Journal of Combinatorics
Volume 11, Number 3, 475–493, 2020

The probability of positivity in symmetric and quasisymmetric functions

Rebecca Patrias* and Stephanie van Willigenburg†

Given an element in a finite-dimensional real vector space, V, that is a nonnegative linear combination of basis vectors for some basis B, we compute the probability that it is furthermore a nonnegative linear combination of basis vectors for a second basis, A. We then apply this general result to combinatorially compute the probability that a symmetric function is Schur-positive (recovering the recent result of Bergeron–Patrias–Reiner), e-positive or h-positive. Similarly we compute the probability that a quasisymmetric function is quasisymmetric Schur-positive or fundamental-positive. In every case we conclude that the probability tends to zero as the degree of the function tends to infinity.

AMS 2000 subject classifications: Primary 05E05; secondary 05E45, 60C05.
Keywords and phrases: Composition tableau, e-positive, fundamental-positive, Kostka number, quasisymmetric Schur-positive, Schur-positive, Young tableau.

arXiv: 1810.11038
*Supported in part by the National Sciences and Engineering Research Council of Canada, CRM-ISM, and the Canada Research Chairs Program.
†Supported in part by the National Sciences and Engineering Research Council of Canada, the Simons Foundation, and the Centre de Recherches Mathématiques, through the Simons-CRM scholar-in-residence program.

1. Introduction

The subject of when a symmetric function is Schur-positive, that is, a nonnegative linear combination of Schur functions, is an active area of research. If a homogeneous symmetric function of degree n is Schur-positive, then it is the image of some representation of the symmetric group $\mathfrak{S}_n$ under the Frobenius characteristic map. Furthermore, if it is a polynomial, then it is the character of a polynomial representation of the general linear group $GL(n, \mathbb{C})$. Consequently, much effort has been devoted to determining when the difference of two symmetric functions is Schur-positive, for example [2, 8, 12, 13, 14, 16, 18, 19]. While this question is still wide open in full generality, there exist well-known examples of Schur-positive functions. These include the product of two Schur functions, skew Schur functions, and the chromatic symmetric function of the incomparability graph of $(3+1)$-free posets [9], the latter of which are further conjectured to be e-positive, that is, a nonnegative linear combination of elementary symmetric functions [23]. One other well-known example that is known to be Schur-positive is the bigraded Frobenius characteristic of the space of diagonal harmonics, which is therefore known to be fundamental-positive, namely, a nonnegative linear combination of fundamental quasisymmetric functions. A conjectured combinatorial formula for the latter [10] was recently proved [6]. This result is better known as the proof of the shuffle conjecture.

Quasisymmetric functions, a natural generalization of symmetric functions, are further related to positivity via representation theory since the 1-dimensional representations of the 0-Hecke algebra map to fundamental quasisymmetric functions under the quasisymmetric characteristic map [7]. There also exist 0-Hecke modules whose quasisymmetric characteristic map images are quasisymmetric Schur functions [24]. Additionally, if a quasisymmetric function is both symmetric and a nonnegative linear combination of quasisymmetric Schur functions, then it is Schur-positive [4]. While nonnegative linear combinations of quasisymmetric functions are not as extensively studied, some progress has been made in this direction, for example [1, 3, 4, 15, 17], and this area is ripe for study.

This paper is structured as follows. In Theorem 2.1, we calculate the probability that an element of a vector space that is a nonnegative linear combination of basis elements is also a nonnegative linear combination of the elements of a second basis, where the bases satisfy certain conditions. In Section 3, we then apply this theorem and compute the probability that a symmetric function is Schur-positive or e-positive in Corollaries 3.4, 3.7, 3.11. We show that these probabilities tend to 0 as the degree of the function tends to infinity in Corollaries 3.6, 3.9, 3.13. We then apply Theorem 2.1 again in Section 4 to compute the probability that a quasisymmetric function is quasisymmetric Schur-positive or fundamental-positive in Corollaries 4.4, 4.7, 4.10, and similarly show these probabilities tend to 0 in Corollaries 4.6, 4.9, 4.12.

2. The probability of vector positivity

Let V be a finite-dimensional real vector space with bases $A = \{A_0, \ldots, A_d\}$ and $B = \{B_0, \ldots, B_d\}$, and suppose further that

$$A_j = \sum_{i \leq j} a_i^{(j)} B_i,$$

where $a_j^{(j)} = 1$ and $a_i^{(j)} \geq 0$. In particular, note that $A_0 = B_0$. We say that $f \in V$ is *A-positive* (respectively, *B-positive*) if f is a nonnegative linear combination of $\{A_0, \ldots, A_d\}$ (respectively, $\{B_0, \ldots, B_d\}$). We would like to answer the following question: What is the probability that if $f \in V$ is B-positive, then it is furthermore A-positive? We denote this probability by $\mathbb{P}(A_i \mid B_i)$ and note that any A-positive $f \in V$ will also necessarily be B-positive.

