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Labeling resolving sets

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Let G be a simple, connected graph, $W \subseteq V(G)$ and $L : W \to \{1, 2, \dots, m\}$. For $v \in V(G)$, define $v[i] = \{d(u, v) | u \in L^{-1}[i]\}$ and $R(v|L) = (v[1], v[2], \dots, v[m])$. The labeling L is m-tracked if for any $v, w \in V(G)$ where $R(v|L) = R(w|L)$, $v = w$. The minimum m such that G has an m-tracked labeling is the graph's tracking dimension ($\mathrm{Trac}(G)$). Tracked labelings are an extension of resolving sets, which were defined independently by Slater (1975) and Harary and Melter (1976). W is a resolving set if for every distinct pair $u, v \in V(G)$, there exists $x \in W$ where $d(x, u) \neq d(x, v)$. Albertson and Collins (1996) defined a labeling of the entire vertex set to be distinguishing if no nontrivial automorphism preserves the labels. The graph's distinguishing number ($\mathrm{Dist}(G)$) is the minimum number of labels needed to have a distinguishing labeling. Tracked labelings must also break the graph's symmetries, and in this way, bridge distinguishing labelings and resolving sets. We show that $\mathrm{Trac}(G) \geq \mathrm{Dist}(G) - 1$. For $n > 5$, we show $\mathrm{Trac}(C_n)$ achieves the lower bound, but for complements of cycles, $\mathrm{Trac}(G) - \mathrm{Dist}(G)$ can be arbitrarily large. For complete multipartite graphs, we show $\mathrm{Dist}(G) - 1 \leq \mathrm{Trac}(G) \leq \mathrm{Dist}(G)$.

1. Introduction

Bob is going to be dropped at a random vertex in the graph in Figure 1, and he'll be tasked with determining his location. Before this happens, Bob

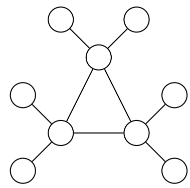

Figure 1: Bob's Graph.

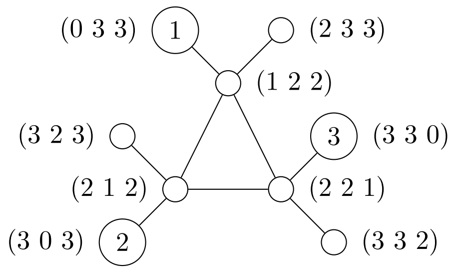

Figure 2: Sample Resolving Set.

can prepare by placing markers on a subset of the vertices. Then, from any specific vertex, he can send signals to the markers, and each marker will return its distance to Bob. Furthermore, each marker will send a unique response, not necessarily unique in its distance to Bob, but unique in its tone so that Bob knows from which marker the distance came.

In Figure 2, we've placed three markers, which are labeled 1, 2, and 3. Next to each vertex, we've attached a vector with the distances to those vertices in order. In this case, all vertices have a unique vector. In other words, with this assignment of markers, Bob would always be able to identify the vertex at which he's located.

When the markers yield unique vectors, the marked subset of vertices is called a resolving set. The minimum number of markers needed is the metric dimension of the graph. This concept was introduced by Harary and Melter in [12] and independently by Slater in [15]. (Slater referred to resolving sets as locating sets and the metric dimension as the location number.)

Let's modify Bob's problem. Suppose Bob has the option of placing multiple markers that return the same tone. For example, Bob could place two blue markers. These markers would collectively return a set of two distances. Bob would know these distances came from blue markers, but would not be able to determine which distance came from which blue marker.

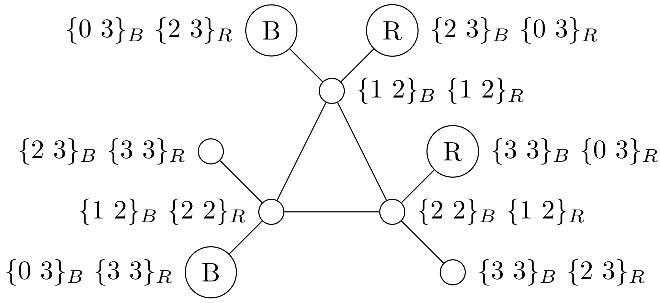

Figure 3: Sample Tracked Labeling.

