_{i=1}^{k} M_i| \geq \frac{(r-1)k-(r-3)}{(r^2-r-2)k-(r^2-3r-2)}|E| + \frac{(r^2-2r-1)k-(r^2-4r+1)}{(r^2-r-2)k-(r^2-3r-2)}|\bigcup_{i=1}^{k-1} M_i|,$$

which leads to

$$\frac{|\bigcup_{i=1}^{k} M_i|}{|E|} \geq 1 - \prod_{i=1}^{k} \frac{(r^2-2r-1)i-(r^2-4r+1)}{(r^2-r-2)i-(r^2-3r-2)},$$

as desired.

Moreover, let C be an edge cut of cardinality r. The formula (10) implies $\sum_{i=1}^{k-1} \chi^{M_i}(C) = k-1$. On the other hand, We can calculate from the formula (12) that $w_k(C) = 1$, and thus $\chi^{M_k}(C) = 1$ by Lemma 2.3. Therefore, $\sum_{i=1}^{k} \chi^{M_i}(C) = k$, as desired.

We next let D be an edge cut of cardinality $r+2$. Clearly, $\chi^{M_k}(D) \leq r+2$. Thus if $\sum_{i=1}^{k-1} \chi^{M_i}(D) \leq r(k-1)$, then $\sum_{i=1}^{k} \chi^{M_i}(D) \leq rk+2$, as desired. By the formula (11) and by parity, we may next assume that $\sum_{i=1}^{k-1}\chi^{M_i}(D) = r(k-1)+2$. By calculation we can get $w_k(D)=1$, and thus $\chi^{M_k}(D)=1$ by Lemma 2.3, which also yields $\sum_{i=1}^{k}\chi^{M_i}(D)\leq rk+2$. This completes the proof of this theorem. □

The following corollary is a direct consequence of this theorem.

Corollary 3.4. *Let r and k be two positive integers with $r\geq 3$. If r is even then*

$$m(r,k)\geq 1-\prod_{i=1}^{k}\frac{(r^2-3r+1)i-(r^2-5r+3)}{(r^2-2r-1)i-(r^2-4r-1)},$$

and if r is odd then

$$m(r,k)\geq 1-\prod_{i=1}^{k}\frac{(r^2-2r-1)i-(r^2-4r+1)}{(r^2-r-2)i-(r^2-3r-2)}.$$

4. Overfull graphs

We start with the following observations.

Observation 4.1. *Let $r\geq 2$ be an integer. Every r-overfull-free r-regular graph is an r-graph.*

Observation 4.2. *A graph G is 2-overfull-free if and only if G is bipartite.*

If G is a graph, then $o(G)$ denotes the number of odd components of G. We will use the following theorem of Tutte.

Theorem 4.3 ([25])**.** *A graph G has a 1-factor if and only of $o(G-S)\leq |S|$ for all $S\subseteq V(G)$.*

Proposition 4.4. *Let $k\geq 2$. If G is a k-overfull graph, then $0\leq s_k(G)\leq k-2$ and $k<\frac{|V(G)|}{|V(G)|-1}\Delta(G)$.*

Proof. Since G is k-overfull, $2|E(G)|>k(|V(G)|-1)$ and $|V(G)|$ is odd. Notice that the two sides of this inequality has the same parity. So, $2|E(G)|\geq k(|V(G)|-1)+2$, that is, $s_k(G)\leq k-2$. Moreover, by Handshaking Lemma, $2|E(G)|=\sum_{v\in V(G)}d_G(v)\leq \Delta(G)|V(G)|$. Combining it with the fact that $2|E(G)|>k(|V(G)|-1)$, we deduce that $k<\frac{|V(G)|}{|V(G)|-1}\Delta(G)$. □

Theorem 4.5. *Let $k\geq 3$ be an integer. Every k-overfull graph contains a $(k-1)$-overfull subgraph.*

Proof. Suppose to the contrary that the statement is not true. Then there is a k-overfull graph G which does not contain a $(k-1)$-overfull subgraph. We may assume that $|V(G)|$ is minimum and according to this property $|E(G)|$ is minimum as well.

It holds $\Delta(G) = k$, since for otherwise G is $(k-1)$-overfull as well, a contradiction.

Claim 4.5.1. *Let H be a proper subgraph of G. If H is of odd order, then $s_k(H) \geq k$.*

By the minimality of G, the subgraph H is not k-overfull. Note that H is of odd order and has maximum degree at most k. Thus, $\frac{2|E(H)|}{|V(H)|-1} \leq k$ and therefore, $s_k(H) = k|V(H)| - 2|E(H)| \geq k|V(H)| - k|V(H)| + k = k$.

Claim 4.5.2. $s_k(G) = k - 2$*, that is,* $2|E(G)| = k(|V(G)| - 1) + 2$.

Choose any edge e of G. By Claim 4.5.1, $s_k(G-e) \geq k$. It follows that $s_k(G) = s_k(G-e) - 2 \geq k - 2$. On the other hand, $s_k(G) \leq k - 2$ by Proposition 4.4. Therefore, $s_k(G) = k - 2$.