In order to calculate $\mathbb{P}(A_i \mid B_i)$, observe that any B-positive $f \in V$ can be written as

$$f = \sum_{i=0}^{d} b_i B_i,$$

where each $b_i \geq 0$, and the set of all B-positive elements of V forms a cone

$$B^+_{\text{cone}} = \left\{ \sum_{i=0}^{d} b_i B_i \mid b_i \in \mathbb{R}_{\geq 0} \right\}.$$

Inside the cone B^+_{cone} is the cone of A-positive elements of V

$$A^+_{\text{cone}} = \left\{ \sum_{i=0}^{d} b_i B_i \mid b_i \in \mathbb{R}_{\geq 0} \text{ and the expression is } A\text{-positive} \right\}.$$

We define $\mathbb{P}(A_i \mid B_i)$ to be the ratio of the volume of the slice of A^+_{cone} defined by

$$A^+_{\text{slice}} = \left\{ \sum_{i=0}^{d} b_i B_i \mid b_i \in \mathbb{R}_{\geq 0}, \text{ the expression is } A\text{-positive, and } \sum_{i=0}^{d} b_i = 1 \right\}$$

to the volume of the slice of B^+_{cone}

$$B^+_{\text{slice}} = \left\{ \sum_{i=0}^{d} b_i B_i \mid b_i \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=0}^{d} b_i = 1 \right\}.$$

We could, equivalently, replace "1" in both definitions with any positive real number and obtain the same ratio. Note that this probability will depend on the choice of bases $\{A_0, \ldots, A_d\}$ and $\{B_0, \ldots, B_d\}$; however, each application of the following theorem (see Corollaries 3.4, 3.7, 3.11, 4.4, 4.7, and 4.10) comes with a natural choice of bases, and the asymptotics we explore (see Corollaries 3.6, 3.9, 3.13, 4.6, 4.9, and 4.12) do not depend on this choice.

The existence of the general statement below was suggested by F. Bergeron and its proof inspired by a conversation with V. Reiner about Schur-positivity.

Theorem 2.1. Let $\{A_0, \ldots, A_d\}$ and $\{B_0, \ldots, B_d\}$ be bases of a finite-dimensional real vector space V such that

$$A_j = \sum_{i \leq j} a_i^{(j)} B_i,$$

where $a_j^{(j)} = 1$ and $a_i^{(j)} \geq 0$, so in particular, $A_0 = B_0$. Then

$$\mathbb{P}(A_i \mid B_i) = \prod_{j=0}^{d} \left(\sum_{i=0}^{j} a_i^{(j)} \right)^{-1}.$$

Proof. Consider

$$B^+_{\text{slice}} = \left\{ \sum_{i=0}^{d} b_i B_i \mid b_i \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=0}^{d} b_i = 1 \right\}$$

and the corresponding slice of A^+_{cone}

$$A^+_{\text{slice}} = \left\{ \sum_{i=0}^{d} b_i B_i \mid b_i \in \mathbb{R}_{\geq 0}, \text{ the expression is } A\text{-positive, and } \sum_{i=0}^{d} b_i = 1 \right\}.$$

Note that B^+_{slice} is the simplex determined by vertices $B_0, \ldots, B_d$. Define vectors $v_1, \ldots, v_d$ by $v_i = B_i - B_0$ for $1 \leq i \leq d$. Then the volume of B^+_{slice}

is by definition

$$\frac{1}{d!}|\det(v_1, \ldots, v_d)|.$$

The simplex A^+_{slice} is determined by vertices $\left\{\left(\sum_i a_i^{(j)}\right)^{-1} A_j\right\}_{0\leq j\leq d}$. To find its volume, we first define vectors $w_1, \ldots, w_d$ by

$$w_j = \frac{1}{\sum_i a_i^{(j)}} A_j - A_0.$$

We then see that

$$\begin{aligned}
w_j &= \frac{1}{\sum_i a_i^{(j)}} A_j - A_0 \\
&= \frac{1}{\sum_i a_i^{(j)}} \left(B_j + a_{j-1}^{(j)} B_{j-1} + \cdots + a_0^{(j)} B_0\right) - B_0 \\
&= \frac{1}{\sum_i a_i^{(j)}} \left(B_j + a_{j-1}^{(j)} B_{j-1} + \cdots + a_0^{(j)} B_0\right) \\
&\quad - \frac{1}{\sum_i a_i^{(j)}} \left(B_0 + a_{j-1}^{(j)} B_0 + \cdots + a_0^{(j)} B_0\right) \\
&= \frac{1}{\sum_i a_i^{(j)}} \left(v_j + a_{j-1}^{(j)} v_{j-1} + \cdots + a_1^{(j)} v_1\right).
\end{aligned}$$

Thus the volume of A^+_{slice}, namely the simplex determined by vertices $\left\{\frac{1}{\sum_i a_i^{(j)}} A_j\right\}$, is

$$\begin{aligned}
\frac{1}{d!}|\det(w_1, \ldots, w_d)| &= \frac{1}{d!}\left|\det\left(\frac{1}{\sum_i a_i^{(1)}} v_1, \frac{1}{\sum_i a_i^{(2)}}(v_2 + a_1^{(2)} v_1),\right.\right. \\
&\qquad \left.\left. \ldots, \frac{1}{\sum_i a_i^{(d)}}(v_d + \cdots + a_1^{(d)} v_1)\right)\right| \\
&= \frac{1}{d!} \prod_j \frac{1}{\sum_i a_i^{(j)}} \left|\det(v_1, \ldots, v_d)\right|.
\end{aligned}$$

The result now follows, since by definition we have that

$$\mathbb{P}(A_i \mid B_i) = \frac{\text{volume of } A^+_{\text{slice}}}{\text{volume of } B^+_{\text{slice}}}.$$ □

3. Probabilities of symmetric function positivity

Before we define the various symmetric functions that will be of interest to us, we need to recall some combinatorial concepts. A *partition* $\lambda = (\lambda_1, \dots, \lambda_k)$ of n, denoted by $\lambda \vdash n$, is a list of positive integers whose *parts* λ_i satisfy $\lambda_1 \geq \cdots \geq \lambda_k$ and $\sum_{i=1}^{k} \lambda_i = n$. If there exists $\lambda_{m+1} = \cdots = \lambda_{m+j} = i$, then we often abbreviate this to i^j. There exist two total orders on partitions of n, which will be useful to us. The first of these is *lexicographic order*, which states that given partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_\ell)$ we say that μ is lexicographically smaller than λ, denoted by $\mu <_{lex} \lambda$, if $\mu \neq \lambda$ and the first i for which $\mu_i \neq \lambda_i$ satisfies $\mu_i < \lambda_i$. The second is the closely related *reverse lexicographic order*, where we say that μ is reverse lexicographically smaller than λ, denoted by $\mu <_{revlex} \lambda$ if and only if $\mu >_{lex} \lambda$.

