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# On even rainbow or nontriangular directed cycles

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Let $G = (V, E)$ be an $n$-vertex edge-colored graph. In 2013, H. Li proved that if every vertex $v \in V$ is incident to at least $(n+1)/2$ distinctly colored edges, then $G$ admits a rainbow triangle. We establish a corresponding result for fixed even rainbow $\ell$-cycles $C_\ell$: if every vertex $v \in V$ is incident to at least $(n+5)/3$ distinctly colored edges, where $n \geq n_0(\ell)$ is sufficiently large, then $G$ admits an even rainbow $\ell$-cycle $C_\ell$. This result is best possible whenever $\ell \not\equiv 0 \pmod 3$. Correspondingly, we also show that for a fixed (even or odd) integer $\ell \geq 4$, every large $n$-vertex oriented graph $\vec{G} = (V, \vec{E})$ with minimum outdegree at least $(n+1)/3$ admits a (consistently) directed $\ell$-cycle $\vec{C}_\ell$. Our latter result relates to one of Kelly, Kühn, and Osthus, who proved a similar statement for oriented graphs with large semi-degree. Our proofs are based on the stability method.

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## 1. Introduction

An *edge-colored graph* is a pair $(G, c)$, where $G = (V, E)$ is a graph and $c : E \to P$ is a function mapping edges to some palette of colors $P$. A subgraph $H \subseteq G$ is a *rainbow* subgraph if the edges of $H$ are distinctly colored by $c$. We consider degree conditions ensuring the existence of rainbow cycles $C_\ell$ in $(G, c)$ of fixed even length $\ell \geq 4$. To that end, a vertex $v \in V$ in an edge-colored graph $(G, c)$ has *c-degree* $\deg_G^c(v)$ given by the number of distinct colors assigned by $c$ to the edges $\{v, w\} \in E$, where we set $\delta^c(G) =$

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$\min_{v\in V} \deg^c_G(v)$. The following result of H. Li [12] motivates the main results of our paper.

**Theorem 1.1** (H. Li, 2013)**.** *Let $(G, c)$ be an $n$-vertex edge-colored graph. If $\delta^c(G) \geq (n+1)/2$, then $(G, c)$ admits a rainbow 3-cycle $C_3$.*

A rainbow $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$ shows that Theorem 1.1 is best possible.

Our first result ensures rainbow cycles of fixed even length.

**Theorem 1.2.** *There exists an absolute constant $\alpha > 0$ so that, for every even integer $\ell \geq 4$, every edge-colored graph $(G, c)$ on $n \geq n_0(\ell)$ many vertices satisfying*

$$\delta^c(G) \geq \begin{cases} \left(\frac{1}{3} - \alpha\right)n & \text{if } \ell \equiv 0 \pmod 3, \\ \frac{n+5}{3} & \text{if } \ell \not\equiv 0 \pmod 3, \end{cases} \tag{1}$$

*admits a rainbow $\ell$-cycle $C_\ell$.*

Theorem 1.2 is best possible for $\ell \not\equiv 0 \pmod 3$, which we verify at the end of the Introduction. We prove Theorem 1.2 in Section 2 using the stability method.

**Remark 1.3.** In a related paper [3], we establish an analogue of Theorem 1.2 for fixed odd integers $\ell \geq 3$. In particular, we show that for large integers $n \geq n_0(\ell)$, H. Li's condition $\delta^c(G) \geq (n+1)/2$ ensures rainbow $\ell$-cycles $C_\ell$ in $(G, c)$, which is again best possible by a rainbow $K_{\lfloor n/2\rfloor,\lceil n/2\rceil}$. □

We also consider an analogue of Theorem 1.2 for *oriented graphs* $\vec{G} = (V, \vec{E})$, i.e., those for which $\vec{E} \subset V \times V$ satisfies the rule that $(u, v) \in \vec{E}$ forbids $(v, u) \in \vec{E}$. Here, we seek a *directed* or consistently oriented $\ell$-cycle $\vec{C}_\ell$, whose vertices $V(\vec{C}_\ell)$ may be ordered $(v_0, \ldots, v_{\ell-1})$ so that $(v_i, v_{i+1}) \in \vec{E}$ for all $i \in \mathbb{Z}_\ell$. In this context, we may take $\ell \geq 4$ to be even or odd.

**Theorem 1.4.** *For every fixed integer $\ell \geq 4$, whether even or odd, every oriented graph $\vec{G} = (V, \vec{E})$ on $n \geq n_0(\ell)$ many vertices with minimum out-degree $\delta^+(\vec{G}) \geq (n+1)/3$ admits a directed $\ell$-cycle $\vec{C}_\ell$.*

We prove Theorem 1.4 in Section 2 using ideas similar to that of Theorem 1.2. Note that Theorem 1.4 is best possible for every $\ell \not\equiv 0 \pmod 3$, as seen by the *blow-up* $\vec{G} = (V, \vec{E})$ of a directed triangle:

let $V = V_0 \cup V_1 \cup V_2$ be a partition,

and let $\vec{E} = (V_0 \times V_1) \cup (V_1 \times V_2) \cup (V_2 \times V_0)$, where $|V_2| \leq |V_1| \leq |V_0| \leq |V_2| + 1$. Here, $\delta^+(\vec{G}) = |V_2| \geq ((n+1)/3) - 1$.

Note that Theorem 1.4 omits the case $\ell = 3$, which is the triangular case of the Caccetta-Häggkvist conjecture (cf. [2, 5]) and is beyond the reach of our methods. We also mention that Theorem 1.4 relates to the following result of Kelly, Kühn, and Osthus [8].

**Theorem 1.5** (Kelly, Kühn, Osthus, 2010). *For every integer $\ell \geq 4$ and for every integer $n \geq 10^{10}\ell$, every $n$-vertex oriented graph $\vec{G} = (V, E)$ with $\delta_0(\vec{G}) = \min\{\delta^+(\vec{G}), \delta^-(\vec{G})\} \geq (n+1)/3$ contains a directed $\ell$-cycle $\vec{C}_\ell$. Moreover, every vertex $v \in V$ belongs to a directed $\ell$-cycle $\vec{C}_\ell$.*

The remainder of this paper is organized as follows. In Section 2, we prove both Theorems 1.2 and 1.4. For these proofs, we need upcoming Lemmas 2.6 and 2.9, which (in a sense made precise later) distinguish whether or not a given context is *extremal*. We prove Lemma 2.6 in Sections 3–5 where we also prove supplemental results needed along the way. We prove Lemma 2.9 in Sections 6–8, where again we prove supplemental results needed along the way. We conclude this Introduction by verifying the sharpness of Theorem 1.2 when $\ell \not\equiv 0 \pmod 3$. To aid in the description of these sharpness examples, we let $K[A, B]$ denote the complete bipartite graph with parts $A$ and $B$, and similarly, for a set $C$ disjoint from $A$ and $B$, we let $K[A, B, C]$ denote the complete tripartite graph with parts $A$, $B$, and $C$.

