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Relations in doubly laced crystal graphs via discrete Morse theory*

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We study the combinatorics of crystal graphs given by highest weight representations of finite simply and doubly laced type, uncovering new relations that exist among crystal operators. Much structure in these graphs has been revealed by local relations given by Stembridge and Sternberg. However, there exist relations among crystal operators that are not implied by Stembridge or Sternberg relations. Viewing crystal graphs as edge colored posets, we use poset topology to study them. Using the lexicographic discrete Morse functions of Babson and Hersh, we relate the Möbius function of a given interval in a crystal poset of simply laced or doubly laced type to the types of relations that can occur among crystal operators within this interval.

More specifically, for a crystal of a highest weight representation of finite simply or doubly laced type, we show that whenever there exists an interval whose Möbius function is not equal to -1, 0, or 1, there must be a relation among crystal operators within this interval not implied by Stembridge or Sternberg relations. As an example of an application, this yields relations among crystal operators in types B_n and C_n that were not previously known. Additionally, by studying the structure of Sternberg relations in the doubly laced case, we prove that crystals of highest weight representations of types B_2 and C_2 are not lattices. Finally, we prove a result under certain conditions regarding the truncation algorithm for lexicographic discrete Morse functions.

Keywords and phrases: Crystals, Möbius function, crystal operators, discrete Morse functions.

1. Introduction

In this paper we study crystal graphs given by highest weight representations of finite simply and doubly laced type. These graphs are equipped with a natural partial ordering (see Section 2 for background information on

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partially ordered sets). This partial order is given by covering relations, denoted $\lessdot$, as follows: we say that $x \lessdot y$ whenever $y = f_i(x)$, where f_i is a so-called crystal operator. We color each of these covering relations with i, giving the crystal the structure of an edge colored poset. We aim to study the structure of these crystal posets. We do so here by trying to understand the relations that can exist among crystal operators.

In [14], Stembridge provides a characterization of crystal graphs coming from highest weight representations in the simply laced case. He gives axioms that can be used to construct the crystal graph corresponding to the crystal of a highest weight representation. These axioms imply a list of local relations that exist among crystal operators. The axioms and relations also hold for crystals coming from highest weight representations in the doubly laced case, but do not provide a complete characterization. In [15], Sternberg proves that there are additional local relations that hold among crystal operators in the doubly laced case. Danilov, Karzanov, and Koshevoy give a characterization of doubly laced crystals in [4]. In spite of this, when viewing these crystal graphs as posets, there exist intervals within the poset where Stembridge relations do not control the structure of the interval, as seen in [8] and this paper. By this, we mean that within the interval, there exist saturated chains that are not connected by some sequence of Stembridge relations, in the sense seen with braid relations in weak order.

The question of what types of relations can exist among crystal operators has been previously studied by Hersh and Lenart in [8] in the simply laced case. They show that for arbitrary intervals in crystals of simply laced type, there exist relations among crystal operators not implied by Stembridge relations. More generally, Hersh and Lenart prove that whenever there is an interval $[u, v]$ in a crystal of finite, simply-laced type with the Möbius function $\mu(u, v) \notin \{-1, 0, 1\}$, then within $[u, v]$ there exists a relation among crystal operators not implied by Stembridge relations. However, the proof technique that is used does not carry over to the doubly laced case.

Here, we prove the analogue of this result for crystals of finite, doubly laced type, which was not previously known, using a tool developed in [1] known as lexicographic discrete Morse functions. These functions have been previously used to study certain classes of posets, see e.g. [12, 18]. By using these for crystal posets, we also give a new proof of the result in the simply laced case. More specifically, we show that if we have an interval $[u, v]$ in a crystal poset of finite simply or doubly laced type such that all relations among crystal operators are implied by Stembridge or Sternberg relations, then the Möbius function of this interval must be equal to $-1, 0$, or 1. We do so by constructing a discrete Morse function on the order complex, $\Delta(u, v)$, with at most one critical cell. We give a procedure for determining if $[u, v]$ has a critical cell, and finding this cell when it exists. If the discrete Morse function has exactly one critical cell, this results in the Möbius function of the interval equalling ± 1, else the Möbius function equals 0.

Danilov, Karzanov, and Koshevoy have studied these crystal posets in case when $n = 2$ in [4, 5]. They show that crystals of highest weight representations of type A_2 are in fact lattices. In the present paper, by studying the structure of the Sternberg relations, we prove that crystals of highest weight representations of types B_2 and C_2 are not lattices. Additionally, using SAGE, we search for intervals in crystal posets with Möbius function not equal to -1,0, or 1. As an example, we present new relations among crystal operators in crystals of types B_3 and C_3.

Our main results, Corollaries 4.21 and 4.28 consider intervals in crystal posets where all relations among crystal operators are implied by Stembridge or Sternberg relations. Now let us describe and illustrate the main ideas of this paper through an example.

The interval $[u, v]$ in Figure 1 is a subposet of the crystal of type A_4 with highest weight $(3, 1, 0, 0)$. We order the saturated chains in our interval according to lexicographic order on their edge label sequences as we travel up each chain from u to v. The critical cells in our lexicographic discrete Morse function come from so-called fully covered saturated chains in the interval. Informally, we have a fully covered saturated chain C from u to v when each rank along C, excluding that of u and v, is covered by a "minimal skipped interval". Roughly speaking, we have a skipped interval from u' to v' consisting of all elements strictly between u' and v' along C if there is a lexicographically earlier chain C' from u' to v'. If there are no strictly smaller skipped intervals between u' and v' then we have a minimal skipped interval. The technique we are using is a generalization of a lexicographic shelling. It differs from lexicographic shellings as we allow our minimal skipped intervals to cover more than one rank.

Consider the chain in bold in our example. This chain has edge label sequence $(4, 3, 2, 2, 3)$. We can see that this saturated chain is fully covered by looking at its minimal skipped intervals. For our first minimal skipped interval, instead of traveling up this chain via the edges labeled 4 and 3, we could have traveled up the lexicographically earlier segment via the edges labeled 3 and then 4. Next, instead of traveling along the edges labeled by the sequence $(3, 2, 2, 3)$, we could have traveled up the lexicographically earlier segment labeled $(2, 3, 3, 2)$. These two minimal skipped intervals cover all proper ranks of our interval and so the chain with label sequence $(4, 3, 2, 2, 3)$ is fully covered. This is the only fully covered saturated chain within $[u, v]$.

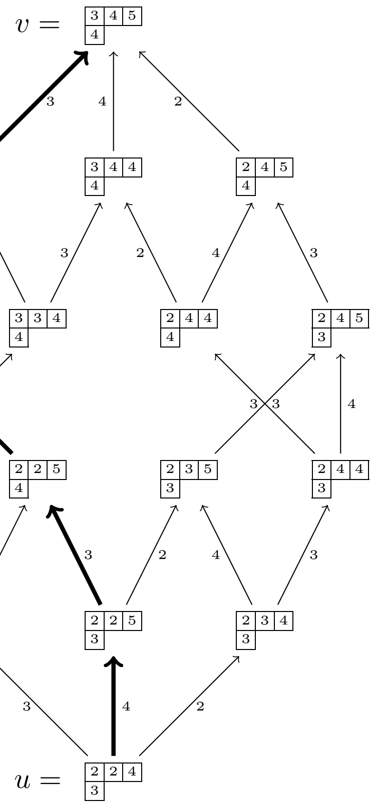

Figure 1: Subposet of type A_4 crystal with highest weight $\lambda = (3, 1, 0, 0)$.

As having a fully covered saturated chain gives rise to a critical cell in our discrete Morse function, we are able to deduce that the Möbius function, $\mu(u, v)$, of our interval is -1.

We give an algorithm for finding a fully covered saturated chain, when one exists, in intervals in crystals of highest weight representations of finite simply and doubly type where all relations are implied by Stembridge or Sternberg relations. In the process, we prove that there is at most one such fully covered saturated chain in any given interval. We note that when a fully covered saturated chain exists, it is not always the lexicographically last chain, though often it is. For such an instance, see Example 4.18.

We give background information on crystals, partially ordered sets, and discrete Morse functions in Section 2. In Section 3, we point out some immediate consequences of the Stembridge axioms for the simply laced and doubly laced cases. Additionally, we use the structure of the degree five Sternberg relation to prove that crystals of highest weight representations of types B_2 and C_2 are not lattices. In Section 4, we construct lexicographic discrete Morse functions for intervals in crystals of highest weight representations of finite simply and doubly laced type where all relations among crystal operators are implied by Stembridge or Sternberg relations. This construction allows us to prove the main result, namely that if there is an interval in a crystal of finite simply or doubly laced type with Möbius function not equal to $-1, 0$, or 1, then there exists a relation among crystal operators within that interval not implied by Stembridge or Sternberg relations. In doing so, we prove a more general result regarding fully covered saturated chains and the truncation algorithm for lexicographic discrete Morse functions. Finally, we give two concrete applications of the main result in Section 5 demonstrating how it can lead to the discovery of new relations among crystal operators via computer search. Specifically, we present new relations among crystal operators in crystals of types B_3 and C_3.

2. Background and terminology

2.1. Crystal bases

Crystals bases are combinatorial structures that give information regarding representations of Lie algebras. Each crystal is associated with a root system Φ that has index set I and weight lattice Λ. Let $\{\alpha_i\}_{i\in I}$ be the set of simple roots and Λ^+ be the set of dominant integral weights. The root systems considered in this paper are all of finite type. For more background on root systems, see [9].

Definition 2.1. For a fixed root system Φ with index set I and weight lattice Λ, a *crystal* of type Φ is a nonempty set $\mathcal{B}$ together with maps

$$e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}, \tag{1a}$$

$$\varepsilon_i, \varphi_i : \mathcal{B} \to \mathbb{Z} \sqcup \{-\infty\}, \tag{1b}$$

$$\mathrm{wt} : \mathcal{B} \to \Lambda, \tag{1c}$$

where $i \in I$ and $0 \notin \mathcal{B}$ is an auxillary element satisfying the following:

(A1) If $x, y \in \mathcal{B}$ then $e_i(y) = x$ if and only if $f_i(x) = y$, and in this case we assume

$$\mathrm{wt}(x) = \mathrm{wt}(y) + \alpha_i, \quad \varepsilon_i(x) = \varepsilon_i(y) - 1, \quad \varphi_i(x) = \varphi_i(y) + 1$$

(A2) We require that

$$\varphi_i(x) = \langle \mathrm{wt}(x), \alpha_i^\vee \rangle + \varepsilon_i(x)$$

for all $x \in \mathcal{B}$ and $i \in I$. In particular, if $\varphi(x) = -\infty$, then $\varepsilon_i(x) = -\infty$ as well. If $\varphi_i(x) = -\infty$ then we require $e_i(x) = f_i(x) = 0$.

The map wt is the *weight map*. The operators e_i, f_i are called *Kashiwara or crystal operators*, and the maps φ_i, ε_i are called *string lengths*.

We will only be referring to crystals of highest weight representations of finite type and our main results will hold for crystals of highest weight representations of finite simply and doubly laced type. For a dominant integral weight $\lambda \in \Lambda^+$, we let $\mathcal{B} = \mathcal{B}_\lambda$ denote the crystal of the irreducible representation $V(\lambda)$ of highest weight λ.

Given any crystal $\mathcal{B}$, we can associate to it a *crystal graph*.