Example 3.1. The partitions of 4 in lexicographic order are

$$(1^4) <_{lex} (2, 1^2) <_{lex} (2^2) <_{lex} (3, 1) <_{lex} (4).$$

Given a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ and commuting variables $\{x_1, x_2, \dots\}$, we define the *monomial symmetric function* m_λ to be

$$m_\lambda = \sum x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$$

where the sum is over all k-tuples $(i_1, \dots, i_k)$ of distinct indices that yield distinct monomials.

Example 3.2. We see that $m_{(2,1)} = x_1^2 x_2 + x_2^2 x_1 + x_1^2 x_3 + x_3^2 x_1 + \cdots$.

The set of all monomial symmetric functions forms a basis for the graded algebra of symmetric functions

$$\mathrm{Sym} = \bigoplus_{n \geq 0} \mathrm{Sym}^n \subseteq \mathbb{R}[[x_1, x_2, \dots]]$$

where $\mathrm{Sym}^0 = \mathrm{span}\{1\}$ and $\mathrm{Sym}^n = \mathrm{span}\{m_\lambda \mid \lambda \vdash n\}$ for $n \geq 1$. Hence each graded piece Sym^n for $n \geq 1$ is a finite-dimensional real vector space with basis $\{m_\lambda \mid \lambda \vdash n\}$.

For our second required basis we need Young diagrams and Young tableaux. Given a partition $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$, we call the array of n left-justified boxes with λ_i boxes in row i from the top, for $1 \leq i \leq k$, the

Young diagram of λ, also denoted by λ. Given a Young diagram we say that T is a *semistandard Young tableau (SSYT)* of *shape* λ if the boxes of λ are filled with positive integers such that

1. the entries in each row weakly increase when read from left to right,
2. the entries in each column strictly increase when read from top to bottom.

Two SSYTs of shape $(2, 1)$ can be seen below in Example 3.3. Given an SSYT T we define the *content* of T, denoted by content(T), to be the list of nonnegative integers

$$\mathrm{content}(T) = (c_1, \ldots, c_{max})$$

where c_i is the number of times that i appears in T and max is the largest integer appearing in T. We say that an SSYT is of *partition content* if $c_1 \geq \cdots \geq c_{max} > 0$. With this in mind, if λ and μ are partitions, then we define the *Schur function* s_λ to be

$$s_\lambda = m_\lambda + \sum_{\mu <_{lex} \lambda} K_{\lambda\mu} m_\mu \tag{3.1}$$

where $K_{\lambda\mu}$ is the number of SSYTs, T, of shape λ and content$(T) = \mu$.

Example 3.3. We see $s_{(2,1)} = m_{(2,1)} + 2m_{(1,1,1)}$ from the following two SSYTs arising from the nonleading term.

1	2
3	

1	3
2	

We now define the *i-th complete homogeneous symmetric function* to be

$$h_i = s_{(i)}$$

and if $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition then we define the *complete homogeneous symmetric function* h_λ to be

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k} = s_{(\lambda_1)} \cdots s_{(\lambda_k)}.$$

Similarly, we define the *i-th elementary symmetric function* to be

$$e_i = s_{(1^i)}$$

and if $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a partition then we define the *elementary symmetric function* e_λ to be

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k} = s_{(1^{\lambda_1})} \cdots s_{(1^{\lambda_k})}.$$

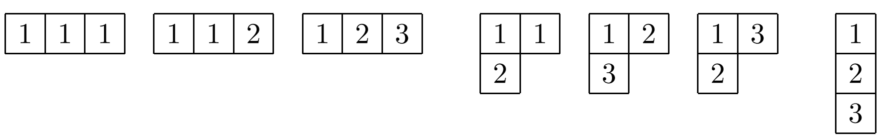
We have that $\{m_\lambda \mid \lambda \vdash n\}$, $\{s_\lambda \mid \lambda \vdash n\}$, $\{h_\lambda \mid \lambda \vdash n\}$ and $\{e_\lambda \mid \lambda \vdash n\}$ are all bases of Sym^n for $n \geq 1$. Additionally, if $f \in \mathrm{Sym}$ is a nonnegative linear combination of elements in these bases, then using the vernacular we say that f is, respectively, *monomial-, Schur-, h- or e-positive*. Also, in the following results, we use the notation $\mathbb{P}_n(\cdot \mid \cdot)$ to denote that the probability is being calculated in Sym^n for $n \geq 1$. Considering its importance, our first result shows the rarity that a monomial-positive symmetric function is furthermore Schur-positive. This statement was previously determined by Bergeron–Patrias–Reiner using a proof method similar to that of Theorem 2.1. The statement without proof is given in [20].

Corollary 3.4. [20] Let $\mathcal{K}_\lambda$ denote the number of SSYTs of shape λ and partition content. Then

$$\mathbb{P}_n(s_\lambda \mid m_\lambda) = \prod_{\lambda \vdash n} (\mathcal{K}_\lambda)^{-1}.$$

Proof. The result follows from Theorem 2.1 by first setting $A_0 = s_{(1^n)} = m_{(1^n)} = B_0$, and ordering the basis elements in increasing order by taking their indices in lexicographic order. Then use Equation (3.1) along with $K_{\lambda\mu} = 0$ if $\lambda <_{lex} \mu$ and $K_{\lambda\lambda} = 1$ [22, Proposition 7.10.5]. □

Example 3.5. For $n = 3$ we have that $\mathcal{K}_{(3)} = 3$, $\mathcal{K}_{(2,1)} = 3$ and $\mathcal{K}_{(1,1,1)} = 1$ from the following SSYTs.



