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Rank of incidence matrix with applications to digraph reconstruction

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The incidence matrix $W_{t\,k}$ is defined as follow: Let V be a finite set, with v elements. Given non-negative integers t, k, $W_{t\,k}$ is the $\binom{v}{t}$ by $\binom{v}{k}$ matrix of 0's and 1's, the rows of which are indexed by the t-element subsets T of V, the columns are indexed by the k-element subsets K of V, and where the entry $W_{t\,k}(T, K)$ is 1 if $T \subseteq K$ and is 0 otherwise.

R.M. Wilson proved that for $t \leq min(k, v-k)$, the rank of $W_{t\,k}$ modulo a prime p is $\sum_{i=0}^{t} \binom{v}{i} - \binom{v}{i-1}$ where p does not divide the binomial coefficient $\binom{k-i}{t-i}$.

In this paper, we begin by giving an analytic expression of the rank of the matrix $W_{t\,k}$ when $t = t_0 + t_1 p + t_2 p^2$, with $t_0, t_1, t_2 \in [0, p-1]$ and we characterize values of t and k such that $\dim Ker({}^tW_{t\,k}) \in \{0, 1\}$. Next, using this result we generalize a result in the (≤ 6)-reconstruction of digraphs due to G. Lopez.

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1. Introduction

We consider the matrix $W_{t\,k}$ defined as follows: Let V be a finite set, with v elements. Given non-negative integers $t \leq k$, let $W_{t\,k}$ be the $\binom{v}{t}$ by $\binom{v}{k}$ matrix of 0's and 1's, the rows of which are indexed by the t-element subsets T of V, the columns are indexed by the k-element subsets K of V, and where the entry $W_{t\,k}(T, K)$ is 1 if $T \subseteq K$ and is 0 otherwise. The matrix transpose of $W_{t\,k}$ is denoted ${}^tW_{t\,k}$. Theorem 1.1, due to Gottlieb [8], shows the rank over the field $\mathbb{Q}$ of $W_{t\,k}$ is $\binom{v}{t}$. On the other hand $rank_p\ W_{t\,k}$ over the field $\mathbb{Z}/p\mathbb{Z}$, is given by Theorem 1.2 below, due to Wilson [17].

Theorem 1.1. *(D.H. Gottlieb [8]) For $t \leq min(k, v-k)$, the rank of $W_{t\,k}$ over the field $\mathbb{Q}$ of rational numbers is $\binom{v}{t}$ and thus $Ker({}^tW_{t\,k}) = \{0\}$.*

Theorem 1.2. *(R.M. Wilson [17]) For $t \le min(k, v-k)$, the rank of $W_{t\,k}$ modulo a prime p is*

$$\sum \binom{v}{i} - \binom{v}{i-1}$$

where the sum is extended over those indices i, $0 \le i \le t$, such that p does not divide the binomial coefficient $\binom{k-i}{t-i}$. In the statement of the theorem, $\binom{v}{-1}$ should be interpreted as zero.

Let k, p be positive integers, the decomposition of $k = \sum_{i=0}^{k(p)} k_i p^i$ in the basis p is also denoted $[k_0, k_1, \ldots, k_{k(p)}]_p$ where $k_{k(p)} \neq 0$ if and only if $k \neq 0$ and $0 \le k_i < p$ for all $0 \le i \le k(p)$.

First, we give an analytic expression of the rank of the matrix $W_{t\,k}$ when $t = [t_0, t_1, t_2]_p$.

Theorem 1.3. *Let p be a prime, $t \le k$ positive integers.*

We assume that $t = [t_0, t_1, t_2]_p$ and $k = [k_0, k_1, \ldots, k_{k(p)}]_p$.

1) If $k_0 \le t_0 - 1$, $k_1 \le t_1$ and $k_2 \le t_2$. Then $rank_p(W_{t\,k}) =$

$$\sum_{i_2=k_2}^{t_2} \sum_{i_1=k_1}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p+k_0}.$$

2) If $k_0 \ge t_0$, $k_1 \le t_1 - 1$ and $k_2 \le t_2$. Then $rank_p(W_{t\,k}) =$

$$\sum_{i_2=k_2}^{t_2} \sum_{i_1=k_1+1}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+(i_1-1)p+k_0}.$$

3) If $k_0 \le t_0 - 1$, $k_1 \ge t_1 + 1$ and $k_2 \le t_2 - 1$. Then $rank_p(W_{t\,k}) =$

$$\sum_{i_2=k_2}^{t_2-1} \sum_{i_1=k_1}^{p-1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p+k_0} + \sum_{i_2=k_2+1}^{t_2} \sum_{i_1=0}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p+k_0}.$$

4) If $k_0 \ge t_0$, $k_1 \ge t_1$ and $k_2 \le t_2 - 1$. Then $rank_p(W_{t\,k}) =$

$$\sum_{i_2=k_2}^{t_2-1} \sum_{i_1=k_1+1}^{p-1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+(i_1-1)p+k_0} + \sum_{i_2=k_2+1}^{t_2} \sum_{i_1=0}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+(i_1-1)p+k_0}.$$

5) If $k_0 \le t_0 - 1$, $k_1 \le t_1$ and $k_2 \ge t_2 + 1$. Then $rank_p(W_{t\,k}) =$

$$\sum_{i_2=0}^{t_2} \sum_{i_1=k_1}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p+k_0}.$$

6) *If* $k_0 \geq t_0$, $k_1 \leq t_1 - 1$ *and* $k_2 \geq t_2 + 1$. *Then* $rank_p(W_{t\,k}) =$

$$\sum_{i_2=0}^{t_2} \sum_{i_1=k_1+1}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+(i_1-1)p+k_0}.$$

7) *If* $k_0 \leq t_0 - 1$, $k_1 \geq t_1 + 1$ *and* $k_2 \geq t_2$. *Then* $rank_p(W_{t\,k}) =$

$$\sum_{i_2=0}^{t_2-1} \sum_{i_1=k_1}^{p-1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p+k_0} + \sum_{i_2=0}^{t_2} \sum_{i_1=0}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p+k_0}.$$

8) *If* $k_0 \geq t_0$, $k_1 \geq t_1$ *and* $k_2 \geq t_2$. *Then* $rank_p(W_{t\,k}) =$

$$\sum_{i_2=0}^{t_2-1} \sum_{i_1=k_1+1}^{p-1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+(i_1-1)p+k_0} + \sum_{i_2=0}^{t_2} \sum_{i_1=0}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+(i_1-1)p+k_0}.$$

As a consequence of Theorem 1.3, we have.

Corollary 1.1. *Let p be a prime number. Let v, t and k be non-negative integers.*

We assume that we have:

1) *Assume* $t < p$

   a) *If* $k_0 \geq t$. *Then*

$$rank_p(W_{t\,k}) = \binom{v}{t} \text{ and } Ker_p({}^tW_{t\,k}) = \{0\}.$$

   b) *If* $k_0 = 0$. *Then*

$$rank_p(W_{t\,k}) = \binom{v}{t} - 1, \dim Ker_p({}^tW_{t\,k}) = 1,$$
$$\text{and } \{(1, 1, \cdots, 1)\} \text{ is a basis of } Ker_p({}^tW_{t\,k}).$$

2) *Assume* $t = t_0 + t_1p$

   a) *If* $k_0 = t_0$ *and* $k_1 \geq t_1$. *Then*

$$Ker_p({}^tW_{t\,k}) = \{0\}.$$

   b) *If* $t = t_1p$ *and* $k_0 = k_1 = 0$. *Then*

$$\dim Ker_p({}^tW_{t\,k}) = 1 \text{ and } \{(1, 1, \cdots, 1)\} \text{ is a basis of } Ker_p({}^tW_{t\,k}).$$

3) *Assume* $t = t_0 + t_1p + t_2p^2$

   a) *If* $k_0 = t_0$, $k_1 = t_1$ *and* $k_2 \geq t_2$. *Then*

$$Ker_p({}^tW_{t\,k}) = \{0\}.$$

   b) *If* $t = t_2p^2$ *and* $k_0 = k_1 = k_2 = 0$. *Then*

$$\dim Ker_p({}^tW_{t\,k}) = 1 \textit{ and } \{(1,1,\cdots,1)\} \textit{ is a basis of } Ker_p({}^tW_{t\,k}).$$

A *directed graph* or simply *digraph* G consists of a finite and nonempty set V of vertices together with a prescribed collection E of ordered pairs of distinct vertices, called the set of the arcs of G. Such a digraph is denoted by $(V(G), E(G))$ or simply (V, E). Given a digraph $G = (V, E)$ with each nonempty subset X of V associate the *subdigraph* $(X, E \cap (X \times X))$ of G induced by X denoted by $G_{\restriction X}$. Given a proper subset X of V, $G_{\restriction V-X}$ is also denoted by $G - X$, and by $G - v$ whenever $X = \{v\}$.