Definition 2.2. A *crystal graph* of some crystal $\mathcal{B}$ is a directed, edge colored (with colors $i \in I$) graph with vertices in $\mathcal{B}$ satisfying the following:

(S1) all monochromatic directed paths have finite length,

(S2) for every vertex $x \in \mathcal{B}$ and $i \in I$, there is at most one edge $z \xrightarrow{i} x$, and dually, at most one edge $x \xrightarrow{i} y$. Here, we say that $z = e_i(x)$ and $y = f_i(x)$.

All examples in this paper will use a well known combinatorial model for crystals where vertices of the crystal graph are represented by tableaux. For a description of this model, see [2, 11].

2.2. Stembridge axioms and Sternberg relations

In [14], Stembridge gives a local characterization of crystals coming from integrable highest weight representations in the simply laced case. In doing so, he provides a list of local relations that exist among crystal operators. He shows that these relations also hold in the doubly laced case, but do not give a complete characterization. In [15], Sternberg shows that for crystals of doubly laced type coming from a highest weight representation, there are additional relations among crystal operators other than those given by the Stembridge axioms. For a complete characterization of doubly laced crystals see [4, 16]. Now, we introduce some notation and the axioms as seen in [14].

Throughout this section we will let $A = (a_{ij})_{i,j \in I}$ be the Cartan matrix of a Kac-Moody algebra $\mathfrak{g}$, where I is a finite index set. We recall the following from [14].

We define the *i-string through x* to be:

$$f_i^{-d}(x) \to \cdots \to f_i^{-1}(x) \to x \to f_i(x) \to \cdots \to f_i^r(x).$$

We can then define the *i-rise of x* to be $\vartheta_i(x) := r$ and the *i-depth of x* to be $\delta_i(x) := -d$. To measure the effect of the crystal operators e_i and f_i on the j-rise and j-depth of each vertex, we define the *difference operators* Δ_i and ∇_i to be:

$$\Delta_i \delta_j(x) = \delta_j(e_i(x)) - \delta_j(x), \quad \Delta_i \vartheta_j(x) = \vartheta_j(e_i(x)) - \vartheta_j(x),$$

whenever $e_i(x)$ is defined, and

$$\nabla_i \delta_j(x) = \delta_j(x) - \delta_j(f_i(x)), \quad \nabla_i \vartheta_j(x) = \vartheta_j(x) - \vartheta_j(f_i(x)),$$

whenever $f_i(x)$ is defined.

Definition 2.3. We say that an edge-colored, directed graph, X, is *A-regular* if the axioms (S1) and (S2) from Definition 2.2 hold as well as (S3)-(S6) and (S5′)-(S6′).

- (S3) For a fixed $x \in X$ and $i, j \in I$ such that $e_i(x)$ is defined, we require $\Delta_i \delta_j(x) + \Delta_i \vartheta_j(x) = a_{ij}$,
- (S4) For a fixed $x \in X$ and $i, j \in I$ such that $e_i(x)$ is defined, we require $\Delta_i \delta_j(x) \leq 0$ and $\Delta_i \vartheta_j(x) \leq 0$.
- (S5) For a fixed $x \in X$ such that $e_i(x)$ and $e_j(x)$ are both defined, we require that $\Delta_i \delta_j(x) = 0$ implies $e_i e_j(x) = e_j e_i(x) \neq 0$ and $\nabla_j \vartheta_i(y) = 0$ where $y = e_i e_j(x) = e_j e_i(x)$.
- (S6) For a fixed $x \in X$ such that $e_i(x)$ and $e_j(x)$ are both defined, we require that $\Delta_i \delta_j(x) = \Delta_j \delta_i(x) = -1$ implies $e_i e_j^2 e_i(x) = e_j e_i^2 e_j(x) \neq 0$ and $\nabla_i \vartheta_j(y) = \nabla_j \vartheta_i(y) = -1$ where $y = e_i e_j^2 e_i(x) = e_j e_i^2 e_j(x)$.

Dually, we have the additional two requirements for X to be *A-regular*,

- (S5′) For a fixed $x \in X$, $\nabla_i \vartheta_j(x) = 0$ implies $f_i f_j(x) = f_j f_i(x) \neq 0$ and $\Delta_j \delta_i(y) = 0$ where $y = f_i f_j(x) = f_j f_i(x)$.
- (S6′) For a fixed $x \in X$, $\nabla_i \vartheta_j(x) = \nabla_j \vartheta_i(x) = -1$ implies $f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x) \neq 0$ and $\Delta_i \delta_j(y) = \Delta_j \delta_i(y) = -1$ where $y = f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x)$.

In [14], Stembridge proves the following:

Theorem 2.4 ([14]). *The crystal graph corresponding to any highest weight representation is A-regular. These axioms characterize crystal graphs in the simply laced case.*

All crystals studied in this paper are such that the Stembridge axioms hold. The axioms only give a complete characterization in the simply laced case.

Definition 2.5. If we have $x \in \mathcal{B}$ such that

$$f_i f_j(x) = f_j f_i(x) \neq 0,$$

then we say there is a *degree two Stembridge relation* upward from x. Similarly, if we have $x \in \mathcal{B}$ such that

$$f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x) \neq 0,$$

then we say that there is a *degree four Stembridge relation* upward from x. Dually, when these relations occur involving the e_i crystal operators, we say we have a degree two or degree four Stembridge relation downward from x, the degree coming from the number of operators.

See Figure 2 for visualizations of the degree two and degree four Stembridge relations.

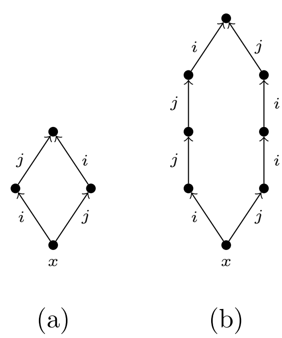

Figure 2: (a) The degree two Stembridge relation, and (b) the degree four Stembridge relation.

We now consider the doubly laced case, i.e. crystals corresponding to the root systems of type B_n and C_n. In [15], Sternberg proves a conjecture of Stembridge by providing a description of the local structure of crystals arising from highest weight representations in the doubly laced case.

Theorem 2.6 ([15]). *Let $\mathcal{B}$ be a crystal coming from a highest weight representation of doubly laced type. Let x be a vertex of $\mathcal{B}$ such that $f_i(x) \neq 0$ and $f_j(x) \neq 0$ where f_i and f_j are two distinct crystal operators. Then exactly one of the following is true:*

1. $f_i f_j(x) = f_j f_i(x)$,
2. $f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x)$,
3. $f_i f_j^3 f_i(x) = f_j f_i f_j f_i f_j(x) = f_j^2 f_i^2 f_j(x)$,
4. $f_i f_j^3 f_i^2 f_j(x) = f_i f_j^2 f_i f_j f_i f_j(x) = f_j f_i^2 f_j^3 f_i(x) = f_j f_i f_j f_i f_j^2 f_i(x)$.

The equivalent statement with the crystal operators e_i and e_j also holds.

For crystals of doubly laced type, there are additional relations besides those seen in Definition 2.5.

Definition 2.7. If we have $x \in \mathcal{B}$ such that

$$f_i f_j^3 f_i(x) = f_j f_i f_j f_i f_j(x) = f_j^2 f_i^2 f_j(x),$$

then we say there is a *degree five Sternberg relation* upward from x. Similarly, if we have $x \in \mathcal{B}$ such that

$$f_i f_j^3 f_i^2 f_j(x) = f_i f_j^2 f_i f_j f_i f_j(x) = f_j f_i^2 f_j^3 f_i(x) = f_j f_i f_j f_i f_j^2 f_i(x),$$

then we say that there is a *degree seven Sternberg relation* upward from x. Dually, when these relations occur involving the e_i's, we say we have a degree five or degree seven Sternberg relation downward from x.

See Figure 3 for visualizations of the degree five and degree seven Sternberg relations.

2.3. Basics of partially ordered sets (posets)

We will now give a brief overview of partially ordered sets, as the main objects of study in this paper are crystal posets.

Definition 2.8. A *partially ordered set* P (or poset) is a set P together with a binary relation $\leq$ such that for all $s, t, u \in P$ we have:

1. reflexivity: $s \leq s$.
2. antisymmetry: if $s \leq t$ and $t \leq s$, then $s = t$.

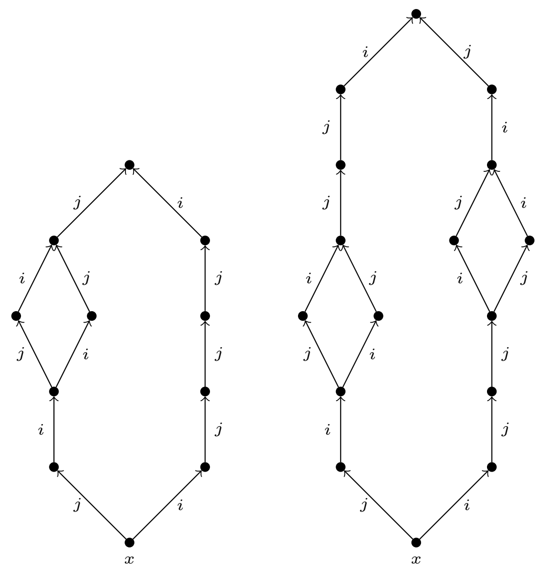

Figure 3: (a) The degree five Sternberg relation, and (b) the degree seven Sternberg relation.

3. transitivity: if $s \leq t$ and $t \leq u$, then $s \leq u$.

We call "$\leq$" a partial order.

Given a subset $Q \subseteq P$, we say that Q is a *subposet* of P if for $s, t \in Q$, we have $s \leq t$ in Q if and only if $s \leq t$ in P. We say that u is covered by v (or v covers u), denoted by $u \lessdot v$ if $u < v$ and there is no element $w \in P$ such that $u < w < v$. We call these *cover relations*. For finite posets (and more generally for locally finite posets), P is generated by such relations. An *interval* $[u, v]$ is a subposet of P defined by $[u, v] = \{s \in P : u \leq s \leq v\}$ whenever $u < v$, Similarly, an *open interval* (u, v) is defined by $(u, v) = \{s \in P : u < s < v\}$. A poset P is *locally finite* if each interval $[u, v]$ is finite. We say that P has a *minimum element*, denoted $\hat{0}$, if there exists an element $\hat{0} \in P$ such that $\hat{0} \leq u$ for all $u \in P$. Similarly, P has a *maximum element*, denoted $\hat{1}$, if there exists an element $\hat{1} \in P$ such that $u \leq \hat{1}$ for all $u \in P$. A *chain* is a poset in which any two elements x and y are comparable (i.e. $x \leq y$ or $y \leq x$). A subset C of P is a chain if it is a chain when considered as a subposet of P. A *saturated chain* from u to v is a series of cover relations $u = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_k = v$. We say that a finite poset is *graded* if for all

$u \leq v$, every saturated chain from u to v has the same number of cover relations, and we call this number the *rank* of the interval $[u, v]$. The *rank of an element* $x \in P$ is the rank of the interval $[\hat{0}, x]$. The *Hasse diagram* of a finite poset P is the graph whose vertices are elements of P with an edge drawn upward from x to y whenever $x \lessdot y$.

For $s, t \in P$, an *upper bound* of s and t is an element v in P such that $v \geq s$ and $v \geq t$. Similarly, a *lower bound* of s and t is an element u such that $u \leq s$ and $u \leq t$. A *least upper bound* for s and t is an element v such that for any w where $s \leq w \leq v$ and $t \leq w \leq v$, we must have $v = w$. We define a greatest lower bound similarly. If two elements have a unique least upper bound it is called a *join*. Similarly, if two elements have a unique greatest lower bound, it is called a *meet*. We denote by $s \vee t$ the join of s and t and $s \wedge t$ the meet of s and t. A poset L in which every two elements have a meet and a join is a *lattice*.