Hence,

$$\mathbb{P}_3(s_\lambda \mid m_\lambda) = \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{1}\right) = \frac{1}{9}.$$

Corollary 3.6. We have that

$$\lim_{n\to\infty} \mathbb{P}_n(s_\lambda \mid m_\lambda) = 0.$$

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of n, and consider the following two fillings of λ. For the first, fill the boxes in the top row with $1, \dots, \lambda_1$ from left to right, the second row with $\lambda_1 + 1, \dots, \lambda_1 + \lambda_2$, etc. For the second, fill the boxes in row i from the top with i for $1 \le i \le k$. For $\lambda \ne (1^n)$, these fillings are distinct, and thus $\mathcal{K}_\lambda \ge 2$. It follows that

$$0 \le \prod_{\lambda \vdash n} (\mathcal{K}_\lambda)^{-1} \le \frac{1}{2^{p(n)-1}},$$

where $p(n)$ denotes the number of partitions of n, and hence

$$0 \le \lim_{n\to\infty} \prod_{\lambda \vdash n} (\mathcal{K}_\lambda)^{-1} \le \lim_{n\to\infty} \frac{1}{2^{p(n)-1}} = 0. \qquad \square$$

Corollary 3.7. Let $\mathcal{E}_\lambda$ be the number of SSYTs with content λ. Then

$$\mathbb{P}_n(e_\lambda \mid s_\lambda) = \mathbb{P}_n(h_\lambda \mid s_\lambda) = \prod_{\lambda \vdash n} (\mathcal{E}_\lambda)^{-1}.$$

Proof. By [22, Proposition 7.10.5 and Corollary 7.12.4] we have that

$$h_\lambda = s_\lambda + \sum_{\mu >_{lex} \lambda} K_{\mu\lambda} s_\mu. \tag{3.2}$$

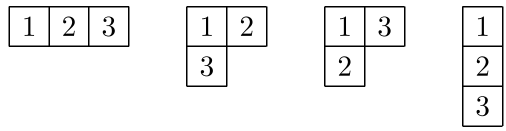
The result for $\mathbb{P}_n(h_\lambda \mid s_\lambda)$ now follows from Equation (3.2) and Theorem 2.1, along with $K_{\mu\lambda} = 0$ if $\mu <_{lex} \lambda$ and $K_{\lambda\lambda} = 1$ [22, Proposition 7.10.5], by setting $A_0 = h_n = s_{(n)} = B_0$ and by ordering the basis elements in increasing order by taking their indices in reverse lexicographic order. The result for $\mathbb{P}_n(e_\lambda \mid s_\lambda)$ now follows from applying the involution ω to that acts as a bijection from Schur-positive functions that are h-positive to Schur-positive functions that are e-positive. It satisfies

$$\omega(h_\lambda) = e_\lambda \text{ and } \omega(s_\lambda) = s_{\lambda'}$$

where λ' is the transpose of λ, that is, the partition whose parts are obtained from λ with maximum part $max(\lambda)$ by letting $\lambda'_i =$ the number of parts of $\lambda \ge i$, for $1 \le i \le max(\lambda)$. $\square$

Example 3.8. For $n = 3$ we have that $\mathcal{E}_{(3)} = 1$, $\mathcal{E}_{(2,1)} = 2$ and $\mathcal{E}_{(1,1,1)} = 4$ from the following SSYTs.

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \cline{1-1} \end{array}$$



Hence,

$$\mathbb{P}_3(e_\lambda \mid s_\lambda) = \mathbb{P}_3(h_\lambda \mid s_\lambda) = \left(\frac{1}{1}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{8}.$$

Corollary 3.9. We have that

$$\lim_{n\to\infty} \mathbb{P}_n(e_\lambda \mid s_\lambda) = \lim_{n\to\infty} \mathbb{P}_n(h_\lambda \mid s_\lambda) = 0.$$

Proof. As in the proof of Corollary 3.6, the result will follow from showing that $\mathcal{E}_\lambda \geq 2$ for all $\lambda \neq (n)$. Indeed, first consider the tableau T of shape $\lambda = (\lambda_1, \ldots, \lambda_k)$ with the boxes in row i from the top filled with i for $1 \leq i \leq k$. Second, consider the tableau of shape $(\lambda_1+1, \lambda_2, \ldots, \lambda_{k-1}, \lambda_k-1)$ obtained from T by moving the rightmost box filled with k from row k to row 1. These are distinct for all $\lambda \neq (n)$, hence $\mathcal{E}_\lambda \geq 2$ for all $\lambda \neq (n)$. □

Remark 3.10. One can form a square matrix with the $K_{\lambda\mu}$ (known as the *Kostka numbers*), where λ and μ vary over all partitions of n, and rows and columns are ordered in lexicographic order. Then $\mathcal{K}_\lambda$ and $\mathcal{E}_\lambda$ may be interpreted as a row sum and as a column sum of this matrix, respectively.

Since elementary symmetric functions are Schur-positive, which in turn are monomial-positive, it is natural to compute the following.

