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Avoiding long Berge cycles II, exact bounds for all n

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Let $\mathrm{EG}_r(n,k)$ denote the maximum number of edges in an n-vertex r-uniform hypergraph with no Berge cycles of length k or longer. In the first part of this work [5], we have found exact values of $\mathrm{EG}_r(n,k)$ and described the structure of extremal hypergraphs for the case when $k-2$ divides $n-1$ and $k \geq r+3$.

In this paper we determine $\mathrm{EG}_r(n,k)$ and describe the extremal hypergraphs for all n when $k \geq r+4$.

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1. Definitions, Berge F subhypergraphs

An r-uniform hypergraph, or simply r-*graph*, is a family of r-element subsets of a finite set. We associate an r-graph $\mathcal{H}$ with its edge set and call its vertex set $V(\mathcal{H})$. Usually we take $V(\mathcal{H}) = [n]$, where $[n]$ is the set of first n integers, $[n] := \{1,2,3,\dots,n\}$. We also use the notation $\mathcal{H} \subseteq \binom{[n]}{r}$.

Definition 1.1. *For a graph F with vertex set $\{v_1,\dots,v_p\}$ and edge set $\{e_1,\dots,e_q\}$, a hypergraph $\mathcal{H}$ contains a* **Berge** *F if there exist distinct vertices $\{w_1,\dots,w_p\} \subseteq V(\mathcal{H})$ and edges $\{f_1,\dots,f_q\} \subseteq E(\mathcal{H})$, such that if $e_i = v_\alpha v_\beta$, then $\{w_\alpha, w_\beta\} \subseteq f_i$.*

Of particular interest to us are Berge cycles and Berge paths.

Definition 1.2. *A* **Berge cycle** *of length ℓ in a hypergraph is a set of ℓ distinct vertices $\{v_1,\dots,v_\ell\}$ and ℓ distinct edges $\{e_1,\dots,e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ with indices taken modulo ℓ.*

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A **Berge path** *of length ℓ in a hypergraph is a set of $\ell+1$ distinct vertices $\{v_1, \dots, v_{\ell+1}\}$ and ℓ distinct hyperedges $\{e_1, \dots, e_\ell\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \le i \le \ell$.*

Let $\mathcal{H}$ be a hypergraph and p be an integer. The *p-shadow*, $\partial_p \mathcal{H}$, is the collection of the p-sets that lie in some edge of $\mathcal{H}$. In particular, we will often consider the 2-shadow $\partial_2 \mathcal{H}$ of a r-uniform hypergraph $\mathcal{H}$. Each edge of $\mathcal{H}$ yields in $\partial_2 \mathcal{H}$ a clique on r vertices.

2. Graphs without long cycles

Theorem 2.1 (Erdős and Gallai [1]). *Let $k \ge 3$ and let G be an n-vertex graph with no cycle of length k or longer. Then $e(G) \le (k-1)(n-1)/2$.*

This bound is the best possible if $n-1$ is divisible by $k-2$. A matching lower bound can be obtained by gluing together complete graphs of sizes $k-1$.

Let $\mathrm{EG}(n,k)$ denote the maximum size of a graph on n vertices such that it does not contain any cycle of length k or longer. Write n in the form of $(k-2)\lfloor \frac{n-1}{k-2} \rfloor + m$ where $1 \le m \le k-2$. Considering an n-vertex graph whose 2-connected blocks are complete graphs of size $k-1$ except one which is a K_m we get

$$\mathrm{EG}(n,k) \ge f(n,k) := \left\lfloor \frac{n-1}{k-2} \right\rfloor \binom{k-1}{2} + \binom{m}{2}. \tag{1}$$

It took some 15 years to prove that equality holds in (1) for all n and $k \ge 3$ (Kopylov [8] and independently Woodall [10]). One of the difficulties is, as Faudree and Schelp [3, 4] observed, that for odd k there are infinitely many extremal graphs very different from the ones above.

Construction 2.2. *Fix $k \ge 4$, $n \ge k$, $\frac{k}{2} > a \ge 1$. Define the n-vertex graph $H_{n,k,a}$ as follows. The vertex set of $H_{n,k,a}$ is partitioned into three sets A, B, C such that $|A| = a$, $|B| = n-k+a$ and $|C| = k-2a$ and the edge set of $H_{n,k,a}$ consists of all edges between A and B together with all edges in $A \cup C$. B is taken to be an independent set.*

When $a \ge 2$, $H_{n,k,a}$ is 2-connected, has no cycle of length k or longer, and

$$e(H_{n,k,a}) = \binom{k-a}{2} + a(n-k+a).$$

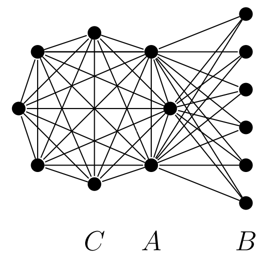


Figure 1: $H_{14,11,3}$.

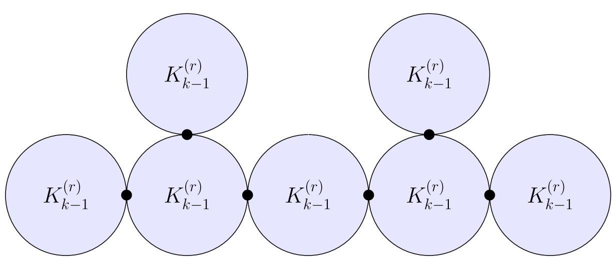
Kopylov and Woodall ([8] and [10]) characterized the structure of the extremal graphs. Namely, either

– the blocks of G are p complete graphs K_{k-1} and a K_m, where $p := \lfloor \frac{n-1}{k-2} \rfloor$, or
– k is odd, $m = (k+1)/2$ or $(k-1)/2$ and q of the blocks of G are K_{k-1}'s and one block is a copy of an $H_{n-q(k-2),k,(k-1)/2}$.

3. Main result: Hypergraphs with no long Berge cycles

Let $\mathrm{EG}_r(n,k)$ denote the maximum size of an r-uniform hypergraph on n vertices that does not contain any Berge cycle of length k or longer. In [5], we proved an analogue of the Erdős–Gallai theorem on cycles for r-graphs.

Theorem 3.1 ([5]). *Let $r \geq 3$ and $k \geq r+3$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length k or longer. Then $e(\mathcal{H}) \leq \frac{n-1}{k-2}\binom{k-1}{r}$. Moreover, equality is achieved if and only if $\partial_2\mathcal{H}$ is connected and for every block D of $\partial_2\mathcal{H}$, $D = K_{k-1}$ and $\mathcal{H}[D] = K_{k-1}^{(r)}$.*



Since a Berge cycle can only be contained in a single block of the 2-shadow $\partial_2\mathcal{H}$, the construction in Theorem 3.1 cannot contain Berge cycles of length k or longer. Thus Theorem 3.1 determines $\mathrm{EG}_r(n,k)$ and describes extremal r-graphs when $k-2$ divides $n-1$ and $k \geq r+3$. Ergemlidze, Győri, Methuku, Salia, Tompkins, and Zamora [2] proved similar results for $k \in \{r+1, r+2\}$. The case of short cycles, $k \leq r$, is different, see [9, 7].

