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Pak-Stanley labeling for central graphical arrangements

Mikhail Mazin* and Joshua Miller

The original Pak-Stanley labeling was defined by Pak and Stanley as a bijective map from the set of regions of an extended Shi arrangement to the set of parking functions. This map was later generalized to other arrangements associated with graphs and directed multigraphs. In these more general cases the map is no longer bijective. However, it was shown that it is surjective to the set of the G-parking functions, where G is the multigraph associated with the arrangement.

This leads to a natural question: when is the generalized Pak-Stanley map bijective? In this paper we answer this question in the special case of central hyperplane arrangements, i.e. the case when all the hyperplanes of the arrangement pass through a common point.

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Introduction

Let $V \subset \mathbb{R}^n$ be given by $x_1 + \ldots + x_n = 0$. Consider an arrangement $\mathcal{A}$ of affine hyperplanes in V, such that every hyperplane of $\mathcal{A}$ is of the form $H^a_{i,j} := \{x_i - x_j = a\}$ for some $i, j \in \{1, \ldots n\}$ and $a > 0$. Let $G_{\mathcal{A}}$ be the associated directed multigraph, defined as follows. The set of vertices of $G_{\mathcal{A}}$ is $\{1, \ldots, n\}$, and the edge $i \to j$ has multiplicity

$$m_{ij} := \#\{a \in \mathbb{R}_{>0} | H^a_{i,j} \in \mathcal{A}\}.$$

Note that one gets $m_{ij} + m_{ji}$ hyperplanes parallel to $\{x_i = x_j\}$ in $\mathcal{A}$, m_{ij} of them on one side of the origin, and m_{ji} of them on the other. Note also that the multigraph $\mathcal{G}_{\mathcal{A}}$ does not determine the combinatorial type of the arrangement $\mathcal{A}$, as one can shift the hyperplanes by changing the constants on the right hand sides of the equations without changing the graph.

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Definition 1. We will call the arrangements of the type described above the *multigraphical arrangements.*

The generalized Pak-Stanley labeling of the regions (connected components of the complement) of a multigraphical arrangement was defined in [4]:

Definition 2. Let R be a region of $\mathcal{A}$. Let $\mathcal{A}_R \subset \mathcal{A}$ be the subset consisting of the hyperplanes that separate R from the origin. We define the label λ_R to be the function $\lambda_R : \{1, \ldots, n\} \to \mathbb{Z}_{\geq 0}$ given by the following formula:

$$\lambda_R(i) := \#\{(a, j) | a \in \mathbb{R}_{>0}, j \in \{1, \ldots, n\}, \text{and } H^a_{i,j} \in \mathcal{A}_R\}.$$

In other words, $\lambda_R(i)$ equals to the number of hyperplanes of the arrangement $\mathcal{A}$ of the form $H^a_{i,j}$ separating the region R from the origin. (Note that here i is fixed, but j and a might vary.)

We will use the notation $\langle \lambda(1), \ldots, \lambda(n) \rangle$ for a label λ. The region R_0 containing the origin is called the *fundamental region.* It is the only region labeled by $\langle 0, \ldots, 0 \rangle$. Note that the labeling can be defined inductively: as one crosses a hyperplane $H^a_{ij} = \{x_i - x_j = a > 0\}$ in the direction **away** from the origin, the ith component of the label is increased by one, while the rest of the components remain unchanged.

Definition 3. Let G be a directed multigraph on a vertex set $\{1, \ldots, n\}$. A function $\lambda : \{1, \ldots, n\} \to \mathbb{Z}_{\geq 0}$ is called a *G-parking function* if for any non-empty subset $I \subset \{1, \ldots, n\}$ there exists a vertex $i \in I$ such that the number of edges $(i \to j) \in E_G$, counted with multiplicity, such that $j \notin I$ is greater than or equal to $\lambda(i)$.

The following results were proved in [3] and [4]:

Theorem 4 ([3, 4])**.** *Let R be any region of a multigraphical arrangement $\mathcal{A}$. Then the corresponding label λ_R is a $\mathcal{G}_{\mathcal{A}}$-parking function.*

Theorem 5 ([3, 4])**.** *Let $\mathcal{A}$ be a multigraphical arrangement, and let λ be any $\mathcal{G}_{\mathcal{A}}$-parking function. Then there exists a region R of $\mathcal{A}$, such that $\lambda_R = \lambda$.*

Combining the above, we get that the generalized Pak-Stanley labeling is a surjective map from the set of regions of $\mathcal{A}$ to the set of $\mathcal{G}_{\mathcal{A}}$-parking functions.