Furthermore, suppose the cost of additional tones (or colors) is prohibitive in comparison to adding multiple markers with the same tone. Bob would like to know how many unique tones he needs.

In Figure 3, we've assigned blue and red markers. When on a given vertex, Bob will receive a blue set of distances and a red set of distances. In Figure 3, we've attached to each vertex these two sets with blue first. Here, the assignment of just two colors is sufficient so that each vertex can be identified by this information.

When the labeling of a subset of vertices is sufficient such that each vertex in the graph has a unique collection of distance sets to the labeled vertices (as in Figure 3), we say the labeling is tracked. The tracking dimension is the minimum number of colors needed for a tracked labeling. (Formal definitions are presented in the next section.)

Tracked labelings identify all vertices through the vertices' distances to the labeled set, but the labeling in Figure 3 suggests a relationship between the labeling and the graph's automorphisms. The labeling breaks the graph's symmetries in the sense that there does not exist a nontrivial automorphism of the graph that preserves the labeling. The idea of breaking symmetries through labeling vertices was formalized by Albertson and Collins [2] when they defined distinguishing labelings. (Note that in their definition, all vertices are labeled.) The minimum number of colors needed for a distinguishing labeling is called the distinguishing number of the graph. There are similar ideas presented in the context of groups from around the same time [13, 14].

The labeled vertices in Figure 3 also have the property that if two automorphisms of the graph agree on the subset of the vertex set that is labeled, then those automorphisms agree on the entire vertex set. When a subset of a vertex set has this property, it has been called a determining set by Boutin [5] and a fixing set by Erwin and Harary [9]. The minimal cardinality of a determining set is the graph's determining number [5], fixing number [9], or rigidity index [10].

Recently, there has been a push to understand the relationship between resolving and determining sets. The efforts are detailed by Bailey and Cameron [4], where they also provide an introduction to distinguishing and determining numbers while connecting both to comparable or equivalent group-theoretic terminology and results.

It has been harder to compare resolving sets to distinguishing labelings. Tracked labelings are a bridge between these concepts.

Formal definitions and a few examples are presented in Section 2. Section 3 explores the relationship between tracked labelings and distinguishing labelings. We show that the tracking dimension of a graph is greater than or equal to one less than the distinguishing number, and we present a class of examples where the difference between the tracked and distinguishing numbers can be arbitrarily large. In Section 4, we show that the tracking dimension for complete multipartite graphs is either the distinguishing number or one less.

2. Basic definitions and examples

We will always work on simple, connected graphs.

Definition 2.1. Let G be a graph and let $W = \{w_1, w_2, \ldots, w_n\} \subseteq V(G)$. For $v \in V(G)$, define $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_n))$. W is a resolving set for G if whenever $r(u|W) = r(v|W)$, $u = v$. The minimum cardinality of a resolving set for G is the metric dimension of G and is denoted $\dim(G)$.

Definition 2.2. Let G be a graph, $W \subseteq V(G)$ and $L : W \to \{1, 2, \ldots, m\}$. For $v \in V(G)$, define $v[i] = \{d(u, v)|u \in L^{-1}[i]\}$ (a multiset) and $R(v|L) = (v[1], v[2], \ldots, v[m])$. The labeling L is m-tracked (or simply, tracked) if whenever $R(u|L) = R(v|L)$, $u = v$. The minimum m such that G has an m-tracked labeling is the tracking dimension of G and is denoted $\mathrm{Trac}(G)$.

Every graph has a tracked labeling since labeling each vertex a unique color is sufficient. In fact, labeling every vertex except one with a unique color is always sufficient. (This implies that $\mathrm{Trac}(G) \leq |V(G)| - 1$.) The following proposition shows that $\mathrm{Trac}(G) \leq \dim(G)$.