### 1.1. Theorem 1.2 is sharp for $\ell \not\equiv 0 \pmod 3$

Fix an integer $\ell \not\equiv 0 \pmod 3$, which in the constructions below can be even or odd. Let $V = V_0 \cup V_1 \cup V_2$ be a partition of an $n$-element set $V$, where for optimality we take $\lfloor n/3 \rfloor = m = |V_2| \leq |V_1| \leq |V_0| \leq |V_2| + 1$. Let $G = (V, E)$ be given by the complete 3-partite graph $K[V_0, V_1, V_2]$. We now distinguish the cases $n, \ell \pmod 3$.

**Case 1 ($n \not\equiv 2$ (mod 3)).** Define $c_+ : E \to V$ by setting, for each $i \in \mathbb{Z}_3$ and $(v_i, v_{i+1}) \in V_i \times V_{i+1}$,

$$c_+(\{v_i, v_{i+1}\}) = v_{i+1}. \tag{2}$$

We say this same edge $e = \{v_i, v_{i+1}\} \in E$ is of *type* $i$, and we write $t(e) = i$ for its type. We write a fixed $\ell$-cycle $C_\ell$ in $G$ by a cyclic ordering $(e_0, e_1, \ldots, e_{\ell-1})$ of its consecutive edges. A consecutive such pair $(e_k, e_{k+1})$ is a *reversal* when $e_k$ and $e_{k+1}$ are of the same type $t(e_k) = t(e_{k+1}) = i \in \mathbb{Z}_3$, where $(e_k, e_{k+1})$ is a *backward reversal* when $e_k \cap e_{k+1} \in V_{i+1}$, and $(e_k, e_{k+1})$ is a *forward reversal* when $e_k \cap e_{k+1} \in V_i$. Since $C_\ell$ is a cycle, the number of backward reversals is the number of forward reversals, and $C_\ell$ admits backward reversals lest $\ell \equiv 0 \pmod 3$. Fix an arbitrary backward reversal $(e_k, e_{k+1})$ of $C_\ell$, where $k \in \mathbb{Z}_\ell$, where $t(e_k) = t(e_{k+1}) = i \in \mathbb{Z}_3$, and where $e_k \cap e_{k+1} = \{v_{i+1}\} \subset V_{i+1}$. Then

$$c_+(e_k) \overset{(2)}{=} v_{i+1} \overset{(2)}{=} c_+(e_{k+1}), \tag{3}$$

whence $C_\ell$ isn't rainbow. We observe from (2) that $\deg_G^{c_+}(v_i) = 1 + |V_{i+1}|$ holds for each fixed $i \in \mathbb{Z}_3$ and for each fixed $v_i \in V_i$. Indeed, an incident edge $e = \{v_i, v_j\} \in E$ is assigned the fixed color $c_+(e) = v_i$ when $v_j \in V_{i-1}$, and is assigned the variable color $c_+(e) = v_j$ among all $|V_{i+1}|$ many possible $v_j \in V_{i+1}$. As such, $\delta^{c_+}(G) = \deg_G^{c_+}(v_1) = m + 1$ is achieved by any vertex $v_1 \in V_1$, while $\lceil (n+5)/3 \rceil = m + 2$ is ensured by $n \not\equiv 2 \pmod 3$. $\square$

**Case 2 ($n \equiv 2$, $\ell \equiv 1$ (mod 3)).** Here, $n \equiv 2 \pmod 3$ ensures that $|V_0| = |V_1| = m + 1$. Fix a perfect matching $M = \{\{x_1, y_1\}, \ldots, \{x_{m+1}, y_{m+1}\}\}$ of $G[V_0, V_1] = K[V_0, V_1]$, where $V_0 = \{x_1, \ldots, x_{m+1}\}$ and $V_1 = \{y_1, \ldots, y_{m+1}\}$, and fix a symbol $\star \notin V$. Define $c_M : E \to \{\star\} \cup V$ by

$$c_M(e) = \begin{cases} c_+(e) & \text{if } e \in E \setminus E_G[V_0, V_1], \\ \star & \text{if } e \in M, \\ x_b & \text{if } e = \{x_a, y_b\} \in E_G[V_0, V_1] \setminus M. \end{cases} \tag{4}$$

We observe from (4) that $(G, c_M)$ is $(m+2)$-color-regular, while $\lceil (n+5)/3 \rceil = m + 3$ is ensured by $n \equiv 2 \pmod 3$. Indeed, as before in Case 1, a vertex $v_2 \in V_2$ has color-degree $\deg_G^{c_M}(v_2) = \deg_G^{c_+}(v_2) = 1 + |V_0| = m + 2$. Less easily, fix $x_a \in V_0$ and fix an incident edge $x_a \in e \in E$. If $e \cap V_2 \neq \emptyset$, then $e$ is assigned the fixed color $c_M(e) = c_+(e) = x_a$, and if $e = \{x_a, y_a\} \in M$, then $e$ is assigned the fixed color $c_M(e) = \star$. Otherwise, $e = \{x_a, y_b\} \in E[V_0, V_1] \setminus M$ for some $y_a \neq y_b \in V_1$, whence $e$ is assigned the variable color $c_M(e) = x_b$ among all $|V_1| - 1 = m$ many possible $y_b \in V_1 \setminus \{y_a\}$. Similarly, fix $y_b \in V_1$, and fix an incident edge $y_b \in e \in E$. If $e = \{x_b, y_b\} \in M$, then $e$ is assigned the fixed color $c_M(e) = \star$, and if $e = \{x_a, y_b\} \in E[V_0, V_1] \setminus M$ for some $x_b \neq x_a \in V_0$, then $e$ is assigned the fixed color $c_M(e) = x_b$. Otherwise, $e = \{y_b, v_2\} \in E[V_1, V_2]$ for some $v_2 \in V_2$, whence $e$ is assigned the variable color $c_M(e) = c_+(e) = v_2$ among all $|V_2| = m$ many possible $v_2 \in V_2$.

We now observe that $(G, c_M)$ avoids rainbow $\ell$-cycles $C_\ell$. For that, fix an $\ell$-cycle $C_\ell = (e_0, \ldots, e_{\ell-1})$ of $G$ with backward reversal $(e_k, e_{k+1})$, where $k \in \mathbb{Z}_\ell$. For $C_\ell$ to be rainbow, we claim that $G$ must assume the color $\star$ within the backward reversal $(e_k, e_{k+1})$. Indeed, let $t(e_k) = t(e_{k+1}) = i \in \mathbb{Z}_3$, and let $e_k \cap e_{k+1} = \{v_{i+1}\} \subset V_{i+1}$. For $C_\ell$ to be rainbow, $i = 0$ is necessary lest (3) holds, so write $v_{i+1} = y_1 \in V_1$. Since $M$ is a matching, at most one of $e_k, e_{k+1} \in M$, but for $C_\ell$ to be rainbow, at least one such containment is necessary (as claimed) lest (4) gives $c_M(e_k) = x_1 = c_M(e_{k+1})$. Now, for $C_\ell$ to be rainbow, the following are necessary:

($a$) $e_k \in M$ implies $(e_{k-1}, e_k)$ is a forward reversal, lest

$$c_M(e_{k-1}) \overset{(4)}{=} c_+(e_{k-1}) \overset{(2)}{=} x_1 \overset{(4)}{=} c_M(e_{k+1});$$

($b$) $e_{k+1} \in M$ implies $(e_{k+1}, e_{k+2})$ is a forward reversal, lest

$$c_M(e_{k+2}) \overset{(4)}{=} c_+(e_{k+2}) \overset{(2)}{=} x_1 \overset{(4)}{=} c_M(e_k).$$