In [3] these results were proved in a more restricted context. In [4] they were generalized to multigraphical arrangements. In the classical case of extended Shi arrangements, one can show the bijectivity of the Pak-Stanley labeling by using the above results and then comparing the cardinalities of the two sets. The bijectivity results can be extended to other families of arrangements (see [2]). However, in general the generalized Pak-Stanley labelings often fail to be injective. Study of the examples suggests that whenever the map is not injective "globally" it is also not injective "locally."

Definition 6. Let $\mathcal{A}$ be a multigraphical arrangement and let $p \in V$ be any point. We say that the Pak-Stanley labeling for $\mathcal{A}$ is locally injective near p if all labels of regions R such that $p \in \overline{R}$ are distinct. Further, if this holds for all $p \in V$, we say that $\mathcal{A}$ is locally injective.

Conjecture 7. *Let $\mathcal{A}$ be a multigraphical arrangement, then the generalized Pak-Stanley map from the set of regions of $\mathcal{A}$ to the set of parking functions is injective if and only if it is injective locally.*

Remark 8. Note that the "only if" part of the conjecture is trivial.

Furthermore, the examples indicate that a stronger form of Conjecture 7 can be made about the proximity of repeated labels.

Conjecture 9. *Let $\mathcal{A}$ be a multigraphical arrangement. Then for a fixed label λ, the closure of the union of all regions labeled by λ is connected.*

It is clear from the definition of the Pak-Stanley labeling that the local injectivity near a point $x \in V$ is a local question. More precisely one has the following fact

Lemma 10. *Let $\mathcal{A}$ be a multigraphical arrangement and $p \in V$ be a point. Let $\mathcal{A}_p \subset \mathcal{A}$ be the subarrangement consistenting of hyperplanes that contain p. Then the Pak-Stanley labeling for $\mathcal{A}$ is locally injective near p if and only if the Pak-Stanley labeling for $\mathcal{A}_p$ is injective.*

Proof. Let $R_0, \ldots, R_N$ be all of the regions of $\mathcal{A}$ whose closures contain the point p. Let also $R_0^p, \ldots, R_N^p$ be the corresponding regions of $\mathcal{A}_p$, i.e. $R_i \subset R_i^p$ for each i. Let $\lambda(R_i)$ denote the Pak-Stanley labeling for $\mathcal{A}$ and $\lambda_p(R_i^p)$ denote the Pak-Stanley labeling for $\mathcal{A}_p$. Let us also assume that the origin belongs to the region R_0^p, so that $\lambda_p(R_0^p) = \langle 0, \ldots, 0\rangle$. According to the inductive definition of the labeling, as one crosses a hyperplane of $\mathcal{A}_p$, the labels for $\mathcal{A}$ and $\mathcal{A}_p$ change in the same way. Therefore, one gets

$$\lambda(R_i) = \lambda_p(R_i^p) + \lambda(R_0)$$

for all i. It follows that $\lambda(R_i) = \lambda(R_j)$ if and only if $\lambda_p(R_i^p) = \lambda_p(R_j^p)$, which concludes the proof. □

The natural question is to characterize the directed multigraphs for which there exist arrangements with bijective labelings. Conjecture 7 and