Our goal in this paper is to determine $\mathrm{EG}_r(n,k)$ *for all* n when $r \geq 3$ and $k \geq r+4$. We also describe the extremal hypergraphs. We *conjecture* that our results below holds for $k = r+3$ too. The tools used here do not seem to be sufficient to verify the conjecture (see the remark at the end of Section 6). The case $n \leq k-1$ is trivial, $\mathrm{EG}_r(n,k) = \binom{n}{r}$. Let $n = (k-2)\lfloor \frac{n-1}{k-2} \rfloor + m$ where $1 \leq m \leq k-2$. Define

$$f_r(n,k) := \left\lfloor \frac{n-1}{k-2} \right\rfloor \binom{k-1}{r} + \begin{cases} m-1 & \text{for } 1 \leq m \leq r, \\ \binom{m}{r} & \text{for } r+1 \leq m \leq k-2. \end{cases} \tag{2}$$

Theorem 3.2. *Let $r \geq 3$ and $k \geq r+4$, and suppose $\mathcal{H}$ is an n-vertex r-graph with no Berge cycle of length k or longer. Then $e(\mathcal{H}) \leq f_r(n,k)$. Moreover, equality is achieved if and only if $\mathcal{H}$ has the structure described in Constructions 4.1 and 4.2 in the next section.*

The structure of the paper is as follows. In the next section (Section 4) we prove the lower bound $\mathrm{EG}_r(n,k) \geq f_r(n,k)$. In Section 5 we recall some tools we developed in [5]: the notion of *representative pairs* and Kopylov's Theorem in a useful form. In Section 6 we introduce one more tool, the notion of $(2,r)$ mixed hypergraphs and propose a more general problem. In Section 7 we prepare the proof by proving a handy upper bound in the case of a 2-connected $\partial_2\mathcal{H}$, and finally in Section 8 we prove our main result, Theorem 3.2.

4. Constructions

In this section we define two classes of r-graphs avoiding Berge cycles of length k or longer (for $k \geq r+2$). Write n in the form of $(k-2)\lfloor \frac{n-1}{k-2} \rfloor + m$ where $1 \leq m \leq k-2$. Let $p := \lfloor \frac{n-1}{k-2} \rfloor$. Let $V = [n]$ be an n-element set (the set of vertices).

Construction 4.1. *In case of $m \geq r+1$, let $V_1, \ldots, V_{p+1}$ be a sequence of subsets of $[n]$ satisfying*

$$|(V_1 \cup \cdots \cup V_{i-1}) \cap V_i| = 1, \tag{3}$$

for all $1 < i \leq p+1$ such that one V_i has m elements and each other V_j has $(k-1)$-elements. Then replace each V_i with a copy of $K^{(r)}_{|V_i|}$, the complete r-uniform hypergraph on it.

Each Berge cycle in the r-uniform families in Construction 4.1 must be contained in one of the V_i's so its length is at most $k-1$. Hence

$$\mathrm{EG}_r(n,k) \geq p\binom{k-1}{r} + \binom{m}{r}$$

for all n, k, and r. We will see in Section 8 that in case of $m \geq r+1$ (and $k \geq r+4 \geq 7$) these are the only extremal hypergraphs.

Construction 4.2. *In case of $m \leq r$, let $\mathcal{V} := \{V_1, \dots, V_p\}$ be a sequence of $(k-1)$-element subsets of $[n]$ such that*

$$|(V_1 \cup \dots \cup V_{i-1}) \cap V_i| \leq 1 \tag{4}$$

for every $i \geq 2$. Let H be the graph whose vertex set is $[n]$ and whose edge set is the union of the edge sets of complete graphs on $V_i \in \mathcal{V}$, so $|E(H)| = p\binom{k-1}{2}$. Then H has a forest-like structure of cliques (i.e., every block of H is a clique), and in particular every cycle is contained in some $V_i \in \mathcal{V}$.

The graph H necessarily consists of m (nonempty) components, with vertex sets $C_1, \dots, C_m$ respectively. Some C_α's could be singletons, and $\bigcup_{\alpha=1}^{m} C_\alpha = V$. Let $H_\alpha := H|C_\alpha$. Define $\mathcal{B}_i$ as the complete r-graph with vertex set V_i, and set $\mathcal{H}_\alpha := \cup\{\mathcal{B}_i : V_i \in \mathcal{V}, V_i \subset C_\alpha\}$, $\mathcal{H} := \cup_{\alpha=1}^{m} \mathcal{H}_\alpha$.

If $m > 1$, let T be a tree with vertex set $[m]$ such that a pair $e = \{\alpha(e), \alpha'(e)\}$ is in $E(T)$ only if the components C_α and $C_{\alpha'}$ in H satisfy $|V(C_\alpha)| + |V(C_{\alpha'})| \geq r$. For each such edge e, we "blow up" e into an r-edge containing vertices of C_α and $C_{\alpha'}$ as follows:

Select the non-empty sets $A(e) \subseteq C_\alpha$ and $A'(e) \subseteq C_{\alpha'}$ so that $|A(e)| + |A'(e)| = r$ and if $|V(C_\alpha)| > 1$ (resp. $|V(C_{\alpha'})| > 1$), then $A(e) \subseteq V_i \subseteq C_\alpha$ for some $V_i \in \mathcal{V}$ (resp. $A'(e) \subseteq V_{i'} \subseteq C_{\alpha'}$ for some $V_{i'} \in \mathcal{V}$). Let $\mathcal{D} := \{A(e) \cup A'(e) : e \in E(T)\}$. Our construction is $\mathcal{H} \cup \mathcal{D}$ (see Figure 2).

By definition, $\mathcal{H} \cup \mathcal{D}$ has no long Berge cycle yielding

$$\mathrm{EG}_r(n,k) \geq |\mathcal{H}| + |\mathcal{D}| = p\binom{k-1}{r} + m - 1$$

for all n, k, and r. Indeed, every edge of $\mathcal{D}$ is a *cut-edge* of the hypergraph $\mathcal{H} \cup \mathcal{D}$, every Berge cycle of $\mathcal{H} \cup \mathcal{D}$ is contained in a single component C_α, even more, it is contained a single V_i.

We will see in Section 8 that in the case of $m \leq r$ (and $k \geq r+4 \geq 7$) these are the only extremal hypergraphs.

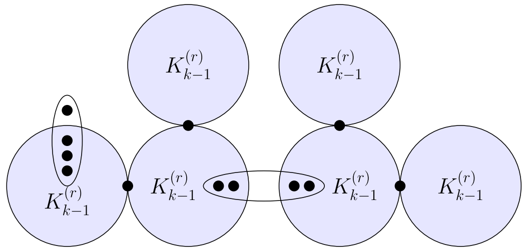


Figure 2: An example of a hypergraph from Construction 4.2.

5. Representative pairs, the structure of Berge F-free hypergraphs

In this section we collect some tools and statements developed and used in [5]. We do not repeat their proofs.

Definition 5.1. *For a hypergraph $\mathcal{H}$, a* **system of distinct representative pairs (SDRP) of** $\mathcal{H}$ *is a set of distinct pairs $A = \{\{x_1, y_1\}, \ldots, \{x_s, y_s\}\}$ and a set of distinct hyperedges $\mathcal{A} = \{f_1, \ldots f_s\}$ of $\mathcal{H}$ such that for all $1 \le i \le s$*

- *$\{x_i, y_i\} \subseteq f_i$, and*
- *$\{x_i, y_i\}$ is not contained in any $f \in \mathcal{H} - \{f_1, \ldots, f_s\}$.*

Lemma 5.2. *Let $\mathcal{H}$ be a hypergraph, let $(A, \mathcal{A})$ be an* SDRP *of $\mathcal{H}$ of maximum size. Let $\mathcal{B} := \mathcal{H} \setminus \mathcal{A}$ and let $B = \partial_2 \mathcal{B}$ be the 2-shadow of $\mathcal{B}$. For a subset $S \subseteq B$, let $\mathcal{B}_S$ denote the set of hyperedges that contain at least one edge of S. Then for all nonempty $S \subseteq B$, $|S| < |\mathcal{B}_S|$.*

Note that $|\mathcal{H}| = |A| + |\mathcal{B}|$.