Either way, $C_\ell$ has further backward reversals (assuming $\star$ again) lest $\ell \equiv 2 \pmod 3$. $\square$

**Case 3** $(n, \ell \equiv 2$ **(mod 3)).** We first slightly alter the graph $G = K[V_0, V_1, V_2]$ above, as follows. Fix $x \in V_0$ and $y \in V_1$ so that $U_0 = V_0 \setminus \{x\}$, $U_1 = V_1 \setminus \{y\}$, and $U_2 = V_2$ all have size $m$. Define $\hat{E}$ by the rule that, for each $\{u, v\} \in \binom{V}{2}$, we put $\{u, v\} \in \hat{E}$ if, and only if,

$$\{y\} \times U_0 \not\ni (u, v) \notin \bigcup_{i \in \mathbb{Z}_3} (U_i \times U_i).$$

In other words, $G = (V, E)$ and $\hat{G} = (V, \hat{E})$ differ only in the $3m = n - 2$ elements among

$$\hat{E} \setminus E = \bigcup \big\{\{x, u_0\} : u_0 \in U_0\big\} \cup \bigcup \big\{\{y, u_1\} : u_1 \in U_1\big\} \quad \text{and}$$
$$E \setminus \hat{E} = \bigcup \big\{\{y, v_0\} : v_0 \in U_0\big\}.$$

Define $\hat{c} : \hat{E} \to \{\star\} \cup V$ by setting, for each $e \in \hat{E}$,

$$\hat{c}(e) = \begin{cases} \star & \text{if } e \in \hat{E} \setminus E, \\ \star & \text{if } e = \{x, u_2\} \in \hat{E} \cap E \text{ for some } u_2 \in U_2, \\ c_+(e) & \text{otherwise.} \end{cases} \tag{5}$$

We observe from (5) that $(\hat{G}, \hat{c})$ is $(m+2)$-color-regular, while $\lceil (n+5)/3) \rceil = m + 3$ is ensured by $n \equiv 2 \pmod 3$. Indeed, fix a vertex $u_0 \in U_0$, and fix an incident edge $u_0 \in e \in \hat{E}$. If $x \in e$, then $e$ is assigned the fixed color $\hat{c}(e) = \star$, and if $e \cap U_2 \neq \emptyset$, then $e$ is assigned the fixed color $\hat{c}(e) = c_+(e) = u_0$. Otherwise, $e = \{u_0, u_1\} \in \hat{E}[U_0, U_1] = E[U_0, U_1]$ for some $u_1 \in U_1$, whence $e$ is assigned the variable color $\hat{c}(e) = c_+(e) = u_1$ among all $|U_1| = m$ many possible $u_1 \in U_1$. Vertices $u_1 \in U_1$ and $u_2 \in U_2$ similarly have $\hat{c}$-degree $m + 2$. For the fixed vertex $x \in V$, fix an incident edge $x \in e \in \hat{E}$. If $e \cap (U_0 \cup U_2) \neq \emptyset$, then $e$ is assigned the fixed color $\hat{c}(e) = \star$, and if $y \in e$, then $e$ is assigned the fixed color $\hat{c}(e) = y$. Otherwise, $e = \{x, u_1\}$ for some $u_1 \in U_1$, whence $e$ is assigned the variable color $\hat{c}(e) = c_+(e) = u_1$ among all $|U_1| = m$ many possible $u_1 \in U_1$. The fixed vertex $y \in V$ similarly has $\hat{c}$-degree $m + 2$. That $(\hat{G}, \hat{c})$ avoids rainbow $\ell$-cycles $C_\ell$ is sketched in the Appendix, when more needed concepts are developed. □

## 2. Proofs of Theorems 1.2 and 1.4

The proofs of Theorems 1.2 and 1.4 are based on the well-known stability method, together with a few elementary results. We present the tools we need in order of increasing technicality.

### 2.1. Elementary tools

Edge-colored graphs $(G, c)$ on a vertex set $V$ correspond to directed graphs $\vec{G} = (V, \vec{E})$, as follows. For each $v \in V$, let $\{v, w_1\}, \ldots, \{v, w_d\} \in E$ be a system of representatives of the color classes of $c$ on edges at $v$, where $d = \deg_G^c(v)$. We put $(v, w_1), \ldots, (v, w_d) \in \vec{E}$, and we say that a directed graph $\vec{G} = (V, \vec{E})$ obtained in this way (which need be neither oriented nor unique) is *associated with* $(G, c)$. Directed graphs $\vec{G} = (V, \vec{E})$ correspond to edge-colored graphs $(G, c)$, as follows. For each $(v, w) \in \vec{E}$, we put $\{v, w\} \in E(G)$ and define $c(\{v, w\}) = w$. Then $(G, c)$ is uniquely determined by $\vec{G}$, although $G = (V, E)$ may be a multigraph. We pause for the following remark.

**Remark 2.1.** In this paper, no directed graph $\vec{G} = (V, \vec{E})$ will allow $\vec{E}$ to be a multiset, nor will $\vec{E}$ consist of any directed loops. When $(v, w) \in \vec{E}$ forbids $(w, v) \in \vec{E}$, then $\vec{G} = (V, \vec{E})$ is an *oriented graph*. When so, the edge-colored graph $(G, c)$ determined by $\vec{G}$ is simple. □

In the contexts above, we make a couple of elementary observations. On the one hand, if $(G, c)$ is an edge-colored graph and $\vec{G} = (V, \vec{E})$ is a directed graph associated with $(G, c)$, then every vertex $v \in V$ has out-degree $\deg^+_{\vec{G}}(v) = \deg^c_G(v)$. On the other hand, if $\vec{G} = (V, \vec{E})$ is an oriented graph and $(G, c)$ is the edge-colored graph determined by $\vec{G} = (V, \vec{E})$, then every vertex $v \in V$ satisfies $\deg^c_G(v) = \deg^+_{\vec{G}}(v) + 1$ when $v$ has positive in-degree in $\vec{G}$, and $\deg^c_G(v) = \deg^+_{\vec{G}}(v)$ otherwise. In these contexts, we next consider the extent to which rainbow cycles of $(G, c)$ relate to directed cycles of $\vec{G}$, and vice versa. We begin with the following elementary but useful observation first noted by H. Li in [12].

**Fact 2.2.** *Let $\vec{G} = (V, \vec{E})$ be an oriented graph, and let $(G, c)$ be the edge-colored graph determined by $\vec{G}$. Every directed $\ell$-cycle $\vec{C}_\ell$ in $\vec{G}$ corresponds to a rainbow $\ell$-cycle $C_\ell$ in $(G, c)$. Moreover, every properly colored $\ell$-cycle $C_\ell$ in $(G, c)$ is, in fact, a rainbow $\ell$-cycle, and corresponds to a directed $\ell$-cycle $\vec{C}_\ell$ in $\vec{G}$.*

In Fact 2.2, the edge-colored graph $(G, c)$ is derived from a given oriented graph $\vec{G} = (V, \vec{E})$, and directed $\ell$-cycles $\vec{C}_\ell$ of $\vec{G}$ are in one-to-one correspondence with rainbow $\ell$-cycles $C_\ell$ of $(G, c)$. However, when $(G, c)$ is given and $\vec{G} = (V, \vec{E})$ is associated with $(G, c)$, the same conclusion need not hold.