Let $G = (V, E)$ be a digraph, for $x \neq y \in V$, $x \longrightarrow_G y$ or $y \longleftarrow_G x$ means $(x, y) \in E$ and $(y, x) \notin E$, $x \underline{\quad}_G y$ means $(x, y) \in E$ and $(y, x) \in E$, $x \ldots_G y$ means $(x, y) \notin E$ and $(y, x) \notin E$. For $X, Y \subseteq V$, $X \longrightarrow_G Y$ (or simply $X \longrightarrow Y$ or $X < Y$ if there is no confusion) signifies that for every $x \in X$ and $y \in Y$, $x \longrightarrow_G y$. For $X, Y \subseteq V$, $X \underline{\quad}_G Y$ and $X \ldots_G Y$ are defined in the same way. Given a digraph $G = (V, E)$, two distinct vertices x and y of G form an *oriented pair* or *directed pair* if either $x \longrightarrow_G y$ or $x \longleftarrow_G y$. Otherwise, $\{x, y\}$ is a *neutral pair*; it is *full* if $x \underline{\quad}_G y$, and *void* when $x \ldots_G y$. A digraph $T = (V, E)$ is a *tournament* whenever $x \longrightarrow_T y$ or $y \longrightarrow_T x$, for all $x \neq y \in V$. A *total order* or a *chain* is a tournament T such that for $x, y, z \in V(T)$, if $x \longrightarrow_T y$ and $y \longrightarrow_T z$ then $x \longrightarrow_T z$. Given a total order $O = (V, E)$, for $x, y \in V$, $x < y$ means $x \longrightarrow_O y$, then O can be denoted by $v_0 < v_1 \cdots < v_{n-1}$ where $n = |V|$.

Given two digraphs $G = (V, E)$ and $G' = (V', E')$, a bijection σ from V onto V' is an *isomorphism* from G onto G' provided that for any $x, y \in V$, $(x, y) \in E$ if and only if $(\sigma(x), \sigma(y)) \in E'$. Two digraphs are then isomorphic if there exists an isomorphism from one onto the other which is denoted by $G \simeq G'$.

Let $G = (V, E)$ be a digraph. A digraph H *embeds* into a digraph G or H is *embeddable* in G, if H is isomorphic to a subdigraph of G. The digraph $G^* = (V, E^*)$, dual of G, is defined by $(x, y) \in E^*$ if $(y, x) \in E$ for all $x \neq y \in V$. A digraph is *self-dual* if it is isomorphic to its dual.

Two digraphs G and G' on the same vertex set V are *hereditarily isomorphic* if for all $X \subseteq V$, $G_{\restriction X}$ and $G'_{\restriction X}$ are isomorphic.

Let k be a non-negative integer, G and G' are $\{k\}$-*hypomorphic* if for every k-element subset K of V, the induced subdigraphs $G'_{\restriction K}$ and $G_{\restriction K}$ are isomorphic. We say that G and G' are $(\leq k)$-*hypomorphic* if G and G' are $\{h\}$-hypomorphic for every integer $h \leq k$. Let $k \leq |V|$ be an integer, the digraphs G and G' are $\{-k\}$-*hypomorphic* if they are $\{|V| - k\}$-hypomorphic. A digraph G is $\{k\}$-reconstructible (resp. $\{-k\}$-reconstructible) if any digraph $\{k\}$-hypomorphic (resp. $\{-k\}$-hypomorphic) to G is isomorphic to it.

A digraph G is $(\leq k)$-reconstructible if any digraph $(\leq k)$-hypomorphic to G is isomorphic to it.

In 1977 P. K. Stockmeyer [15] showed that the tournaments are not, in general, $\{-1\}$-reconstructible, invalidating the conjecture of Ulam [16] for digraphs. Then, M. Pouzet [2, 3] proposed the $\{-k\}$-reconstruction problem of digraphs. P. Ille [9], in 1988, established that a digraph with at least 11 vertices $\{-5\}$-reconstructible. G. Lopez and C. Rauzy [12, 13], in 1992, showed that a digraph with at least 10 vertices is $\{-4\}$-reconstructible. In 1972, G. Lopez [10, 11], proved that the digraphs are (≤ 6)-reconstructible.

The incidence matrix is used in many reconstruction problems. For example J. Dammak, G. Lopez, M. Pouzet and H. Si Kaddour, in 2009, have used this matrix in a hypomorphy up to complementation problems [6]. As well, A. Ben Amira, J. Dammak and H. Si Kaddour, in 2014, have used this matrix in many construction of graphs and tournaments problems [1]. In this paper we use the previous results of incidence matrix Theorem 1.3 to prove Theorem 1.5 which is a generalization of Theorem 1.4.

Theorem 1.4. *([10, 11]) The digraphs are* (≤ 6)*-reconstructible.*

Using the incidence matrix, we give a version modulo a prime of Theorem 1.4. To introduce this version we should define some digraphs of cardinality 5 which are not self-dual.
$\alpha_5^+ = \{\{v_1, v_2, v_3, t_1, t_2\}, \{(v_1, v_2), (v_1, t_2), (v_2, t_2), (t_2, v_3), (v_3, v_1), (v_3, v_2), (v_3, t_1), (t_1, t_2), (t_2, t_1), (t_1, v_1), (t_1, v_2)\}\}$, $\beta_5^+ = \{\{v_1, v_2, v_3, t_1, t_2\}, \{(v_1, v_2), (v_1, t_2), (v_2, t_2), (t_2, v_3), (v_3, v_1), (v_3, v_2), (v_3, t_1), (t_1, v_1), (t_1, v_2)\}\}$, $\gamma_5^+ = \{\{v_1, t_1, t_2, t_3, t_4\}, \{(v_1, t_2), (v_1, t_3), (t_2, t_3), (t_3, t_4), (t_4, 1), (t_4, t_1), (t_1, t_3), (t_3, t_1), (t_1, v_1)\}\}$, $\alpha_5^- = (\alpha_5^+)^*$, $\beta_5^- = (\beta_5^+)^*$ and $\gamma_5^- = (\gamma_5^+)^*$. Obviously, α_5^+, β_5^+ and γ_5^+ are not self-dual.

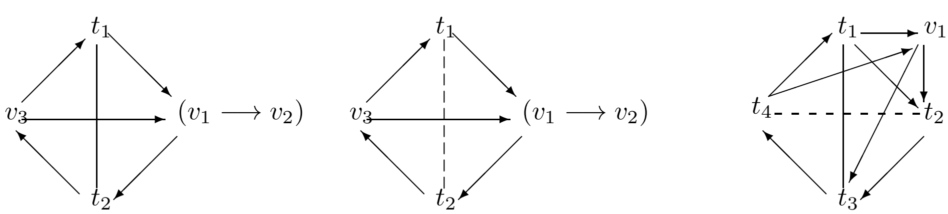

Figure 1: α_5^+, β_5^+ and γ_5^+.

We set β_6^+ the tournament defined in the set of vertices $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$ as follow, $v_0 < v_1 < v_2$, $v_3 < v_4$, $v_5 \longrightarrow \{v_0, v_1, v_2\} \longrightarrow \{v_3, v_4\} \longrightarrow v_5$.

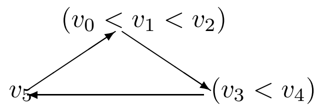

Figure 2: β_6^+.

According to these digraphs and for a digraph $G = (V, E)$, we denote the following sets and their cardinals, that will be used in the hypothesis of Theorem 1.5.

$$A_5^+(G) := \{X \subset V : G_{\restriction X} \simeq \alpha_5^+\},\ A_5^-(G) := \{X \subset V : G_{\restriction X} \simeq \alpha_5^-\},$$
$$B_5^+(G) := \{X \subset V : G_{\restriction X} \simeq \beta_5^+\},\ B_5^-(G) := \{X \subset V : G_{\restriction X} \simeq \beta_5^-\},$$
$$C_5^+(G) := \{X \subset V : G_{\restriction X} \simeq \gamma_5^+\},\ C_5^-(G) := \{X \subset V : G_{\restriction X} \simeq \gamma_5^-\},$$
$$a_5^+(G) :=\mid A_5^+(G) \mid,\ a_5^-(G) :=\mid A_5^-(G) \mid,\ b_5^+(G) :=\mid B_5^+(G) \mid,$$
$$b_5^-(G) :=\mid B_5^-(G) \mid,\ c_5^+(G) :=\mid C_5^+(G) \mid \text{ and } c_5^-(G) :=\mid C_5^-(G) \mid.$$
$$A_6^+(G) := \{X \subset V : G_{\restriction X} \simeq \beta_6^+\},\ a_6^+(G) := |A_6^+(G)|.$$

Theorem 1.5. *Let G, G' be two $\{4\}$-hypomorphic digraphs on the same set V of v vertices. Let p be a prime number and $k = [k_0, k_1, \dots]_p$ be an integer; $6 \leq k \leq v - 6$.*

If one of the following conditions is satisfied,

1) $a_5^+(G_{\restriction K}) = a_5^+(G'_{\restriction K})$, $b_5^+(G_{\restriction K}) = b_5^+(G'_{\restriction K})$, $c_5^+(G_{\restriction K}) = c_5^+(G'_{\restriction K})$ and $a_6^+(G_{\restriction K}) = a_6^+(G'_{\restriction K})$, for all k-elements subset K of V.

2) $a_5^+(G_{\restriction K}) \equiv a_5^+(G'_{\restriction K})$ (mod p), $b_5^+(G_{\restriction K}) \equiv b_5^+(G'_{\restriction K})$ (mod p), $c_5^+(G_{\restriction K}) \equiv c_5^+(G'_{\restriction K})$ (mod p) and $a_6^+(G_{\restriction K}) \equiv a_6^+(G'_{\restriction K})$ (mod p), for all k-elements subset K of V, $p \geq 7$ and ($k_0 \geq 6$ or $k_0 = 0$).

Then G and G' are hereditarily isomorphic.