Proposition 2.3. *If $W \subseteq V(G)$ and $L : W \to \{1, 2, \ldots, m\}$ are such that L is m-tracked, then W is a resolving set for G.*

Proof. Let $W \subseteq V(G)$ and $L : W \to \{1, 2, \ldots, m\}$ be such that L is m-tracked. Let $u, v \in V(G)$ where $r(u|W) = r(v|W)$. Then, $d(u, w) = d(v, w)$ for all $w \in W$. This implies $v[i] = u[i]$ for all $i \in \{1, 2, \ldots, m\}$, and $R(u|L) = R(v|L)$. Since L is m-tracked, $u = v$. Therefore, W is a resolving set for G. □

Definition 2.4. Let $f : V(G) \to \{1, 2, \ldots, d\}$. The labeling f is d-distinguishing if the only $\phi \in \mathrm{Aut}(G)$ such that $f(v) = f(\phi(v))$ for all $v \in V(G)$ is the identity. The minimum d such that G has a d-distinguishing labeling is the distinguishing number of G and is denoted $\mathrm{Dist}(G)$.

Definition 2.5. Let $W \subseteq V(G)$. W is a determining set for G if whenever $\phi, \psi \in \mathrm{Aut}(G)$ such that $\phi(v) = \psi(v)$ for all $v \in W$, $\phi(v) = \psi(v)$ for all $v \in V(G)$. The determining number, denoted $\mathrm{Det}(G)$, is the minimum cardinality of a determining set for G.

Resolving sets are determining sets [5, 9, 11]. (We'll use this fact in the proof of Theorem 3.4.)

In the introduction, we noted that tracked labelings are related to resolving sets, distinguishing labelings and determining sets. Although it's not the focus of this article, it's worth noting that tracked labelings are also related to resolving partitions as defined by Chartrand, Salehi and Zhang in [8]. For $S \subseteq V(G)$, let $d(v, S) = \min\{d(v, u) | u \in S\}$. Let $P = \{S_1, S_2, \ldots, S_k\}$ be an ordered partition of $V(G)$. For each $v \in V(G)$, define $r(v|P) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$. P is a resolving partition if $r(u|P) = r(v|P)$ implies $u = v$. The minimum k needed for a resolving partition is the partition dimension of the graph.

If G has a resolving partition $P = \{S_1, S_2, \ldots, S_k\}$, then the labeling $L : V(G) \to \{1, 2, \ldots, k\}$ where $L(v) = i$ when $v \in S_i$ is a tracked labeling. Thus, the partition dimension of a graph is an upper bound for the graph's tracking dimension.

We now find the tracking dimensions for cycles.

Theorem 2.6. *For $n > 5$, $\mathrm{Trac}(C_n) = 1$.*

Proof. In this proof, all arithmetic will be mod n. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ where v_i is adjacent to v_{i+1}.

Let $W = \{v_1, v_2, v_4\}$ and let $L : W \to \{1\}$. Then $R(v_1|L) = \{0, 1, 3\}$, $R(v_2|L) = \{0, 1, 2\}$, $R(v_3|L) = \{1, 1, 2\}$, and $R(v_4|L) = \{0, 2, 3\}$.

Imagine $v_1, v_2, \ldots, v_n$ are arranged clockwise in the plane. When $n > 7$, the vertices can be split into four categories. The first is v_1 through v_4, the cases detailed above. The second we'll call the "counterclockwise" group because the shortest paths from these vertices to all labeled vertices will travel counterclockwise. The third group will be the "clockwise" group because the shortest paths from these vertices to all labeled vertices will travel clockwise. Finally, there is what we'll call the "middle" group. For these vertices, the shortest path to v_1 will travel clockwise and the shortest path to v_4 will travel counterclockwise. (If $n \leq 7$, the clockwise and counterclockwise categories are empty.) We now find $R(v_i|L)$ for the last three categories.