**Fact 2.3.** *Let $(G, c)$ be an $n$-vertex edge-colored graph, and let $\vec{G} = (V, \vec{E})$ be a directed graph associated with $(G, c)$. Then $\vec{G}$ admits at most $(1/2)(\ell - 1)n^{\ell-1}$ many directed $\ell$-cycles $\vec{C}_\ell$ that were not rainbow in $(G, c)$.*

**Remark.** We apply Fact 2.3 with $\ell$ fixed, and from the $O(n^{\ell-1})$ bound, we only ever use $o(n^\ell)$. □

*Proof of Fact 2.3.* Consider a directed $\ell$-cycle $\vec{C}_\ell = (v_0, v_1, \ldots, v_{\ell-1})$ in $\vec{G}$, where $v_{i+1} \in N^+_{\vec{G}}(v_i)$ for all $i \in \mathbb{Z}_\ell$. When $\vec{C}_\ell$ was not rainbow in $(G, c)$, some distinct pair $(j, k) \in \mathbb{Z}_\ell \times \mathbb{Z}_\ell$ satisfies $c(\{v_j, v_{j+1}\}) = c(\{v_k, v_{k+1}\})$. Owing to the construction of $\vec{G}$ from $(G, c)$, the vertex $v_{k+1} \in N^+_{\vec{G}}(v_k)$ is uniquely determined by $c(\{v_j, v_{j+1}\})$. There are at most $\ell(\ell-1)(n)_{\ell-1}$ many vertex sequences $(v_0, v_1, \ldots, v_{\ell-1})$ with the property that, for some distinct pair $(j, k) \in \mathbb{Z}_\ell \times \mathbb{Z}_\ell$, the vertex $v_{k+1}$ depends uniquely on $j$. Since each directed $\ell$-cycle $\vec{C}_\ell$ of $\vec{G}$ has $2\ell$ many symmetries, the bound in Fact 2.3 follows. □

The following concept is central throughout the remainder of the paper.

**Definition 2.4** ($\lambda$-extremal)**.** Fix $\lambda \geq 0$, an $n$-vertex directed graph $\vec{G} = (V, \vec{E})$, and an edge-colored graph $(G, c)$ with vertex set $V$ and edge set $E$. We say that

1. $\vec{G}$ is $\lambda$-*extremal* if there exists a partition $V = V_0 \cup V_1 \cup V_2$ where for all $i \in \mathbb{Z}_3$,

$$e_{\vec{G}}(V_i, V_{i+1}) \geq \left(\tfrac{1}{9} - \lambda\right) n^2, \tag{6}$$

where $e_{\vec{G}}(V_i, V_{i+1})$ denotes the number of edges $(v_i, v_{i+1}) \in \vec{E} \cap (V_i \times V_{i+1})$;

2. $(G, c)$ is $\lambda$-*extremal* if there exists a partition $V = V_0 \cup V_1 \cup V_2$ on which some directed graph $\vec{G} = (V, \vec{E})$ associated with $(G, c)$ is $\lambda$-extremal.

In these contexts, $V = V_0 \cup V_1 \cup V_2$ is said to be $\lambda$-*extremal* for $\vec{G}$ or $(G, c)$.

We conclude our elementary tools with the following fact. In its proof, we use the notation $(x)_k = x(x-1)\cdots(x-k+1)$ for the *falling factorial.*

**Fact 2.5.** *For all* $0 \le \lambda \le 1/(28)$, *and for every positive integer* $\ell \equiv 0 \pmod 3$, *the following hold:*

1. *Every $\lambda$-extremal $n$-vertex directed graph $\vec{G} = (V, \vec{E})$ has $\Omega(n^\ell)$ many directed $\ell$-cycles $\vec{C}_\ell$.*
2. *Every $\lambda$-extremal $n$-vertex edge-colored graph $(G, c)$ has $\Omega(n^\ell)$ many rainbow $\ell$-cycles $C_\ell$.*

*Proof of Fact 2.5.* Fix $0 \le \lambda \le 1/(28)$ and fix a positive integer $\ell \equiv 0 \pmod 3$. To prove Statement (1), set $k = \ell/3$, and let $\vec{G} = (V, \vec{E})$ be an $n$-vertex directed graph with $\lambda$-extremal vertex partition $V = V_0 \cup V_1 \cup V_2$. Let $\vec{H}$ be a blow-up of the directed triangle on $V = V_0 \cup V_1 \cup V_2$, whose edges consist of $(V_0 \times V_1) \cup (V_1 \times V_2) \cup (V_2 \times V_0)$. Then, $\vec{H}$ admits precisely $(|V_0|)_k \times (|V_1|)_k \times (|V_2|)_k$ many directed $\ell$-cycles $\vec{C}_\ell$ meeting each of $V_0$, $V_1$, and $V_2$ exactly $k$ times. The number of these cycles having some edge $\vec{e} = (v_0, v_1)$ of $\vec{H} \setminus \vec{G}$, where $v_0 \in V_0$ and $v_1 \in V_1$, is at most

$$\left(|V_0||V_1| - \left(\tfrac{1}{9} - \lambda\right) n^2\right) |V_2| \times (|V_0| - 1)_{k-1} \times (|V_1| - 1)_{k-1} \times (|V_2| - 1)_{k-1}.$$

More generally, the number of these cycles having some edge $\vec{e}$ of $\vec{H} \setminus \vec{G}$ is at most

$$\left(3 - \left(\tfrac{1}{9} - \lambda\right) \frac{n^3}{|V_0||V_1||V_2|}\right) (|V_0|)_k \times (|V_1|)_k \times (|V_2|)_k.$$

Thus, $\vec{G}$ admits at least

$$\left(\left(\tfrac{1}{9} - \lambda\right) \frac{n^3}{|V_0||V_1||V_2|} - 2\right) (|V_0|)_k \times (|V_1|)_k \times (|V_2|)_k$$

many directed $\ell$-cycles $\vec{C}_\ell$. Since $|V_0||V_1||V_2| \le n^3/(27)$ holds by convexity, $\vec{G}$ admits at least

$$(1 - 27\lambda)(|V_0|)_k \times (|V_1|)_k \times (|V_2|)_k = \Omega(n^\ell)$$

many directed $\ell$-cycles $\vec{C}_\ell$, where we used $\lambda \le 1/(28)$.

For Statement (2), let $(G, c)$ be an $n$-vertex $\lambda$-extremal edge colored graph, and let $\vec{G} = (V, \vec{E})$ be a directed graph associated with $(G, c)$ which has $\lambda$-extremal partition $V = V_0 \cup V_1 \cup V_2$. Let $\vec{F} \subseteq \vec{G}$ consist of all $(v_i, v_{i+1}) \in \vec{E}$ where $v_i \in V_i$ and $v_{i+1} \in V_{i+1}$ for $i \in \mathbb{Z}_3$. Then $\vec{F}$ is an oriented graph with $\lambda$-extremal partition $V = V_0 \cup V_1 \cup V_2$ which, by Statement (1), admits $\Omega(n^\ell)$ directed $\ell$-cycles $\vec{C}_\ell$. Fact 2.3 ensures that $\Omega(n^\ell) - o(n^\ell)$ of these directed cycles correspond to rainbow cycles in $(G, c)$, because the edge-colored graph $F$ determined by $\vec{F}$ is, by construction, a subgraph of $G$. $\square$

## 2.2. Stability results

In what follows, we distinguish between whether or not a given structure is $\lambda$-extremal (cf. Definition 2.4).