2. Rank of the matrix $W_{t\,k}$ and kernel of ${}^tW_{t\,k}$

The notation $a \mid b$ (resp. $a \nmid b$) means a divides b (resp. a does not divide b).

Theorem 2.1. *(Lucas's Theorem [7]) Let p be a prime number, t, k be positive integers, $t \leq k$, $t = [t_0, t_1, \dots, t_{t(p)}]_p$ and $k = [k_0, k_1, \dots, k_{k(p)}]_p$. Then*

$$\binom{k}{t} = \prod_{i=0}^{t(p)} \binom{k_i}{t_i} \ (mod\ p),\ \ where\ \binom{k_i}{t_i} = 0 \ if\ t_i > k_i.$$

As a consequence of Theorem 2.1, we have.

Corollary 2.1. *Let p be a prime number, t, k be positive integers, $t \leq k$, $t = [t_0, t_1, \ldots, t_{t(p)}]_p$ and $k = [k_0, k_1, \ldots, k_{k(p)}]_p$. Then*

$$p | \binom{k}{t} \text{ if and only if there is } i \in [0, t(p)] \text{ such that } t_i > k_i.$$

Proof. We assume that $t_i \leq k_i < p$, for all $i \in [0, t(p)]$, we will prove that $p \nmid \binom{k}{t}$.

We have $t_i!(k_i - t_i)! \binom{k_i}{t_i} = k_i!$ and $p \nmid k_i!$, then $p \nmid \binom{k_i}{t_i}$ for all $i \in [0, t(p)]$.

From Theorem 2.1, $\binom{k}{t} = \prod_{i=0}^{t(p)} \binom{k_i}{t_i}$ $(mod\ p)$, then $p \nmid \binom{k}{t}$. Inversely, we assume that there exist $i \in [0, t(p)]$, such that $t_i > k_i$, so from Theorem 2.1 $\binom{k_i}{t_i} = 0$ and $\binom{k}{t} = \prod_{i=0}^{t(p)} \binom{k_i}{t_i}$ (mod p), then $p | \binom{k}{t}$. □

Lemma 2.1. *Let p be a prime number, t, k be positive integers, $t \leq k$, $t = [t_0, t_1, \ldots, t_{t(p)}]_p$ and $k = [k_0, k_1, \ldots, k_{k(p)}]_p$.*

We have $p \nmid \binom{k_i}{t_i}$ $t_i \leq k_i \leq p$ and $\binom{k_i}{t_i} = 0$ if $t_i > k_i$.

Proof. The proof follow immediately from Corollary 2.1. □

To prove Theorem 1.3, we use the following lemma:

Lemma 2.2. *Let p be a prime, t, k and i be positive integers, $i \leq t \leq k$, $t = [t_0, t_1, \ldots, t_{t(p)}]_p$, $k = [k_0, k_1, \ldots, k_{k(p)}]_p$ and $i = [i_0, i_1, \ldots, i_{i(p)}]_p$.*

$p \nmid \binom{(k-i)_0}{(t-i)_0}$ if and only if

1. *$k_0 < t_0$ and $i_0 \in [k_0 + 1, t_0]$.*
2. *$k_0 \geq t_0$ and $i_0 \notin [t_0 + 1, k_0]$.*

Proof.

1. $k_0 < t_0$
   (a) If $i_0 \in [0, k_0]$ then $(t - i)_0 = t_0 - i_0 > k_0 - i_0 = (k - i)_0$. From Lemma 2.1, we have $p \mid \binom{(k-i)_0}{(t-i)_0}$ then $p \mid \binom{k-i}{t-i}$.
   (b) If $i_0 \in [k_0 + 1, t_0]$ then $(k - i)_0 = p + k_0 - i_0 \geq t_0 - i_0 = (t - i)_0$. From Lemma 2.1, we have $p \nmid \binom{(k-i)_0}{(t-i)_0}$.
   (c) If $i_0 \in [t_0+1, p-1]$ then $(t-i)_0 = p+t_0-i_0 > p+k_0-i_0 = (k-i)_0$. From Lemma 2.1, we have $p \mid \binom{(k-i)_0}{(t-i)_0}$ then $p \mid \binom{k-i}{t-i}$.
2. $k_0 \geq t_0$
   (a) If $i_0 \in [0, t_0]$ then $(k - i)_0 = k_0 - i_0 \geq t_0 - i_0 = (t - i)_0$. From Lemma 2.1, we have $p \nmid \binom{(k-i)_0}{(t-i)_0}$.

(b) If $i_0 \in [t_0+1, k_0]$ then $(t-i)_0 = p + t_0 - i_0 > k_0 - i_0 = (k-i)_0$. From Lemma 2.1, we have $p \mid \binom{(k-i)_0}{(t-i)_0}$ then $p \mid \binom{k-i}{t-i}$.

(c) If $i_0 \in [k_0+1, p-1]$ then $(k-i)_0 = p+k_0-i_0 \geq p+t_0-i_0 = (t-i)_0$. From Lemma 2.1, we have $p \nmid \binom{(k-i)_0}{(t-i)_0}$. ☐

Lemma 2.3. *Let p be a prime, t, k and i be positive integers, $i \leq t \leq k$, $t = [t_0, t_1, \ldots, t_{t(p)}]_p$, $k = [k_0, k_1, \ldots, k_{k(p)}]_p$ and $i = [i_0, i_1, \ldots, i_{i(p)}]_p$.*

$p \nmid \binom{(k-i)_0}{(t-i)_0}$ and $p \nmid \binom{(k-i)_1}{(t-i)_1}$ if and only if

1) $k_0 < t_0$ and $i_0 \in [k_0+1, t_0]$.

a) $k_1 - 1 < t_1$ and $i_1 \in [k_1, t_1]$.

b) $k_1 - 1 \geq t_1$ and ($i_1 \in [0, t_1]$ or $i_1 \in [k_1, p-1]$).

2) $k_0 \geq t_0$ and $i_0 \in [0, t_0]$.

a) $k_1 < t_1$ and $i_1 \in [k_1+1, t_1]$.

b) $k_1 \geq t_1$ and ($i_1 \in [0, t_1]$ or $i_1 \in [k_1+1, p-1]$).

3) $k_0 \geq t_0$ and $i_0 \in [k_0+1, p-1]$.

a) $k_1 - 1 < t_1 - 1$ and $i_1 \in [k_1, t_1 - 1]$.

b) $k_1 - 1 \geq t_1 - 1$ and ($i_1 \in [0, t_1 - 1]$ or $i_1 \in [k_1, p-1]$).

Proof.

1) As $k_0 < i_0 \leq t_0$, we have $k-i = [k_0 - i_0 + p, \ldots]_p$ and $t-i = [t_0 - i_0, \ldots]_p$. In Lemma 2.2, we replace k_0 by $k_1 - 1$ and t_0 by t_1 we have

a) Assume $k_1 - 1 < t_1$.

i) If $i_1 \in [0, k_1 - 1]$ then $(t-i)_1 = t_1 - i_1 > k_1 - i_1 - 1 = (k-i)_1$. From Lemma 2.1, we have $p \mid \binom{(k-i)_1}{(t-i)_1}$.

ii) If $i_1 \in [k_1, t_1]$ then $(k-i)_1 = p + k_1 - i_1 - 1 \geq t_1 - i_1 = (t-i)_1$. From Lemma 2.1, we have $p \nmid \binom{(k-i)_1}{(t-i)_1}$.

iii) If $i_1 \in [t_1+1, p-1]$ then $(t-i)_1 = p + t_1 - i_1 > p + k_1 - i_1 - 1 = (k-i)_1$. From Lemma 2.1, we have $p \mid \binom{(k-i)_1}{(t-i)_1}$.

b) Assume $k_1 - 1 \geq t_1$.

i) If $i_1 \in [0, t_1]$ then $(k-i)_1 = k_1 - i_1 - 1 \geq t_1 - i_1 = (t-i)_1$. From Lemma 2.1, we have $p \nmid \binom{(k-i)_1}{(t-i)_1}$.

ii) If $i_1 \in [t_1+1, k_1-1]$ then $(t-i)_1 = p+t_1-i_1 > k_1-i_1-1 = (k-i)_1$. From Lemma 2.1, we have $p \mid \binom{(k-i)_1}{(t-i)_1}$.

iii) If $i_1 \in [k_1, p-1]$ then $(k-i)_1 = p+k_1-i_1-1 \geq p+t_1-i_1 = (t-i)_1$. From Lemma 2.1, we have $p \nmid \binom{(k-i)_1}{(t-i)_1}$.

2) As $i_0 \leq t_0 \leq k_0$, we have $k-i = [k_0 - i_0, \dots]_p$ and $t-i = [t_0 - i_0, \dots]_p$. In Lemma 2.2, we replace k_0 by k_1 and t_0 by t_1 we have the result.
3) As $t_0 \leq k_0 < i_0$, we have $k-i = [k_0 - i_0 + p, \dots]_p$ and $t-i = [t_0 - i_0 + p, \dots]_p$. In Lemma 2.2, we replace k_0 by $k_1 - 1$ and t_0 by $t_1 - 1$ we the result. ☐

Lemma 2.4. *Let p be a prime, t, k and i be positive integers, $i \leq t \leq k$, $t = [t_0, t_1, t_2]_p$, $k = [k_0, k_1, \dots, k_{k(p)}]_p$ and $i = [i_0, i_1, i_2]_p$.*
$p \nmid \binom{(k-i)}{(t-i)}$ if and only if

1) $k_0 < t_0$, $k_1 - 1 < t_1$, $i_0 \in [k_0+1, t_0]$ *and* $i_1 \in [k_1, t_1]$.
a) $k_2 \leq t_2$ *and* $i_2 \in [k_2, t_2]$.
b) $k_2 \geq t_2 + 1$ *and* $i_2 \in [0, t_2]$.