The *Möbius function*, μ of a poset P is a function $\mu : P \times P \to \mathbb{Z}$ defined recursively as follows: $\mu(u, u) = 1$, for all $u \in P$, $\mu(u, v) = -\sum_{u \leq t < v} \mu(u, t)$, for all $u < v \in P$, and $\mu(u, v) = 0$ otherwise. Given a poset P, the *order complex* $\Delta(P)$ is the abstract simplicial complex whose i-dimensional faces are the chains $x_0 < x_1 < \cdots < x_i$ of P. Let $\Delta(u, v)$ denote the order complex of the subposet consisting of the open interval (u, v).

One reason to be interested in the order complex of a poset is the connection between the Möbius function of a poset P and the Euler characteristic of the order complex $\Delta(P)$, discussed e.g. in [13, 17].

Theorem 2.9. *Let P be a poset with $\hat{0}$ and $\hat{1}$. Then $\mu(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P))$.*

The posets that we study in this paper come from crystals. More specifically, we study the crystal graphs of crystals of highest weight representations. We view these crystal graphs as posets with exactly the cover relations $u \lessdot v$ for $v = f_i(u)$ for some $i \in I$. This extends transitively to a partial order on the crystal graph, namely $u \leq v$ whenever there is a directed path from u to v. We color the edge of the covering relation given by $f_i(u) = v$ with the color i. This gives the structure of an edge-colored poset. We call these posets *crystal posets*. Note that the crystal graph is the Hasse diagram of the crystal poset. The following definition will be useful later.

Definition 2.10. Given $[u, v] \subseteq \mathcal{B}$, for $\mathcal{B}$ a crystal poset, let $C = u \lessdot x_1 \lessdot \cdots \lessdot x_m \lessdot v$ be a saturated chain from u to v. The *edge label sequence* of C is the tuple $(\beta(u \lessdot x_1), ..., \beta(x_m \lessdot v))$ where $\beta(x_k \lessdot x_{k+1}) = i$ if $x_{k+1} = f_i(x_k)$.

2.4. Discrete Morse functions

Discrete Morse theory was introduced in [6] by Forman as a tool to study the homotopy type and homology groups of (primarily finite) CW complexes.

In [3], Chari gave a combinatorial reformulation in the case of regular CW complexes in terms of acyclic matchings on their face posets, which is what we will use in this paper. A matching on the Hasse diagram of a face poset is *acyclic* if the directed graph obtained by directing matching edges upward and all other edges downward has no directed cycles. It is shown, for example in [7], that whenever a face poset has an acyclic matching, there is a nonempty set of associated discrete Morse functions on the corresponding complex.

In this paper, we will apply discrete Morse theory to simplicial complexes associated to crystal posets. Let Δ be a simplicial complex. We denote a d-simplex α by $\alpha^{(d)}$.

Definition 2.11. A *discrete Morse function* on a simplicial complex Δ is a function $f : \Delta \to \mathbb{R}$ such that for each d-dimensional simplex, $\alpha^{(d)} \in \Delta$,

1. $|\{\beta^{(d+1)} \supseteq \alpha | f(\beta) \leq f(\alpha)\}| \leq 1$,
2. $|\{\gamma^{(d-1)} \subseteq \alpha | f(\gamma) \geq f(\alpha)\}| \leq 1$.

We are interested in finding critical cells of discrete Morse functions.

Definition 2.12. A simplex α is called a *critical cell* if $|\{\beta^{(d+1)} \supseteq \alpha | f(\beta) \leq f(\alpha)\}| = 0$ and $|\{\gamma^{(d-1)} \subseteq \alpha | f(\gamma) \geq f(\alpha)\}| = 0$. Equivalently, a simplex α is called a critical cell if it is left unmatched by the matching on the face poset.

One of the reasons discrete Morse functions are useful is the following theorem.

Theorem 2.13 ([6])**.** *Suppose Δ is a simplicial complex with a discrete Morse function. Then Δ is homotopy equivalent to a CW-complex with exactly one cell of dimension d for each critical cell of dimension d with respect to this choice of discrete Morse function.*

We deviate slightly from Forman's conventions in a way that is typical in combinatorics. We allow the empty set to be in the domain of our discrete Morse function f, as well as in the face posets on which we construct acyclic matchings. By doing so, we must express our results in terms of reduced Euler characteristic and reduced homology.

Remark 1. From Theorem 2.13, rephrased to use reduced Betti numbers and Morse numbers, we can immediately deduce that if a discrete Morse function has exactly one critical cell of dimension i and no other critical cells, then our original simplicial complex is homotopy equivalent to an i-dimensional sphere.

In [1], Babson and Hersh introduced lexicographic discrete Morse functions as a tool to study the topology of order complexes of partially ordered sets with $\hat{0}$ and $\hat{1}$. This is what we will use to study crystal posets.

Before we describe how to construct lexicographic discrete Morse functions, we explain some of the useful properties they will have. Because we attach the facets by lexicographic order on saturated chains, the lexicographic discrete Morse functions will have relatively few critical cells. If the attachment of the facet corresponding to some saturated chain does not change the homotopy of the subcomplex of our order complex built so far, then this step does not introduce any critical cells. Additionally, each facet can contribute at most one critical cell. We describe these critical cells using minimal skipped intervals, which will be discussed shortly.

We now review lexicographic discrete Morse functions in general. This will rely on a notion of rank within a chain that does not require the poset to be graded. However, in this paper, the crystal posets we are interested in are graded by the weight function, as seen in Lemma 4.1, simplifying the grading in a chain.

Given a poset P graded of rank n, let β be an integer labeling on the edges of the Hasse diagram of P such that $\beta(u \lessdot v) \neq \beta(u \lessdot w)$ whenever $v \neq w$. Each facet of $\Delta(P)$ corresponds to a saturated chain, $\hat{0} \lessdot u_1 \lessdot \cdots \lessdot u_k \lessdot \hat{1}$ in P. For each saturated chain we read off the label sequence $(\beta(\hat{0} \lessdot u_1), \beta(u_1 \lessdot u_2), \cdots, \beta(u_k \lessdot \hat{1}))$ and order these lexicographically. This labeling gives rise to a total order on the facets $F_1, ..., F_k$ of the order complex. By virtue of the fact that we attach facets in a lexicographic order, each maximal face in $\overline{F}_j \cap (\cup_{i<j} \overline{F}_i)$ has rank set of the form $1, ..., i, j, ..., n$ for $j > i + 1$ i.e. it omits a single interval of consecutive ranks. We call this rank interval $[i + 1, j - 1]$ a *minimal skipped interval* of F_j with support $i + 1, ..., j - 1$ and height $j - i - 1$. For a given facet F_j, we call the collection of minimal skipped intervals the *interval system of* F_j.

Remark 2. In order to determine the minimal skipped intervals for a given saturated chain M corresponding to some facet F_j, we consider each cover relation $u \lessdot v$ as we travel up M. At each cover relation $u \lessdot v$, we check if there is a lexicographically earlier cover relation $u \lessdot v'$ upward from u. If so, we obtain a maximal face in $\overline{F}_j \cap (\cup_{i<j} \overline{F}_i)$, and hence a minimal skipped

interval, by taking the intersection of $\overline{F}_j$ with the closure of any facet $F_{i'}$ that includes $u \lessdot v'$, that agrees with F_j below u and agrees with F_j above $w \in F_j$ for some $w > v'$ of minimal rank.

When our poset has some natural labeling, like that of our crystal posets, it is often possible to classify its minimal skipped intervals.

Any face in $\overline{F}_j \setminus (\cup_{i<j} \overline{F}_i)$ must include at least one rank from each of the minimal skipped intervals of F_j. For each j, an acyclic matching on the set of faces in $\overline{F}_j \setminus (\cup_{i<j} \overline{F}_i)$ is constructed in [1] in terms of the interval system. The union of these matchings is acyclic on the entire Hasse diagram of the face poset of the order complex of P, and therefore give rise to a family of discrete Morse functions. For more about this acyclic matching, see [7].

A facet F_j will contribute a critical cell if and only if the interval system of F_j covers all ranks in F_j after the truncation algorithm described below. In this case we say that the corresponding saturated chain is *fully covered*. The dimension of such a critical cell is one less than the number of minimal skipped intervals in the interval system after the truncation algorithm. This truncation algorithm is needed when the interval system of some facet F_j covers all ranks but there are overlapping minimal skipped intervals. Otherwise the truncated system equals the original system.

Remark 3. In actuality, we study the order complexes of the proper parts of our posets; if P has a $\hat{0}$ and $\hat{1}$ then $\Delta(P)$ is contractible as it is a cone. We use the $\hat{0}$ and $\hat{1}$ in the lexicographic discrete Morse functions in a bookkeeping role. More specifically, $\hat{0}$ and $\hat{1}$ are needed to record the labels of covering relations upward from $\hat{0}$ and upward towards $\hat{1}$. In particular, when we refer to fully covered saturated chains, the ranks of $\hat{0}$ and $\hat{1}$ are not covered.

For the truncation algorithm, we begin with our interval system, I, and initialize the truncated system, which we call J, to be the empty set. Then, we repeatedly move the minimum interval in I to the truncated system J and truncate all other elements of I to eliminate any overlap with the minimum interval in I being moved to J at this step. Here, by minimum we mean the minimal skipped interval containing the element of smallest rank. Next, remove any intervals in I that are no longer minimal. We repeat this until there are no longer any minimal skipped intervals in I. We call the truncated, minimal intervals obtained by this algorithm the *J-intervals of F_j*. By construction, these are non-overlapping. If the J-intervals cover all ranks of F_j, then F_j contributes a critical cell. We get this critical cell by taking the lowest rank element of each of the J-intervals. Otherwise F_j does not contribute any critical cells. For a more detailed background on lexicographic discrete Morse functions, see [7].

3. Consequences of the Stembridge axioms

In this section, we deduce consequences of the Stembridge axioms regarding relations among crystal operators in both the simply laced and doubly laced cases. The axioms give restrictions on which Stembridge/Sternberg relations can occur among two given crystal operators for crystals coming from highest weight representations. In addition, we prove that crystals of types B_2 and C_2 are not lattices due to the asymmetry of the degree five Sternberg relation.

As we are studying crystals coming from representations, all crystal graphs are A-regular. In the simply laced case, all off diagonal entries of the Cartan matrix are either equal to -1 or 0. Therefore, by axioms (S3) and (S4) we have that for any vertex x in a crystal graph of simply laced type, there are only three possibilities for the triples $(a_{ij}, \Delta_i\delta_j(x), \Delta_i\vartheta_j(x))$, namely $(0,0,0), (-1,-1,0)$ or $(-1,0,-1)$. Hence, by axioms (S5)-(S6) and (S5′)-(S6′), we have the following result.

Proposition 3.1. *Let $\mathcal{B}$ be the crystal of a representation of simply laced type. Let $x \in \mathcal{B}$ such that $f_i(x) \neq 0$ and $f_j(x) \neq 0$. Then we have:*

1. *If the (i,j) entry of the Cartan matrix is 0, then $f_i f_j(x) = f_j f_i(x)$.*
2. *If the (i,j) entry of the Cartan matrix is -1, then either $f_i f_j(x) = f_j f_i(x)$ or $f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x)$.*

In the doubly laced case, namely crystals of representations of types B_n and C_n, there is an off diagonal entry of the Cartan matrix that is equal to -2. This is either the $(n, n-1)$ entry or the $(n-1, n)$ entry. Therefore, we have the following.