**Lemma 2.6.** *For all $\lambda > 0$, there exists $\alpha = \alpha(\lambda) > 0$ so that for all integers $\ell \ge 4$, there exists an integer $n_0 = n_0(\lambda, \alpha, \ell) \ge 1$ so that whenever $\vec{G}$ is an oriented graph on $n \ge n_0$ many vertices satisfying*

$$\delta^+(\vec{G}) \ge \begin{cases} \left(\frac{1}{3} - \alpha\right) n & \text{if } \ell \ne 5, \\ \frac{n+1}{3} & \text{if } \ell = 5, \end{cases} \tag{7}$$

*then $\vec{G}$ is $\lambda$-extremal or $\vec{G}$ admits a closed directed $\ell$-walk $\vec{W}_\ell$.*

We prove Lemma 2.6 in Sections 3–4. We apply Lemma 2.6 in the following convenient form.

**Corollary 2.7** (the non-extremal case)**.** *In the context of Lemma 2.6, the following statements hold:*

1. *If $\ell = 5$ and $\vec{G}$ is not $\lambda$-extremal, then $\vec{G}$ contains a directed 5-cycle $\vec{C}_5$;*
2. *If $\ell \ne 5$ and $\vec{G}$ is not $\lambda$-extremal, then $\vec{G}$ contains $\Omega(n^\ell)$ many directed $\ell$-cycles $\vec{C}_\ell$.*

*Moreover, for even integers $\ell$, Statement (2) above holds when $\vec{G}$ is allowed to be a directed graph.*

Note that Statement (1) of Corollary 2.7 restates the conclusion of Lemma 2.6 when $\ell = 5$, since the only closed directed 5-walk $\vec{W}_5$ is the 5-cycle $\vec{C}_5$. It is standard to derive Statement (2) of Corollary 2.7 from Lemma 2.6 by using a suitable regularity lemma. We sketch such a proof below.

**Remark 2.8.** In the context of Lemma 2.6, let $\vec{G}$ be an oriented graph on $n \geq n_0(\lambda, \ell)$ many vertices which satisfies (7), where $\ell \neq 5$. We may apply Lemma 3.2 from [7] to obtain a regular partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ of $\vec{G}$ with cluster digraph $\vec{R}$, where $\vec{R}$ may not be oriented. Nonetheless, Lemma 3.2 guarantees that $\vec{R}$ admits an oriented spanning subgraph $\vec{Q} \subseteq \vec{R}$, where $\delta^+(\vec{Q})/t$ can be taken arbitrarily close to $\delta^+(\vec{G})/n$, and where $\delta^-(\vec{Q})/t$ can be taken arbitrarily close to $\delta^-(\vec{G})/n$. As such, if the oriented graph $\vec{G}$ is not $\lambda$-extremal, then the oriented graph $\vec{Q}$ isn't $\lambda'$-extremal for some suitably small $0 < \lambda' \leq \lambda$. Lemma 2.6 then guarantees that $\vec{Q}$ admits a closed directed $\ell$-walk $\vec{W}_\ell$. Applying a counting lemma to the system of pairs $(V_i, V_j)$ corresponding to the edges of $\vec{W}_\ell$ guarantees $\Omega(n^\ell)$ many directed $\ell$-cycles $\vec{C}_\ell$.

When $\ell$ is even, $\vec{G}$ need not be oriented. Here, we may apply Lemma 3.1 of [1] to obtain a regular partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ of $\vec{G}$ with cluster digraph $\vec{R}$. Again, if $\vec{G}$ is not $\lambda$-extremal, then $\vec{R}$ is not $\lambda'$-extremal for some suitably small $0 < \lambda' \leq \lambda$. If $\vec{R}$ is, in fact, an oriented graph, then we proceed identically to the above. Assume that $\vec{R}$ admits a 2-cycle, i.e., a closed 2-walk $\vec{W}_2$. Since $\ell$ is even, the pair $(V_i, V_j)$ corresponding to $\vec{W}_2$ admits $\Omega(n^\ell)$ many directed $\ell$-cycles $\vec{C}_\ell$. $\square$

We continue with an extremal counterpart to Corollary 2.7.

**Lemma 2.9** (the extremal case)**.** *There exists an absolute constant $\lambda_0 > 0$ so that, for all $0 < \lambda \leq \lambda_0$ and for all integers $\ell \geq 4$ not divisible by three, there exists an integer $n_0 = n_0(\lambda_0, \lambda, \ell) \geq 1$ so that whenever $(G, c)$ is a $\lambda$-extremal edge colored graph on $n \geq n_0$ many vertices, the following hold:*

1. *If $\ell \neq 5$ and $\delta^c(G) \geq (n+5)/3$ (cf. (1)), then $(G, c)$ admits a rainbow $\ell$-cycle $C_\ell$;*
2. *If $\delta^c(G) \geq (n+4)/3$, then $(G, c)$ admits a properly colored $\ell$-cycle $C_\ell$.*

We prove Lemma 2.9 in Sections 6–8. We proceed to the proofs of Theorems 1.2 and 1.4, which are formal consequences of Corollary 2.7 and Lemma 2.9.

### 2.3. Proof of Theorem 1.2

To define the absolute constant $\alpha > 0$ promised by Theorem 1.2, we consider auxiliary parameters. Let $\lambda_{\text{Lem. 2.9}}$ be the absolute constant $\lambda_0$ guaranteed by Lemma 2.9. Set

$$\lambda = \min\left\{\tfrac{1}{28},\ \lambda_{\text{Lem. 2.9}}\right\}, \tag{8}$$

which is suitably small for an application of Fact 2.5. With $\lambda > 0$ given in (8), let

$$\alpha = \alpha_{\text{Lem. 2.6}}(\lambda) > 0 \tag{9}$$

be the constant guaranteed by Lemma 2.6, which we take to be the constant promised by Theorem 1.2.

Fix an even integer $\ell \geq 4$. Let $(G, c)$ be an $n$-vertex edge-colored graph satisfying (1), where in all that follows we assume that $n \geq n_0(\lambda, \alpha, \ell)$ is sufficiently large. To prove Theorem 1.2, we distinguish between the cases of whether or not $(G, c)$ is $\lambda$-extremal, where $\lambda$ is given in (8).