Claim 4.5.3. *For every $z \in V(G)$, the graph $G - z$ has a 1-factor.*

Let $G' = G - z$. Then $s_k(G') = s_k(G) + d_G(z) - (k - d_G(z)) = k - 2 + 2d_G(z) - k = 2d_G(z) - 2 \leq 2k - 2$.

Suppose to the contrary that G' does not have a 1-factor. By Theorem 4.3, there is $S \subseteq V(G')$ such that $o(G' - S) > |S|$. Let $O_1, \ldots, O_n$ be the odd components of $G' - S$. Since $G - z$ has even order, n and $|S|$ have the same parity. Thus, $n \geq |S| + 2$.

With Claim 4.5.1 it follows that $s_k(O_i) \geq k$. Hence, $|\partial_G(S)| \geq \sum_{i=1}^{n} s_k(O_i) - s_k(G') \geq nk - 2k + 2 = k(n-2) + 2 \geq k|S| + 2$, a contradiction.

We now deduce the statement. If G is regular, then $s_k(G) = 0 = k - 2$. So $k = 2$, a contradiction. Hence, there is $z \in V(G)$ such that $d_G(z) < k$. By Claim 4.5.3, $G - z$ has a 1-factor F. Let $G' = G - F$. Then $\Delta(G') = k - 1$, $|E(G')| = |E(G)| - \frac{1}{2}(|V(G)| - 1)$, and $|V(G')| = |V(G)|$. Hence, $\frac{2|E(G')|}{|V(G')|-1} = \frac{2|E(G)|-(|V(G)|-1)}{|V(G)|-1} = \frac{2|E(G)|}{|V(G)|-1} - 1 > k - 1$. This contradicts our assumption that G does not contain a $(k-1)$-overfull subgraph and the statement is proved. □

The following corollaries are immediate consequences of Theorem 4.5. The first one has the same flavor as a result of Vizing [27] that a class 2 graph with chromatic index k contains critical subgraphs with chromatic index t for every $t \in \{2, \ldots, k\}$.

Corollary 4.6. *Let $k \geq 2$ be an integer and G be a graph. If G is k-overfull, then G contains a t-overfull subgraph for every $t \in \{2, \dots, k\}$.*

Corollary 4.7. *Let $k \geq 2$ be an integer and G be a graph. If G is k-overfull-free, then G is t-overfull-free for every $t \geq k$.*

Corollary 4.8. *Let $2 \leq k \leq r$ be integers and G be an r-regular graph. If G is k-overfull-free, then G is an r-graph and G can be decomposed into a $(r-k)$-graph that is class 1 and a k-graph.*

Proof. By Corollary 4.7, G is r-overfull-free and further, by Observation 4.1, G is an r-graph. Let F_1 be a 1-factor of G. Consider $G - F_1$. If $k = r$, then we are done. Hence, we may assume $k \leq r - 1$. Similarly, we can deduce that $G - F_1$ is an $(r-1)$-graph having a 1-factor F_2. Continue as above till $G' = G - \bigcup_{i=1}^{r-k} F_i$. Then G' and $G'' = (V(G), \bigcup_{i=1}^{r-k} F_i)$ is the desired decomposition. □

Corollary 4.8 gives a sufficient condition for an r-graph decomposable into a r_1-graph and a r_2-graph for some r_1 and r_2. It also shows that for any t-overfull-free r-graph with $2 \leq t \leq r$, we can obtain a better lower bound of $m(r, k, G)$ than the one of $m(r, k)$. More precisely, for $k \leq r - t$, take k pairwise disjoint 1-factors of the class 1 graph from the decomposition by Corollary 4.8, which gives $m(r, k, G) = \frac{k}{r}$. For $k > r - t$, applying Theorem 3.3 to the t-graph from the decomposition by Corollary 4.8 gives k 1-factors, which together with any $r - t$ many pairwise disjoint 1-factors of the class 1 graph from the decomposition leads to a better lower bound for (m, k, G).

Moreover, Corollary 4.8 confirms the following classical result of König [11].

Theorem 4.9. *Let $k \geq 0$ be an integer. Every k-regular bipartite graph is class 1.*

Proof. For $k \in \{0, 1\}$, the proof is trivial. For $k \geq 2$, let G be a k-regular bipartite graph. By Observation 4.2, G is 2-overfull-free. By Corollary 4.8, G is decomposable into a class 1 subgraph and a 1-factor. Thus, G is class 1. □

Acknowledgements

The authors are grateful to Giuseppe Mazzuoccolo for his useful comments and suggestions on 1-factor covering. The authors are also grateful to two anonymous referees for their very careful reading and helpful comments.