Corollary 3.11. Let $\mathcal{M}_\lambda$ be the number of (0,1)-matrices with row sum λ and column sum a partition. Then

$$\mathbb{P}_n(e_\lambda \mid m_\lambda) = \prod_{\lambda \vdash n} (\mathcal{M}_\lambda)^{-1}.$$

Proof. Let $A_0 = e_n = m_{(1^n)} = B_0$. Order the basis elements of A in increasing order by taking their indices in reverse lexicographic order. Order the basis elements of B in increasing order by taking the transpose, as in the proof of Corollary 3.7, of their indices in reverse lexicographic order. By [22, Proposition 7.4.1 and Theorem 7.4.4] we have that

$$e_\lambda = m_{\lambda'} + \sum_{\mu' <_{revlex} \lambda} M_{\lambda\mu} m_\mu,$$

where μ' is the transpose of μ as in the proof of Corollary 3.7, and $M_{\lambda\mu}$ is the number of (0,1)-matrices whose row sums give the parts of λ and whose column sums give the parts of μ. The result now follows from Theorem 2.1 along with $M_{\lambda\mu} = 0$ if $\lambda <_{lex} \mu'$ and $M_{\lambda\lambda'} = 1$ [22, Theorem 7.4.4]. □

Example 3.12. For $n = 3$ we have that $\mathcal{M}_{(3)} = 1$, $\mathcal{M}_{(2,1)} = 4$ and $\mathcal{M}_{(1,1,1)} = 10$ from the six 3×3 permutation matrices, the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and the following four matrices and their transposes.

$$\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Hence,

$$\mathbb{P}_3(e_\lambda \mid m_\lambda) = \left(\frac{1}{1}\right)\left(\frac{1}{4}\right)\left(\frac{1}{10}\right) = \frac{1}{40}.$$

Corollary 3.13. We have that

$$\lim_{n\to\infty} \mathbb{P}_n(e_\lambda \mid m_\lambda) = 0.$$

Proof. As before, in the proof of Corollary 3.6, the result will follow if we show that $\mathcal{M}_\lambda \geq 2$ for all $\lambda = (\lambda_1, \ldots, \lambda_k) \neq (n)$. Consider the matrix where the first λ_1 columns have a 1 in row 1 and 0's everywhere else, the next λ_2 columns have a 1 in row 2 and 0's everywhere else, the next λ_3 columns have a 1 in row 3 and 0's everywhere else, etc. We obtain a second valid (0,1)-matrix by swapping column λ_1 with column $\lambda_1 + 1$. □

4. Probabilities of quasisymmetric function positivity

We now turn our attention to quasisymmetric functions, and again begin by recalling pertinent combinatorial concepts. A *composition* $\alpha = (\alpha_1, \ldots, \alpha_k)$ of n, denoted by $\alpha \vDash n$, is a list of positive integers whose *parts* α_i sum to n. Observe that every composition α determines a partition $\lambda(\alpha)$, which is obtained by reordering the parts of α into weakly decreasing order. Also recall the bijection between compositions of n and subsets of $[n-1] = \{1, \ldots, n-1\}$. Namely, given $\alpha = (\alpha_1, \ldots, \alpha_k) \vDash n$, its corresponding set is $\mathrm{set}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\} \subseteq [n-1]$. Conversely, given $S = \{s_1, \ldots, s_{k-1}\} \subseteq [n-1]$ its corresponding composition is $\mathrm{comp}(S) = (s_1, s_2 - s_1, \ldots, n - s_{k-1}) \vDash n$. Lastly, the empty set is in bijection with (n). We again use the abbreviation i^j to mean j consecutive parts equal to i, and extend the definition of lexicographic order from the previous section for partitions to encompass compositions. We then use this extension to define a total order on compositions of n. Given compositions α, β we say $\beta \blacktriangleleft \alpha$ if $\lambda(\beta) <_{lex} \lambda(\alpha)$ or $\lambda(\beta) = \lambda(\alpha)$ and $\beta <_{lex} \alpha$.

Example 4.1. The compositions of 4 in $\blacktriangleleft$ order are

$$(1^4) \blacktriangleleft (1^2,2) \blacktriangleleft (1,2,1) \blacktriangleleft (2,1^2) \blacktriangleleft (2^2) \blacktriangleleft (1,3) \blacktriangleleft (3,1) \blacktriangleleft (4).$$

There is also a partial order on compositions of n, which will be useful later. Given compositions α, β we say that α is a *proper coarsening* of β (or β is a *proper refinement* of α) denoted by $\beta \prec \alpha$ if we can obtain α by nontrivially adding together adjacent parts of β. For example, $(1,2,1) \prec (1,3)$. Observe that $\beta \prec \alpha$ if and only if $\mathrm{set}(\alpha) \subset \mathrm{set}(\beta)$.

Now, similar to the previous section, given a composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ and commuting variables $\{x_1, x_2, \ldots\}$ we define the *monomial quasisymmetric function* M_α to be

$$M_\alpha = \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_1} \cdots x_{i_k}^{\alpha_k}$$

and the *fundamental quasisymmetric function* F_α to be

$$F_\alpha = M_\alpha + \sum_{\beta \prec \alpha} M_\beta.$$

Example 4.2. We compute $M_{(2,1)} = x_1^2 x_2 + x_1^2 x_3 + \cdots$ and $F_{(2,1)} = M_{(2,1)} + M_{(1,1,1)}$.

The set of monomial quasisymmetric functions or the set of fundamental quasisymmetric functions forms a basis for the graded algebra of quasisymmetric functions

$$\mathrm{QSym} = \bigoplus_{n \geq 0} \mathrm{QSym}^n \subseteq \mathbb{R}[[x_1, x_2, \ldots]]$$

where $\mathrm{QSym}^0 = \mathrm{span}\{1\}$ and $\mathrm{QSym}^n = \mathrm{span}\{M_\alpha \mid \alpha \vDash n\} = \mathrm{span}\{F_\alpha \mid \alpha \vDash n\}$ for $n \geq 1$. Hence each graded piece QSym^n for $n \geq 1$ is a finite-dimensional real vector space with basis $\{M_\alpha \mid \alpha \vDash n\}$ or $\{F_\alpha \mid \alpha \vDash n\}$.