2) $k_0 < t_0$, $k_1 - 1 \geq t_1$, $i_0 \in [k_0+1, t_0]$ *and* $i_1 \in [0, t_1]$.
a) $k_2 \leq t_2 - 1$ *and* $i_2 \in [k_2+1, t_2]$.
b) $k_2 \geq t_2$ *and* $i_2 \in [0, t_2]$.

3) $k_0 < t_0$, $k_1 - 1 \geq t_1$, $i_0 \in [k_0+1, t_0]$ *and* $i_1 \in [k_1, p-1]$.
a) $k_2 \leq t_2 - 1$ *and* $i_2 \in [k_2, t_2 - 1]$.
b) $k_2 \geq t_2$ *and* $i_2 \in [0, t_2 - 1]$.

4) $k_0 \geq t_0$, $k_1 < t_1$, $i_0 \in [0, t_0]$ *and* $i_1 \in [k_1+1, t_1]$.
a) $k_2 \leq t_2$ *and* $i_2 \in [k_2, t_2]$.
b) $k_2 \geq t_2 + 1$ *and* $i_2 \in [0, t_2]$.

5) $k_0 \geq t_0$, $k_1 \geq t_1$, $i_0 \in [0, t_0]$ *and* $i_1 \in [0, t_1]$.
a) $k_2 \leq t_2 - 1$ *and* $i_2 \in [k_2+1, t_2]$.
b) $k_2 \geq t_2$ *and* $i_2 \in [0, t_2]$.

6) $k_0 \geq t_0$, $k_1 \geq t_1$, $i_0 \in [0, t_0]$ *and* $i_1 \in [k_1+1, p-1]$.
a) $k_2 \leq t_2 - 1$ *and* $i_2 \in [k_2, t_2 - 1]$.
b) $k_2 \geq t_2$ *and* $i_2 \in [0, t_2 - 1]$.

7) $k_0 \geq t_0$, $k_1 - 1 < t_1 - 1$, $i_0 \in [k_0+1, p-1]$ *and* $i_1 \in [k_1, t_1 - 1]$.
a) $k_2 \leq t_2$ *and* $i_2 \in [k_2, t_2]$.
b) $k_2 \geq t_2 + 1$ *and* $i_2 \in [0, t_2]$.

8) $k_0 \geq t_0$*,* $k_1 - 1 \geq t_1 - 1$*,* $i_0 \in [k_0 + 1, p - 1]$ *and* $i_1 \in [0, t_1 - 1]$*.*

a) $k_2 \leq t_2 - 1$ *and* $i_2 \in [k_2 + 1, t_2]$*.*

b) $k_2 \geq t_2$ *and* $i_2 \in [0, t_2]$*.*

9) $k_0 \geq t_0$*,* $k_1 - 1 \geq t_1 - 1$*,* $i_0 \in [k_0 + 1, p - 1]$ *and* $i_1 \in [k_1, p - 1]$*.*

a) $k_2 \leq t_2 - 1$ *and* $i_2 \in [k_2, t_2 - 1]$*.*

b) $k_2 \geq t_2$ *and* $i_2 \in [0, t_2 - 1]$*.*

Proof. As $i \leq t$, we have $i_2 \leq t_2$.

1) As $k_0 < i_0 \leq t_0$ and $k_1 - 1 < i_1 \leq t_1$, we have $k - i = [k_0 - i_0 + p, k_1 - i_1 + p \dots]_p$ and $t - i = [t_0 - i_0, t_1 - i_1 \dots]_p$. In Lemma 2.2, we replace k_0 by $k_2 - 1$ and t_0 by t_2 we have

   a) Assume $k_2 \leq t_2$.

      i) If $i_2 \in [0, k_2 - 1]$ then $(t - i)_2 = t_2 - i_2 > k_2 - i_2 - 1 = (k - i)_2$. From Lemma 2.1, we have $p \mid \binom{(k-i)_2}{(t-i)_2}$, then $p \mid \binom{(k-i)}{(t-i)}$.

      ii) If $i_2 \in [k_2, t_2]$ then $(k - i)_2 = p + k_2 - i_2 - 1 \geq t_2 - i_2 = (t - i)_2$. From Lemma 2.1, we have $p \nmid \binom{(k-i)_2}{(t-i)_2}$, then $p \nmid \binom{(k-i)}{(t-i)}$.

   b) Assume $k_2 \geq t_2 + 1$.

      i) If $i_2 \in [0, t_2]$ then $(k - i)_2 = k_2 - i_2 - 1 \geq t_2 - i_2 = (t - i)_2$. From Lemma 2.1, we have $p \nmid \binom{(k-i)_2}{(t-i)_2}$, then $p \nmid \binom{(k-i)}{(t-i)}$.

2) As $k_0 < i_0 \leq t_0$, $i_1 \leq t_1 \leq k_1 - 1$, we have $k - i = [k_0 - i_0 + p, k_1 - i_1 \dots]_p$ and $t - i = [t_0 - i_0, t_1 - i_1 \dots]_p$. In Lemma 2.2, we replace k_0 by k_2 and t_0 by t_2 we have the result.

3) As $k_0 < i_0 \leq t_0$, $t_1 \leq k_1 - 1 \leq i_1$, we have $k - i = [k_0 - i_0 + p, k_1 - i_1 + p \dots]_p$ and $t - i = [t_0 - i_0, t_1 - i_1 + p \dots]_p$. In Lemma 2.2, we replace k_0 by $k_2 - 1$ and t_0 by $t_2 - 1$ we have the result.

4) As $i_0 \leq t_0 \leq k_0$, $k_1 < i_1 \leq t_1$, we have $k - i = [k_0 - i_0, k_1 - i_1 + p \dots]_p$ and $t - i = [t_0 - i_0, t_1 - i_1 \dots]_p$. In Lemma 2.2, we replace k_0 by $k_2 - 1$ and t_0 by t_2 we have the result.

5) As $i_0 \leq t_0 \leq k_0$, $i_1 \leq t_1 \leq k_1$, we have $k - i = [k_0 - i_0, k_1 - i_1 \dots]_p$ and $t - i = [t_0 - i_0, t_1 - i_1 \dots]_p$. In Lemma 2.2, we replace k_0 by k_2 and t_0 by t_2 we have the result.

6) As $i_0 \leq t_0 \leq k_0$, $t_1 \leq k_1 < i_1$, we have $k - i = [k_0 - i_0, k_1 - i_1 + p \dots]_p$ and $t - i = [t_0 - i_0, t_1 - i_1 + p \dots]_p$. In Lemma 2.2, we replace k_0 by $k_2 - 1$ and t_0 by $t_2 - 1$ we have the result.

7) As $t_0 \leq k_0 < i_0$, $k_1 - 1 < i_1 \leq t_1 - 1$, we have $k - i = [k_0 - i_0 + p, k_1 - i_1 + p \dots]_p$ and $t - i = [t_0 - i_0 + p, t_1 - i_1 \dots]_p$. In Lemma 2.2, we replace k_0 by $k_2 - 1$ and t_0 by t_2 we have the result.

8) As $t_0 \le k_0 < i_0$, $i_1 \le t_1 - 1 \le k_1 - 1$, we have $k - i = [k_0 - i_0 + p, k_1 - i_1 \dots]_p$ and $t - i = [t_0 - i_0 + p, t_1 - i_1 \dots]_p$. In Lemma 2.2, we replace k_0 by k_2 and t_0 by t_2 we have the result.
9) As $t_0 \le k_0 < i_0$, $t_1 - 1 \le k_1 - 1 < i_1$, $i_0 \in [k_0 + 1, p - 1]$, $i_1 \in [k_1, p - 1]$, we have $k - i = [k_0 - i_0 + p, k_1 - i_1 + p \dots]_p$ and $t - i = [t_0 - i_0 + p, t_1 - i_1 + p \dots]_p$. In Lemma 2.2, we replace k_0 by $k_2 - 1$ and t_0 by $t_2 - 1$ we have the result. □

Proof of Theorem 1.3. Let p be a prime number, t, k be positive integers, $t \le min(k, v - k)$, $t = [t_0, t_1, t_2]_p$ and $k = [k_0, k_1, \dots, k_{k(p)}]_p$.