Proposition 3.2. *Let $\mathcal{B}$ be the crystal of a representation of simply laced type. Let $x \in \mathcal{B}$ such that $f_i(x) \neq 0$ and $f_j(x) \neq 0$. Then we have:*

1. *If the (i,j) entry of the Cartan matrix is 0, then $f_i f_j(x) = f_j f_i(x)$.*
2. *If the (i,j) entry of the Cartan matrix is -1, then either $f_i f_j(x) = f_j f_i(x)$ or $f_i f_j^2 f_i(x) = f_j f_i^2 f_j(x)$.*
3. *If the (i,j) entry of the Cartan matrix is -2, then we either have a degree two Stembridge relation, degree four Stembridge relation, or a Sternberg relation upward from x.*

Remark 4. This says that in the doubly laced case, the degree five and degree seven Sternberg relations can only occur among the crystal operators f_{n-1} and f_n. In the simply laced case, degree four Stembridge relations can only occur among certain crystal operators. In type A_n, degree four Stembridge relations can only occur among consecutively indexed operators, i.e. f_k and f_{k+1}. In type D_n, we also may have a degree four Stembridge relation among f_{n-2} and f_n but not f_{n-1} and f_n.

Crystals of rank two algebras are often of particular interest. This is due to the result seen in [10] which says that a crystal graph with a unique maximal vertex is the crystal graph of some representation if and only if it decomposes as the disjoint union of crystals of representations relative to the rank two subalgebras corresponding to each pair of edge colors. Therefore, we now consider crystals of type B_2 and C_2. In [5], it is shown that crystals of type A_2 are lattices. We show that this result does not carry over to the doubly laced case.

Theorem 3.3. *Crystals of highest weight representations of types B_2 and C_2 are not lattices.*

Proof. This follows from the asymmetry of the degree five Sternberg relations. Let $\mathcal{B}$ be the crystal of a highest weight representation of type B_2 or C_2. Let $x \in \mathcal{B}$ such that there is a degree five Sternberg relation upward from x. Then we have $y \in \mathcal{B}$ such that

$$y = f_1 f_2^3 f_1(x) = f_2 f_1 f_2 f_1 f_2(x) = f_2^2 f_1^2 f_2(x),$$

or

$$y = f_2 f_1^3 f_2(x) = f_1 f_2 f_1 f_2 f_1(x) = f_1^2 f_2^2 f_1(x).$$

In either case, we have that $e_1(y) \neq 0$ and $e_2(y) \neq 0$. As a result, there must be a Stembridge or Sternberg relation downward from y. Hence, $e_1(y)$ and $e_2(y)$ will have two distinct, incomparable greatest lower bounds, one coming from the Stembridge or Sternberg relation downward from y and the other being x.

Similarly, if there exists $y \in \mathcal{B}$ such that there is a degree five Sternberg relation downward from y, then there will exist two vertices that have two distinct, incomparable least upper bounds. Hence, highest weight representations of types B_2 and C_2 are not lattices. □

4. Connections between the Möbius function of a poset and relations among crystal operators

In this section we consider crystal posets coming from highest weight representations of simply and doubly laced Cartan type. We prove that whenever there is an interval $[u, v]$ in such a crystal poset whose Möbius function,

$\mu(u, v)$, is not equal to $-1, 0$ or 1, then there must be a relation among crystal operators within $[u, v]$ not implied by Stembridge or Sternberg relations. We do so by proving the contrapositive. By "implied" we mean that there exists two saturated chains that are not connected by a sequence of Stembridge or Sternberg relations. Hersh and Lenart showed this result in [8] for crystals of highest weight representations of finite simply laced type. However, the proof used there does not extend to the doubly laced case. In this section, we extend the result to crystals of finite doubly laced type, and in doing so, give a new proof for crystals of finite simply laced type. We first develop properties of crystal graphs.

Lemma 4.1. *Let $\mathcal{B}$ be the crystal graph of a crystal of type Φ given by a highest weight representation. Let $u, v \in \mathcal{B}$ such that $u < v$. Any saturated chain from u to v uses the same multiset of edge labels. Moreover, we can determine this multiset from $wt(u)$ and $wt(v)$.*

Proof. Recall that if $y = f_i(x)$ then $\mathrm{wt}(y) = \mathrm{wt}(x) - \alpha_i$ where α_i is the i^{th} simple root of the root system Φ. Since $u < v$, there exists some sequence of crystal operators $f_{i_1}, f_{i_2}, ..., f_{i_k}$ such that $v = f_{i_k} \cdots f_{i_2} f_{i_1}(u)$. Then we have,

$$\mathrm{wt}(v) = \mathrm{wt}(u) - \sum_{j=1}^{k} \alpha_{i_j}.$$

Suppose by way of contradiction that there exists another distinct sequence of crystal operators $f_{l_1}, f_{l_2}, ..., f_{l_m}$ such that $v = f_{l_m} \cdots f_{l_2} f_{l_1}(u)$. Then we have

$$\mathrm{wt}(u) - \mathrm{wt}(v) = \sum_{j=1}^{k} \alpha_{i_j} = \sum_{n=1}^{m} \alpha_{l_n}.$$

Since the set of simple roots $\{\alpha_i\}_{i\in I}$ is a basis, we must have that $\{\alpha_{i_1}, ..., \alpha_{i_k}\} = \{\alpha_{l_1}, ..., \alpha_{l_m}\}$. Therefore, the same crystal operators are used with the same multiplicities along any saturated chain from u to v. In addition, by writing the vector $\mathrm{wt}(u)-$ $\mathrm{wt}(v)$ as a linear combination of the simple roots, we can see exactly how many times each crystal operator f_i is applied along any saturated chain from u to v. □

Remark 5. This implies that crystal posets are graded since every saturated chain in a given interval $[u, v]$ will have the same length.

With Lemma 4.1 in mind, we have the following definition.

Definition 4.2. Let $\mathcal{B}$ be the crystal graph of a crystal of type Φ given by a highest weight representation and let $[u, v] \subseteq \mathcal{B}$. The *multiset of edge labels of* $[u, v]$ is the multiset of edge labels of any saturated chain C from u to v.

To prove our main result, we will show that for intervals $[u, v] \subseteq \mathcal{B}$ of simply laced (respectively, doubly laced) type with the property that all relations among crystal operators are implied by Stembridge (respectively, Stembridge or Sternberg) relations, we must have that $\mu(u, v) \in \{-1, 0, 1\}$. We do so by constructing a lexicographic discrete Morse function on the order complex $\Delta(u, v)$ that has at most one critical cell. Recall that a saturated chain from u to v contributes a critical cell for $\Delta(u, v)$ if and only if it is fully covered. Therefore, we will give a method to find the unique fully covered saturated chain in the given interval $[u, v]$ when such a chain exists. We lexicographically order the edge label sequences of saturated chains in order to construct the lexicographic discrete Morse function.

Definition 4.3. Let $\mathcal{B}$ be the crystal of a highest weight representation and let $[u, v] \subseteq \mathcal{B}$. If all relations among crystal operators within $[u, v]$ are implied by Stembridge relations, then we say that $[u, v]$ is a *Stembridge only interval.* Similarly, if all relations among crystal operators are implied by Stembridge or Sternberg relations, then we say that $[u, v]$ is a *Stembridge and Sternberg only interval.*

We note that Stembridge and Sternberg only intervals will only appear in crystals of doubly laced type while Stembridge only intervals may appear in either simply laced or doubly laced crystals.

Throughout this section, we assume that all intervals are either Stembridge only or Stembridge and Sternberg only intervals. Doing so allows us to control the structure of minimal skipped intervals and construct lexicographic discrete Morse functions. We have that each minimal skipped interval (as described in Remark 2) in a lexicographic discrete Morse function will arise from a Stembridge or Sternberg relation. Hence, all minimal skipped intervals will be of the forms seen in Figure 4 and Figure 5.

In the case where $\mathcal{B}$ is the crystal of a highest weight representation of simply laced type (more generally, when we consider a Stembridge only interval $[u, v]$), all minimal skipped intervals are of the form seen in Figure 4. Assume $i < j$. The saturated chain in red, namely the chain $x \lessdot u_0 \lessdot y$ in the left figure and $x \lessdot u_0 \lessdot u_1 \lessdot u_2 \lessdot y$ in the right figure, represent the pieces of the Stembridge relation that may be on a fully covered saturated chain. This is because it is the lexicographically second chain. The lexicographically earlier chain, (with vertices labeled by the v_i,) will give rise to a minimal skipped interval. In the left figure, the minimal skipped interval covers the single rank corresponding to the vertex u_0. In the right figure, the minimal skipped interval covers the ranks corresponding to the vertices u_0, u_1, and u_2.

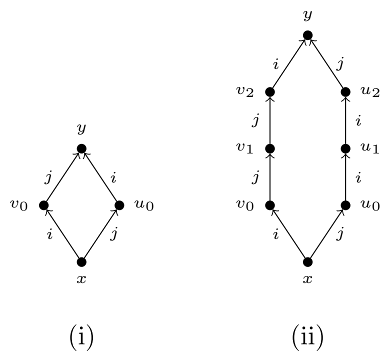

Figure 4: Structure of minimal skipped intervals in simply laced case.

When a minimal skipped interval arises from a Stembridge relation (as in Figure 4), we say the minimal skipped interval *involves* the crystal operators f_i and f_j, (e.g. the minimal skipped intervals in Figure 4 involves the crystal operators f_i and f_j). We remark that the possible values for i and j depend on the type of the crystal. For example, if the crystal is of type A_n, then a degree four Stembridge relation can only involve f_i and f_{i+1}. We discussed in Section 3 when a degree four Stembridge relation can occur for the different types, i.e. the possible values of i and j for our minimal skipped intervals.

When $\mathcal{B}$ is the crystal of a highest weight representation of doubly laced type, in addition to the Stembridge relations, minimal skipped intervals may also arise from the degree five or degree seven Sternberg relations. By Proposition 3.2, we know the degree five and degree seven Sternberg relations can only occur upward from some vertex x if $f_{n-1}(x) \neq 0$ and $f_n(x) \neq 0$. Therefore, we say that minimal skipped intervals arising from Sternberg relations involve the crystal operators f_{n-1} and f_n. The possible Sternberg relations are shown below. The saturated chains with vertices labeled by the u_i (which we marked with red), represent the piece of the Sternberg relation that may be on a fully covered saturated chain, as described above in the simply laced case.