In order to define our third and final basis of QSym we need composition diagrams and composition tableaux. Given a composition $\alpha = (\alpha_1, \ldots, \alpha_k) \vDash n$, we call the array of n left-justified boxes with α_i boxes in row i from the top, for $1 \leq i \leq k$, the *composition diagram* of α, also denoted by α. Given a composition diagram $\alpha \vDash n$, we say τ is a *semistandard composition tableau (SSCT)* of *shape* α if the boxes of α are filled with positive integers such that

1. the entries in each row weakly decrease when read from left to right,
2. the entries in the leftmost column strictly increase when read from top to bottom,

3. if we denote the box in τ that is in the i-th row from the top and j-th column from the left by $\tau(i,j)$, then if $i<j$ and $\tau(j,m)\le\tau(i,m-1)$ then $\tau(i,m)$ exists and $\tau(j,m)<\tau(i,m)$.

Furthermore, if each of the numbers $1,\dots,n$ appears exactly once, then we say that τ is a *standard composition tableau (SCT)*. Intuitively we can think of the third condition as saying that if $a\le b$ then $a<c$ in the following array of boxes.

b	c
	a

Given an SSCT τ we define the *content* of τ, denoted by content(τ), to be the list of nonnegative integers

$$\mathrm{content}(\tau)=(c_1,\dots,c_{max})$$

where c_i is the number of times that i appears in τ and max is the largest integer appearing in τ. We say that an SSCT is of *composition content* if $c_i\neq 0$ for all $1\le i\le max$. Given an SCT τ of shape $\alpha\vDash n$, we define its *descent set* to be

$$\mathrm{Des}(\tau)=\{i\mid i+1 \text{ is weakly right of } i\}\subseteq[n-1]$$

and define its *descent composition* to be

$$\mathrm{comp}(\tau)=\mathrm{comp}(\mathrm{Des}(\tau))\vDash n.$$

We can now define our final basis both in terms of monomial and fundamental quasisymmetric functions, respectively. The first formula is [11, Theorem 6.1] with [11, Proposition 6.7] applied to it, and the second is [11, Theorem 6.2] with [11, Proposition 6.8] applied to it.

If α and β are compositions, then we define the *quasisymmetric Schur function* $\mathcal{S}_\alpha$ to be

$$\mathcal{S}_\alpha=M_\alpha+\sum_{\beta\blacktriangleleft\alpha}K^c_{\alpha\beta}M_\beta \tag{4.1}$$

where $K^c_{\alpha\beta}$ is the number of SSCTs, τ, of shape α and content$(\tau)=\beta$. It is also given by

$$\mathcal{S}_\alpha=F_\alpha+\sum_{\beta\blacktriangleleft\alpha}d_{\alpha\beta}F_\beta \tag{4.2}$$

where $d_{\alpha\beta}$ is the number of SCTs, τ, of shape α and $\mathrm{comp}(\tau) = \beta$.

Example 4.3. $\mathcal{S}_{(1,2)} = M_{(1,2)} + M_{(1,1,1)} = F_{(1,2)}$ from the following SSCT, which is also an SCT, arising from the nonleading term in the first equality.

$$\begin{array}{|c|c}\hline 1 & \multicolumn{1}{|c}{} \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline \end{array}$$

We have that, in addition to $\{M_\alpha \mid \alpha \vDash n\}$ and $\{F_\alpha \mid \alpha \vDash n\}$, $\{\mathcal{S}_\alpha \mid \alpha \vDash n\}$ is a basis of QSym^n for $n \geq 1$, and if $f \in \mathrm{QSym}$ is a nonnegative linear combination of such basis elements, then we refer to f respectively as being *monomial quasisymmetric-, fundamental-* or *quasisymmetric Schur-positive*. We also use the notation $\mathbb{P}_n(\cdot \mid \cdot)$ to denote that the probability is being calculated in QSym^n for $n \geq 1$. Our first result is reminiscent of the probability that a monomial-positive symmetric function is furthermore Schur-positive in Corollary 3.4.

Corollary 4.4. Let $\mathcal{K}^c_\alpha$ be the number of SSCTs of shape α and composition content. Then

$$\mathbb{P}_n(\mathcal{S}_\alpha \mid M_\alpha) = \prod_{\alpha \vDash n} (\mathcal{K}^c_\alpha)^{-1}.$$

Proof. The result follows from Theorem 2.1 by first setting $A_0 = \mathcal{S}_{(1^n)} = M_{(1^n)} = B_0$, and ordering the basis elements in increasing order by taking their indices in $\blacktriangleleft$ order. Then use Equation (4.1) along with $K^c_{\alpha\beta} = 0$ if $\alpha \blacktriangleleft \beta$ and $K^c_{\alpha\alpha} = 1$ [11, Proposition 6.7]. $\square$

Example 4.5. For $n = 3$ we have that $\mathcal{K}^c_{(3)} = 4$, $\mathcal{K}^c_{(2,1)} = 2$, $\mathcal{K}^c_{(1,2)} = 2$ and $\mathcal{K}^c_{(1,1,1)} = 1$ from the following SSCTs.