**Case 1 (($G, c$) is $\lambda$-extremal).** In this case, we apply Fact 2.5 or Lemma 2.9 to $(G, c)$. Assume first that $\ell \equiv 0 \pmod 3$. By our choice of $\lambda \leq 1/(28)$ from (8), Statement (2) of Fact 2.5 guarantees $\Omega(n^\ell)$ many rainbow $\ell$-cycles $C_\ell$ in $(G, c)$. Assume now that $\ell \not\equiv 0 \pmod 3$. By our choice of $\lambda \leq \lambda_{\text{Lem. 2.9}}$ from (8), Statement (1) of Lemma 2.9 guarantees a rainbow $\ell$-cycle $C_\ell$ in $(G, c)$. (Note: $\ell \neq 5$ by the parity of $\ell$.) □

**Case 2 (($G, c$) is not $\lambda$-extremal).** In this case, we will indirectly apply Fact 2.3 and Corollary 2.7 to $(G, c)$. For that, let $\vec{G} = (V, \vec{E})$ be any directed graph associated with $(G, c)$, where necessarily $\vec{G}$ is not $\lambda$-extremal, and where $\delta^+(\vec{G}) = \delta^c(G) \geq \left(\frac{1}{3} - \alpha\right) n$ is ensured by (1). By our choice of $\alpha = \alpha_{\text{Lem. 2.6}}(\lambda)$ in (9) (and $\ell \neq 5$), Statement (2) of Corollary 2.7 guarantees $\Omega(n^\ell)$ many directed $\ell$-cycles $\vec{C}_\ell$ in $\vec{G}$. Fact 2.3 then guarantees that at least one of these corresponds to a rainbow $\ell$-cycle $C_\ell$ in $(G, c)$. □

### 2.4. Proof of Theorem 1.4

We again use the auxiliary constants $\lambda > 0$ and $\alpha > 0$ determined in (8) and (9). Fix an integer $\ell \geq 4$. Let $\vec{G} = (V, \vec{E})$ be an $n$-vertex oriented graph satisfying $\delta^+(\vec{G}) \geq (n+1)/3$, where in all that follows we assume that $n \geq n_0(\lambda, \alpha, \ell)$ is sufficiently large. Let $\vec{H} \subseteq \vec{G}$ be maximally induced w.r.t. satisfying $\delta^-(\vec{H}) \geq 1$, and set $U = V(\vec{H})$. Note that every $u \in U$

satisfies $\deg^+_{\vec{H}}(u) = \deg^+_{\vec{G}}(u)$. Consequently, $|U| = \Omega(n)$ can be taken as large as needed since the number $e(\vec{H})$ of edges of $\vec{H}$ satisfies

$$\binom{|U|}{2} \geq e(\vec{H}) \geq |U|\delta^+(\vec{H}) \geq |U|\delta^+(\vec{G}) \implies |U| \geq 2\delta^+(\vec{G}) \geq 2n/3.$$

We now distinguish between the cases of whether or not $\vec{H}$ is $\lambda$-extremal, where $\lambda$ is determined in (8).

**Case 1 ($\vec{H}$ is not $\lambda$-extremal).** In this case, we apply Corollary 2.7 to $\vec{H}$, which is possible on account that $\delta^+(\vec{H}) \geq \delta^+(\vec{G}) \geq (n+1)/3 \geq ((1/3)-\alpha)n$, for $\alpha = \alpha_{\text{Lem. 2.7}}$ in (9). Whether or not $\ell = 5$, Corollary 2.7 guarantees a directed $\ell$-cycle $\vec{C}_\ell$ in $\vec{H}$, where $\vec{C}_\ell$ also appears in $\vec{G} \supseteq \vec{H}$. □

**Case 2 ($\vec{H}$ is $\lambda$-extremal).** In this case, we will apply Fact 2.5 to $\vec{H}$ or we will indirectly apply Fact 2.2 and Lemma 2.9 to $\vec{H}$. Assume first that $\ell \equiv 0 \pmod 3$. By our choice of $\lambda \leq 1/(28)$ in (8), Statement (1) of Fact 2.5 guarantees $\Omega(n^\ell)$ many directed $\ell$-cycles $\vec{C}_\ell$ in $\vec{H}$, each of which also appears in $\vec{G} \supseteq \vec{H}$. Assume now that $\ell \not\equiv 0 \pmod 3$. Let $(H, c)$ be the edge-colored graph determined by $\vec{H}$, where $H$ has vertex set $U = V(\vec{H})$. Since every vertex $u \in U$ has positive in-degree in $\vec{H}$, we have that

$$\deg^c_H(u) = 1 + \deg^+_{\vec{H}}(u) \geq \tfrac{n+4}{3}.$$

By our choice of $\lambda \leq \lambda_{\text{Lem. 2.9}}$ in (8), Statement (2) of Lemma 2.9 guarantees a properly colored $\ell$-cycle $C_\ell$ in $(H, c)$. Since $(H, c)$ was determined by the oriented graph $\vec{H}$, Fact 2.2 guarantees that $C_\ell$ corresponds to a directed $\ell$-cycle $\vec{C}_\ell$ in $\vec{H}$, which also appears in $\vec{G} \supseteq \vec{H}$. □

## 3. Proof of Lemma 2.6

Lemma 2.6 is a formal consequence of the following two propositions (recall $\delta_0(\vec{G})$ from Theorem 1.5).

**Proposition 3.1.** *For all $\beta > 0$, there exists $\alpha = \alpha(\beta) > 0$ so that for every integer $\ell \geq 4$, there exists an integer $n_0 = n_0(\beta, \alpha, \ell) \geq 1$ so that the following holds. Let $\vec{G} = (V, \vec{E})$ be an oriented graph satisfying (7) on $n \geq n_0$ many vertices. If $\vec{G}$ admits no closed directed $\ell$-walk, then $\vec{G}$ admits an induced subgraph $\vec{H} = \vec{G}[U]$ on $|U| = m \geq (1-\beta)n$ many vertices which satisfies*

$$\delta_0(\vec{H}) \geq \left(\tfrac{\delta^+(\vec{G})}{n} - \beta\right)m. \tag{10}$$

**Proposition 3.2.** *For all $\lambda_0 > 0$, there exists $\beta = \beta(\lambda_0) > 0$ so that for every integer $\ell \geq 4$, there exists an integer $m_0 = m_0(\lambda_0, \beta, \ell) \geq 1$ so that the following holds. Let $\vec{H}$ be an oriented graph on $m \geq m_0$ vertices which admits no closed directed $\ell$-walk, but which satisfies $\delta_0(\vec{H}) \geq ((1/3) - \beta)m$. Then $\vec{H}$ is $\lambda_0$-extremal.*

The proof of Proposition 3.1 is not too difficult, and will be given later in this section. The proof of Proposition 3.2 is more involved, and will be postponed to the following section.

### 3.1. Proof of Lemma 2.6

Let $\lambda > 0$ be given. To define the constant $\alpha = \alpha(\lambda) > 0$ promised by Lemma 2.6, we consider several auxiliary constants. First, set $\lambda_0 = \lambda/2$, and let

$$\beta_{\text{Prop. 3.2}} = \beta_{\text{Prop. 3.2}}(\lambda_0) > 0 \tag{11}$$

be the constant guaranteed by Proposition 3.2. Second, set

$$\beta = \tfrac{1}{2}\min\{\lambda_0, \beta_{\text{Prop. 3.2}}\}. \tag{12}$$

Third, let

$$\alpha_{\text{Prop. 3.1}} = \alpha_{\text{Prop. 3.1}}(\beta) > 0 \tag{13}$$

be the constant guaranteed by Proposition 3.1. We define

$$\alpha = \min\{\alpha_{\text{Prop. 3.1}}, \beta\} \tag{14}$$

to be the constant promised by Lemma 2.6. Let an integer $\ell \geq 4$ be given. Let $\vec{G} = (V, \vec{E})$ be an $n$-vertex oriented graph satisfying (7) with $\alpha$ in (14), where in all that follows we assume that $n \geq n_0(\lambda, \alpha, \ell)$ is sufficiently large. We assume that $\vec{G}$ admits no closed directed $\ell$-walk, and establish that $\vec{G}$ is $\lambda$-extremal.