First, we consider the counterclockwise. Suppose $n > 7$. Let $4 < i \leq n/2 + 1$. Then $i - 1 \leq n/2$. Since $(i-1) + (1-i) = n$, $1 - i \geq i - 1$. Then, for $4 < i \leq n/2 + 1$, $R(v_i|L) = \{i-4, i-2, i-1\}$ where the set is ordered from least to greatest. Clearly, within the counterclockwise category, if $i \neq j$, $R(v_i|L) \neq R(v_j|L)$.

Next, we consider the clockwise. Suppose $n > 7$. Let $n/2 + 4 \leq i < n$. Then $n/2 \leq i-4$. Since $(4-i)+(i-4) = n$, $4-i \leq i-4$. Then, for $n/2+4 \leq i < n$, $R(v_i|L) = \{1-i, 2-i, 4-i\}$ where the set is ordered from least to greatest. Clearly, within the clockwise category, if $i \neq j$, $R(v_i|L) \neq R(v_j|L)$.

For vertices in the clockwise category, $R(v|L)$ takes the form $\{a, a+1, a+3\}$ (where a is the smallest distance in the multiset). For vertices in the counterclockwise category, $R(v|L)$ takes the form $\{a, a+2, a+3\}$. Thus, if u is in the clockwise category and v is in the counterclockwise category, $R(u|L) \neq R(v|L)$.

Finally, we handle the middle. If n is even, there exist two i such that $n/2 + 1 < i < n/2 + 4$. Those are $i = n/2 + 2, n/2 + 3$. If $i = n/2 + 2$, we can calculate the distances to the labeled vertices as follows. $1 - (n/2 + 2) = n/2 - 1$, $2 - (n/2 + 2) = n/2$, and $(n/2 + 2) - 4 = n/2 - 2$. Thus, $R(v_{n/2+2}|L) = \{n/2 - 2, n/2 - 1, n/2\}$.

If $i = n/2 + 3$, we can calculate the distances as follows. $1 - (n/2+3) = n/2 - 2$, $2 - (n/2 + 3) = n/2 - 1$, and $(n/2 + 3) - 4 = n/2 - 1$. Thus, $R(v_{n/2+3}|L) = \{n/2 - 2, n/2 - 1, n/2 - 1\}$.

For these two middle vertices, $R(v|L)$ takes the form $\{a, a+1, a+2\}$ and $\{a, a+1, a+1\}$. Both patterns are different from those in either the clockwise or counterclockwise categories.

If n is odd, there are three i such that $n/2 + 1 < i < n/2 + 4$. If $i = n/2 + 3/2$, we can calculate the distances as follows. $(n/2 + 3/2) - 4 = n/2 - 5/2$, $(n/2 + 3/2) - 2 = n/2 - 1/2$, $1 - (n/2 + 3/2) = n/2 - 1/2$. Thus, $R(v_{n/2+3/2}|L) = \{n/2 - 5/2, n/2 - 1/2, n/2 - 1/2\}$.

If $i = n/2 + 5/2$, we can calculate the distances as follows. $(n/2 + 5/2) - 4 = n/2 - 3/2$, $1 - (n/2 + 5/2) = n/2 - 3/2$, and $2 - (n/2 + 5/2) = n/2 - 1/2$. Thus, $R(v_{n/2+5/2}|L) = \{n/2 - 3/2, n/2 - 3/2, n/2 - 1/2\}$.

If $i = n/2 + 7/2$, we can calculate the distances as follows. $(n/2 + 7/2) - 4 = n/2 - 1/2$, $1 - (n/2 + 7/2) = n/2 - 5/2$, and $2 - (n/2 + 7/2) = n/2 - 3/2$. Thus, $R(v_{n/2+7/2}|L) = \{n/2 - 5/2, n/2 - 3/2, n/2 - 1/2\}$.