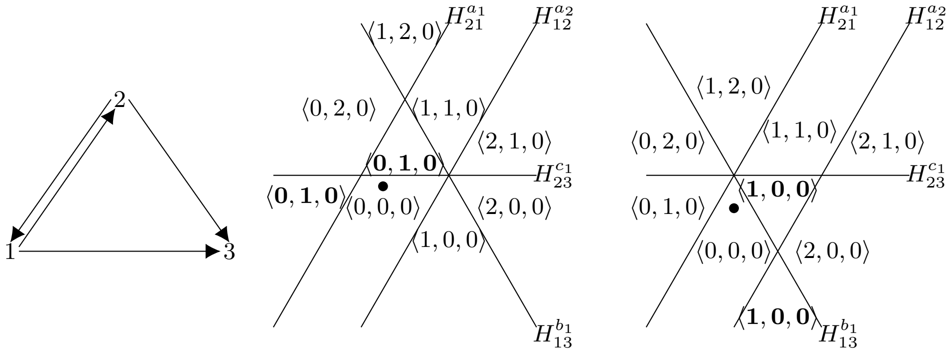


Figure 1: In [6] Baker shows that the graph on the left does not emit an arrangement with a bijective labeling despite satisfying conditions listed in Theorem 11. We illustrate this with two arrangements (center and right) corresponding to the graph. In the first arrangement (center) the label $\langle 0, 1, 0\rangle$ appears twice, while in the second arrangement (right) the label $\langle 1, 0, 0\rangle$ appears twice. One can modify the arrangements by changing the positive constants a_1, a_2, b_1, and c_1 on the right hand sides of the equations of $H_{12}^{a_1}, H_{12}^{a_2}, H_{12}^{b_1}$, and $H_{12}^{c_1}$, but one cannot get rid of both duplicates at the same time (see [6] for details).

Observation 10 motivates studying this question in the special case of central hyperplane arrangements, i.e. arrangements for which all the hyperplanes pass through a common point.

In this paper we answer this question for the special case of central affine multigraphical arrangements, which correspond to acyclic digraphs, by giving necessary and sufficient conditions on the digraph such that the labeling is injective.

Baker, in [6], has been working on generalizing to arbitrary multigraphical arrangements in the $n = 3$ case. She noticed that arrangements with a bijective labeling had a corresponding graph that satisfied the following.

Theorem 11. *Suppose $\mathcal{A}$ is a multigraphical arrangement with a bijective Pak-Stanley labeling and $\mathcal{G}_{\mathcal{A}}$ is the corresponding graph, then for $i, j, k \in V$ with $m_{ij} \neq 0$, $m_{ik} \neq 0$, then $m_{jk} + m_{kj} \geq m_{ij} + m_{ik} - 1$.*

In her thesis, she showed that the above criterion was necessary, but not sufficient for a graph to yield a bijective labeling by giving several families of graphs that satisfy the conditions of Theorem 11 but do not emit arrangements with a bijective labeling.

If $\mathcal{A}$ is a central multigraphical arrangement, then $\mathcal{G}_{\mathcal{A}}$ is a simple acyclic digraph, and the condition of Theorem 11 reduces to the following: if both edges $i \to j$ and $i \to k$ are in $\mathcal{G}_{\mathcal{A}}$ then $m_{kj} + m_{jk} > 1$, i.e. either $j \to k$ or $k \to j$ is also in $\mathcal{G}_{\mathcal{A}}$. The main result of this paper is that in this case the condition is not only necessary for the bijectivity of the Pak-Stanley labeling of $\mathcal{A}$, but also sufficient (see Theorem 15).

1. Central affine multigraphical arrangements

In the case of central multigraphical arrangements, the arrangement is fully determined by the corresponding multigraph (up to a global shift). We start by characterizing the multigraphs corresponding to central arrangements.

Theorem 12. *Let $\mathcal{A}$ be a central multigraphical arrangement, then the corresponding multidigraph is simple and acyclic. Vice versa, if G is a simple acyclic digraph, then there exists a central multigraphical arrangement $\mathcal{A}$, such that $\mathcal{G}_{\mathcal{A}} = G$.*

Proof. Let $\mathcal{A}$ be a central multigraphical arrangement such that all hyperplanes intersect at the point $c = (c_1, c_2, \ldots, c_n)$. Since all hyperplanes $H^a_{i,j}$ intersect at c, then we can have at most one $H^a_{i,j}$ for each pair i, j. Moreover, if we have a hyperplane $H^a_{i,j}$ then we cannot have a hyperplane of the form $H^b_{j,i}$, because they would also be parallel. Thus the digraph $\mathcal{G}_{\mathcal{A}}$ is simple.

Assume that $\mathcal{G}_{\mathcal{A}}$ contains the cycle $i_0 \to i_1 \to \cdots \to i_k \to i_0$. It then follows that the hyperplanes corresponding to the edges in the cycle exhibit

$$\begin{array}{rcl} x_{i_0} - x_{i_1} & = & a_1 > 0 \\ x_{i_1} - x_{i_2} & = & a_2 > 0 \\ \vdots & \vdots & \vdots \\ x_{i_{k-1}} - x_{i_k} & = & a_k > 0 \\ x_{i_k} - x_{i_0} & = & a_{k+1} > 0 \end{array}$$

Since each hyperplane passes through the point c all these equations are satisfied at $\mathbf{x} = \mathbf{c}$. After taking the sum of the above equations we see that $0 = \sum_{i=1}^{k+1} a_i$ which contradicts the assumption that the $a_i > 0$ for all i. Thus $\mathcal{G}_{\mathcal{A}}$ is acyclic.