Lemma 5.3. *Let $\mathcal{H}$ be a hypergraph and let $(A, \mathcal{A})$ be an* SDRP *of $\mathcal{H}$ of maximum size. Let $\mathcal{B} := \mathcal{H} \setminus \mathcal{A}$, $B = \partial_2 \mathcal{B}$, and let G be the graph on $V(\mathcal{H})$ with edge set $A \cup B$. If G contains a copy of a graph F, then $\mathcal{H}$ contains a Berge F on the same base vertex set.*

In this paper, we only use the previous lemma in the case that F is a cycle or path. I.e., if the longest Berge cycle (path) in $\mathcal{H}$ is of length ℓ, then the longest cycle (path) in G is also of length at most ℓ.

Definition. For a natural number α and a graph G, the *α-disintegration* of a graph G is the process of iteratively removing from G the vertices with degree at most α until the resulting graph has minimum degree at least $\alpha+1$ or is empty. This resulting subgraph $H(G, \alpha)$ will be called the $(\alpha+1)$-*core* of G. It is well known (and easy) that $H(G,\alpha)$ is unique and does not depend on the order of vertex deletion.

The following theorem is a consequence of Kopylov's Theorem [8] on the structure of graphs without long cycles. We state it in the form that we need.

Theorem 5.4 ([8], see also Theorem 5.1 in [5])**.** *Let $k \geq 5$ and let $t = \lfloor \frac{k-1}{2} \rfloor$. Suppose that G is an n-vertex graph with no cycle of length at least k. If G is 2-connected and $n \geq k$ then there exists a subset $S \subset V(G)$, $s := |S|$, $k - t \leq s \leq k-2$ (i.e., $2 \leq k - s \leq t$), such that the vertices of $V \setminus S$ can be removed by a $(k-s)$-disintegration.*

Lemma 5.5 ([5], Lemma 5.3)**.** *Let w, $r \geq 2$ and let $\mathcal{H}$ be a w-vertex r-graph. Let $\overline{\partial_2 \mathcal{H}}$ denote the family of pairs of $V(\mathcal{H})$ not contained in any member of $\mathcal{H}$ (i.e., the complement of the 2-shadow). Then*

$$|\mathcal{H}| + |\overline{\partial_2 \mathcal{H}}| \leq \begin{cases} \binom{w}{2} & \text{for } 2 \leq w \leq r+2, \\ \binom{w}{r} & \text{for } r+2 \leq w. \end{cases}$$

Moreover, equality holds if and only if

- *$w > r+2$ and $\mathcal{H}$ is complete, or*
- *$w = r+2$ and either $\mathcal{H}$ or $\overline{\partial_2 \mathcal{H}}$ is complete.*

We say a graph G is *hamilton-connected* if for any $x, y \in V(G)$, G contains a path from x to y that covers $V(G)$.

Lemma 5.6 ([6], Theorem 5)**.** *Let G be an n-vertex graph with minimum degree $\delta(G) \geq 2$. If $e(G) \geq \binom{n-1}{2} + 2$ then G is hamilton-connected unless G is obtained from K_{n-1} by adding a vertex of degree 2.*

6. Maximal mixed hypergraphs

One of our tools is the notion of *mixed hypergraphs*. For $r \geq 3$, a $(2,r)$ *mixed hypergraph* is a triple $\mathcal{M} = (A, \mathcal{B}, V)$, where V is a vertex set, A is the edge set of a graph, $\mathcal{B}$ is an r-graph (i.e., $A \subseteq \binom{V}{2}$, $\mathcal{B} \subseteq \binom{V}{r}$) such that $A \cup \mathcal{B}$ satisfies the *Sperner property*: there is no $a \in A$, $b \in \mathcal{B}$ with $a \subset b$. We often will denote the 2-shadow $\partial_2 \mathcal{B}$ by B.

Let $m_r(n,k)$ denote the maximum size of a mixed hypergraph $\mathcal{M}$ on n vertices such that $\partial_2 \mathcal{M}$ does not contain any cycle of length k or longer.

Lemma 6.1.

$$\mathrm{EG}_r(n,k) \leq m_r(n,k).$$

Proof. Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices with no Berge cycle of length k or longer ($k \geq r+3 \geq 6$) with $EG_r(n,k)$ edges. Let $(A, \mathcal{A})$ be an SDRP of $\mathcal{H}$ of maximum size. Let $\mathcal{B} := \mathcal{H} \setminus \mathcal{A}$, $B = \partial_2 \mathcal{B}$. By definition, $\mathcal{M} := (A, \mathcal{B}, V)$ is a $(2,r)$ mixed hypergraph ($r \geq 3$) with vertex set V. By Lemma 5.3 the graph G with edge set $A \cup B$ does not contain a cycle of length k or longer. Hence

$$\mathrm{EG}_r(n,k) = |\mathcal{H}| = |A| + |\mathcal{B}| \leq m_r(n,k). \qquad \square$$

We will show that these two functions are very close to each other and determine $m_r(n,k)$ for all n (when $k \geq r+4$, $r \geq 3$). We need more definitions and constructions.

A sequence of sets $\mathcal{S} = (V_1, \ldots, V_p)$, $V_i \subseteq V$, is called a (linear) *hypergraph forest* with vertex set V if

$$|(V_1 \cup \cdots \cup V_{i-1}) \cap V_i| \leq 1 \tag{5}$$

holds for each $2 \leq i \leq p$. To avoid trivialities we usually suppose that $|V_i| \geq 2$ for each i. If $\sum_i (|V_i| - 1) = |V| - 1$ then equality holds in (5) for all i, and we call $\mathcal{S}$ a *hypergraph tree.*

Construction 6.2. *Write n in the form of $(k-2)\lfloor \frac{n-1}{k-2} \rfloor + m$ where $1 \leq m \leq k-2$. Let $p := \lfloor \frac{n-1}{k-2} \rfloor$. In case of $m = 1$, let $V_1, \ldots, V_p$ be a sequence of $(k-1)$-element subsets of $[n]$ forming a hypergraph tree. In case of $2 \leq m \leq k-2$, let $V_1, \ldots, V_{p+1}$ be a sequence of subsets of $[n]$ satisfying (5) such that one V_i has m elements and each other has $(k-1)$-elements. Finally, put either a copy of $K^{(r)}_{|V_i|}$ or $K_{|V_i|}$ into each V_i.*

Each cycle in the 2-shadow of any $(2,r)$ mixed family in Construction 6.2 must be contained in one of the V_i's, so its length is at most $k-1$. Taking the largest possible mixed hypergraph of this type we get

$$m_r(n,k) \geq f_r^+(n,k) := \left\lfloor \frac{n-1}{k-2} \right\rfloor \binom{k-1}{r} + \begin{cases} \binom{m}{2} & \text{for } 1 \leq m \leq r+1, \\ \binom{m}{r} & \text{for } r+2 \leq m \leq k-2. \end{cases} \tag{6}$$

Theorem 6.3. *Let $r \geq 3$ and $k \geq r+4$, and suppose $\mathcal{M}$ is an n-vertex $(2,r)$ mixed hypergraph with no cycle of length k or longer in $\partial_2 \mathcal{M}$. Then $|\mathcal{M}| \leq f_r^+(n,k)$. Moreover, equality is achieved if and only if $\mathcal{M}$ has the structure described in Construction 6.2 above.*

Remark. This is one point that does not hold for $k = r + 3$, because in that case every SDRP is simply a graph, $\mathcal{B} = \emptyset$, and according to Kopylov's Theorem 5.4, there are more extremal graphs than in Construction 6.2.