For these three middle vertices, $R(v|L)$ takes the form $\{a, a+2, a+2\}$, $\{a, a, a+1\}$ and $\{a, a+1, a+2\}$. All three patterns are distinct from each other and from those in either the clockwise or counterclockwise categories.

Lastly, we must show that v_1, v_2, v_3, and v_4 yield distinct $R(v|L)$. All are distinct from each other. The labeled vertices v_1, v_2, and v_4 will be the only vertices with a distance of 0 to a labeled vertex. The last vertex, v_3, will be the only vertex with distance 1 to two labeled vertices. Thus, $R(v|L)$ for these four vertices will be distinct from those in the clockwise, counterclockwise and middle categories. □

The previous theorem shows that for $n > 5$, C_n has tracking dimension 1, but clearly, labeling one or two vertices in C_n with a single color would not be sufficient. For the proof, we used a specific pattern of labeling v_1, v_2, and v_4. Those familiar with distinguishing labelings will recognize this pattern. If those three vertices are labeled 1, and the other vertices are labeled 2, we would have a distinguishing labeling of C_n. For $n > 5$, $\text{Dist}(C_n) = 2$ [2]. Since it will come up in the proof of Theorem 3.5, let us also mention that a graph and its complement must have the same distinguishing number, and so, for $n > 5$, $\text{Dist}(\overline{C_n}) = 2$ as well.

Distinguishing labelings are somewhat surprising in that they require more colors for C_n when $n < 5$. $\text{Dist}(C_n) = 3$ for $n = 3, 4, 5$. The same phenomenon occurs for tracked labelings. For each of these graphs, it's a simple exercise to show that labeling two adjacent vertices with distinct colors will be tracked, while any labeling with only one color will not be tracked.

When working with complete graphs, we again see a link between tracked and distinguishing labelings. It's a simple exercise to show that $\text{Trac}(K_n) = n - 1$, where one vertex is left unlabeled. Again, if the unlabeled vertex is colored a new color, the labeling is distinguishing. Clearly, there is a relationship between distinguishing and tracked labelings, and in the next section, we explore that connection.

3. Relating distinguishing and tracked labelings

To relate tracked and distinguishing labelings, we will use the definition of distinguishing on a subset of $V(G)$ as defined by Albertson and Boutin [1]. Recall that for $W \subseteq V(G)$, the pointwise stabilizer of W is $\text{Stab}(W) = \{\phi \in \text{Aut}(G) | \phi(v) = v, \forall v \in W\}$.

Definition 3.1. Let $W \subseteq V(G)$ and $L : W \to \{1, 2, \dots, m\}$. L is m-distinguishing if whenever $\phi \in \text{Aut}(G)$ and $L(v) = L(\phi(v))$ for all $v \in W$, then $\phi \in \mathrm{Stab}(W)$. If there exists an m-distinguishing labeling of W, we say W is m-distinguishable.

Theorem 3.2. *Let $W \subseteq V(G)$ and $L : W \to \{1, 2, \ldots, m\}$. If L is m-tracked, then W is m-distinguishable.*

Proof. Let $\phi \in \mathrm{Aut}(G)$ where $L(\phi(v)) = L(v)$ for all $v \in W$. Using that ϕ preserves labels and that all automorphisms preserve distances, for each $v \in V(G)$ and $i \in \{1, 2, \ldots, m\}$, $v[i] = \phi(v)[i]$. Thus, $R(v|L) = R(\phi(v)|L)$ for all $v \in W$. Since ϕ is m-tracked, $\phi(v) = v$ for all $v \in W$. Therefore, $\phi \in \mathrm{Stab}(W)$. □

When an m-tracked labeling L is defined on a proper subset, W, of the vertex set, we must be careful to say L distinguishes the subset W and not that L distinguishes the whole graph G. We can only conclude that L is distinguishing on G if $W = V(G)$.

For our next theorem, we'll use the following result, which is Theorem 3 in [1].