Now, given an acyclic digraph $G = (V, E)$, with $V = \{1, \ldots, n\}$, one can assume without loss of generality that the edges are oriented in an increasing way. We create the corresponding arrangement $\mathcal{A}$ by: for every edge $(i \to j) \in E$ create the hyperplane $H^{j-i}_{i,j} = \{x_i - x_j = j - i\}$. Consider the following point $c \in V$:

$$c = \left(\frac{n+1}{2}, \ldots, \frac{n+1}{2}\right) - (1, 2, \ldots, n)$$

We immediately see that the point c lies in the intersection of all the hyperplanes since $c_i - c_j = j - i$ for all $1 \leq i < j \leq n$. Therefore the graph G has a corresponding central multigraphical arrangement. $\square$

Let $\mathcal{A}$ be a central multigraphical arrangement, and let $\mathcal{A}'$ be the linear arrangement obtained from $\mathcal{A}$ by shifting all the hyperplanes so that they pass through the origin. Let G be the simple graph obtained from $\mathcal{G}_{\mathcal{A}}$ by removing the orientations on the edges. Then it is well-known that the acyclic orientations of G are in one to one correspondence with the regions of $\mathcal{A}'$. The bijection is constructed as follows. Given a region R of $\mathcal{A}'$ and an edge $i - j$ of G, we orient it $i \to j$ if and only if $x_i < x_j$ at every point of R.

The regions of the original arrangement $\mathcal{A}$ are simply the regions of $\mathcal{A}'$ shifted by a vector. Therefore, they are also in bijection with the acyclic orientations of G, or *acyclic reorientations* of $\mathcal{G}_{\mathcal{A}}$.

Theorem 13. *The fundamental region of $\mathcal{A}$ corresponds to the original orientation of $\mathcal{G}_{\mathcal{A}}$, and crossing a hyperplane $H^a_{i,j} \in \mathcal{A}$ switches the orientation of the corresponding edge between i and j.*

Proof. Let R_0 be the fundamental region of the arrangement $\mathcal{A}$, and let $\mathcal{A}'$ be the corresponding linear arrangement. Let $c = (c_1, \ldots, c_n)$ be in the intersection of all the hyperplanes of the arrangement $\mathcal{A}$. Then it follows that $-c$ belongs to the corresponding region $R' = R_0 - c$ of $\mathcal{A}'$. Therefore, if $H^a_{i,j} \in \mathcal{A}$ and the edge $i \to j$ is the corresponding edge in $\mathcal{G}_{\mathcal{A}}$, then at c we have $c_i - c_j = a$, in particular we have that $c_i > c_j$. It then follows that at $-c \in R'$ that we have $-c_i < -c_j$. Thus, in the orientation corresponding to R' we also get the edge oriented as $i \to j$.

Finally, crossing a hyperplane $H^a_{i,j}$ corresponds to crossing the hyperplane $x_i = x_j$ of the linear arrangement $\mathcal{A}'$, which switches the orientation of the corresponding edge. $\square$

Lemma 14. *The Pak-Stanley labels for the arrangement $\mathcal{A}$ can be computed in terms of acyclic reorientations of the graph $\mathcal{G}_{\mathcal{A}}$. More precisely, for a region R of $\mathcal{A}$ the label $\lambda_R(i)$ equals to the number of edges of $\mathcal{G}_{\mathcal{A}}$ leading from i, such that their orientations got switched in the reorientation corresponding to R.*

Proof. For an arrangement $\mathcal{A}$ the Pak-Stanley label for a region R is calculated by counting the number of hyperplanes of the form $H^a_{i,j}$ separating R