Remark 6. Note that, unlike in the simply laced case, the chain within the Sternberg relations that is a candidate to be a part of a fully covered saturated chain is not always lexicographically last. This is due to the degree

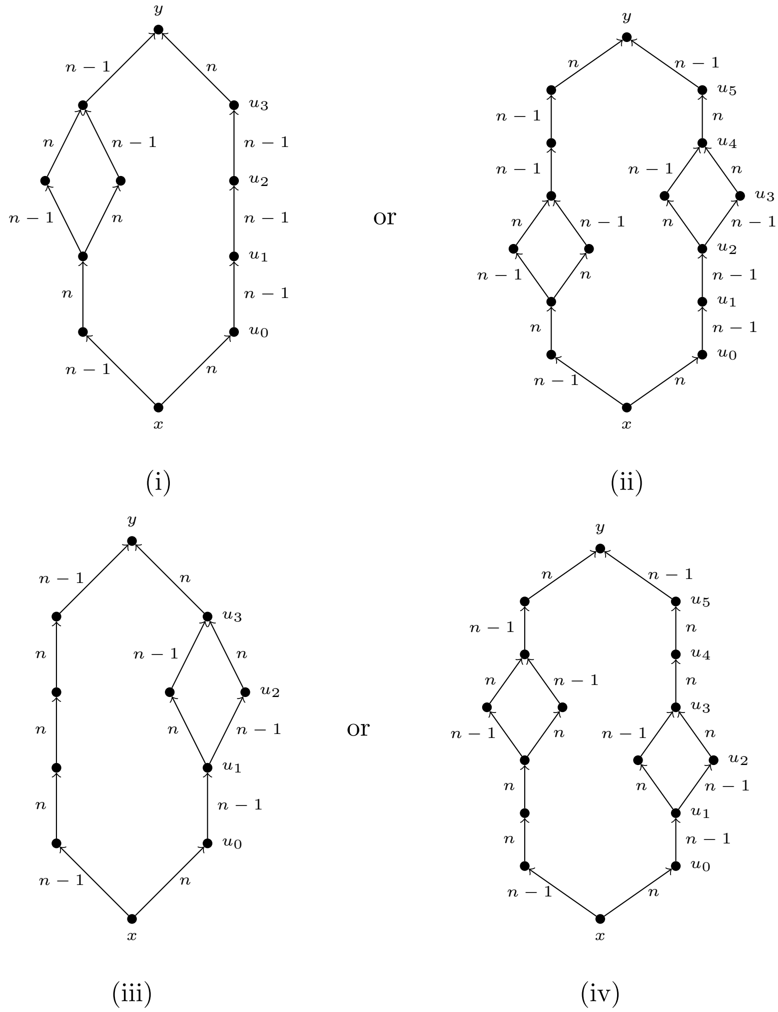

Figure 5: Additional minimal skipped intervals in doubly laced case.

two Stembridge relations sitting inside the degree five and degree seven Sternberg relations.

We now give a series of type dependent definitions which are needed to describe the algorithm used to search for a fully covered saturated chain. We will use these definitions to define what we will call a greedily maximal chain. We then show any fully covered saturated chain is greedily maximal and that there is at most one greedily maximal saturated chain in a given interval.

Definition 4.4. Let $[u, v]$ be an interval in a highest weight crystal of type A_n, B_n, or C_n. Let x be a vertex along a saturated chain C in $[u, v]$ and $f_i(x)$ also belong to C. Suppose there is a minimal skipped interval for the interval system of C involving the crystal operators f_i and f_l beginning at x. Let I' be the multiset of indices of crystal operators that need to be applied along C from $f_i(x)$ to v. We say that f_j is the *maximal operator for* f_i *at* x if

$$j = \max\{k \mid k \in I' \text{ and } k < i\}.$$

Remark 7. This is well defined since there is a finite choice of crystal operators and we can always determine which crystal operators will be used along any saturated chain by Lemma 4.1. It should be noted that j need not equal l.

We wish to extend the idea of maximal operators to the remaining types. We will see that to define greedily maximal saturated chains, we need to adjust our definition of maximal operators. The main difference is that in all types except A_n, we can have a degree four Stembridge relation among non-consecutively indexed crystal operators. To take this into account, we define special vertices. We begin with type D_n.

Definition 4.5. Let $\mathcal{B}$ be the crystal of a highest weight representation of type D_n. Let $[u, v] \subseteq \mathcal{B}$ be a Stembridge only interval. Let x be a vertex on a saturated chain C from u to v. We say that x is an *(n,n-2)-special vertex* in C if there is an edge labeled n, upward from x along C which is the start of a minimal skipped interval for the interval system of C and $n \in I$ where I is the multiset of edge labels for $[f_n(x), v]$.

Maximal operators for these special vertices behave differently.

Definition 4.6. Let $[u, v]$ be an interval in a highest weight crystal of type D_n. Let x be a vertex along a saturated chain C in $[u, v]$ and $f_i(x)$ also belong to C. Suppose there is a minimal skipped interval for the interval system of C involving the crystal operators f_i and f_l beginning at x. Let I' be the multiset of indices of crystal operators that need to be applied along C from $f_i(x)$ to v. If $i = n$ and $n \in I'$, then f_{n-2} is defined to be the maximal operator for f_n at x. Else, we say that f_j is the *maximal operator for* f_i *at* x if

$$j = \max\{k \mid k \in I' \text{ and } k < i\}.$$

We now move on to the exceptional types E_6, E_7, and E_8.

Definition 4.7. Let $\mathcal{B}$ be the crystal of a highest weight representation of type E_6. Let $[u,v] \subseteq \mathcal{B}$ be a Stembridge only interval. Let x be a vertex on a saturated chain C from u to v. We say that x is a *(6,3)-special vertex* if there is an edge labeled 6 upward from x along C which is the start of a minimal skipped interval for the interval system of C and $6 \in I$ where I is the multiset of edge labels for $[f_6(x), v]$.

Definition 4.8. Let $[u,v]$ be an interval in a highest weight crystal of type E_6. Let x be a vertex along a saturated chain C in $[u,v]$ and $f_i(x)$ also belong to C. Suppose there is a minimal skipped interval for the interval system of C involving the crystal operators f_i and f_l beginning at x. Let I' be the multiset of indices of crystal operators that need to be applied along C from $f_i(x)$ to v. If $i = 6$ and $6 \in I'$, then f_3 is defined to be the maximal operator for f_n at x. Else, we say that f_j is the *maximal operator for f_i at x* if

$$j = \max\{k \mid k \in I' \text{ and } k < i\}.$$

Similarly, we have the following for type E_7.

Definition 4.9. Let $\mathcal{B}$ be the crystal of a highest weight representation of type E_7. Let $[u,v] \subseteq \mathcal{B}$ be a Stembridge only interval. Let x be a vertex on a saturated chain C from u to v. We say that x is a *(7,3)-special vertex* if there is an edge labeled 7 upward from x along C which is the start of a minimal skipped interval for the interval system of C and $7 \in I$ where I is the multiset of edge labels for $[f_7(x), v]$.

Definition 4.10. Let $[u,v]$ be an interval in a highest weight crystal of type E_7. Let x be a vertex along a saturated chain C in $[u,v]$ and $f_i(x)$ also belong to C. Suppose there is a minimal skipped interval for the interval system of C involving the crystal operators f_i and f_l beginning at x. Let I' be the multiset of indices of crystal operators that need to be applied along C from $f_i(x)$ to v. If $i = 7$ and $7 \in I'$, then f_3 is defined to be the maximal operator for f_n at x. Else, we say that f_j is the *maximal operator for f_i at x* if

$$j = \max\{k \mid k \in I' \text{ and } k < i\}.$$

Finally, we consider type E_8.

Definition 4.11. Let $\mathcal{B}$ be the crystal of a highest weight representation of type E_8. Let $[u,v] \subseteq \mathcal{B}$ be a Stembridge only interval. Let x be a vertex on a saturated chain C from u to v. We say that x is a *(8,5)-special vertex* if there is an edge labeled 8 upward from x along C which is the start of a minimal skipped interval for the interval system of C and $8 \in I$ where I is the multiset of edge labels for $[f_8(x), v]$.

Definition 4.12. Let $[u, v]$ be an interval in a highest weight crystal of type E_8. Let x be a vertex along a saturated chain C in $[u, v]$ and $f_i(x)$ also belong to C. Suppose there is a minimal skipped interval for the interval system of C involving the crystal operators f_i and f_l beginning at x. Let I' be the multiset of indices of crystal operators that need to be applied along C from $f_i(x)$ to v. If $i = 8$ and $8 \in I'$, then f_5 is defined to be the maximal operator for f_n at x. Else, we say that f_j is the *maximal operator for* f_i *at* x if

$$j = \max\{k \mid k \in I' \text{ and } k < i\}.$$

We now define what a greedily maximal chain is. We will prove that any fully covered saturated chain must be greedily maximal.

Definition 4.13. A saturated chain C is *greedily maximal* if for each minimal skipped interval involving f_i and f_j, $(i < j)$, f_i is the maximal operator for f_j.

In order to prove our main result connecting the Möbius function of an interval $[u, v]$ with relations among crystal operators within this interval, we first prove a series of lemmas. We will first show that any fully covered saturated chain must be greedily maximal. Then we will show that in the intervals we are interested in, namely Stembridge only and Stembridge and Sternberg only intervals, there is at most one greedily maximal saturated chain. Finally, we give an algorithm to find the greedily maximal chain.

We begin by proving the following lemma for crystals of highest weight representations of all types. The main idea from this proof is used in several proofs throughout the rest of this section.

Lemma 4.14. *Let $[u, v] \subseteq \mathcal{B}$ be a Stembridge only or a Stembridge and Sternberg only interval, for $\mathcal{B}$ the crystal of a highest weight representation. Let*

$$j = \max\{k \mid k \textit{ is in the multiset of edge labels of } (u, v)\},$$

then f_j must be the first operator applied along a fully covered saturated chain, i.e. j must appear first in the edge label sequence of any fully covered saturated chain.

Proof. Suppose by way of contradiction that there is a fully covered saturated chain, C, from u to v such that f_j is not the first operator applied along C. Consider the first occurrence of the crystal operator f_j as we proceed upward along C from u towards v, namely the first edge colored j. By definition of j, the label k on the edge immediately preceding the edge colored j on C satisfies $k < j$. Since all Stembridge and Sternberg relations involve exactly two crystal operators and all minimal skipped intervals in $[u, v]$ arise from Stembridge or Sternberg relations, the rank corresponding to the vertex labeled x (see Figure 6) on the fully covered saturated chain C will not be covered by any minimal skipped intervals, as we justify next. If

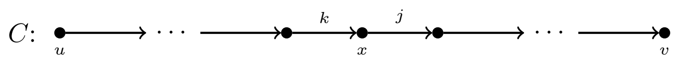

Figure 6.

the rank corresponding to the vertex labeled x was covered by some minimal skipped interval, the corresponding Stembridge or Sternberg relation must involve the crystal operators f_k and f_j. However, since $k < j$, this piece of the Stembridge or Sternberg relation along C will be lexicographically earlier than the piece with edge label sequence (j, k). Hence, we will not have a minimal skipped interval covering the rank corresponding to the vertex x. This contradicts the saturated chain C being fully covered. □

The interval systems for Stembridge and Sternberg only intervals behave differently than those for Stembridge only intervals. Namely, no minimal skipped intervals will overlap in the Stembridge only intervals case, but this does not carry over to the Stembridge and Sternberg only intervals case. We begin with the Stembridge only intervals. We note that any result proven for Stembridge only intervals proves the result for crystals of simply laced type whereas we need to prove analogous results for Stembridge and Sternberg only intervals to extend to the doubly laced case.

Lemma 4.15. *Let $[u, v] \subseteq \mathcal{B}$ where $\mathcal{B}$ is the crystal of a highest weight representation. Assume $[u, v]$ is a Stembridge only interval. Let C be a saturated chain in $[u, v]$. Then no two minimal skipped intervals in the interval system of C overlap, i.e. no two minimal skipped intervals cover a common rank.*

Proof. Let I be the interval system for C. Any minimal skipped interval in I is of the form seen in Figure 4. The first type of minimal skipped interval coming from the degree two Stembridge relation covers exactly one rank. Therefore, any minimal skipped interval arising from this relation cannot overlap with another minimal skipped interval. Hence, we restrict our attention to minimal skipped intervals that arise from the degree four Stembridge relation $f_i f_j f_j f_i(x) = f_j f_i f_i f_j(x)$ where $i < j$.