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 2 & 2 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c}\hline 1 & \multicolumn{1}{|c}{} \\ \hline 2 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|c}\hline 1 & \multicolumn{1}{|c}{} \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

Hence,

$$\mathbb{P}_3(\mathcal{S}_\alpha \mid M_\alpha) = \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{1}\right) = \frac{1}{16}.$$

Corollary 4.6. We have that

$$\lim_{n\to\infty} \mathbb{P}_n(\mathcal{S}_\alpha \mid M_\alpha) = 0.$$

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a composition of n and consider the following two fillings of α. For the first, fill the boxes of α such that the boxes in the bottom row contain $n, n-1, \ldots, n+1-\alpha_k$ from left to right, the next row up $n-\alpha_k, n-\alpha_k-1, \ldots, n+1-\alpha_k-\alpha_{k-1}$ etc. For the second, fill the boxes in row i from the top with i, for $1 \leq i \leq k$. For $\alpha \neq (1^n)$ these fillings are distinct, and thus $\mathcal{K}^c_\alpha \geq 2$. Since the number of compositions of n is 2^{n-1} it follows that

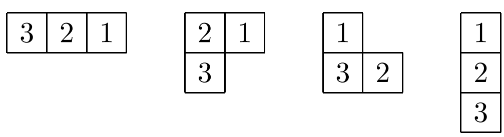
$$0 \leq \lim_{n\to\infty} \prod_{\alpha \vDash n} (\mathcal{K}^c_\alpha)^{-1} \leq \lim_{n\to\infty} \frac{1}{2^{2^{n-1}-1}} = 0. \qquad \square$$

Corollary 4.7. Let $\mathcal{D}_\alpha$ be the number of SCTs of shape α. Then

$$\mathbb{P}_n(\mathcal{S}_\alpha \mid F_\alpha) = \prod_{\alpha \vDash n} (\mathcal{D}_\alpha)^{-1}.$$

Proof. The result follows from Theorem 2.1 by first setting $A_0 = \mathcal{S}_{(1^n)} = F_{(1^n)} = B_0$ and ordering the basis elements in increasing order by taking their indices in $\blacktriangleleft$ order. Then use Equation (4.2) along with $d_{\alpha\beta} = 0$ if $\alpha \blacktriangleleft \beta$ and $d_{\alpha\alpha} = 1$ [11, Proposition 6.8]. $\square$

Example 4.8. For $n = 3$ we have that $\mathcal{D}_{(3)} = 1$, $\mathcal{D}_{(2,1)} = 1$, $\mathcal{D}_{(1,2)} = 1$ and $\mathcal{D}_{(1,1,1)} = 1$ from the following SSCTs.



Hence,

$$\mathbb{P}_3(\mathcal{S}_\alpha \mid F_\alpha) = 1.$$

Corollary 4.9. We have that

$$\lim_{n\to\infty} \mathbb{P}_n(\mathcal{S}_\alpha \mid F_\alpha) = 0.$$

Proof. By [5, Theorem 4.4], we know that $\mathcal{S}_\alpha = F_\alpha$ if and only if $\alpha = (m, 1^{\epsilon_1}, 2, 1^{\epsilon_2}, \ldots, 2, 1^{\epsilon_k})$, where $m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ ($m = 0$ is understood to mean it does not appear in the composition), $k \in \mathbb{N}_0$, $\epsilon_i \in \mathbb{N} = \{1, 2, \ldots\}$ for $i \in [k-1]$, and $\epsilon_k \in \mathbb{N}_0$.

Let $\mathcal{A}_n$ be the set of compositions of n *not* in the set of compositions described above. Note that if $\alpha \in \mathcal{A}_n$ then $\mathcal{D}_\alpha \geq 2$. Also note that if $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathcal{A}_n$, then $(\alpha_1, \ldots, \alpha_k + 1), (\alpha_1, \ldots, \alpha_k, 1) \in \mathcal{A}_{n+1}$.

Hence $2|\mathcal{A}_n| \leq |\mathcal{A}_{n+1}|$. Using this repeatedly, along with $|\mathcal{A}_4| = 2$ since $\mathcal{A}_4 = \{(1,3),(2,2)\}$, yields that for $n \geq 5$

$$2^{n-3} \leq |\mathcal{A}_n|.$$

Hence it follows that

$$0 \leq \lim_{n\to\infty} \prod_{\alpha \vDash n} (\mathcal{D}_\alpha)^{-1} \leq \lim_{n\to\infty} \frac{1}{2^{|\mathcal{A}_n|}} \leq \lim_{n\to\infty} \frac{1}{2^{2^{n-3}}} = 0. \qquad \square$$

We end with the most succinct of our formulas, namely the probability that a quasisymmetric monomial-positive function is furthermore fundamental-positive.

Corollary 4.10.

$$\mathbb{P}_n(F_\alpha \mid M_\alpha) = \frac{1}{(n-1)2^{n-2}}$$

Proof. Recall that

$$F_\alpha = M_\alpha + \sum_{\beta \prec \alpha} M_\beta,$$

and that $\beta \prec \alpha$ if and only if $\mathrm{set}(\alpha) \subset \mathrm{set}(\beta)$. Letting $A_0 = F_{(1^n)} = M_{(1^n)} = B_0$ and ordering the basis elements in increasing order by taking their indices in $\blacktriangleleft$ order, Theorem 2.1 gives that

$$\mathbb{P}_n(F_\alpha \mid M_\alpha) = \prod_{\alpha \vDash n} \left(\sum_{\beta \preceq \alpha} 1 \right)^{-1}.$$

Now

$$\prod_{\alpha \vDash n} \left(\sum_{\beta \preceq \alpha} 1 \right) = \prod_{S \subseteq [n-1]} \left(\sum_{T \supseteq S} 1 \right) = \prod_{S \subseteq [n-1]} \left(2^{n-1-|S|} \right) = 2^{\sum_{S \subseteq [n-1]} (n-1-|S|)},$$

and

$$\sum_{S \subseteq [n-1]} (n-1-|S|) = \sum_{T \subseteq [n-1]} |T| = \sum_{k=0}^{n-1} k \binom{n-1}{k} = (n-1)2^{n-2},$$

where the last equality is [21, Chapter 1 Exercise 2(b)]. $\square$

Example 4.11. For $n = 3$ we have that $\mathbb{P}_3(F_\alpha \mid M_\alpha) = \frac{1}{4}$.