Since $\vec{G}$ admits no closed directed $\ell$-walk, and by our choice of $\alpha \leq \alpha_{\text{Prop. 3.1}}$ in (13) and (14), Proposition 3.1 guarantees that $\vec{G}$ admits an induced subgraph $\vec{H} = \vec{G}[U]$ on $|U| = m \geq (1-\beta)n$ (cf. (12)) many vertices for which

$$\delta_0(\vec{H}) \overset{(10)}{\geq} \Big(\frac{\delta^+(\vec{G})}{n} - \beta\Big)m \overset{(7)}{\geq} \Big(\frac{1}{3} - \alpha - \beta\Big)m \overset{(14)}{\geq} \Big(\frac{1}{3} - 2\beta\Big)m \overset{(12)}{\geq} \Big(\frac{1}{3} - \beta_{\text{Prop. 3.2}}\Big)m.$$

Since $\vec{H}$ admits no closed directed $\ell$-walk, and by our choice of $\beta_{\text{Prop. 3.2}}$ in (11), Proposition 3.2 guarantees that $\vec{H}$ is $\lambda_0$-extremal. Let $U = V(\vec{H}) = U_0 \cup U_1 \cup U_2$ be any $\lambda_0$-extremal partition of $\vec{H}$ (cf. Definition 2.4), and let $V = V(\vec{G}) = V_0 \cup V_1 \cup V_2$ be any partition satisfying $U_i \subseteq V_i$ for each $0 \le i \le 2$. Then, for each $i \in \mathbb{Z}_3$,

$$e_{\vec{G}}(V_i, V_{i+1}) \ge e_{\vec{G}}(U_i, U_{i+1}) = e_{\vec{H}}(U_i, U_{i+1}) \\ \overset{\text{Prop. 3.2}}{\ge} \left(\tfrac{1}{9} - \lambda_0\right) m^2 \overset{\text{Prop. 3.1}}{\ge} \left(\tfrac{1}{9} - \lambda_0\right) (1-\beta)^2 n^2 \overset{(12)}{\ge} \left(\tfrac{1}{9} - \lambda\right) n^2,$$

where we also used $\lambda = 2\lambda_0$. Thus, $V = V_0 \cup V_1 \cup V_2$ is a $\lambda$-extremal partition of $\vec{G}$, as desired.

## 3.2. Proof of Proposition 3.1

Let $\beta > 0$ be given. Define

$$\alpha = \beta^6/(96). \tag{15}$$

Let integer $\ell \ge 4$ be given. Let $\vec{G} = (V, \vec{E})$ be an $n$-vertex oriented graph satisfying (7), where in all that follows, we take $n \ge n_0(\beta, \alpha, \ell)$ to be sufficiently large. Assume that $\vec{G}$ admits no closed directed $\ell$-walks. The subgraph $\vec{H} = \vec{G}[U]$ desired in (10) is induced on the following vertices of large in-degree:

$$U = V_{\text{high}} = \left\{ v \in V : \deg^-_{\vec{G}}(v) \ge \delta^+(\vec{G}) - n(\beta^2/2) \right\}. \tag{16}$$

To see that $\vec{H} = \vec{G}[V_{\text{high}}]$ satisfies (10), we use the following claim (whose proof we defer for a moment).

**Claim 3.3.** $\Delta^-(\vec{G}) \le \delta^+(\vec{G}) + n(\beta^3/4)$, *where* $\Delta^-(\vec{G})$ *denotes the maximum in-degree in* $\vec{G}$.

Using Claim 3.3, we will verify that $|U| = |V_{\text{high}}| = m \ge (1-\beta)n$. Indeed, with $V_{\text{low}} = V \setminus V_{\text{high}}$,

$$n\delta^+(\vec{G}) \le \sum_{u \in V} \deg^+_{\vec{G}}(u) = \sum_{v \in V} \deg^-_{\vec{G}}(v) = \sum_{w \in V_{\text{low}}} \deg^-_{\vec{G}}(v) + \sum_{x \in V_{\text{high}}} \deg^-_{\vec{G}}(w) \\ \overset{(16)}{<} |V_{\text{low}}| \left( \delta^+(\vec{G}) - \tfrac{1}{2}\beta^2 n \right) + |V_{\text{high}}| \Delta^-(\vec{G})$$

$$\overset{\text{Clm. 3.3}}{\leq} |V_{\text{low}}| \left(\delta^+(\vec{G}) - \tfrac{1}{2}\beta^2 n\right) + |V_{\text{high}}| \left(\delta^+(\vec{G}) + \tfrac{1}{4}\beta^3 n\right),$$

from which $2|V_{\text{low}}| \leq \beta|V_{\text{high}}| \leq \beta n$ and $|V_{\text{high}}| \geq (1-(\beta/2))n$ follow. By construction, both

$$\delta^+(\vec{H}) \geq \delta^+(\vec{G}) - |V_{\text{low}}| \geq \delta^+(\vec{G}) - \tfrac{1}{2}\beta n \geq \left(\tfrac{\delta^+(\vec{G})}{n} - \beta\right)n \geq \left(\tfrac{\delta^+(\vec{G})}{n} - \beta\right)m,$$

$$\text{and} \qquad \delta^-(\vec{H}) \geq \min\left\{ \deg^-_{\vec{G}}(v) : v \in V_{\text{high}} \right\} - |V_{\text{low}}|$$

$$\overset{(16)}{\geq} \delta^+(\vec{G}) - \tfrac{1}{2}\beta^2 n - \tfrac{1}{2}\beta n \geq \left(\tfrac{\delta^+(\vec{G})}{n} - \beta\right)m$$

hold, as promised in (10). Thus, it remains to prove Claim 3.3, where we will use the following fact.

**Fact 3.4.** *Let $R, S \subset V$ be some disjoint pair with sizes $|R| \geq \Delta^-(\vec{G})$ and $|S| \geq \delta^+(\vec{G})$, where $(S,R)$ admits no path $s \to v \to r$ in $\vec{G}$ with $s \in S$ and $r \in R$. Then $\Delta^-(\vec{G}) \leq \delta^+(\vec{G}) + n(\beta^3/4)$ (cf. Claim 3.3).*

*Proof of Fact 3.4.* Let $R, S \subset V$ be given as above. Fix $S_0 \subseteq S$ with $|S_0| = \delta^+(\vec{G})$ and set $S_1 = N^+_{\vec{G}}(S_0)$. Then $N^+_{\vec{G}}(S_1) \cap R = \emptyset$. Set $S_2 = N^+_{\vec{G}}(S_1) \setminus S_0$ so that $R$, $S_0$ and $S_2$ are pairwise disjoint. Thus

$$\Delta^-(\vec{G}) + \delta^+(\vec{G}) + |S_2| \leq |R| + |S_0| + |S_2| \leq n. \tag{17}$$