Obviously, we have $\sum_{i=\alpha}^{\beta} \binom{v}{i} - \binom{v}{i-1} = \binom{v}{\beta} - \binom{v}{\alpha-1}$

1. We have $k_0 \le t_0 - 1$, $k_1 \le t_1$ and $k_2 \le t_2$, then from 1)a) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [k_2, t_2]$, $i_1 \in [k_1, t_1]$ and $i_0 \in [k_0 + 1, t_0]$. From Theorem 1.2, $rank(W_{t\,k})$
$$= \sum_{i_2=k_2}^{t_2} \sum_{i_1=k_1}^{t_1} \sum_{i_0=k_0+1}^{t_0} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1}$$
$$= \sum_{i_2=k_2}^{t_2} \sum_{i_1=k_1}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p+k_0}.$$
2. We have $k_0 \ge t_0$, $k_1 \le t_1 - 1$ and $k_2 \le t_2$, then from 4)a) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [k_2, t_2]$, $i_1 \in [k_1 + 1, t_1]$ and $i_0 \in [0, t_0]$, from 7)a) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [k_2, t_2]$, $i_1 \in [k_1, t_1 - 1]$ and $i_0 \in [k_0 + 1, p - 1]$. From Theorem 1.2, $rank(W_{t\,k})$
$$= \sum_{i_2=k_2}^{t_2} \sum_{i_1=k_1+1}^{t_1} \sum_{i_0=0}^{t_0} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1} +$$
$$\sum_{i_2=k_2}^{t_2} \sum_{i_1=k_1}^{t_1-1} \sum_{i_0=k_0+1}^{p-1} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1}$$
$$= \sum_{i_2=k_2}^{t_2} \sum_{i_1=k_1+1}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p-1} +$$
$$\sum_{i_2=k_2}^{t_2} \sum_{i_1=k_1}^{t_1-1} \binom{v}{i_2p^2+i_1p+p-1} - \binom{v}{i_2p^2+i_1p+k_0}$$
$$= \sum_{i_2=k_2}^{t_2} \sum_{i_1=k_1+1}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+(i_1-1)p+p-1} +$$
$$\sum_{i_2=k_2}^{t_2} \sum_{i_1=k_1+1}^{t_1} \binom{v}{i_2p^2+(i_1-1)p+p-1} - \binom{v}{i_2p^2+(i_1-1)p+k_0}$$
$$= \sum_{i_2=k_2}^{t_2} \sum_{i_1=k_1+1}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+(i_1-1)p+k_0}.$$

3. We have $k_0 \leq t_0 - 1$, $k_1 \geq t_1 + 1$ and $k_2 \leq t_2 - 1$, then from 2)a) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [k_2+1, t_2]$, $i_1 \in [0, t_1]$ and $i_0 \in [k_0+1, t_0]$ and from 3)a) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [k_2, t_2-1]$, $i_1 \in [k_1, p-1]$ and $i_0 \in [k_0+1, t_0]$. From Theorem 1.2, $rank(W_{t\,k})$
$$= \sum_{i_2=k_2+1}^{t_2} \sum_{i_1=0}^{t_1} \sum_{i_0=k_0+1}^{t_0} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1} +$$
$$\sum_{i_2=k_2}^{t_2-1} \sum_{i_1=k_1}^{p-1} \sum_{i_0=k_0+1}^{t_0} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1}$$
$$= \sum_{i_2=k_2+1}^{t_2} \sum_{i_1=0}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p+k_0} +$$
$$\sum_{i_2=k_2}^{t_2-1} \sum_{i_1=k_1}^{p-1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p+k_0}.$$
4. We have $k_0 \geq t_0$, $k_1 \geq t_1$ and $k_2 \leq t_2 - 1$, then from 5)a) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [k_2+1, t_2]$, $i_1 \in [0, t_1]$ and $i_0 \in [0, t_0]$, from 6)a) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [k_2, t_2-1]$, $i_1 \in [k_1+1, p-1]$ and $i_0 \in [0, t_0]$, from 8)a) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [k_2+1, t_2]$, $i_1 \in [0, t_1-1]$ and $i_0 \in [k_0+1, p-1]$ and from 9)a) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [k_2, t_2-1]$, $i_1 \in [k_1, p-1]$ and $i_0 \in [k_0+1, p-1]$. From Theorem 1.2, $rank(W_{t\,k})$
$$= \sum_{i_2=k_2+1}^{t_2} \sum_{i_1=0}^{t_1} \sum_{i_0=0}^{t_0} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1} +$$
$$\sum_{i_2=k_2}^{t_2-1} \sum_{i_1=k_1+1}^{p-1} \sum_{i_0=0}^{t_0} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1}$$
$$+ \sum_{i_2=k_2+1}^{t_2} \sum_{i_1=0}^{t_1-1} \sum_{i_0=k_0+1}^{p-1} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1} +$$
$$\sum_{i_2=k_2}^{t_2-1} \sum_{i_1=k_1}^{p-1} \sum_{i_0=k_0+1}^{p-1} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1}$$
$$= \sum_{i_2=k_2+1}^{t_2} \sum_{i_1=0}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p-1} +$$
$$\sum_{i_2=k_2}^{t_2-1} \sum_{i_1=k_1+1}^{p-1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p-1}$$
$$+ \sum_{i_2=k_2+1}^{t_2} \sum_{i_1=0}^{t_1-1} \binom{v}{i_2p^2+i_1p+p-1} - \binom{v}{i_2p^2+i_1p+k_0} +$$
$$\sum_{i_2=k_2}^{t_2-1} \sum_{i_1=k_1}^{p-1} \binom{v}{i_2p^2+i_1p+p-1} - \binom{v}{i_2p^2+i_1p+k_0}$$

$$= \sum_{i_2=k_2+1}^{t_2} \sum_{i_1=0}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p-1} +$$
$$\sum_{i_2=k_2+1}^{t_2} \sum_{i_1=0}^{t_1-1} \binom{v}{i_2p^2+i_1p+p-1} - \binom{v}{i_2p^2+i_1p+k_0}$$
$$+ \sum_{i_2=k_2}^{t_2-1} \sum_{i_1=k_1+1}^{p-1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p-1} +$$
$$\sum_{i_2=k_2}^{t_2-1} \sum_{i_1=k_1}^{p-1} \binom{v}{i_2p^2+i_1p+p-1} - \binom{v}{i_2p^2+i_1p+k_0}$$
$$= \sum_{i_2=k_2}^{t_2-1} \sum_{i_1=k_1+1}^{p-1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+(i_1-1)p+k_0} +$$
$$\sum_{i_2=k_2+1}^{t_2} \sum_{i_1=0}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+(i_1-1)p+k_0}.$$

5. We have $k_0 \le t_0 - 1$, $k_1 \le t_1$ and $k_2 \ge t_2 + 1$, then from 1)b) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [0, t_2]$, $i_1 \in [k_1, t_1]$ and $i_0 \in [k_0 + 1, t_0]$. From Theorem 1.2, $rank(W_{t\,k})$
$$= \sum_{i_2=0}^{t_2} \sum_{i_1=k_1}^{t_1} \sum_{i_0=k_0+1}^{t_0} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1} =$$
$$\sum_{i_2=0}^{t_2} \sum_{i_1=k_1}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p+k_0}.$$
6. We have $k_0 \ge t_0$, $k_1 \le t_1 - 1$ and $k_2 \ge t_2 + 1$, then from 4)b) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [0, t_2]$, $i_1 \in [k_1 + 1, t_1]$ and $i_0 \in [0, t_0]$, from 7)b) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [0, t_2]$, $i_1 \in [k_1, t_1 - 1]$ and $i_0 \in [k_0 + 1, p - 1]$. From Theorem 1.2, $rank(W_{t\,k})$
$$= \sum_{i_2=0}^{t_2} \sum_{i_1=k_1+1}^{t_1} \sum_{i_0=0}^{t_0} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1} +$$
$$\sum_{i_2=0}^{t_2} \sum_{i_1=k_1}^{t_1-1} \sum_{i_0=k_0+1}^{p-1} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1}$$
$$= \sum_{i_2=0}^{t_2} \sum_{i_1=k_1+1}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p-1} +$$
$$\sum_{i_2=0}^{t_2} \sum_{i_1=k_1}^{t_1-1} \binom{v}{i_2p^2+i_1p+p-1} - \binom{v}{i_2p^2+i_1p+k_0}$$
$$= \sum_{i_2=0}^{t_2} \sum_{i_1=k_1+1}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+(i_1-1)p+p-1} +$$
$$\sum_{i_2=0}^{t_2} \sum_{i_1=k_1+1}^{t_1} \binom{v}{i_2p^2+(i_1-1)p+p-1} - \binom{v}{i_2p^2+(i_1-1)p+k_0}$$
$$= \sum_{i_2=0}^{t_2} \sum_{i_1=k_1+1}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+(i_1-1)p+k_0}.$$