The following corollary follows from Corollary 4.10.

Corollary 4.12. We have that

$$\lim_{n\to\infty} \mathbb{P}_n(F_\alpha \mid M_\alpha) = 0.$$

Acknowledgements

The authors would like to thank the referees for thoughtful comments, François Bergeron, Vic Reiner, and Sébastien Labbé for conversations that led to fruitful research directions, and LaCIM where some of the research took place.

References

[1] P. Alexandersson and R. Sulzgruber, *P-partitions and p-positivity*, arXiv:1807.02460v2.

[2] F. Bergeron, R. Biagioli and M. Rosas, *Inequalities between Littlewood-Richardson coefficients*, J. Combin. Theory Ser. A 113 (2006) 567–590. MR2216453

[3] C. Bessenrodt, K. Luoto and S. van Willigenburg, *Skew quasisymmetric Schur functions and noncommutative Schur functions*, Adv. Math. 226 (2011) 4492–4532. MR2770457

[4] C. Bessenrodt, V. Tewari and S. van Willigenburg, *Littlewood-Richardson rules for symmetric skew quasisymmetric Schur functions*, J. Combin. Theory Ser. A 137 (2016) 179–206. MR3403520

[5] C. Bessenrodt and S. van Willigenburg, *Multiplicity free Schur, skew Schur, and quasisymmetric Schur functions*, Ann. Comb. 17 (2013) 275–294. MR3056768

[6] E. Carlsson and A. Mellit, *A proof of the shuffle conjecture*, J. Amer. Math. Soc. 31 (2018) 661–697. MR3787405

[7] G. Duchamp, D. Krob, B. Leclerc and J.-Y. Thibon, *Fonctions quasi-symétriques, fonctions symétriques non-commutatives, et algèbres de Hecke à* $q = 0$, C. R. Math. Acad. Sci. Paris 322 (1996) 107–112. MR1373744

[8] S. Fomin, W. Fulton, C.-K. Li and Y.-T. Poon, *Eigenvalues, singular values, and Littlewood-Richardson coefficients*, Amer. J. Math. 127 (2005) 101–127. MR2115662

[9] V. Gasharov, *Incomparability graphs of (3+1)-free posets are s-positive*, Discrete Math. 157 (1996) 193–197. MR1417294

[10] J. Haglund, M. Haiman, N. Loehr, J. Remmel and A. Ulyanov, *A combinatorial formula for the character of the diagonal coinvariants*, Duke Math. J. 126 (2005) 195–232. MR2115257

[11] J. Haglund, K. Luoto, S. Mason and S. van Willigenburg, *Quasisymmetric Schur functions*, J. Combin. Theory Ser. A 118 (2011) 463–490. MR2739497

[12] R. King, T. Welsh and S. van Willigenburg, *Schur positivity of skew Schur function differences and applications to ribbons and Schubert classes*, J. Algebraic Combin. 28 (2008) 139–167. MR2420783

[13] A. Kirillov, *An invitation to the generalized saturation conjecture*, Publ. Res. Inst. Math. Sci. 40 (2004) 1147–1239. MR2105706

[14] T. Lam, A. Postnikov and P. Pylyavskyy, *Schur positivity and Schur log-concavity*, Amer. J. Math. 129 (2007) 1611–1622. MR2369890

[15] T. Lam and P. Pylyavskyy, *P-partition products and fundamental quasi-symmetric function positivity*, Adv. in Appl. Math. 40 (2008) 271–294. MR2402171

[16] P. McNamara, *Necessary conditions for Schur-positivity*, J. Algebraic Combin. 28 (2008) 495–507. MR2443283

[17] P. McNamara, *Comparing skew Schur functions: a quasisymmetric perspective*, J. Comb. 5 (2014) 51–85. MR3173797

[18] P. McNamara and S. van Willigenburg, *Positivity results on ribbon Schur function differences*, European J. Combin. 30 (2009) 1352–1369. MR2514658

[19] A. Okounkov, *Log-concavity of multiplicities with application to characters of* $\mathrm{U}(\infty)$, Adv. Math. 127 (1997) 258–282. MR1448715

[20] R. Patrias, *What is Schur positivity and how common is it?*, arXiv:1809.04448v1. MR3953520

[21] R. Stanley, *Enumerative Combinatorics. Volume 1*, Wadsworth and Brooks/Cole (1986). MR0847717

[22] R. Stanley, *Enumerative Combinatorics. Volume 2*, Cambridge University Press (1999). MR1676282

[23] R. Stanley and J. Stembridge, *On immanants of Jacobi-Trudi matrices and permutations with restricted position*, J. Combin. Theory Ser. A 62 (1993) 261–279. MR1207737

[24] V. Tewari and S. van Willigenburg, *Modules of the 0-Hecke algebra and quasisymmetric Schur functions*, Adv. Math. 285 (2015) 1025–1065. MR3406520

Rebecca Patrias
Laboratoire de Combinatoire et d'Informatique Mathématique
Université du Québec à Montréal
Montréal QC H3C 3P8
Canada
E-mail address: patriasr@lacim.ca

Stephanie van Willigenburg
Department of Mathematics
University of British Columbia
Vancouver BC V6T 1Z2
Canada
E-mail address: steph@math.ubc.ca

Received March 10, 2019