References

[1] Bonisoli, A. and Cariolaro, D. (2007). Excessive factorizations of regular graphs, Graph Theory in Paris, *in: Thrend Math.* Birkhäuser, Basel, 73–84. MR2279168

[2] Cariolaro, D. and Rizzi, R. (2016). On the complexity of computing the excessive [B]-index of a graph. *J. Graph Theory* **82**(1) 65–74. MR3477033

[3] Edmonds, J. (1965). Maximum matching and a polyhedron with (0,1)-vertices. *J. Res. Nat. Bur. Standards Sect. B* **69** 125–130. MR0183532

[4] Fouquet, J. L. and Vanherpe, J. M. (2009). On the perfect matching index of bridgeless cubic graphs. *arXiv:0904.1296.* MR2574471

[5] Fulkerson, D. R. (1971). Blocking and anti-blocking pairs of polyhedra. *J. Combin. Theory Ser. B* **1** 168–194. MR0294149

[6] Jakobsen, I. T. (1975). On critical graphs with respect to edge-colouring, *in: A. Hajnal, R. Rado, V. T. Sós (eds.) Infinite and Finite Sets.* North Holland, Amsterdam, 927–934. MR0389643

[7] Jensen, T. R. and Toft, B. (1995). *Graph Coloring Problems.* John Wiley & Sons, Inc. MR1304254

[8] Jin, L. and Steffen, E. (2017). Petersen cores and the oddness of cubic graphs. *J. Graph Theory* **84** 109–120. MR3601120

[9] Kaiser, T. and Král, D. and Norine, S. (2006). Unions of perfect matching in cubic graphs, Topics in Discrete Mathematics, *in: Algorithms Combin.* **26**. Springer, Berlin, 225–230. MR2249273

[10] Kierstead, H. A. (1984). On the chromatic index of multigraphs without large triangles. *J. Combin. Theory Ser. B* **36** 156–160. MR0746546

[11] König, D. (1931). Graphs and matrices. *Mat. Fiz. Lapok.* **38** 116–119.

[12] Mazzuoccolo, G. (2011). The equivalence of two conjectures of Berge and Fulkerson. *J. Graph Theory* **68** 125–128. MR2833954

[13] Mazzuoccolo, G. (2013). An upper bound for the excessive index of an r-graph. *J. Graph Theory* **73** 377–385. MR3065110

[14] Mazzuoccolo, G. (2013). Covering a cubic graph with perfect matchings. *Discrete Math.* **313** 2292–2296. MR3084274

[15] Mazzuoccolo, G. (2014). On the excessive [m]-index of a tree. *Discrete Applied Math.* **162** 264–270. MR3128529

[16] Mazzuoccolo, G. and Young, M. (2011). Graphs of arbitrary excessive class. *Discrete Math.* **311** 32–37. MR2737965

[17] Patel V. (2006). Unions of perfect matchings in cubic graphs and implications of the Berge-Fulkerson conjecture. *CDAM Research Report*, LSE-CDAM-2006-06.

[18] Petersen, J. (1891). Die Theorie der regulären Graphs. *Acta Math.* **15** 193–220. MR1554815

[19] Rajasingh, I. and Bharati, R. and Muthumalai, A. and Shanthi, A. S. (2013). On excessive index of certain networks. *Theor. Comput. Sci.* **501** 34–40. MR3090089

[20] Rizzi, R. (1999). Indecomposable r-graphs and some other counterexamples. *J. Graph Theory* **32** 1–15. MR1704172

[21] Scheide, D. (2010). Graph edge colouring: Tashkinov trees and Goldberg's conjecture. *J. Combin. Theory Ser. B* **100** 68–96. MR2563514

[22] Seymour, P. D. (1979). On multicolourings of cubic graphs, and conjectures of Fulkerson and Tutte. *Proc. London Math. Soc.* **38** 423–460. MR0532981

[23] Steffen, E. (2015). 1-factor and cycle covers of cubic graphs. *J. Graph Theory* **78** 195–206. MR3298703

[24] Stiebitz, M. and Scheide, D. and Toft, B. and Favrholdt, L. M. (2012). *Edge Graph Coloring: Vizing's Theorem and Goldberg's Conjecture*. John Wiley & Sons, Inc., Hoboken, NJ. MR2975974

[25] Tutte, W. T. (1947). The factorization of linear graphs. *J. London Math. Soc.* **22** 107–111. MR0023048

[26] Vizing, V. G. (1964). On an estimate of the chromatic class of a p-graph. *Diskretnyi Analiz* **3** 25–30. MR0180505

[27] Vizing, V. G. (1965). Critical graphs with given chromatic class. *Diskretnyi Analiz* **5** 9–17. MR0200202

[28] Zhang, C. Q. (1997). *Integer Flows and Cycle Covers of Graphs.* Marcel Dekker Inc. New York, Basel, Hong Kong. MR1426132

Ligang Jin
Department of Mathematics,
Zhejiang Normal University,
Yingbin Road 688,
321004 Jinhua,
China
E-mail address: ligang.jin@zjnu.cn

Eckhard Steffen
Paderborn Center for Advanced Studies
and Institute for Mathematics,
Paderborn University,
Warburger Str. 100,
33098 Paderborn,
Germany
E-mail address: es@upb.de

Received September 1, 2018