Theorem 3.3. *Let $W \subseteq V(G)$. W is an m-distinguishable determining set if and only if there exists an $(m+1)$-distinguishing labeling of G.*

We can now use the distinguishing number of G to provide a lower bound for its tracking dimension.

Theorem 3.4. $\mathrm{Trac}(G) \geq \mathrm{Dist}(G) - 1$.

Proof. Let $\mathrm{Trac}(G) = m$. Then, for some $W \subseteq V(G)$, there exists $L : W \to \{1, 2, \ldots, m\}$ that is m-tracked. By Theorem 3.2, W is m-distinguishable. Since L is m-tracked, W is a resolving set (Proposition 2.3). Resolving sets are determining sets. By Theorem 3.3, there exists an $(m+1)$-distinguishing labeling of G. So, $\mathrm{Dist}(G) \leq m + 1$. □

The lower bound in Theorem 3.4 for $\mathrm{Trac}(G)$ is achieved by C_n. In general, there is no limit to the difference between $\mathrm{Trac}(G)$ and $\mathrm{Dist}(G)$. To establish this result, we'll use the complements of cycles.

We'll use the following definition in the proof of the next theorem and a later result. For $W \subseteq V(G)$, $S \subseteq W$, and $L : W \to \{1, 2, \ldots, m\}$, define the multiset $L[S] = \{L(u) | u \in S\}$. If S is empty, we consider $L[S]$ to be the empty set.

Theorem 3.5. *For each $k \in \mathbb{N}$, there exists an n where $(\mathrm{Trac}(\overline{C_n}) - \mathrm{Dist}(\overline{C_n})) > k$.*

Proof. Fix $n > 5$, and let $m = \mathrm{Trac}(\overline{C_n})$. We'll show that $(m+2)(m+1)^2 \geq 2n$. Since $\mathrm{Dist}(\overline{C_n}) = 2$ for $n > 5$, this will establish the conclusions of our theorem.

For $v \in V(\overline{C_n})$, let v^i be the set of vertices distance i from v. For each $v \in V(\overline{C_n})$, $|v^0| = 1$, $|v^1| = n - 3$, and $|v^2| = 2$.

Let $W \subseteq V(\overline{C_n})$ and $L : W \to \{1, 2, \ldots, m\}$ be such that L is m-tracked. If for $u, v \in V(\overline{C_n})$, $L[u^0 \cap W] = L[v^0 \cap W]$ and $L[u^2 \cap W] = L[v^2 \cap W]$, then $L[u^1 \cap W] = L[v^1 \cap W]$. In this case, $R(u|L) = R(v|L)$. Thus, if $u \neq v$, either $L[u^0 \cap W] \neq L[v^0 \cap W]$ or $L[u^2 \cap W] \neq L[v^2 \cap W]$.

Including "undefined" as an option, there are $m + 1$ ways to label v and $\binom{m+2}{2}$ ways to label v^2. This means that there are $(m+2)(m+1)^2/2$ distinct choices for labeling v and v^2 combined. Since for any pair of distinct vertices u and v, $L[u^0 \cap W] \neq L[v^0 \cap W]$ or $L[u^2 \cap W] \neq L[v^2 \cap W]$, we can conclude $(m + 2)(m + 1)^2 \geq 2n$. $\square$

Since $\mathrm{Trac}(C_n) = 1$ for $n > 5$, Theorem 3.5 shows that the difference between the tracking dimension of a graph and its complement can be arbitrarily large.

Note that for $n > 3$, $\mathrm{Det}(C_n) = \mathrm{Det}(\overline{C_n}) = 2$ [5]. So, Theorem 3.5 shows that $\mathrm{Trac}(G) - \mathrm{Det}(G)$ can be arbitrarily large. In [5], Boutin asks if $\dim(G) - \mathrm{Det}(G)$ can be arbitrarily large. Cáceres et al. [7] demonstrated that it could be using a family of trees. Since $\dim(G) \geq \mathrm{Trac}(G)$, Theorem 3.5 provides another class of graphs where $\dim(G) - \mathrm{Det}(G)$ can be arbitrarily large.