7. We have $k_0 \leq t_0 - 1$, $k_1 \geq t_1 + 1$ and $k_2 \geq t_2$, then from 2)b) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [0, t_2]$, $i_1 \in [0, t_1]$ and $i_0 \in [k_0 + 1, t_0]$ and from 3)b) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [0, t_2 - 1]$, $i_1 \in [k_1, p-1]$ and $i_0 \in [k_0 + 1, t_0]$. From Theorem 1.2, $rank(W_{t\,k})$
$$= \sum_{i_2=0}^{t_2} \sum_{i_1=0}^{t_1} \sum_{i_0=k_0+1}^{t_0} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1} +$$
$$\sum_{i_2=0}^{t_2-1} \sum_{i_1=k_1}^{p-1} \sum_{i_0=k_0+1}^{t_0} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1}$$
$$= \sum_{i_2=0}^{t_2} \sum_{i_1=0}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p+k_0} +$$
$$\sum_{i_2=0}^{t_2-1} \sum_{i_1=k_1}^{p-1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p+k_0}.$$
8. We have $k_0 \geq t_0$, $k_1 \geq t_1$ and $k_2 \geq t_2$, then from 5)b) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [0, t_2]$, $i_1 \in [0, t_1]$ and $i_0 \in [0, t_0]$, from 6)b) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [0, t_2 - 1]$, $i_1 \in [k_1 + 1, p-1]$ and $i_0 \in [0, t_0]$, from 8)b) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [0, t_2]$, $i_1 \in [0, t_1 - 1]$ and $i_0 \in [k_0 + 1, p-1]$ and from 9)b) of Lemma 2.4, $p \nmid \binom{k-i}{t-i}$ if and only if $i_2 \in [0, t_2 - 1]$, $i_1 \in [k_1, p-1]$ and $i_0 \in [k_0 + 1, p-1]$. From Theorem 1.2, $rank(W_{t\,k})$
$$= \sum_{i_2=0}^{t_2} \sum_{i_1=0}^{t_1} \sum_{i_0=0}^{t_0} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1} +$$
$$\sum_{i_2=0}^{t_2-1} \sum_{i_1=k_1+1}^{p-1} \sum_{i_0=0}^{t_0} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1}$$
$$+ \sum_{i_2=0}^{t_2} \sum_{i_1=0}^{t_1-1} \sum_{i_0=k_0+1}^{p-1} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1} +$$
$$\sum_{i_2=0}^{t_2-1} \sum_{i_1=k_1}^{p-1} \sum_{i_0=k_0+1}^{p-1} \binom{v}{i_2p^2+i_1p+i_0} - \binom{v}{i_2p^2+i_1p+i_0-1}$$
$$= \sum_{i_2=0}^{t_2} \sum_{i_1=0}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p-1} +$$
$$\sum_{i_2=0}^{t_2-1} \sum_{i_1=k_1+1}^{p-1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p-1}$$
$$+ \sum_{i_2=0}^{t_2} \sum_{i_1=0}^{t_1-1} \binom{v}{i_2p^2+i_1p+p-1} - \binom{v}{i_2p^2+i_1p+k_0} +$$
$$\sum_{i_2=0}^{t_2-1} \sum_{i_1=k_1}^{p-1} \binom{v}{i_2p^2+i_1p+p-1} - \binom{v}{i_2p^2+i_1p+k_0}$$

$$= \sum_{i_2=0}^{t_2}\sum_{i_1=0}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p-1} +$$
$$\sum_{i_2=0}^{t_2}\sum_{i_1=0}^{t_1-1} \binom{v}{i_2p^2+i_1p+p-1} - \binom{v}{i_2p^2+i_1p+k_0}$$
$$+ \sum_{i_2=0}^{t_2-1}\sum_{i_1=k_1+1}^{p-1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+i_1p-1} +$$
$$\sum_{i_2=0}^{t_2-1}\sum_{i_1=k_1}^{p-1} \binom{v}{i_2p^2+i_1p+p-1} - \binom{v}{i_2p^2+i_1p+k_0}$$
$$= \sum_{i_2=0}^{t_2-1}\sum_{i_1=k_1+1}^{p-1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+(i_1-1)p+k_0} +$$
$$\sum_{i_2=0}^{t_2}\sum_{i_1=0}^{t_1} \binom{v}{i_2p^2+i_1p+t_0} - \binom{v}{i_2p^2+(i_1-1)p+k_0} \qquad \square$$

3. Proof of Theorem 1.5

Let $k \geq 1$ be an integer and G be a digraph. G is $\{k\}$-*monomorphic* if $G_{\restriction X} \simeq G_{\restriction Y}$ for all k-element subsets X and Y of V.

Lemma 3.1. *([14]) Let v, t, k be three integers, $t \leq min(k, v-k)$ and G and G' be two graphs on the same set V of v vertices. If G and G' are $\{k\}$-hypomorphic (resp. G is $\{k\}$-monomorphic) then G and G' are $\{t\}$-hypomorphic (resp. G is $\{t\}$-monomorphic).*

Let $G = (V, E)$ and $G' = (V, E')$ be two digraph. G and G' are $\{2\}$-hypomorphic if and only if, for all $x, y \in V$, if $x \underline{\quad}_G y$ (resp. $x \ldots_G y$), then $x \underline{\quad}_{G'} y$ (resp. $x \ldots_{G'} y$) and if $\{x, y\}$ is an oriented pair in G then $\{x, y\}$ is oriented in G'.

From Lemma 3.1, follow immediately this result.

Corollary 3.1. *If $G = (V, E)$ and $G' = (V, E')$ are $\{4\}$-hypomorphic digraphs and $|V| \geq 7$, then G and G' are (≤ 4)-hypomorphic.*

A 3-*cycle* is a tournament isomorphic to $C_3 = (\{v_0, v_1, v_2\}, \{(v_0, v_1), (v_1, v_2), (v_2, v_0)\})$.

Lemma 3.2.

1) *Every digraph G with at least 7 vertices contains a restriction of cardinality 5 not isomorphic to α_5^+, nor β_5^+, nor γ_5^+.*
2) *Every digraph G with at least 9 vertices contains a restriction of cardinality 6 not isomorphic to β_6^+.*

Proof.

1) By contradiction, we assume that $G_{\restriction X} \simeq \alpha_5^+$ (resp. $G_{\restriction X} \simeq \beta_5^+$ or $G_{\restriction X} \simeq \gamma_5^+$) for all 5-element subsets X, so G is $\{5\}$-monomorphic. From Lemma 3.1, we deduce G is (≤ 2)-monomorphic, then G is a tournament, or G is the full graph, or G is the empty graph. A contradiction.
2) By contradiction, we assume that $G_{\restriction X} \simeq \beta_6^+$ for all 6-element subsets X, so G is $\{6\}$-monomorphic. From Lemma 3.1, we deduce G is (≤ 3)-monomorphic. As β_6^+ embeds at least a 3-cycle and a 3-chain. A contradiction. □

A *flag* is a digraph isomorphic to $(\{v_0, v_1, v_2\}, \{(v_1, v_0), (v_0, v_2), (v_2, v_0)\})$ or to its dual.

A *full peak* is a digraph isomorphic to $(\{v_0, v_1, v_2\}, \{(v_1, v_0), (v_2, v_0), (v_1, v_2), (v_2, v_1)\})$ or to its dual.

A *void peak* is a digraph isomorphic to $(\{v_0, v_1, v_2\}, \{(v_1, v_0), (v_2, v_0)\})$ or to its dual.

A 3-*consecutivity* is a digraph isomorphic to $(\{v_0, v_1, v_2\}, \{(v_0, v_1), (v_1, v_2)\})$ or to $(\{v_0, v_1, v_2\}, \{(v_0, v_1), (v_1, v_2), (v_2, v_0), (v_0, v_2)\})$.

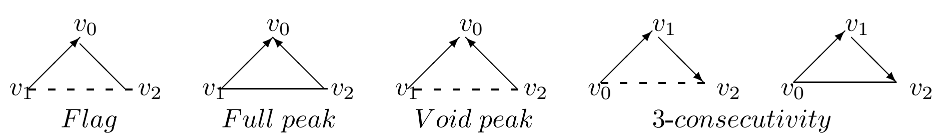

Figure 3: Flag, Full peak, Void peak, 3-consecutivity.

Let $G = (V, E)$ and $G' = (V, E')$ be two (≤ 2)-hypomorphic digraphs. Denote $\mathfrak{D}_{G,G'}$ the binary relation on V such that: for $x \in V$, $x\mathfrak{D}_{G,G'}x$; and for $x \neq y \in V$, $x\mathfrak{D}_{G,G'}y$ if there exists a sequence $x_0 = x, ..., x_n = y$ of elements of V satisfying $(x_i, x_{i+1}) \in E$ if and only if $(x_i, x_{i+1}) \notin E'$, for all i, $0 \leq i \leq n-1$. The relation $\mathfrak{D}_{G,G'}$ is an equivalence relation called *the difference relation*, its classes are called *difference classes*. Let $D_{G,G'}$ denote the set of difference classes. The $x = x_0, x_1, \ldots, x_n = y$ as above, are referred to as $D_{G,G'}$-paths.

The families S_n and $\mathcal{E}(S_n)$ Let $n \geq 1$ be an integer. The integers below are considered modulo $2n$. An *element of the family* $\mathcal{E}(S_n)$ is a digraph, not a tournament that embeds neither peaks nor diamonds nor adjacent neutral pairs. The morphology of such a family is described by G. Lopez and C.

Rauzy [12]. First we introduce a *sub family* S_n of the family $\mathcal{E}(S_n)$. For $n = 1$, an element of the family S_1 is a digraph on 2 vertices with a neutral pair. For $n \geq 2$, *an element of the family* S_n is a digraph isomorphic to $g_n = (\{t_1, \ldots, t_{2n}\}, E_n)$, where g_n is defined by, $\{t_i, t_j\}$ is a neutral pair of g_n if and only if $j = i+n$ and $t_i \longrightarrow_{g_n} t_j$ if there exists $k \in \{1, \ldots, n-1\}$ such that $j = i + k$. The two neutral pairs $\{t_i, t_{i+n}\}$ and $\{t_{i+1}, t_{i+n+1}\}$ are called *successive* for every $i \in \{1, 2, \ldots, n-1\}$. *An element of the family* $\mathcal{E}(S_n)$ is a digraph isomorphic to the digraph G_n, where G_n is obtained from g_n by adding mutually disjoint sets $s_1, s_2, \ldots, s_{2n}$ (the set s_i is called a *sector* and it could be empty) to the vertex set $\{t_1, t_2, \ldots, t_{2n}\}$ of g_n satisfying the following conditions:

- **(i)** $G_n[\{t_1, t_2, \ldots, t_{2n}\}] = g_n$ and for all $i \in \{1, 2, \ldots, 2n\}$, the subdigraph $G_n[s_i \cup \{t_i, t_{i+1}\}]$ is a finite chain such that $t_i \rightarrow_{G_n} s_i$ and $s_i \rightarrow_{G_n} t_{i+1}$.
- **(ii)** For $i \in \{1, 2, \ldots, 2n\}$, $\{t_i, t_{i+n}\}$ are the only neutral pairs of G_n.
- **(iii)** For $i, j \in \{1, 2, \ldots, 2n\}$, $s_i \rightarrow_{G_n} t_j$ if there exists $k \in \{1, \ldots, n\}$ such that $j = i + k$.
- **(iv)** For $i, j \in \{1, 2, \ldots, 2n\}$, $s_i \rightarrow_{G_n} s_j$ if there exists $k \in \{1, 2 \ldots, n-2, n-1\}$ such that $j = i + k$.