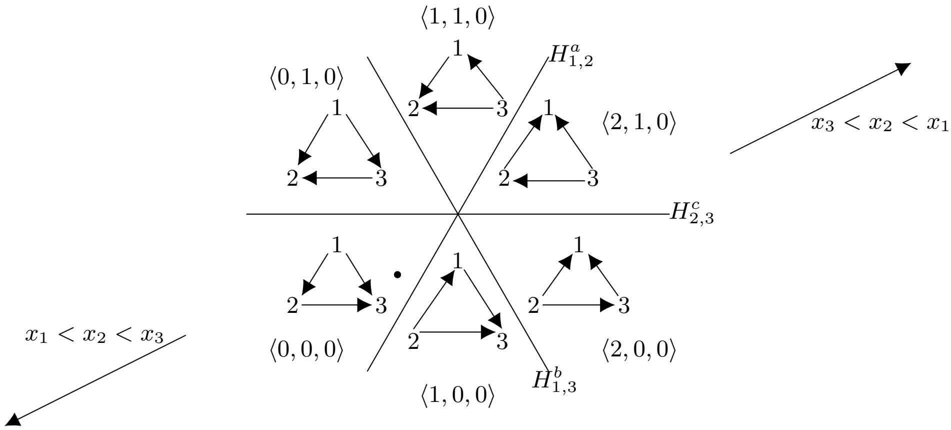


Figure 2: We consider the central arrangement corresponding to the digraph $\mathcal{G}_{\mathcal{A}} = (1 \to 2, 1 \to 3, 2 \to 3)$. The regions of the arrangement are labeled by the corresponding reorientations and the generalized Pak-Stanley labels. Note that the fundamental region is labeled by $\mathcal{G}_{\mathcal{A}}$ and $\langle 0,0,0\rangle$, and as we cross the hyperplanes the orientations of the corresponding edges switch. Moreover, as we cross the hyperplane $H^a_{i,j}$ in a direction away from the origin, the ith entry of the Pak-Stanley label increases by 1.

from the origin and increasing the value $\lambda_R(i)$ accordingly. However, Theorem 13 implies that as we cross a hyperplane $H^a_{i,j}$ we reorient the edge from $(i \to j)$ to $(j \to i)$, so it follows that $\lambda_R(i)$ is the number of edges of $\mathcal{G}_{\mathcal{A}}$ leading from i that get reoriented in the graph corresponding to R. □

Now we are ready to prove our main theorem:

Theorem 15. *Let $V = \{1, 2, \dots, n\}$ and $G = (V, E)$ be an acyclic directed graph on n vertices with edges oriented in the increasing way. Then the hyperplane arrangement corresponding to G produces duplicate Pak-Stanley labelings if and only if there exists $1 \le k < i < j \le n$ such that $(k \to i), (k \to j) \in E$ and $(i \to j) \notin E$.*

Proof of Theorem 15. $\Rightarrow$) Assume that G produces duplicate Pak-Stanley labelings and for the sake of contraction assume that no such i, j, k exists. Since labelings correspond to acyclic reorientations of G, let $G' = (V, E')$ and $G'' = (V, E'')$ be such reorientations.

Since reorientations are in correspondence with labelings then there is an edge $k \to i$ of $\mathcal{G}_{\mathcal{A}}$ that is reoriented as $i \to k$ in G' but not in G''. Moreover since the labels are equal, then there must also be another edge

Figure 3: Here we see the two reorientations of the graph G, G' and G'', and the corresponding cycles created depending on the orientation of the edge $i \to j$.

emanating from k, say edge $k \to j$, such that it is reoriented as $j \to k$ in G'' but not in G'. In other words, the duplicate labeling implies that we have edges $(i \to k), (k \to j) \in E'$ and $(k \to i), (j \to k) \in E''$.

Let k be the largest integer such that this occurs. Since k is the largest possible, it follows that all edges between vertices p, q where $p, q > k$ are oriented in the same way in both reorientations. Without loss of generality we can assume that $i < j$. This gives arise to two cases depending on whether or not the edge from $i \to j$, is oriented as $i \to j$ or $j \to i$ in both G' and G''. If we have the edge $i \to j$ then in G'' we have the cycle $k \to i \to j \to k$, a contradiction since G-parking functions arise from acyclic reorientations. Otherwise we have the edge $j \to i$, but as before we have the cycle $k \to j \to i \to k$ in G' (see Figure 3).

$\Leftarrow$) The easiest way to produce the acyclic reorientations, G' and G'', is reordering the vertices and reorienting the edges so that they point in the increasing direction after considering the new vertex order. For the reoriented graph $G' = (V, E')$ we reorder the vertices of G' as follows

$$1 \prec \ldots \prec k-1 \prec k+1 \prec \ldots \prec i-1 \prec i+1 \prec \ldots \prec j-1 \prec i \prec k \prec j \prec \ldots \prec n.$$

In other words, for G' we move the vertices $k+1, \ldots, i-1, i+1, \ldots, j-1$ to the left so that they precede vertex k, and then switch vertices k and i. Note that as we reorder the vertices, the only edges that are reversed are

(1) $(k \to p) \in E$ such that:
$p \in \{k+1, \ldots, i-1\}$, or
$p \in \{i+1, \ldots, j-1\}$, or
$p = i$

(2) $(i \to p) \in E$ such that: $p \in \{i+1, \ldots, j-1\}$.