4. Tracked labelings of complete multipartite graphs

A labeling of a complete multipartite graph will be distinguishing if it meets the following two criteria. First, any two vertices within a partite set must have different colors since there is an automorphism that swaps the two vertices while leaving the rest of the graph fixed. Second, given two partite sets of equal size, there must be a vertex in one that was assigned a color that was not used in the other. If the two partite sets had exactly the same assignment of colors, the automorphism that swaps the two partite sets and fixes everything else would preserve the labels.

After considering tracked labelings for cycles and complete graphs, one might suspect we can find tracked labelings for complete multipartite graphs by finding distinguishing labelings where we treat "unlabeled" as another color.

It is true that for a tracked labeling, a partite set of size n might only require $n - 1$ colors where one vertex is left unlabeled. However, in tracked labelings, “unlabeled” is not just another color. Consider $K_{4,3}$. We can label the partite set of size 4 with colors $\{1, 2, 3\}$ while leaving the fourth vertex unlabeled, and we can label the partite set of size 3 with $\{1, 2, 3\}$, but we cannot do both. If we did, $R(u|L) = R(v|L)$ when $L(u) = L(v)$. With this labeling, the labeled vertices are blind to the existence of the unlabeled vertex.

Tracked labelings must meet two criteria very similar to that of distinguishing labelings. First, within each partite set, each vertex must have a unique label, where “unlabeled” is an option. Second, two partite sets cannot have the same set of labels, where “unlabeled” is not considered an element of the set of labels.

The above two criteria are captured in the following theorem.

Theorem 4.1. *Let G be complete multipartite with partite sets $\{P_1, P_2, \ldots, P_n\}$, let $W \subseteq V(G)$, and let $L : W \to \{1, 2, \ldots, m\}$. L is m-tracked if and only if both of the following statements are true.*

(i) For each i, $|W \cap P_i| \geq |P_i| - 1$, and for $u, v \in (W \cap P_i)$, $L(u) \neq L(v)$.
(ii) $L[W \cap P_i] = L[W \cap P_j]$ implies $i = j$.

Proof. Suppose conditions (i) and (ii) are satisfied. Suppose u and v exist in the same partite set. If $u, v \in W$, condition (i) guarantees $L(u) \neq L(v)$, and thus, $R(u|L) \neq R(v|L)$. Condition (i) guarantees at most one of u and v are not in W. If, for example $u \notin W$, then u is not distance 0 from any vertex in W. Again, $R(u|L) \neq R(v|L)$.

Now, let $u \in P_i$ and $v \in P_j$ where $i \neq j$. Each vertex is distance 0 or 2 from only the vertices in its partite set. Condition (ii) guarantees $L[W \cap P_i] \neq L[W \cap P_j]$. As we did in the proof of Theorem 3.5, let v^d be the set of vertices distance d from v. We have either $L[u^0 \cap W] \neq L[v^0 \cap W]$ or $L[u^2 \cap W] \neq L[v^2 \cap W]$. Thus, $R(u|L) \neq R(v|L)$.

Since for any $u, v \in G$, $R(u|L) \neq R(v|L)$, L is m-tracked.

Now suppose L is m-tracked. Let u and v be in the same partite set. For any other vertex w, $d(u, w) = d(v, w)$. Thus, if both $u, v \notin W$, or $u, v \in W$ and $L(u) = L(v)$, $R(u|L) = R(v|L)$. Therefore, condition (i) must be satisfied.