Suppose we have a vertex $x \in C$ such that there is a minimal skipped interval for the interval system of C beginning at x coming from a degree four Stembridge relation. If there exists another minimal skipped interval that overlaps with the one arising from the degree four Stembridge relation beginning at x, then using the notation from Figure 4, it must either begin at the vertex u_0 or the vertex u_1. Since $[u, v]$ is a Stembridge only interval, if we have a minimal skipped interval beginning at u_0 or u_1, it must come from a degree two or degree four Stembridge relation involving f_i and f_j. In fact, it must come from a degree four Stembridge relation. If not, the minimal skipped interval arising from the degree four Stembridge relation beginning at x for the interval system of C would not be minimal.

However, we cannot have a minimal skipped interval beginning at u_0 because the lexicographically last chain in a degree four Stembridge relation does not have an edge label sequence beginning with i, i, j. We also cannot have a minimal skipped interval beginning at u_1 since we have f_i being applied before f_j and therefore we would only see the lexicographically earlier piece of a Stembridge relation on C. As a result, this will not give rise to a minimal skipped interval. Therefore, no two minimal skipped intervals in the interval system of C will overlap. □

Remark 8. Lemma 4.15 tells us that if we have a fully covered saturated chain in a Stembridge only interval in a crystal of a highest weight representation, then the truncation algorithm will not need to be performed.

Now, we prove that any fully covered saturated chain must be greedily maximal, in the sense of Definition 4.13. We begin with the proof in type A_n. The ideas for the proofs of the other types are similar but require slightly more care. We include the proof for type D_n. The proofs for types E_6, E_7, and E_8 are analogous. We will use this to prove that if there is a fully covered saturated chain in a given interval, then this chain is unique.

Lemma 4.16. *Let $\mathcal{B}$ be the crystal of a highest weight representation of type A_n and $[u, v] \subseteq \mathcal{B}$ be a Stembridge only interval, then any fully covered saturated chain in $[u, v]$ is greedily maximal.*

Proof. Let C be a fully covered saturated chain from u to v and let I be the interval system for C. Let $x \in C$ such that the rank of x is the last rank covered by some minimal skipped interval in I. Since C is fully covered, by Lemma 4.15, x must be the start of a new minimal skipped interval for I. Suppose the first edge along C in this minimal skipped interval is labeled i. Let j be the index such that f_j is the maximal operator for f_i at x. Assume by way of contradiction that the minimal skipped interval upward from x

involves f_i and f_k where $k \neq j$. Since f_k is not the maximal operator for f_i at x, we know that $k < j$.

We note that since $i > j > k$, we cannot have $k = i - 1$. This implies the minimal skipped interval involving f_i and f_k arises from a degree two Stembridge relation, $f_k f_i(x) = f_i f_k(x)$. Therefore, the next time there is an edge colored j upward from x to v along C, the edge below it on C will have label strictly less than j by definition of maximal operator. This contradicts C being fully covered via the same argument as the proof of Lemma 4.14. Namely, there will exist a vertex along the saturated chain C that is not contained in any minimal skipped interval. □

We now prove the analogous result for type D_n.

Lemma 4.17. *Suppose that $[u, v] \subseteq \mathcal{B}$ is a Stembridge only interval, for $\mathcal{B}$ a crystal of a highest weight representation of type D_n, then any fully covered saturated chain in $[u, v]$ is greedily maximal.*

Proof. Recall, for crystals coming from highest weight representations of type D_n, for all y such that $f_n(y) \neq 0$ and $f_i(y) \neq 0$, we have that $f_n f_i(y) = f_i f_n(y)$ unless $i = n - 2$. In the case $i = n - 2$, it is possible we have $f_n f_{n-2}^2 f_n(y) = f_{n-2} f_n^2 f_{n-2}(y)$.

Suppose by way of contradiction that the fully covered saturated chain C is not greedily maximal. Therefore, there exists a vertex x that is the start of a minimal skipped interval involving f_i and f_j with $i > j$, where f_j is not the maximal operator for f_i at x. As can be seen in Proposition 3.1, each crystal operator f_k can be involved in a degree four Stembridge relation with at most one crystal operator f_l, where $l < k$. Additionally, with the exception of f_n and f_{n-2}, all degree four Stembridge relations involve consecutively indexed operators, i.e. f_k and f_{k+1}. Therefore, the case where x is a $(n, n-2)$-special vertex needs to be treated separately.

Assume x is a $(n, n-2)$-special vertex in C and the minimal skipped interval upward from x involves f_n and f_j. Since we are assuming for contradiction that C is not greedily maximal, we must have that f_{n-2} is not the maximal operator for f_n at x (i.e. $j \neq n-2$). Then the minimal skipped interval must arise from a degree two Stembridge relation since f_n commutes with all other operators. Consider the next edge labeled n proceeding upwards along C. Since n is the largest possible edge label occurring on saturated chains from u to v, the edge in C below the edge colored n will have label k for some $k \in [n-1]$. By the nature of Stembridge relations, the rank of the vertex between the k edge and the n edge must be uncovered as seen in the proof of Lemma 4.14.

Therefore, the only way to have C be a fully covered saturated chain is if the maximal operator for f_n at x is f_{n-2}. This is because if the minimal skipped interval for C beginning at x comes from a degree four Stembridge relation involving f_n and f_{n-2}, then the next time there is a vertex y on C such that $f_n(y)$ is also along C, it is the start of a new minimal skipped interval and the rank of y is contained in a previous minimal skipped interval. For any minimal skipped interval that does not begin with an $(n, n-2)$-special vertex, the proof is analogous to the type A_n case from Lemma 4.16. □

We now demonstrate via example the ideas of Lemma 4.17.

Example 4.18. Consider the type D_3 crystal $\mathcal{B}$ of shape $(2,1,1)$ and the interval $[u,v]$ shown in Figure 7 where

$$u = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{3} \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}, \qquad v = \begin{array}{|c|c|}\hline 2 & \bar{2} \\ \hline \bar{3} \\ \cline{1-1} \bar{1} \\ \cline{1-1}\end{array}.$$

One can check that $[u,v]$ is a Stembridge only interval. By Lemma 4.14, we know any fully covered saturated chain begins with the application of f_3. By weight considerations, it follows that f_3 needs to be applied again to get from $f_3(u)$ to v. Hence, u is a $(3,1)$-special vertex, so f_1 is the maximal operator for f_3 at u. Therefore, the fully covered saturated chain begins $u \lessdot f_3(u) \lessdot f_1 f_3(u)$. The first minimal skipped interval comes from the Stembridge relation $f_1 f_3^2 f_1(u) = f_3 f_1^2 f_3(u)$. The next minimal skipped interval comes from the Stembridge relation $f_2 f_3(f_1^2 f_3(u)) = f_3 f_2(f_1^2 f_3(u))$. Therefore, the chain C with label sequence $(3,1,1,3,2)$ is fully covered. We note that C is not the lexicographically last chain in this interval. The lexicographically last chain has edge label sequence $(3,2,1,1,3)$.

We now state the corresponding lemma for the exceptional types E_6, E_7, and E_8.

Lemma 4.19. *Suppose that $[u,v] \subseteq \mathcal{B}$ is a Stembridge only interval, for $\mathcal{B}$ a crystal of a highest weight representation of type E_6, E_7, or E_8, then any fully covered saturated chain in $[u,v]$ is greedily maximal.*

Proof. This proof is analogous to the proof of Lemma 4.17 with $(6,3)$-special vertices playing the role of $(n, n-1)$-special vertices for type E_6, $(7,3)$-special vertices playing the role of $(n, n-1)$-special vertices for type E_7, and $(8,5)$-special vertices playing the role of $(n, n-1)$-special vertices for type E_8. □

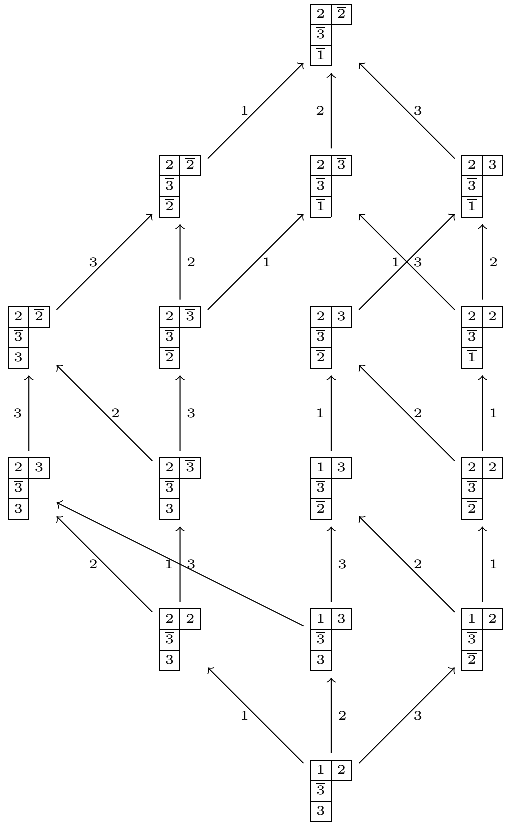

Figure 7: Type D_n greedily maximal saturated chain.

Lemmas 4.16, 4.17, and 4.19 say that for all finite simply laced types, any fully covered saturated chain is greedily maximal. We now give a description of how to find the unique fully covered saturated chain in Stembridge only intervals in crystals of highest weight representations, when it exists.

Theorem 4.20. *Let* $[u, v] \subseteq \mathcal{B}$ *be a Stembridge only interval, for* $\mathcal{B}$ *the crystal of a highest weight representation. Then, there is at most one fully covered saturated chain in* $[u, v]$*.*

Proof. From Lemma 4.14, we know that in order to have a fully covered saturated chain, the chain must start with the application of the crystal operator f_k where

$$k = \max\{i \mid i \text{ is in the multiset of edge labels of } [u, v]\}.$$

Moreover, this says that if $f_k(u) = 0$, then there is no fully covered saturated chain in $[u, v]$. Assuming now that $f_k(u) \neq 0$, we next need to apply f_j where f_j is the maximal operator for f_k at u because any fully covered saturated chain is greedily maximal. If a fully covered saturated chain exists, then it begins with the relations $u \lessdot f_k(u) \lessdot f_j f_k(u)$. In order for the rank of the vertex $f_k(u)$ to be covered by a minimal skipped interval, the chain $u \lessdot f_k(u) \lessdot f_j f_k(u)$ must be contained within a Stembridge relation. In particular, this can only happen if $f_j(u) \neq 0$. If $f_j(u) = 0$, then there is no fully covered saturated chain in this interval because the rank of the vertex $f_k(u)$ will be uncovered. If $f_j(u) \neq 0$, then C must contain the lexicographically later chain in the Stembridge relation upward from u, involving f_k and f_j.

We repeat the process above, beginning at the last uncovered rank. More specifically, this minimal skipped interval described above either ends with the application of f_k (in the case where we have a degree four Stembridge relation between f_k and f_j) or f_j (in the case where we have a degree two Stembridge relation between f_k and f_j). We then see if the maximal operator for the final operator in the previous relation is contained in a Stembridge relation with the final operator beginning at the vertex of the last uncovered rank. If not, there is no fully covered saturated chain in this interval. We continue this process until we reach v. If there is a saturated chain from u to v that is greedily maximal, then we have a fully covered saturated chain. Note that since we chose maximal operators at each step, this chain is uniquely described. □

Using this result, we can say something about the Möbius function of the interval.

Corollary 4.21. *For an interval as above, we have* $\mu(u, v) \in \{-1, 0, 1\}$.