7. Inequalities

Let $k \geq 5$ and let $t = \lfloor \frac{k-1}{2} \rfloor$, $r \geq 3$, and $k \geq r + 3$. In this section most of the time we suppose that $k \geq r + 4$, but almost all inequalities hold for the case $k = r + 3$, too.

Let $\mathcal{M} = (A, \mathcal{B}, V)$ be a $(2, r)$ mixed hypergraph such that $G := A \cup B$ is an n-vertex graph with no cycle of length at least k.

Suppose that $A \cup B$ is 2-connected and $n \geq k$. Theorem 5.4 implies that for some $k - t \leq s \leq k - 2$ (i.e., $2 \leq k - s \leq t$) there exist an s-element set $S \subset V$ such that

(7) *the vertices of $A \cup B \setminus S$ can be removed by a $(k - s)$-disintegration.*

For the edges of A and $\mathcal{B}$ contained in S we use Lemma 5.5 to see that

$$|A[S]| + |\mathcal{B}[S]| \leq \max \left\{ \binom{s}{2}, \binom{s}{r} \right\}.$$

In the $(k-s)$-disintegration steps, we iteratively remove vertices with degree at most $(k - s)$ until we arrive to S. When we remove a vertex v with degree $\ell \leq (k - s)$ from G, a of its incident edges are from A, and the remaining $\ell - a$ incident edges eliminate at most $\binom{\ell-a}{r-1}$ hyperedges from $\mathcal{B}$ containing v. Therefore v contributes at most $a + \binom{\ell-a}{r-1}$ to $|A| + |\mathcal{B}|$. Since the function $a + \binom{\ell-a}{r-1}$ is convex (for nonnegative integers a) it takes its maximum at either $a = 0$ or $a = \ell$, and since $\ell \leq k - s$ we obtain that

(8)
$$|A| + |\mathcal{B}| \leq u_r(n, k, s) := \max \left\{ \binom{s}{2}, \binom{s}{r} \right\} + (n - s) \max \left\{ k - s, \binom{k - s}{r - 1} \right\}.$$

In the rest of this section we give upper bounds for $u_r(n, k, s)$. The following inequalities can be obtained by some elementary estimates on binomial coefficients. The main result of this section is the following lemma.

Lemma 7.1. *If $r \geq 3$, $k \geq r + 4$, $k - t \leq s \leq k - 2$, and $n \geq k$, then $u_r(n, k, s) \leq f_r(n, k) - \binom{r}{2}$.*

Proof. When s is a variable taking only nonnegative integer values, and r, n and k are fixed, the functions $\binom{s}{2}$, $\binom{s}{r}$, $(n-s)(k-s)$ and $(n-s)\binom{k-s}{r-1}$ are convex. So their maximums and sums, in particular, $u_r(n,k,s)$, are convex, too. We obtain that

$$\max_{k-t\le s\le k-2} u_r(n,k,s) = \max\left\{u_r(n,k,k-2), u_r(n,k,k-t)\right\}.$$

Our first observation is that (for $r\ge 3$, $k\ge r+3$) if $n\ge k-2+s$, then

$$u_r(n,k,s) = u_r(n-k+2,k,s) + (k-2)\max\left\{k-s, \binom{k-s}{r-1}\right\}. \tag{9}$$

□

Claim 7.2. *For $2\le k-s\le t$ and $k\ge r+4$,*

$$(k-2)\max\left\{k-s, \binom{k-s}{r-1}\right\} < \binom{k-1}{r} - \binom{r}{2}. \tag{10}$$

Proof of Claim 7.2. Because of the convexity of the left-hand side (in variable s), it is enough to check the cases $s\in\{k-t,k-2\}$ (i.e., $k-s\in\{t,2\}$, respectively). We have three cases to consider: when $k-s=2$, when $k-s=t$ and $r\ge k-s$, and finally when $k-s=t$ and $3\le r\le t-1$. Substituting $k-s=2$ and $k-s=t$ into the left-hand side of (10), we get $(k-2)2$ and $(k-2)t$, respectively. Then (for $k\ge 7$) we have

$$(k-2)t < \binom{k-1}{3} - \binom{k-4}{2} \le \binom{k-1}{r} - \binom{r}{2}.$$

This settles the first two cases.

In the case $k-s=t$ and $r<k-s$, we need the following inequality (for $3\le r<t$):

$$(k-2)\binom{t}{r-1} < \binom{k-1}{r} - \binom{r}{2}.$$

We prove the following stronger inequality (for $3\le r<t$), because we will use it again.

$$\binom{k-t}{r} + (k-3)\binom{t}{r-1} < \binom{k-1}{r} - \binom{r}{2}. \tag{11}$$

Since $\binom{k-t}{r}\ge\binom{t+1}{r}\ge\binom{t}{r-1}$ (for $t\ge r$), equation (11) completes the proof of (10).

Returning to the proof of (11) note that (since $2 \le r-1 \le t-2$)

$$\binom{r}{2} < \binom{r+1}{2} \le \binom{t}{2} \le \binom{t}{r-1}.$$

So (11) is implied by the inequality below.

$$\binom{k-t}{r} + (k-2)\binom{t}{r-1} \le \binom{k-1}{r}. \tag{12}$$

We give a purely combinatorial proof of (12).

Define four r-graphs with vertex set $[k-1]$.

$$\begin{aligned}
\mathcal{F}_0 &:= \binom{[k-1]}{r},\\
\mathcal{F}_1 &:= \binom{[k-t]}{r},\\
\mathcal{F}_2 &:= \left\{ e \cup \{i\} : e \in \binom{[t]}{r-1}, i \in [k-t+1, k-1] \right\}, \text{ and}\\
\mathcal{F}_3 &:= \left\{ f \cup \{j\} : f \in \binom{[k-t, k-1]}{r-1}, j \in [k-t-1] \right\}.
\end{aligned}$$

Their sizes are $\binom{k-1}{r}$, $\binom{k-t}{r}$, $(t-1)\binom{t}{r-1}$, and $(k-t-1)\binom{t}{r-1}$ respectively. We claim that $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ are disjoint. Indeed, $|A \cap [k-t]| \le r-1$ holds for every $A \in \mathcal{F}_2 \cup \mathcal{F}_3$, so $A \notin \mathcal{F}_1$. Also, if $A \in \mathcal{F}_2$ then $|A \cap [k-t-1]| = r-1 > 1$ so $A \notin \mathcal{F}_3$. Since the families $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ are disjoint subfamilies of $\mathcal{F}_0$, we have $|\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| \le |\mathcal{F}_0|$. This completes the proof of (12). ☐

Claim 7.3. *For $k \le n \le 2k-3$ and $k \ge r+4$, one has $u_r(n,k,k-2) \le f_r(n,k) - \binom{r}{2}$.*

Proof. We have

$$\begin{aligned}
u_r(n,k,k-2) = &\quad \max\left\{\tbinom{k-2}{2}, \tbinom{k-2}{r}\right\} + (n-k+2)\max\left\{2, \tbinom{2}{r-1}\right\}\\
= &\quad \binom{k-2}{r} + 2(n-k+2).
\end{aligned}$$

We will show

$$\binom{k-2}{r} + 2(n-k+2) \le \binom{k-1}{r} + (n-k+1) - \binom{r}{2} \quad \left(\le f_r(n,k) - \binom{r}{2} \right).$$

Since $k \geq 7$, we have

$$\left(\binom{k-2}{r} + 2(n-k+2)\right) - \left((n-k+1) - \binom{r}{2}\right)$$
$$= \binom{k-2}{r} + (n-k+3) + \binom{r}{2}$$
$$\leq \binom{k-2}{r} + k + \binom{k-4}{2} \leq \binom{k-2}{r} + \binom{k-2}{2} \leq \binom{k-1}{r}.$$

□

Claim 7.4. *For $k \leq n \leq 2k-t-3$, $r \geq t$ and $k \geq r+4$,*

$$u_r(n,k,k-t) < \binom{k-1}{r} + (n-k+1) - \binom{r}{2}.$$

Note that the right-hand side is at most $f_r(n,k) - \binom{r}{2}$.