Finally, we'll show that condition (ii) must be satisfied. Suppose $i \neq j$ and $L[W \cap P_i] = L[W \cap P_j]$. Then, there exist $u \in P_i$ and $v \in P_j$ where $L[u^0 \cap W] = L[v^0 \cap W]$. Since $L[u^0 \cap W] = L[v^0 \cap W]$ and $L[W \cap P_i] = L[W \cap P_j]$, $L[u^2 \cap W] = L[v^2 \cap W]$. $L[u^1 \cap W] = L[W] - L[W \cap P_i]$ and $L[v^1 \cap W] = L[W] - L[W \cap P_j]$. Since $L[W \cap P_i] = L[W \cap P_j]$, $L[u^1 \cap W] = L[v^1 \cap W]$. This implies $R(u|L) = R(v|L)$ and the labeling is not m-tracked. $\square$

Suppose G, W, and L are defined as in Theorem 4.1 and that L is tracked. Suppose $\max\{|P_i||i = 1, 2, \ldots n\} = K$ and let $|k|$ denote the number of partite sets of size k. If all of the partite sets in G are of size 1, condition (ii) yields that G is $(n-1)$-tracked (where n is the number of partite sets).

Suppose $K \geq 2$. Condition (i) allows for a partite set of size K to be labeled with K or $K-1$ vertices. Since partite sets of size less than K cannot use K labels, it is best to exhaust those options first. There are $\binom{m}{K}$ ways to label a partite set of size K with m labels. If $|K| > \binom{m}{K}$, we'll say the partite sets of size K borrow $|K| - \binom{m}{K}$ labelings with $K-1$ colors. When $K \geq 2$, condition (ii) requires m to be large enough so that $\binom{m}{K-1} \geq (|K| - \binom{m}{K})$.

For $2 \leq k \leq K$, we let $B(k) = \max\{0, |k| - \binom{m}{k}\}$. In other words, $B(k)$ is the number of $k-1$ labelings borrowed to label partite sets of size k. Condition (ii) forbids the use of any labelings with $k-1$ colors on partite sets of size $k-1$ that were borrowed to label partite sets of size k. This leads to the following corollary.

Corollary 4.2. *Let G be a complete multipartite graph where the maximum cardinality of its partite sets is $K \geq 2$. Let $|k|$ denote the number of partite sets of size k. There exists an m-tracked labeling of G if and only if*

$$|k| \leq \begin{cases} \binom{m}{k} + B(k), & \textit{if } k = K \\ \binom{m}{k} + B(k) - B(k+1), & \textit{if } 1 < k < K \\ \binom{m}{k} - B(k+1) + 1, & \textit{if } k = 1 \end{cases}$$

where for $2 \leq k \leq K$, $B(k) = \max\{0, |k| - \binom{m}{k}\}$.

G has an m-distinguishing labeling if for each $k \in \{1, 2, \ldots, K\}$, $|k| \leq \binom{m}{k}$. So, if L is an m-distinguishing labeling of G, the conditions on $|k|$ in Corollary 4.2 are satisfied with no borrowing necessary. L must also be m-tracked. Together with Theorem 3.4, when G is a complete multipartite graph, $\mathrm{Dist}(G) - 1 \leq \mathrm{Trac}(G) \leq \mathrm{Dist}(G)$. It's a simple exercise to show that the left bound is realized when $G = K_{2,2}$ and the right bound is satisfied when $G = K_{3,2,2,2}$.

5. Future directions

The available literature on distinguishing labelings and resolving sets provides a roadmap for investigations into tracked labelings. One obvious example is to find formulas or bounds for the tracking dimension of special classes of graphs, such as the Kneser graphs. (See [1] and [3] for relevant results on distinguishing labelings and resolving sets, respectively.)

In addition, it would be interesting to know when graph properties influence the potential differences between the tracking dimension and the determining number, distinguishing number or metric dimension.

When assigning tracked labelings, what are the possible sizes of the label classes? In particular, when a graph has tracking dimension m, how small can the label classes be when defining an m-tracked labeling?

Finally, what would be the influence of allowing multiple markers on the same vertex? For example, for a given graph with tracking dimension 3, if we allowed up to two markers per vertex, can we reduce the number of colors needed to distinguish all vertices to 2. Note that when red and blue markers have been placed on vertices, no more than one per vertex, adding a red and blue marker to an unlabeled vertex is not the same as giving that vertex a new color.

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