Proof. This follows from the correspondence of the reduced Euler characteristic of the order complex of an open interval with the Möbius function of the interval. More specifically, we have the following:

$$\mu(u, v) = \tilde{\chi}(\Delta(u, v)) = \tilde{\chi}(\Delta^M(u, v)),$$

where $\Delta^M(u, v)$ is the CW-complex obtained from the discrete Morse function. Since there is at most one fully covered saturated chain, the discrete Morse function has at most one critical cell. In this case, the cell complex is homotopy equivalent to a sphere with the same dimension as the dimension of the critical cell. Hence, the reduced Euler characteristic will be ± 1 when there is a fully covered saturated chain, and 0 otherwise. □

Remark 9. The converse of Corollary 4.21 is not true. There exist intervals $[u, v]$ in crystals of highest weight representations of simply laced type such that $\mu(u, v) \in \{-1, 0, 1\}$ where the relations among crystal operators are not implied by Stembridge relations.

In practice, we use the contrapositive of Corollary 4.21 to search for new relations among crystal operators as will be seen for the doubly laced case in Section 5. We state it here as a corollary.

Corollary 4.22. *Let $[u, v] \in \mathcal{B}$, for $\mathcal{B}$ the crystal of a highest weight representation of finite simply laced type. If $\mu(u, v) \notin \{-1, 0, 1\}$, then there exists a relation among crystal operators that is not implied by Stembridge relations.*

We now consider crystals of types B_n and C_n and so we also need to consider Stembridge and Sternberg only intervals. For these crystals, it is possible to have minimal skipped intervals that overlap.

Lemma 4.23. *Suppose $\mathcal{B}$ is the crystal of a highest weight representation of type B_n or C_n. Let $[u, v] \subseteq \mathcal{B}$ be a Stembridge and Sternberg only interval. If all minimal skipped intervals arise from Stembridge relations or degree five Sternberg relations, then there is no overlap among minimal skipped intervals.*

Proof. The minimal skipped intervals arising from degree two and degree four Stembridge relations remain non-overlapping in the doubly laced case by the same argument used in Lemma 4.15. Therefore, we only need to show that if there is a minimal skipped interval for some saturated chain C that comes from a degree five Sternberg relation, then no other minimal skipped intervals overlap with it. The argument is analogous to that of the degree four Stembridge case.

First, suppose that there is a minimal skipped interval coming from (i) in Figure 5. In this case, the minimal skipped interval covers the ranks of the vertices $\{u_0, u_1, u_2, u_3\}$. Therefore, we just need to show that no minimal skipped intervals begin at u_0, u_1, or u_2. In each of these cases, the minimal skipped interval would have to involve the crystal operators f_{n-1} and f_n. However, this cannot happen because no saturated chain within a Stembridge or Sternberg relation begins with multiple applications of f_{n-1}, i.e. no edge label sequence begins $(n-1, n-1, ...)$. Hence, there is no minimal skipped interval that begins at u_0 or u_1. In addition, there is no minimal skipped interval beginning at u_2 because any Stembridge or Sternberg relation upward from u_2 must involve f_{n-1} and f_n. However, since f_{n-1} is being applied before f_n, due to the lexicographic ordering of chains, we would not have a minimal skipped interval here.

Next, we consider case (iii) from Figure 5. As with before, the only possibility for overlap occurs if a minimal skipped interval begins at u_0, u_1, or u_2 and it would need to involve the crystal operators f_{n-1} and f_n. But as before, no saturated chain within a Stembridge or Sternberg relation begins with the repeated application of a single crystal operator so we cannot have a new minimal skipped interval beginning at u_0 or u_2. Also, any chain in a Stembridge or Sternberg relation involving the operators f_{n-1} and f_n beginning with f_{n-1}, will be the lexicographically earlier chain within that Stembridge or Sternberg relation. As a result, there will be no overlap among minimal skipped intervals coming from a degree five Sternberg relation. □

While there is no overlap among minimal skipped intervals coming from Stembridge relations and degree five Sternberg relations, there can be overlap with a minimal skipped interval coming from a degree seven Sternberg relation. However, we show that if the interval system of a fully covered saturated chain is overlapping, then the truncated interval system still covers all ranks. To do so, we prove a general fact about truncated interval systems for lexicographic discrete Morse functions. See Section 2.4 for background on the truncation algorithm.

Let P be an edge labeled poset and let $[u, v]$ be an interval in P. We prove that in certain cases if a saturated chain C from u to v is fully covered by the I-intervals but there is overlap among minimal skipped intervals, then the J-intervals will also fully cover C. We will order our I-interval system $I = \{I_1, ..., I_m\}$ so that the lowest rank elements sequentially increase in rank.

Theorem 4.24. *Let P be an edge labeled poset and let $[u, v] \subseteq P$. Suppose we have constructed a lexicographic discrete Morse function on $[u, v]$. Let C be a saturated chain from u to v that is fully covered by its I-interval system with the following properties:*

(1) Every minimal skipped interval in I either covers exactly one rank or covers at least three ranks,

(2) For two minimal skipped intervals I_k and I_{k+1}, either $I_k \cap I_{k+1} = \emptyset$ or $I_k \cap I_{k+1}$ contains exactly one element, i.e. any two minimal skipped intervals can overlap on at most one rank.

In this case, after truncation the J-intervals fully cover C.

Proof. We aim to prove that after the truncation algorithm, the J-intervals cover all ranks of C. To do so, we examine what happens at each step of the algorithm. Note that if a minimal skipped interval I_j covers exactly one rank, it cannot overlap with any other intervals and thus will also be a J-interval.

To begin, we set $J_1 = I_1$ since I_1 has the element of minimal rank among all I-intervals. We then need to truncate any I-intervals that overlap with I_1. If $I_1 \cap I_2 = \emptyset$, then set $J_2 = I_2$. Otherwise, if $I_1 \cap I_2 \neq \emptyset$, then there is exactly one rank in this intersection. In this case, we remove the vertex of this rank from I_2 to get an interval I_2' with one fewer element than I_2. If I_2' is still minimal, it becomes a J-interval. We assumed each minimal skipped interval that may have overlap had at least three elements and can overlap with other elements in at most one rank. Therefore, $|I_2'| \geq 2$ and at most one of these elements is contained in another minimal skipped interval. As a result, all remaining minimal skipped intervals in the I-interval system are still minimal so there are none to throw out. We set $J_2 = I_2'$.

We now repeat the process considering I_3. If $I_2 \cap I_3 = \emptyset$, set $J_3 = I_3$. Otherwise, if $I_2 \cap I_3 \neq \emptyset$, there is at most one element in this intersection. We remove this element from I_3 to get I_3' and by the same argument as before this is not contained in any other minimal skipped interval in I.

Continuing this process gives nonoverlapping J-intervals that fully cover the saturated chain C as desired. □

We use this result in the next lemma.

Lemma 4.25. *Suppose $\mathcal{B}$ is the crystal of a highest weight representation of type B_n or C_n. Let $[u, v] \subseteq \mathcal{B}$ be a Stembridge and Sternberg only interval. Let C be a saturated chain from u to v such that its interval system covers all ranks, but with overlap among minimal skipped intervals. Then C remains fully covered after the truncation algorithm.*

Proof. From Lemma 4.23, we know that there is no overlap between minimal skipped intervals that arise from Stembridge relations or degree five Sternberg relations. Therefore, we restrict our attention to fully covered saturated chains that have a minimal skipped interval arising from a degree seven Sternberg relation. Let C be one such fully covered saturated chain.

Suppose the minimal skipped interval for C coming from the degree seven Sternberg relation is of the form seen in Figure 5 (ii). Say this minimal skipped interval begins at a vertex $x \in C$. If the number of times f_n is applied to get from x to v is greater than three, then in order for C to be fully covered, there is overlap among minimal skipped intervals. Similarly, if the minimal skipped interval comes from the degree seven Sternberg relation seen in Figure 5 (iv) and the number of times f_n is applied to get from x to v

is greater than four, then there is overlap among minimal skipped intervals. To see why the previous two statements are true, note that in either case, the label sequence of the degree seven Sternberg relation that is contained in C ends with $n-1$. If f_n still needs to be applied along C to reach v, the rank of the first vertex y in C such that $f_n(y)$ is in C is not be contained in a minimal skipped interval unless there is overlap. This is because the edge along C below y is labeled i for some $i \in [n-1]$. The rank of the vertex y is uncovered by the same argument seen in Lemma 4.14. To remedy this, there must exist a minimal skipped interval begin with the application of f_n. However, this can only happen if there is overlap.

In either case, the overlap among minimal skipped intervals will include only the rank of the vertex u_5 from Figure 5. The proof of why this is the case is analogous to that seen in Lemma 4.23. Since all minimal skipped intervals arise from Stembridge or Sternberg relations, a new minimal skipped interval can only arise off of the degree seven Sternberg relation if it starts at the vertex u_4. Depending on how many times f_n is applied from u to v, the minimal skipped interval may be a degree four Stembridge, degree five or degree seven Sternberg relation. Note that it cannot arise from a degree two Stembridge relation. If this were the case, the original degree seven minimal skipped interval would not in fact be minimal.

Note that each minimal skipped interval either covers exactly one rank or at least three ranks. Additionally, any two minimal skipped intervals are either disjoint, or overlap at exactly one rank. Therefore, by Theorem 4.24, if the I-intervals cover all ranks, then the J-intervals do as well. □

The proof that fully covered saturated chains are greedily maximal in the doubly laced case is analogous to the proof for type A_n. The only difference is that there are possibly overlapping intervals. These only occur with degree seven Sternberg relations, which always involve the crystal operators f_{n-1} and f_n. As a result, in this case, the minimal skipped intervals will always involve the maximal operator for f_n. Therefore, we have the following result.

Lemma 4.26. *Let $\mathcal{B}$ be the crystal of a highest weight representation of type B_n or C_n and $[u, v] \subseteq \mathcal{B}$ be a Stembridge and Sternberg only interval. Then any fully covered saturated chain in $[u, v]$ is greedily maximal.*

Proof. As stated above, the proof is analogous to that of Lemma 4.16. □

We now prove for any interval that is Stembridge and Sternberg only in a highest weight crystal of doubly laced type, there is at most one fully covered saturated chain. The proof is analogous to the simply laced type seen in Theorem 4.20.

Theorem 4.27. *Let $[u, v] \subseteq \mathcal{B}$ be a Stembridge and Sternberg only interval, for $\mathcal{B}$ the crystal of a highest weight representation of finite doubly laced type. Then there is at most one fully covered saturated chain in $[u, v]$.*

Proof. Let C be a saturated chain from u to v. If there is no overlap among the minimal skipped intervals in the interval system of C, then the argument from Theorem 4.20 applies directly. The only difference for doubly laced crystals is that overlap can occur with minimal skipped intervals that arise from degree seven Sternberg relations. Hence, we need only to consider fully covered saturated chains where there is a minimal skipped interval that arises from a degree seven Sternberg relation.

Let C be one such chain. Suppose x is a vertex in C such that there is a minimal skipped interval for the interval system of C beginning at x coming from a degree seven Sternberg relation. In this case, we check if there is overlap among minimal skipped intervals as described in Lemma 4.25. Recall that this overlap can occur at exactly one place. In this case, we travel up C until the end of the last minimal skipped interval with an overlap. From there, we once again look for the maximal operator as in the proof of Theorem 4.20. At each step, there is a unique choice, therefore a fully covered saturated chain from u to v is unique, if it exists. □

Once again, having at most one fully covered saturated chain in a Stembridge and Sternberg only interval allows us to say something about the Möbius function.