Proof. We have

$$\begin{aligned} u_r(n,k,k-t) = {} & \max\left\{\tbinom{k-t}{2}, \tbinom{k-t}{r}\right\} + (n-k+t)\max\left\{t, \tbinom{t}{r-1}\right\} \\ = {} & \binom{k-2}{2} + (n-k+t)t. \end{aligned}$$

Moreover

$$\left(\binom{k-t}{2} + (n-k+t)t\right) - \left((n-k+1) - \binom{r}{2}\right)$$
$$= \binom{k-t}{2} + (t-1)(n-k+t) + (t-2) + \binom{r}{2}$$
$$\leq \binom{k-t}{2} + (t-1)(k-3) + (t-2) + \binom{k-4}{2} < \binom{k-1}{3} \leq \binom{k-1}{r}.$$

□

Claim 7.5. *For $k \leq n \leq 2k-t-3$, $r < t$ and $k \geq r+4$,*

$$u_r(n,k,k-t) < \binom{k-1}{r} - \binom{r}{2}.$$

Note that the right-hand side is at most $f_r(n,k) - \binom{r}{2}$.

Proof. We have

$$\begin{aligned} u_r(n,k,k-t) &= \max\left\{\binom{k-t}{2}, \binom{k-t}{r}\right\} + (n-k+t)\max\left\{t, \binom{t}{r-1}\right\} \\ &= \binom{k-t}{r} + (n-k+t)\binom{t}{r-1} \\ &\le \binom{k-t}{r} + (k-3)\binom{t}{r-1}. \end{aligned}$$

Here the right-hand side is less than $\binom{k-1}{r} - \binom{r}{2}$ by (11). □

Proof of Lemma 7.1. Because of the convexity of $u_r(n,k,s)$ (in the variable s), it is enough to check the cases $k-s \in \{2,t\}$. Note that for $n_1, n_2 \ge 2$

$$f_r(n_1,k) + f_r(n_2,k) \le f_r(n_1+n_2-1,k) \tag{13}$$

$$f_r^+(n_1,k) + f_r^+(n_2,k) \le f_r^+(n_1+n_2-1,k) \tag{14}$$

and here equalities hold for $n_2 = k-1$. (If we define $f_r(1,k) = f_r^+(1,k) = 0$, then we can use (13), (14) for these values, too).

If $s=2$ and $k \le n \le 2k-3$, then Claim 7.3 yields $u_r(n,k,k-2) \le f_r(n,k) - \binom{r}{2}$. For $n \ge 2k-2$ we use (9), then the induction hypothesis $u_r(n-k+2,k,k-2) \le f_r(n-k+2,k)$, and then Claim 7.2 (equation (10)) implies that

$$\begin{aligned} u_r(n,k,k-2) &= u_r(n-k+2,k,k-2) + (k-2)2 \\ &< f_r(n-k+2,k) + \binom{k-1}{r} - \binom{r}{2} \\ &= f_r(n-k+2,k) + f_r(k-1,k) - \binom{r}{2} \\ &= f_r(n,k) - \binom{r}{2}, \end{aligned}$$

and we are done.

When $k-s=t$ the proof is similar. For $k \le n \le 2k-t-3$, Claim 7.4 and Claim 7.5 yield $u_r(n,k,k-t) < f_r(n,k) - \binom{r}{2}$. For $n \ge 2k-t-2$, we use (9), then the induction hypothesis $u_r(n-k+t,k,k-t) \le f_r(n-k+t,k)$, and then Claim 7.2 (equation (10)) implies that

$$\begin{aligned} u_r(n,k,k-t) &= u_r(n-k+2,k,k-2)+(k-2)\max\left\{t,\binom{t}{r-1}\right\} \\ &< f_r(n-k+2,k)+\binom{k-1}{r}-\binom{r}{2} \\ &= f_r(n-k+2,k)+f_r(k-1,k)-\binom{r}{2} \\ &= f_r(n,k)-\binom{r}{2}. \end{aligned}$$

□

8. Proofs of the main results

In this section we first prove Theorem 6.3 and then Theorem 3.2 for all $n \geq k$ (and $r \geq 3$, $k \geq r+4$).

8.1. Proof of Theorem 6.3 about mixed hypergraphs

Let $\mathcal{M} = (A, \mathcal{B}, V)$ be a $(2,r)$ mixed hypergraph such that $G := A \cup B$ is an n-vertex graph with no cycle of length at least k ($B := \partial_2\mathcal{B}$ and $A \cap B = \emptyset$). Let $V_1, V_2, \ldots, V_q$ be the vertex sets of the standard (and unique) decomposition of G into blocks of sizes $n_1, n_2, \ldots, n_q$. Then the graph $A \cup B$ restricted to V_i, denoted by G_i, is either a 2-connected graph or a single edge (in the latter case $n_i = 2$), each edge from $A \cup B$ is contained in a single G_i, and $\sum_{i=1}^{q}(n_i - 1) \leq (n-1)$. This decomposition yields a decomposition of $A = A_1 \cup A_2 \cup \cdots \cup A_q$ and $B = B_1 \cup B_2 \cup \cdots \cup B_q$, $A_i \cup B_i = E(G_i)$. If an edge $e \in B_i$ is contained in $f \in \mathcal{B}$, then $f \subseteq V_i$ (because f induces a 2-connected graph K_r in B), so the block-decomposition of G naturally extends to $\mathcal{B}$, $\mathcal{B}_i := \{f \in \mathcal{B} : f \subseteq V_i\}$ and we have $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_q$, and $B_i = \partial_2\mathcal{B}_i$. By definition, G has no cycle of length k or longer, so the same is true for each G_i. Suppose that the size of $A \cup \mathcal{B}$ is as large as possible, $\mathcal{M}$ is extremal, $|\mathcal{M}| = m_r(n,k)$.

Lemma 5.5 implies that for $n_i \leq k-1$,

$$|A_i| + |\mathcal{B}_i| \leq \max\left\{\binom{n_i}{2},\binom{n_i}{r}\right\} = f_r^+(n_i,k),$$

and equality holds only if A_i is the complete graph (and $\mathcal{B}_i = \emptyset$) or $\mathcal{B}_i$ is the r-uniform complete r-graph (and $A_i = \emptyset$).