To produce the reorientation that corresponds to $G'' = (V, E'')$ we reorder the vertices of G'' as follows:

$$1 \prec \cdots \prec k-1 \prec k+1 \prec \cdots \prec i-1 \prec i+1$$
$$\prec \cdots \prec j-1 \prec j \prec k \prec i \prec j+1 \prec \cdots \prec n.$$

In other words, for G'' we move the vertices $k+1, \ldots, i-1, i+1, \ldots, j-1$ so that they precede vertex k, but now we move vertex j two places to the left so that it precedes k instead of switching vertices k and i. This time the following edges are reoriented

(1) $(k \to p) \in E$ such that:
$p \in \{k+1, \ldots, i-1\}$, or
$p \in \{i+1, \ldots, j-1\}$, or
$p = j$

(2) $(i \to p) \in E$ such that: $p \in \{i+1, \ldots, j-1\}$.

Note that $(i \to j) \notin E$ by assumption, therefore it does not need to be reoriented.

We conclude that both $G' = (V, E')$ and $G'' = (V, E'')$ produce the labeling

$$\tau = \langle 0, \ldots, 0, \overset{k\text{th}}{(N+1)}, 0, \ldots, 0, \overset{i\text{th}}{(K)}, 0, \ldots, 0 \rangle$$

where

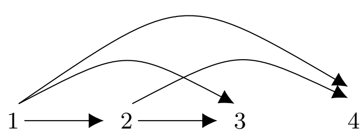
$$N = \#\{(k \to p) \in E : p \in \{k+1, \ldots, i-1\} \cup \{i+1, \ldots, j-1\}\}$$

and

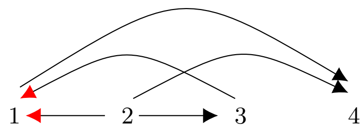
$$K = \#\{(i \to p) \in E : p \in \{i+1, \ldots, j-1\}\}.$$

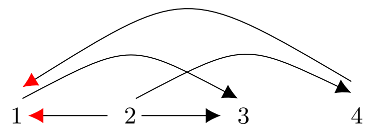
□

Example 16. *Consider the following graph* $G = (V, E)$ *where the vertex and edge sets are given by* $V = \{1, 2, 3, 4\}$ *and* $E = \{(1 \to 2), (1 \to 3), (1 \to 4), (2 \to 3), (2 \to 4)\}$.

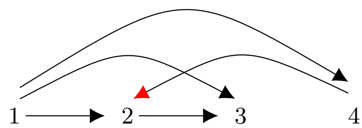


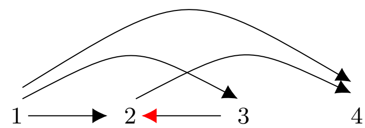
In this example we see that $(1 \to 3)$ *and* $(1 \to 4)$*, but* $(3 \to 4) \notin E$*, so Theorem 15 implies that there should exist two reorientations* G' *and* G'' *that produce the same Pak-Stanley labeling. Consider the following reorientations*





These two reorientations of $G_{\mathcal{A}}$ *produce the label* $\langle 2, 0, 0, 0\rangle$*. Similarly for* $(2 \to 3), (2 \to 4) \in E$*, but* $(3 \to 4) \notin E$ *there will be duplicates*





These two reorientations of G *produce the duplicate label* $\langle 0, 1, 0, 0\rangle$*. Actually, this graph produces four more duplicate labelings*

$$\{\langle 1, 1, 0, 0\rangle, \langle 2, 1, 0, 0\rangle, \langle 1, 2, 0, 0\rangle, \langle 3, 1, 0, 0\rangle\}.$$

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Mikhail Mazin
Mathematics Department
Kansas State University
Cardwell Hall
Manhattan, KS 66506
USA
E-mail address: mmazin@math.ksu.edu

Joshua Miller
Mathematics Department
Kansas State University
Cardwell Hall
Manhattan, KS 66506
USA
E-mail address: jgm019@ksu.edu

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