Corollary 4.28. *For an interval as above, $\mu(u, v) \in \{-1, 0, 1\}$.*

Proof. The proof is completely analogous to that of Corollary 4.21. □

As in the simply laced case, we use the contrapositive of Corollary 4.28 to search for new relations among crystal operators. We state this here as a corollary. For examples illustrating this result, see Section 5.

Corollary 4.29. *Let $[u, v] \in \mathcal{B}$, for $\mathcal{B}$ the crystal of a highest weight representation of type B_n or C_n. If $\mu(u, v) \notin \{-1, 0, 1\}$, then there exists a relation among crystal operators that is not implied by Stembridge or Sternberg relations.*

5. New relations in crystals of doubly laced type

While trying to find new relations among crystal operators is a difficult task, computing the Möbius function of a given interval is algorithmic and efficient. Specifically, we use SAGE to search for intervals among crystals of

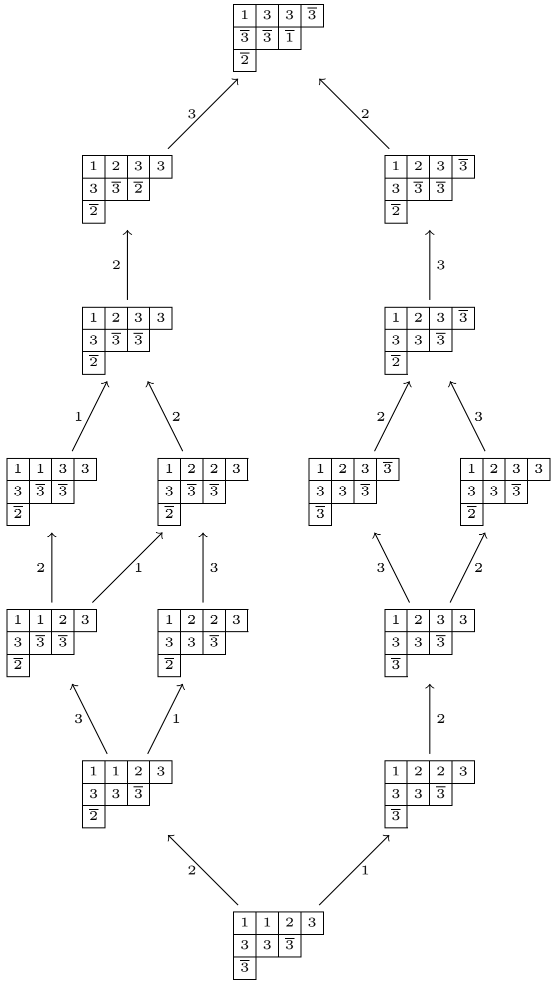

Figure 8: New relation in type C_3 crystal $\mathcal{B}_{(4,3,1)}$.

finite classical type with Möbius function not equal to −1, 0, or 1. In general, it is not obvious how to search for new relations among crystal operators. By establishing a relationship between the Möbius function of an interval within our crystal posets and relations among crystal operators within this interval, we have a computational and algorithmic tool to find new relations.

We have found multiple new relations among crystal operators in crystals of type B_n and C_n. We do so by examining intervals where the Möbius function is not equal to $-1, 0$ or 1. See Figure 8 above for an example of a

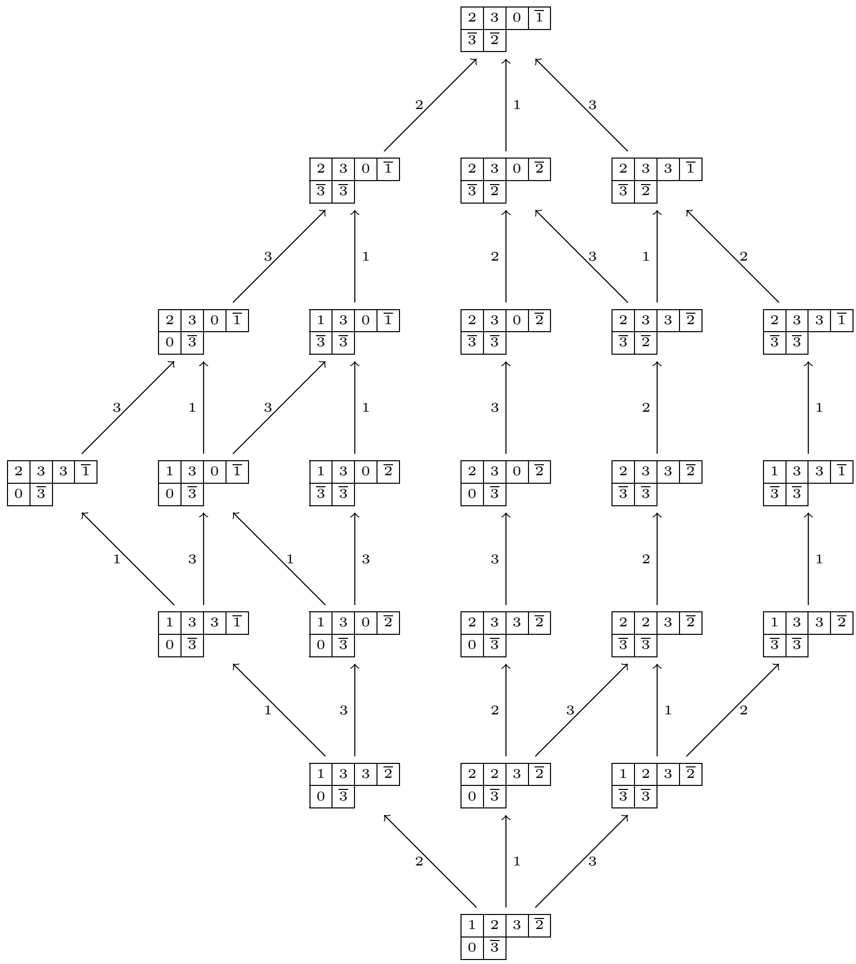

Figure 9: New relation in type B_3 crystal $\mathcal{B}_{(4,2)}$.

new relation among crystal operators found in the type C_3 crystal $\mathcal{B}_{(4,3,1)}$ of shape $\lambda = (4, 3, 1)$, namely we have $x \in \mathcal{B}_{(4,3,1)}$ such that:

$$\begin{aligned} f_2 f_3^2 f_2^2 f_1(x) &= f_2 f_3 f_2 f_3 f_2 f_1(x) = f_3 f_2^2 f_3 f_1 f_2(x) = f_3 f_2 f_1 f_2 f_3 f_2(x) \\ &= f_3 f_2^2 f_1 f_3 f_2(x) \end{aligned}$$

See Figure 9 for an example of a new relation among crystal operators found in the type B_3 crystal $\mathcal{B}_{(4,2)}$ of shape $\lambda = (4, 2)$. Note that the open interval (u, v) has exactly two connected components. It is clear from Figure

9 that there is no way to move from the saturated chain with label sequence $(2,3,3,1,1,2)$ to the saturated chain with label sequence $(1,2,3,3,2,1)$ using only Stembridge and Sternberg relations. Therefore, this interval gives a new relation among crystal operators. Namely, we have $u \in \mathcal{B}_{(4,2)}$ such that:

$$f_2 f_1^2 f_3^2 f_2(u) = f_1 f_2 f_3^2 f_2 f_1(u)$$

We note that there are many intervals in crystals of highest weight representations of finite type where the Möbius function is not equal to $-1, 0$ or 1 that have yet to be explored. It is likely that there are many unknown relations in the doubly laced case still to be discovered. This paper gives a tool to find these.

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References

[1] Eric Babson and Patricia Hersh. "Discrete Morse functions from lexicographic orders". In: *Trans. Amer. Math. Soc* 357 (2005), pp. 509–534. MR2095621

[2] Daniel Bump and Anne Schilling. *Crystal Bases: Representations and Combinatorics*. World Scientific, 2017. MR3642318

[3] Manoj K. Chari. "On discrete Morse functions and combinatorial decompositions". In: *Discrete Math.* 217.1-3 (2000). Formal power series and algebraic combinatorics (Vienna, 1997), pp. 101–113. ISSN: 0012-365X. DOI: 10.1016/S0012-365X(99)00258-7. URL: https://doi.org/10.1016/S0012-365X(99)00258-7. MR1766262

[4] V. I. Danilov, A. V. Karzanov, and G. A. Koshevoy. "B_2-crystals: axioms, structure, models". In: *J. Combin. Theory Ser. A* 116.2 (2009), pp. 265–289. ISSN: 0097-3165. DOI: 10.1016/j.jcta.2008.06.002. URL: https://doi.org/10.1016/j.jcta.2008.06.002. MR2475017

[5] V. I. Danilov, A. V. Karzanov, and G. A. Koshevoy. “Combinatorics of regular A_2-crystals”. In: *J. Algebra* 310 (2007), pp. 218–234. MR2307791

[6] Robin Forman. “Morse theory for cell complexes”. In: *Adv. Math* 134 (1998), pp. 90–145. MR1612391

[7] Patricia Hersh. “On optimizing discrete Morse functions”. In: *Advances in Appl. Math.* 35 (2005), pp. 294–322. MR2164921

[8] Patricia Hersh and Cristian Lenart. “From the weak Bruhat order to crystal posets”. In: *Math. Z.* 286.3-4 (2017), pp. 1435–1464. ISSN: 0025-5874. DOI: 10.1007/s00209-016-1808-5. URL: https://doi.org/10.1007/s00209-016-1808-5. MR3671583

[9] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Graduate Texts in Mathematics. Springer-Verlag, 1972. MR0323842

[10] Seok-Jin Kang et al. “Affine crystals and vertex models”. In: *Infinite analysis, Part A, B (Kyoto, 1991)*, Vol. 16. Adv. Ser. Math. Phys. World Sci. Publ., River Edge, NJ, 1992, pp. 449–484. MR1187560

[11] Masaki Kashiwara and Toshiki Nakashima. “Crystal graphs for representations of the q-analogue of classical Lie algebras”. In: *J. Algebra* 165.2 (1994), pp. 295–345. ISSN: 0021-8693. DOI: 10.1006/jabr.1994.1114. URL: https://doi.org/10.1006/jabr.1994.1114. MR1273277

[12] Peter R. W. McNamara and Bruce E. Sagan. “The Möbius function of generalized subword order”. In: *Adv. Math.* 229.5 (2012), pp. 2741–2766. ISSN: 0001-8708. DOI: 10.1016/j.aim.2012.01.019. URL: https://doi.org/10.1016/j.aim.2012.01.019. MR2889144

[13] Gian-Carlo Rota. “On the foundations of combinatorial theory. I. Theory of Möbius functions”. In: *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 2 (1964), pp. 340–368. MR0174487

[14] John R. Stembridge. “A local characterization of simply laced crystals”. In: *Transactions of the American Mathematical Society* 355 (2003), pp. 4807–4823. MR1997585

[15] Philip Sternberg. “On the local structure of doubly laced crystals”. In: *Journal of Combinatorial Theory, Series A* 114 (2007), pp. 809–824. MR2333134

[16] S. Tsuchioka. "A local characterization of B_2 regular crystals". *ArXiv e-prints* (Oct. 2017). arXiv: 1710.09622 [math.QA].

[17] Michelle L. Wachs. "Poset topology: tools and applications". In: *Geometric combinatorics*. Vol. 13. IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI, 2007, pp. 497–615. MR2383132

[18] Robert Willenbring. "The Möbius function of generalized factor order". In: *Discrete Math.* 313.4 (2013), pp. 330–347. ISSN: 0012-365X. DOI: 10.1016/j.disc.2012.11.002. URL: https://doi.org/10.1016/j.disc.2012.11.002. MR3004467

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