Lemma 7.1 implies that in the case $n_i \geq k$

$$|A_i| + |\mathcal{B}_i| \leq f_r(n_i, k) - \binom{r}{2} \leq f_r^+(n_i, k) - \binom{r}{2}. \tag{15}$$

Adding up these inequalities for all $1 \leq i \leq q$ and applying (14), we get

$$\sum_i (|A_i| + |\mathcal{B}_i|) \leq \sum_i f^+(n_i, k) \leq f_r^+(1 + \sum_i (n_i - 1), k) \leq f_r^+(n, k). \tag{16}$$

Since $f_r^+(n, k) \leq m_r(n, k)$, here equality holds in each term. Consequently $n_i < k$ for each i, and all but at most one of them should be $k-1$. Otherwise we can use the inequality

$$\max\left\{\binom{a}{2}, \binom{a}{r}\right\} + \max\left\{\binom{b}{2}, \binom{b}{r}\right\}$$
$$< \max\left\{\binom{a-1}{2}, \binom{a-1}{r}\right\} + \max\left\{\binom{b+1}{2}, \binom{b+1}{r}\right\}$$

which holds for all $1 < a \leq b < k-1$ (and $3 \leq r$, $r+4 \leq k$). (The inequality $f(a)+f(b) \leq f(a-1)+f(b+1)$ holds for every convex function f, and here equality holds only if the four points $(a-1, f(a-1))$, $(a, f(a))$, $(b, f(b))$, and $(b+1, f(b+1))$ are lying on a line). So $\mathcal{M}$ is a linear tree formed by cliques, as described in Construction 6.2. □

8.2. Proof of Theorem 3.2 for $m > r + 1$

Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices with no Berge cycle of length k or longer ($r \geq 3$, $k \geq r+4$). Suppose that $|\mathcal{H}|$ is maximal, $|\mathcal{H}| = \mathrm{EG}_r(n, k)$. We have $f_r(n, k) \leq \mathrm{EG}_r(n, k)$ by Constructions 4.1 and 4.2.

Let $(A, \mathcal{A})$ be an SDRP of $\mathcal{H}$ of maximum size. Let $\mathcal{B} := \mathcal{H} \setminus \mathcal{A}$, $B = \partial_2 \mathcal{B}$. By Lemma 5.3 the graph G with edge set $A \cup B$ does not contain a cycle of length k or longer. In other words, $\mathcal{M} = (A, \mathcal{B}, V)$ is a $(2, r)$ mixed hypergraph such that $G := A \cup B$ is an n-vertex graph with no cycle of length at least k. Then Theorem 6.3 implies that

$$|A| + |\mathcal{B}| \leq f_r^+(n, k). \tag{17}$$

Since $n = (k-2)p + m$ where $1 \leq m \leq k-2$ and $m \geq r+2$ we have $f_r^+(n, k) = f_r(n, k)$ by (2) and (6). We obtained that $\mathrm{EG}_r(n, k) = f_r(n, k)$, as claimed.

Equality can hold in (17) only if $\mathcal{M}$ has the clique-tree structure with vertex sets $V_1, V_2, \ldots, V_{p+1}$, described in Construction 6.2. In the case of $m \geq r+3$ each block is a complete r-uniform hypergraph, so Construction 6.2 and Construction 4.1 coincide, and we are done.

In the case $m = r+2$, Theorem 6.3 implies that all but one block define complete r-graphs and for one of them, say V_ℓ, $\mathcal{M}|V_\ell$ could be either K_{r+2} or $K_{r+2}^{(r)}$. If $\mathcal{M}|V_\ell = K_{r+2}^{(r)}$, then $\mathcal{H}|V_\ell = K_{r+2}^{(r)}$, so $\mathcal{A} = \emptyset$, $\mathcal{B} = \mathcal{H}$ and we are done. Consider the other case, $\mathcal{M}|V_\ell = K_{r+2}^{(2)}$, i.e., $A = G|V_\ell$ is a complete graph (and $\mathcal{B} = \cup_{i \neq \ell} K_{k-1}^{(r)}[V_i]$). We claim that $\mathcal{H}|V_\ell = K_{r+2}^{(r)}$ which completes the proof in this subsection.

Suppose, on the contrary, that there exists an $f_i \in \mathcal{A}$ such that $\{x_i, y_i\} \subset V_\ell$, $\{x_i, y_i, z_i\} \subset f_i$ such that $z_i \notin V_\ell$. One of the pairs of $x_i z_i$ and $y_i z_i$ is not an edge of G, say it is $x_i z_i$. Then removing $x_i y_i$ from A and replacing it by $x_i z_i$, one obtains an SDRP A' ($\mathcal{A}$ and $\mathcal{B}$ are unchanged). In this case, $E(G') = E(G) \backslash \{x_i y_i\} \cup \{x_i z_i\}$ has a different structure (not a tree of cliques), so it could not be optimal by Theorem 6.3. Therefore such f_i does not exist, i.e., $f_i \subset V_\ell$. In other words $\mathcal{A} \subseteq K_{k-1}^{(r)}[V_i]$. Since $|A| = \binom{r+2}{2} = \binom{r+2}{r}$, $\mathcal{A}$ is a complete r-graph on V_ℓ. □

8.3. Proof of Theorem 3.2 for $m \leq r+1$, preparations

This is a continuation of the previous two subsections.

Consider an extremal $\mathcal{H}$ (i.e., $|\mathcal{H}| = \mathrm{EG}_r(n,k) \geq f_r(n,k)$) with $\mathcal{A}, \mathcal{B}$, A, B, and G as defined in previous subsection. Let G have blocks $G_1, \ldots, G_q$ of G with vertex sets $V_1, \ldots, V_q$ where $|V_i| = n_i \geq 2$. As we have seen in (15) and (16),

$$|\mathcal{H}| = \sum_i (|A_i| + |\mathcal{B}_i|) \leq f_r^+(n,k) - \binom{r}{2} \tag{18}$$

if for any i, $n_i \geq k$. For $m = r+1$, here the right-hand side is

$$p\binom{k-1}{r} + \binom{m}{2} - \binom{r}{2} < p\binom{k-1}{r} + \binom{r+1}{r} = f_r(n,k).$$

Similarly in the case $m \leq r$, the right-hand side is

$$p\binom{k-1}{r} + \binom{m}{2} - \binom{r}{2} < p\binom{k-1}{r} + m - 1 = f_r(n,k).$$

So from now on, we may suppose that $n_i \leq k-1$ for all i.

Claim 8.1. *There are exactly p blocks V_i of size $k-1$, $n_i = k-1$.*

Proof. For $1 \le x \le k-1$, define $f(x) := f_r^+(x,k) = \max\{\binom{x}{2}, \binom{x}{r}\}$. Let $f(x_1,\dots,x_q) := \sum_i f(x_i)$. We want to estimate $f(n_1,\dots,n_q)$, so define $x_i := n_i$. Let $n' := 1 + \sum_i (n_i - 1)$; we have $n' \le n$. In case of $2 \le x_i \le x_j < k-1$ we are going to replace x_i by $x_i - 1$ and x_j by $x_j + 1$. During this process f never decreases and it ends when all but one x_i's become 1 or $k-1$. Then the value of f is exactly $f_r^+(n',k)$ and since $\sum_{1\le i\le q} x_i = \sum_i n_i = n' + q - 1$ is unchanged, in the last step our sequence contains $(k-1)$ exactly p times.

If the number of $(k-1)$'s is unchanged, then there is nothing to prove. Otherwise, after some step the pair x and $k-2$ ($2 \le x \le k-2$) is replaced by $(x-1)$ and $(k-1)$. Then the value of f increased by $f(k-1) + f(x-1) - f(k-2) - f(x)$. Since $f(k-1) = \binom{k-1}{r}$ and $f(k-2) = \binom{k-2}{r}$ the increment is

$$\binom{k-1}{r} + \max\{\binom{x-1}{2}, \binom{x-1}{r}\} - \binom{k-2}{r} - \max\{\binom{x}{2}, \binom{x}{r}\}.$$

This is at least

$$\binom{k-2}{r-1} - \max\{x-1, \binom{x-1}{r-1}\} \ge \binom{k-2}{r-1} - \binom{k-3}{r-1}$$
$$= \binom{k-3}{r-2} \ge \binom{r+1}{r-2} > \binom{r}{2}.$$

In this case $|\mathcal{H}| < f_r^+(n',k) - \binom{r}{2} \le f_r(n',k) \le f_r(n,k)$, a contradiction. ☐

Claim 8.2. *If a block V_i is of size $k-1$, then $e(B_i) \ge \binom{k-2}{2} + r - 1$.*

Proof. If $|\mathcal{B}_i| > \binom{k-2}{r}$ then the Kruskal-Katona Theorem (or a simple double counting) implies that $|\partial_2 \mathcal{B}_i| \ge \binom{k-2}{2} + r - 1$, and we are done.