A *diamond* is a tournament isomorphic to $\delta^+ = (\{v_0, v_1, v_2, v_3\}, \{(v_0, v_1), (v_1, v_2), (v_2, v_0), (v_0, v_3), (v_1, v_3), (v_2, v_3)\})$, called *a positive diamond*, or to its dual $\delta^- = (\delta^+)^*$, called *negative diamond*. A tournament T is called a *diamond-free tournament* if none of its subtournaments is a diamond.

Lemma 3.3.

1. *Two* (≤ 6)*-hypomorphic digraphs are hereditarily isomorphic.*
2. *Let* G *and* G' *be two digraphs. If for all* $C \in D_{G,G'}$ C *is an interval of* G *and* G'*, and* $G'_{\restriction C}$*,* $G_{\restriction C}$ *are hereditarily isomorphic, then* G *and* G' *are hereditarily isomorphic.*

Proof. Let $C \in D_{G,G'}$.

1. Let G and G' be two (≤ 6)-hypomorphic digraphs. For all $K \subseteq V$, $G_{\restriction K}$ and $G'_{\restriction K}$ are (≤ 6)-hypomorphic. So, from Theorem 1.4, $G_{\restriction K}$ and $G'_{\restriction K}$ are isomorphic.
2. Let $K \subseteq V$. As $K = \displaystyle\bigcup_{C \in D_{G,G'}} K \cap C$ and $G'_{\restriction C}$, $G_{\restriction C}$ are hereditarily isomorphic, then $G'_{\restriction K \cap C} \simeq G_{\restriction K \cap C}$ and $K \cap C$ is an interval of $G_{\restriction K}$ and $G'_{\restriction K}$. So, $G_{\restriction K}$ and $G'_{\restriction K}$ are isomorphic. □

Lemma 3.4. *[12] Let* G *and* G' *be two* (≤ 4)*-hypomorphic digraphs and* $C \in D_{G,G'}$*.*

1. *If $G_{\restriction C}$ is a tournament, then $G_{\restriction C}$ is a diamond-free tournament.*
2. *If $G_{\restriction C}$ has no 3-cycles, then $G_{\restriction C}$ is either a chain or a near-chain or a consecutivity or a cycle.*
3. *If $G_{\restriction C}$ has a 3-cycle and $G_{\restriction C}$ is not a tournament, then there exists an integer $n \geq 1$ such that $G_{\restriction C}$ is an element of $\mathcal{E}(S_n)$.*
4. *C is an interval of G and G'. Hence, if $G'_{\restriction C'}$ and $G_{\restriction C'}$ are isomorphic for each $C' \in D_{G,G'}$, then G and G' are isomorphic.*
5. *Neither peaks nor flags and no diamonds are embeddable in the subdigraphs $G_{\restriction C}$ and $G'_{\restriction C}$.*
6. *Every 3-consecutivity (resp. 3-cycle) in $G_{\restriction C}$ is reversed in $G'_{\restriction C}$.*

As a consequence from Lemma 3.4, we have:

Corollary 3.2. *Let G and G' be two (≤ 4)-hypomorphic digraphs, and $C \in D_{G,G'}$.*

1. *If $G_{\restriction C}$ is neither a diamond-free tournament nor an element of $\mathcal{E}(S_n)$, then $G'_{\restriction C}$ and $G_{\restriction C}$ are hereditarily isomorphic.*
2. *If $G_{\restriction C}$ is either a diamond-free tournament or an element of $\mathcal{E}(S_n)$, then $G'_{\restriction C}$ and $G^*_{\restriction C}$ are hereditarily isomorphic.*

Lemma 3.5. *([5]) Let T and T' be two (≤ 4)-hypomorphic tournaments on at least 5 vertices. Then, T and T' are (≤ 5)-hypomorphic.*

Lemma 3.6. *([4]) Let T and T' be two (≤ 5)-hypomorphic tournaments defined on a vertex set V such that for all $X \subseteq V$; if $T_{\restriction X}$ is isomorphic to β_6^+ or to β_6^-, then $T'_{\restriction X}$ is isomorphic to $T_{\restriction X}$. Thus T and T' are (≤ 6)-hypomorphic.*

Lemma 3.7. *Let G and G' be two (≤ 4)-hypomorphic digraphs defined on a vertex set V. Let $C \in D_{G,G'}$ such that $G_{\restriction C}$ is an element of $\mathcal{E}(S_n)$ and for all $X \subseteq C$; if $G_{\restriction X}$ is isomorphic to α_5^+ or to α_5^- or to β_5^+ or to β_5^- or to γ_5^+ or to γ_5^-, then $G'_{\restriction X}$ is isomorphic to $G_{\restriction X}$. Thus $G_{\restriction C}$ and $G'_{\restriction C}$ are (≤ 6)-hypomorphic.*

Proof.

Fact 3.1. *We have $G_{\restriction C}$ does not embeds α_5^+, α_5^-, β_5^+, β_5^-, γ_5^+ and γ_5^-. Indeed, if there exist $X \subset C$, such that $G_{\restriction X}$ is isomorphic to α_5^+ or to α_5^- or to β_5^+ or to β_5^- or to γ_5^+ or to γ_5^-, then from Lemma 3.4, every 3-consecutivity and 3-cycle of $G_{\restriction C}$ are reversed in $G'_{\restriction C}$, then $G'_{\restriction X\}} \simeq G^*_{\restriction X}$, thus α_5^+ or α_5^- or β_5^+ or β_5^- or γ_5^+ or γ_5^- is self dual, that is impossible.*

We have $n \leq 3$. Indeed, if $n \geq 4$ then $G_{\restriction\{t_1,t_{1+n},t_2,t_3,t_{4+n}\}} \simeq \alpha_5^+$ or β_5^+, that contradict Fact 3.1.

1) If $n = 3$, then $G_{\restriction C} \in S_3$ and it's neutral pairs have the same type.
Indeed if $\{t_1, t_4\}$, $\{t_2, t_5\}$, $\{t_3, t_6\}$ are 3 neutral pairs of $G_{\restriction C}$. Without loss of generality, we assume that there is x in the sector s_1, then $G_{\restriction\{t_1,t_4,x,t_2,t_6\}} \simeq \alpha_5^+$ or β_5^+ that contradict Fact 3.1. Thus $G_{\restriction C} \in S_3$ and from the fact that neither γ_5^+ nor γ_5^- are embeddable in the subdigraph $G_{\restriction C}$, the neutral pairs are all of the same type.
2) If $n = 2$, then $G_{\restriction C} \in S_2$ or $G_{\restriction C} \in \mathcal{E}(S_2)$ and its two neutral pairs have the same type and its sectors are empty except one of cardinality 1.
Indeed if $\{t_1, t_3\}$, $\{t_2, t_4\}$ are 2 neutral pairs of $G_{\restriction C}$.
Case 1. If a_1, b_1 in the sector s_1, then $G_{\restriction\{t_1,a_1,b_1,t_3,t_4\}} \simeq \alpha_5^+$ or β_5^+.
Case 2. If $a_1 \in s_1$ and $a_2 \in s_2$ then $G_{\restriction\{a_1,t_2,a_2,t_3,t_4\}} \simeq \alpha_5^+$ or β_5^+.
Case 3. If $a_1 \in s_1$, $a_3 \in s_3$ and $a_1 \longrightarrow_G a_3$, then $G_{\restriction\{a_1,t_2,t_3,a_3,t_4\}} \simeq \alpha_5^+$ or β_5^+.
All this cases contradict the Fact 3.1.
Since neither γ_5^+ nor γ_5^- are embeddable in the subdigraph $G_{\restriction C}$ and from the 3 cases, $G_{\restriction C} \in S_2$ or $G_{\restriction C} \in \mathcal{E}(S_2)$ and its two neutral pairs have the same type and its sectors are empty except one of cardinality 1.
3) If $n = 1$, then $G_{\restriction C}$ is either a near-chain, or an element of $\mathcal{E}(S_1)$ on 5 vertices with sectors $s_1 = \{b_1, c_1\}$ and $s_2 = \{b_2\}$ such that $G_{\{b_1,b_2,c_1\}}$ is a 3-cycle, or an element of $\mathcal{E}(S_1)$ on 4 vertices.

Clearly, in all of this cases, $G'_{\restriction C}$ and $G_{\restriction C}$ are (≤ 6)-hypomorphic. □

In the rest of this paper $G = (V, E)$, $G' = (V, E')$ are supposed to be (≤ 4)-hypomorphic digraph. Under the same hypothesis of Theorem 1.5, we have the following results.