If $|\mathcal{B}_i| \le \binom{k-2}{r}$ then we use Lovász' version of the Kruskal-Katona theorem. Write $|\mathcal{B}_i|$ in the form of $\binom{x}{r}$, where x is a real number $0 \le x \le k-2$ and (only in this paragraph) $\binom{x}{r}$ is defined as the *real* polynomial $x(x-1)\dots(x-r+1)/r!$ for $x \ge r-1$ and 0 otherwise. We obtain $|\partial_2 \mathcal{B}_i| \ge \binom{x}{2}$. Since A_i and B_i are disjoint, we have $|A_i| \le \binom{k-1}{2} - \binom{x}{2}$. So,

$$|A_i| + |\mathcal{B}_i| \le \binom{k-1}{2} - \binom{x}{2} + \binom{x}{r} \tag{19}$$

holds for some $0 \leq x \leq k-2$. In this range the right-hand side (as a polynomial of variable x) is maximized at $x = k-2$. Hence (19) yields

$$|A_i| + |\mathcal{B}_i| \leq \binom{k-1}{2} - \binom{k-2}{2} + \binom{k-2}{r}.$$

Here the right-hand side is less than $\binom{k-1}{r} - \binom{r}{2}$ which (as we have seen in (18)) leads to the contradiction $|\mathcal{H}| < f_r(n,k)$. □

Claim 8.3. *If a block V_i is of size $k-1$, then $\mathcal{B}_i = K_{k-1}^{(r)}$, a complete r-graph.*

Proof. Suppose that there exists an r-set $f \subset V_i$, $f \notin \mathcal{H}$. Consider the hypergraph $\mathcal{H} \cup \{f\}$. By the maximality of $\mathcal{H}$, $\mathcal{H} \cup \{f\}$ contains a Berge cycle C of length at least k, say with base vertices $\{v_1, \ldots, v_\ell\}$ and edges $\{f_1, \ldots, f_\ell\}$ where $f_\ell = f$ (and so $v_1, v_\ell \in V_i$). Since $|V_i| = k-1$, there is a base vertex of C not contained in V_i. Therefore we may pick a segment P of C (a Berge path in $\mathcal{H}$) say $\{v_a, v_{a+1}, \ldots, v_b\}$, $\{f_a, \ldots, f_{b-1}\}$ such that $v_a, v_b \in V_i$ but $\{v_{a+1}, \ldots, v_{b-1}\} \cap V_i = \emptyset$.

Since each r-edge in $\mathcal{B}_i$ yields a clique of order r in B_i, we have $\delta(B_i) \geq r-1 \geq 2$. By Claim 8.2 and Lemma 5.6, B_i is hamilton-connected unless $r = 3$ and B_i is a clique on $k-2$ vertices with a vertex x of degree 2. If the latter holds, then for a neighbor y of x, the edge xy is contained in exactly one triangle in B_i. But then xy can only be contained in one r-edge of $\mathcal{B}$, contradicting Lemma 5.2. So we may assume B_i has a hamilton path between any two vertices, in particular by Lemma 5.3, there is a Berge path P' of length $k-2$ from x_a to x_b containing all $k-1$ vertices of V_i as base vertices and using only the edges from $\mathcal{B}_i$. The cycle $P \cup P'$ is a Berge cycle in $\mathcal{H}$ of length at least k, a contradiction. Therefore such an edge f cannot exist, $\mathcal{H}|V_i = K_{k-1}^{(r)}$.

Finally, there is no A-edge in V_i. If $\{x,y\} \subset V_i$ is an A-edge, then no $\mathcal{B}$-edge can contain $\{x,y\}$. So all the $\binom{k-3}{r-2}$ $(\geq k-3)$ subsets of V_i of size r and containing xy should belong to $\mathcal{A}$. Therefore V_i must contain at least as many A-edges. But $|A_i| \leq k-1-r (= \binom{k-1}{2} - \binom{k-2}{2} - r + 1)$ by Claim 8.2. □

8.4. Proof of Theorem 3.2 for $m \leq r+1$, the end

This is a continuation of the previous three subsections.

Consider an extremal r-graph $\mathcal{H}$ on the n-element vertex set V (i.e., $|\mathcal{H}| = \mathrm{EG}_r(n,k) \geq f_r(n,k)$) where $n = p(k-2)+m$, $1 \leq m \leq r+1$. Using the definitions of $\mathcal{A}, \mathcal{B}, A, B, G, V_1, \ldots, V_q$ from the previous subsection, we define a different split of $\mathcal{H}$.

Let $\mathcal{V} := \{V_i : |V_i| = k-1\}$. By Claim 8.1, $|\mathcal{V}| = p$. Let H be the graph whose edge set is the union of the complete graphs on $V_i \in \mathcal{V}$, so $|E(H)| = p\binom{k-1}{2}$ and it has a forest like structure of cliques (i.e., every cycle in H is contained in some $V_i \in \mathcal{V}$). Let $C_1, \ldots, C_m$ be the vertex sets of the connected components of H. The graph H necessarily consists of m (nonempty) components, $\cup C_\alpha = V$ $(1 \leq \alpha \leq m)$, some of them could be singletons. Let $H_\alpha := H|C_\alpha$, $\mathcal{H}_\alpha := \cup\{\mathcal{B}_i : V_i \in \mathcal{V}, V_i \subset C_\alpha\}$, and $\mathcal{D} := \mathcal{H} \setminus (\cup\mathcal{H}_\alpha)$. Note that every edge of H used to be a B-edge, $\mathcal{H}_\alpha \subseteq \mathcal{B}$ for all $1 \leq \alpha \leq m$, and $\mathcal{D}$ is the set of edges in $\mathcal{H}$ not contained in some $K^{(r)}_{k-1}$.

Our main observation is the following which is implied by Claim 8.3.

Claim 8.4. *If $x, y \in C_\alpha$, $x \neq y$ then there exists an x-y Berge path of length at least $k-2$ consisting only of $\mathcal{H}_\alpha$ edges. Moreover, if $xy \notin E(H_\alpha)$ then there exists such a path of length at least $2k-4$.*

Proof. Suppose that $f, f' \in \mathcal{D}$, $(f \neq f')$, $x_\alpha \in C_\alpha \cap f$, $x'_\alpha \in C_\alpha \cap f'$, $x_\beta \in C_\beta \cap f$, and $x'_\beta \in C_\beta \cap f'$, $(\alpha \neq \beta)$, then

$$x_\alpha = x'_\alpha \text{ and } x_\beta = x'_\beta. \tag{20}$$

For example, if $x_\alpha \neq x'_\alpha$ and $x_\beta \neq x'_\beta$, then there is a Berge path P_α of length at least $(k-2)$ connecting x_α with x'_α, $P_\alpha \subset \mathcal{H}_\alpha$ and another Berge path P_β of length at least $(k-2)$ connecting x_β with x'_β, $P_\beta \subset \mathcal{H}_\beta$, and these, together with f and f' form a Berge cycle of length at least $2k-2$, a contradiction. The case $|\{x_\alpha, x_\beta\} \cap \{x'_\alpha, x'_\beta\}| = 1$ is similar: we find a Berge cycle in $\mathcal{H}$ of length at least k. □

The same proof, and the second half of Claim 8.4 imply that

$$\partial_2\mathcal{H}|V_\alpha = H_\alpha. \tag{21}$$

In other words, if $f \in \mathcal{H} \setminus \mathcal{H}_\alpha$ then

$$|f \cap C_\alpha| \geq 2 \text{ implies that } \exists V_i \in \mathcal{V},\ V_i \subseteq C_\alpha \text{ such that } C_\alpha \cap f = V_i \cap f.$$

Indeed, otherwise there are $x, y \in f$ and a Berge x,y-path in $\mathcal{H}_i$ of length at least $2k-4$, which together with f form a Berge cycle of length at least $2k-3$.

For a subset $S \subseteq V$, define $\varphi(S)$ as the set of indices $1 \leq \alpha \leq m$ for which $S \cap C_\alpha \neq \emptyset$. Equation (20) can be restated as follows

$$\text{if } \{\alpha, \beta\} \subseteq \varphi(f) \cap \varphi(f') \text{ then } C_\alpha \cap f = C_\alpha \cap f' \text{ is a singleton,} \tag{22}$$

and similarly for β. This implies that $\varphi(f) \neq \varphi(f')$ for $f \neq f'$, $f, f' \in \mathcal{D}$. Even more, the family $\{\varphi(f) : f \in \mathcal{D}\}$ has the Sperner property. This means that for $f, f' \in \mathcal{D}$ with $f \neq f'$, one cannot have $\varphi(f) \subsetneq \varphi(f')$. Indeed, $|\varphi(f)| < r$ implies that there exists a C_α with $|C_\alpha \cap f| \geq 2$, equation (21) implies that $|\varphi(f)| \geq 2$ for every $f \in \mathcal{D}$, so there exists a $\beta \in \varphi(f)$, $\alpha \neq \beta$. But then $\{\alpha, \beta\} \subseteq \varphi(f) \cap \varphi(f')$ and (22) implies that $|C_\alpha \cap f| = 1$, a contradiction.

The following claim on the intersection structure of the edges in $\mathcal{D}$ is a generalization of (22) which can be considered as the case $\ell = 2$. (Technically, two hyperedges sharing at least two vertices form a Berge cycle of length 2.)

Claim 8.5. *Let $\mathcal{F} =: \varphi(\mathcal{D}) = \{\varphi(f) : f \in \mathcal{D}\}$. Suppose that $\{\alpha_1, \ldots, \alpha_\ell\} \subset \{1, \ldots, m\}$ and $\varphi(f_1), \ldots, \varphi(f_\ell) \in \mathcal{F}$ form a Berge cycle in $\mathcal{F}$. Then for each j, the sets $C_{\alpha_j} \cap f_j = C_{\alpha_j} \cap f_{j-1}$ are singletons.*

Proof. Otherwise, we can relabel $j := 1$ and find two distinct vertices x_1 and x_1' such that $x_1 \in C_{\alpha_1} \cap \varphi(f_1)$ and $x_1' \in C_{\alpha_1} \cap \varphi(f_\ell)$. Furthermore, let $x_i, x_i' \in C_{\alpha_i}$ such that $\{x_{i-1}, x_i\} \subset f_i$ for all $1 \leq i \leq \ell$ ($x_0 := x_\ell$, etc.), P_i a Berge path in $\mathcal{H}_i$ connecting x_i with x_i'. These paths could be empty (if $x_i = x_i'$) but by Claim 8.4 we can choose P_1 so that its length is at least $k-2$. Then $f_1, P_1, f_2, P_2, \ldots, f_\ell, P_\ell$ form a cycle of length at least k, a contradiction. □

Case 1: there exists an f such that $|\varphi(f)| = r$. Then $m \geq r$, so $m \in \{r, r+1\}$. If $m = r$, then (because of the Sperner property) $|\mathcal{D}| = 1 < m-1$, a contradiction. So assume $m = r+1$. Let $\alpha := [m] \setminus \varphi(f)$. We have $\alpha \in \varphi(f')$ for all other $f' \in \mathcal{D}$. Since $|\mathcal{D}| \geq r+1 > 3$, there are at least two more $f_2 \neq f_3 \in \mathcal{D} \setminus \{f\}$.

Consider first the case that $|C_\alpha \cap \varphi(f_2)| \geq 2$ for some $f_2 \in \mathcal{D}$. The Sperner property implies that there are distinct $\alpha_2, \alpha_3 \in [m] \setminus \alpha$ such that $\alpha_2 \in \varphi(f_2) \setminus \varphi(f_3)$ and $\alpha_3 \in \varphi(f_3) \setminus \varphi(f_2)$. Then $\alpha, \alpha_2, \alpha_3$ with the hyperedges $\varphi(f_2)$, $\varphi(f)$, and $\varphi(f_3)$ form a Berge cycle. However this cycle does not satisfy Claim 8.5. So from now on, we may suppose that $|C_\alpha \cap \varphi(f')| = 1$ for all $f' \in \mathcal{D} \setminus \{f\}$.

Suppose that there exists an $f_2 \in \mathcal{D}$ and an $\alpha_2 \in [m]$ such that $|C_{\alpha_2} \cap \varphi(f_2)| \geq 2$ (necessarily $\alpha_2 \neq \alpha$). Again Sperner property implies that there is an $\alpha_3 \in [m] \setminus \alpha$ such that $\alpha_3 \in \varphi(f_3) \setminus \varphi(f_2)$ (so we have $\alpha_3 \neq \alpha_2$). Then $\alpha, \alpha_2, \alpha_3$ with the hyperedges $\varphi(f_2)$, $\varphi(f)$, and $\varphi(f_3)$ form a Berge cycle. However this cycle does not satisfy Claim 8.5. So from now on, we may suppose that $|C_{\alpha'} \cap \varphi(f')| = 1$ for all $f' \in \mathcal{D}$ and all $\alpha' \in [m]$.

Since $|\mathcal{D}| \geq r+1$ and $[m]$ has exactly $r+1$ r-subsets, $\varphi(\mathcal{D})$ is a complete r-graph. Its hyperedges form many Berge cycles, so Claim 8.5 implies that $\mathcal{D}$ itself is isomorphic to $K_{r+1}^{(r)}$. Thus $\mathcal{H}$ is as in Construction 4.1.

Case 2: $|\varphi(f)| < r$ for all $f \in \mathcal{D}$. In this case every $f \in \mathcal{D}$ has an $\alpha(f) \in [m]$ such that $|C_{\alpha(f)} \cap f| \geq 2$. For every $f \in \mathcal{D}$, choose another element $\beta(f) \in \varphi(f)$ ($\beta(f) \neq \alpha(f)$) and consider the graph $T := \{\{\alpha(f), \beta(f)\} : f \in \mathcal{D}\}$. By Claim 8.5 the graph T has no cycle, and the maximality of $|\mathcal{H}|$ implies that $e(T) = |\mathcal{D}| \geq m-1$. So T is a tree. Since T is a tree, one cannot replace an edge $\{\alpha(f), \beta(f)\}$ by the 3-edge $\{\alpha(f), \beta(f), \gamma(f)\}$ without creating a cycle in the resulting hypergraph and thus violating Claim 8.5. So $\varphi(f) = \{\alpha(f), \beta(f)\}$, and (21) implies that the structure of $\mathcal{D}$ is as in Construction 4.2. This completes the proof of Theorem 3.2. □

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