Lemma 3.8. *1)* $A_5^+(G) = A_5^+(G')$, $B_5^+(G) = B_5^+(G')$, $C_5^+(G) = C_5^+(G')$.
2) $A_6^+(G) = A_6^+(G')$.

Proof. Let $t \in \{5, 6\}$. Let $T_1, T_2, \ldots, T_{\binom{v}{t}}$ be an enumeration of the t-elements subsets of V. Let $K_1, K_2, \ldots, K_{\binom{v}{k}}$ be an enumeration of the k-elements subsets of V.

1) Let w_G^a be the row matrix $(g_1^a, g_2^a, \ldots, g_{\binom{v}{t}}^a)$ where $g_i^a = 1$ if $G_{\restriction T_i} \simeq \alpha_5^+$, 0 otherwise.
Let w_G^b be the row matrix $(g_1^b, g_2^b, \ldots, g_{\binom{v}{t}}^b)$ where $g_i^b = 1$ if $G_{\restriction T_i} \simeq \beta_5^+$, 0 otherwise.
Let w_G^c be the row matrix $(g_1^c, g_2^c, \ldots, g_{\binom{v}{t}}^c)$ where $g_i^c = 1$ if $G_{\restriction T_i} \simeq \gamma_5^+$, 0 otherwise.

We have $w^a_G W_{5\ k} = (a^+_5(G_{\restriction K_1}), a^+_5(G_{\restriction K_2}), \ldots, a^+_5(G_{\restriction K_{\binom{v}{k}}}))$, $w^b_G W_{5\ k} = (b^+_5(G_{\restriction K_1}), b^+_5(G_{\restriction K_2}), \ldots, b^+_5(G_{\restriction K_{\binom{v}{k}}}))$ and $w^c_G W_{5\ k} = (c^+_5(G_{\restriction K_1}), c^+_5(G_{\restriction K_2}), \ldots, c^+_5(G_{\restriction K_{\binom{v}{k}}}))$. And we do the same for G'.

(a) Since $a^+_5(G_{\restriction K_i}) = a^+_5(G'_{\restriction K_i})$, $b^+_5(G_{\restriction K_i}) = b^+_5(G'_{\restriction K_i})$ and $c^+_5(G_{\restriction K_i}) = c^+_5(G'_{\restriction K_i})$ for all $i \in [1, \binom{v}{k}]$, then $w^a_G - w^a_{G'} \in Ker_{\mathbb{Q}}({}^tW_{5\ k})$, $w^b_G - w^b_{G'} \in Ker_{\mathbb{Q}}({}^tW_{5\ k})$ and $w^c_G - w^c_{G'} \in Ker_{\mathbb{Q}}({}^tW_{5\ k})$. From Theorem 1.1, $Ker_{\mathbb{Q}}({}^tW_{5\ k}) = \{0\}$, then $w^a_G = w^a_{G'}$, $w^b_G = w^b_{G'}$ and $w^c_G = w^c_{G'}$. Thus $A^+_5(G) = A^+_5(G')$, $B^+_5(G) = B^+_5(G')$ and $C^+_5(G) = C^+_5(G')$.

(b) Since $a^+_5(G_{\restriction K_i}) \equiv a^+_5(G'_{\restriction K_i})\ (mod\ p)$, $b^+_5(G_{\restriction K_i}) \equiv b^+_5(G'_{\restriction K_i})\ (mod\ p)$ and $c^+_5(G_{\restriction K_i}) \equiv c^+_5(G'_{\restriction K_i})\ (mod\ p)$ for all $i \in [1, \binom{v}{k}]$, $w^a_G - w^a_{G'} \in Ker({}^tW_{5\ k})$, $w^b_G - w^b_{G'} \in Ker_p({}^tW_{5\ k})$ and $w^c_G - w^c_{G'} \in Ker_p({}^tW_{5\ k})$.
Case 1. $p \geq 7$, $t = 5 = [5]_p$, $k = [k_0, \ldots]_p$ and $t_0 = 5 \leq k_0$, then from 1.a) of Corollary 1.1, $Ker_p({}^tW_{5\ k}) = \{0\}$ (mod p). Thus $A^+_5(G) = A^+_5(G')$, $B^+_5(G) = B^+_5(G')$ and $C^+_5(G) = C^+_5(G')$.
Case 2. $p \geq 7$, $t = 5 = [5]_p$ and $k_0 = 0$, then from 1.b) of Corollary 1.1 there is $\lambda_1, \lambda_2, \lambda_3 \in \{0, 1, -1\}$ such that $w^a_G - w^a_{G'} = \lambda_1(1, 1 \ldots, 1)$, $w^b_G - w'^b_G = \lambda_2(1, 1 \ldots, 1)$, and $w^c_G - w^c_{G'} = \lambda_3(1, 1 \ldots, 1)$. From 1) of Lemma 3.2 there exist X_1, X_2 and X_3 of cardinality 5 such that $G_{\restriction X_1} \not\simeq \alpha^+_5$, $G_{\restriction X_2} \not\simeq \beta^+_5$ and $G_{\restriction X_3} \not\simeq \gamma^+_5$, then $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus $A^+_5(G) = A^+_5(G')$, $B^+_5(G) = B^+_5(G')$ and $C^+_5(G) = C^+_5(G')$.

2) Let w^a_G be the row matrix $(g^a_1, g^a_2, \ldots, g^a_{\binom{v}{t}})$ where $g^a_i = 1$ if $G_{\restriction T_i} \simeq \beta^+_6$, 0 otherwise.
We have $w^a_G W_{6\ k} = (a^+_6(G_{\restriction K_1}), a^+_6(G_{\restriction K_2}), \ldots, a^+_6(G_{\restriction K_{\binom{v}{k}}}))$. And we do the same for G'.

(a) Since $a^+_6(G_{\restriction K_i}) = a^+_6(G'_{\restriction K_i})$ for all $i \in [1, \binom{v}{k}]$, then $w^a_G - w^a_{G'} \in Ker({}^tW_{6\ k})$. From Theorem 1.1, $Ker({}^tW_{6\ k}) = \{0\}$, then $w^a_G = w^a_{G'}$. Thus $A^+_6(G) = A^+_6(G')$.

(b) Since $a^+_6(G_{\restriction K_i}) \equiv a^+_6(G'_{\restriction K_i})\ (mod\ p)$ for all $i \in [1, \binom{v}{k}]$, then $w^a_G - w^a_{G'} \in Ker({}^tW_{6\ k})$.
Case 1. $p \geq 7$, $t = 6 = [6]_p$, $k = [k_0, \ldots]_p$ and $t_0 = 6 \leq k_0$, from 1.a) of Corollary 1.1 $Ker({}^tW_{6\ k}) = \{0\}\ (mod\ p)$. Thus $A^+_6(G) = A^+_6(G')$.
Case 2. $p \geq 7$, $t = 6 = [6]_p$ and $k_0 = 0$, from 1.b) of Corollary 1.1 there is $\lambda \in \{0, 1, -1\}$ such that $w^a_G - w^a_{G'} = \lambda(1, 1 \ldots, 1)$. From 2) of Lemma 3.2, there exist X of cardinality 6 such that $G_{\restriction X} \not\simeq \beta^+_6$ then $\lambda = 0$. Thus $A^+_6(G) = A^+_6(G')$. □

Lemma 3.9. *Let $C \in D_{G,G'}$. $G_{\restriction C}$ and $G'_{\restriction C}$ do not embeds α_5^+, α_5^-, β_5^+, β_5^-, γ_5^+, γ_5^-, β_6^+, and β_6^-.*

Proof. By contradiction, we assume that there is S such that $G_{\restriction S}$ is isomorphic to an element of the set $\{\alpha_5^+, \alpha_5^-, \beta_5^+, \beta_5^-, \gamma_5^+, \gamma_5^-, \beta_6^+, \beta_6^-\}$. From Lemma 3.4, every 3-consecutivity and 3-cycle in $G_{\restriction C}$ are reversed in $G'_{\restriction C}$, then $G'_{\restriction C} \simeq G^*_{\restriction C}$. From Lemma 3.8, $G'_{\restriction S} \simeq G_{\restriction S}$, so $G_{\restriction S} \simeq G^*_{\restriction S}$, a contradiction. □

Proof of Theorem 1.5. Let $C \in D_{G,G'}$. From Corollary 3.2, we can assume that $G_{\restriction C}$ is a diamond free tournament or an element $\mathcal{E}(S_n)$.

Case 1. $G_{\restriction C}$ is a diamond free tournament. From Lemma 3.9, $G_{\restriction C}$ and $G'_{\restriction C}$ do not embed β_6^+ and β_6^-. From Lemma 3.5, $G_{\restriction C}$ and $G'_{\restriction C}$ are (≤ 5)-hypomorphic, so by Lemma 3.6, $G_{\restriction C}$ and $G'_{\restriction C}$ are (≤ 6)-hypomorphic.

Case 2. $G_{\restriction C}$ is an element $\mathcal{E}(S_n)$. From Lemma 3.9, $G_{\restriction C}$ and $G'_{\restriction C}$ do not embed α_5^+, α_5^-, β_5^+, β_5^-, γ_5^+ and γ_5^-, so by Lemma 3.7, $G_{\restriction C}$ and $G'_{\restriction C}$ are (≤ 6)-hypomorphic, then, from Lemma 3.3, G and G' are hereditarily isomorphic. □

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