We double-count the number $e_{\vec{G}}(S_1, S_2)$ of edges from $S_1$ to $S_2$. On the one hand,

$$e_{\vec{G}}(S_1, S_2) \leq |S_2|\Delta^-(\vec{G}) \overset{(17)}{\leq} \Delta^-(\vec{G}) \left(n - \Delta^-(\vec{G}) - \delta^+(\vec{G})\right). \tag{18}$$

On the other hand,

$$\begin{aligned} e_{\vec{G}}(S_1, S_2) &\geq |S_1|\delta^+(\vec{G}) - e_{\vec{G}}(S_1, S_0) \\ &\geq |S_1|\delta^+(\vec{G}) - \left(|S_0||S_1| - e_{\vec{G}}(S_0, S_1)\right) \\ &= e_{\vec{G}}(S_0, S_1) \geq |S_0|\delta^+(\vec{G}) = \left(\delta^+(\vec{G})\right)^2, \end{aligned} \tag{19}$$

where we twice used that $|S_0| = \delta^+(\vec{G})$. Comparing (18) and (19), we infer

$$\left(\Delta^-(\vec{G})\right)^2 - \left(n - \delta^+(\vec{G})\right)\Delta^-(\vec{G}) + \left(\delta^+(\vec{G})\right)^2 \leq 0$$

$$\implies \qquad \Delta^-(\vec{G}) \le \frac{1}{2}\left(n - \delta^+(\vec{G}) + \sqrt{\left(n - 3\delta^+(\vec{G})\right)\left(n + \delta^+(\vec{G})\right)}\right)$$
$$\overset{(7)}{\le} \tfrac{1}{2}\left(n - \delta^+(\vec{G}) + n\sqrt{6\alpha}\right) = \delta^+(\vec{G}) + \tfrac{1}{2}\left(n - 3\delta^+(\vec{G}) + n\sqrt{6\alpha}\right)$$
$$\overset{(7)}{\le} \delta^+(\vec{G}) + \tfrac{1}{2}n\left(3\alpha + \sqrt{6\alpha}\right) \overset{(15)}{\le} \delta^+(\vec{G}) + n\sqrt{6\alpha},$$

and so our choice of $\alpha = \beta^6/(96)$ from (15) completes the proof of Fact 3.4. □

We now prove Claim 3.3.

**Proof of Claim 3.3** Assume, on the contrary, that

$$\Delta^-(\vec{G}) > \delta^+(\vec{G}) + \tfrac{1}{4}\beta^3 n. \tag{20}$$

Then Fact 3.4 ensures that

(21) *every disjoint pair $R, S \subset V$ with $|R| \ge \Delta^-(\vec{G})$ and $|S| \ge \delta^+(\vec{G})$ admits a directed path $s \to v \to r$ in $\vec{G}$ with $s \in S$ and $r \in R$.*

Fix $x_{\max} \in V$ satisfying $\deg^-_{\vec{G}}(x_{\max}) = \Delta^-(\vec{G})$. We distinguish several cases of $\ell \ge 4$.

**Case 1 ($\ell = 4$).** Set $R = N^-_{\vec{G}}(x_{\max})$ and $S = N^+_{\vec{G}}(x_{\max})$, which are disjoint and satisfy $|R| = \Delta^-(\vec{G})$ and $|S| \ge \delta^+(\vec{G})$. Then (21) guarantees a directed 4-cycle $(x_{\max}, s, v, r, x_{\max})$, which contradicts that $\vec{G}$ admits no closed directed 4-walks. In other words, (20) must be false when $\ell = 4$. □

**Case 2 ($\ell = 5$).** We use the following peculiar observation, proven in a moment:

(22) *if $x_{\max} \to y \to z \to a$ is a directed path in $\vec{G}$, then $(x_{\max}, a) \notin \vec{E}$, $(x_{\max}, z) \notin \vec{E}$, and $(y, a) \notin \vec{E}$.*

Using (22), $N^+_{\vec{G}}(x_{\max})$ is an independent set whose every fixed element $y \in N^+_{\vec{G}}(x_{\max})$ has an independent out-neighborhood $N^+_{\vec{G}}(y)$ which is disjoint from $N^+_{\vec{G}}(x_{\max})$. Thus, for $z \in N^+_{\vec{G}}(y)$ fixed, it must be that $N^+_{\vec{G}}(z) \cap$

$N^+_{\vec{G}}(x_{\max}) \neq \emptyset$ since otherwise $N^+_{\vec{G}}(x_{\max}) \cup N^+_{\vec{G}}(y) \cup N^+_{\vec{G}}(z) \subseteq V$ is a disjoint union with

$$\deg^+_{\vec{G}}(x_{\max}) + \deg^+_{\vec{G}}(y) + \deg^+_{\vec{G}}(z) \geq 3\delta^+(\vec{G}) \geq n+1 \qquad \text{(recall } \ell = 5 \text{ in (7))}.$$

On the other hand, $N^+_{\vec{G}}(z) \cap N^+_{\vec{G}}(x_{\max}) \neq \emptyset$ violates (22), and so (20) is false when $\ell = 5$.

To see (22), we first observe that

$$N^+_{\vec{G}}(a) \cap N^-_{\vec{G}}(x_{\max}) = \emptyset, \tag{23}$$

since $b \in N^+_{\vec{G}}(a) \cap N^-_{\vec{G}}(x_{\max})$ would give the directed 5-cycle $(x_{\max}, y, z, a, b, x_{\max})$, which would contradict that $\vec{G}$ admits no closed directed 5-walks. Now, (23) forbids $(x_{\max}, a) \in \vec{E}$, since otherwise we set $R = N^-_{\vec{G}}(x_{\max})$ and $S = N^+_{\vec{G}}(a)$ and use (21) to guarantee a directed 5-cycle $(x_{\max}, a, s, v, r, x_{\max})$. We next observe that

$$N^+_{\vec{G}}(a) \cap N^+_{\vec{G}}(x_{\max}) \neq \emptyset, \tag{24}$$

since otherwise (23) gives that $N^+_{\vec{G}}(a) \cup N^+_{\vec{G}}(x_{\max}) \cup N^-_{\vec{G}}(x_{\max}) \subseteq V$ is a disjoint union with

$$\deg^+_{\vec{G}}(a) + \deg^+_{\vec{G}}(x_{\max}) + \deg^-_{\vec{G}}(x_{\max}) \overset{(20)}{>} 3\delta^+(\vec{G}) \overset{(7)}{\geq} n+1.$$

Using (24), fix $b \in N^+_{\vec{G}}(a) \cap N^+_{\vec{G}}(x_{\max})$. Then

$$N^+_{\vec{G}}(b) \cap N^-_{\vec{G}}(x_{\max}) \neq \emptyset, \tag{25}$$

as otherwise we set $R = N^-_{\vec{G}}(x_{\max})$ and $S = N^+_{\vec{G}}(b)$ and use (21) to guarantee a directed 5-cycle $(x_{\max}, b, s, v, r, x_{\max})$. Using (25), we fix $c \in N^+_{\vec{G}}(b) \cap N^-_{\vec{G}}(x_{\max})$, which forbids $(x_{\max}, z) \in \vec{E}$ lest $(x_{\max}, z, a, b, c, x_{\max})$ is a directed 5-cycle. Similarly $(y, a) \notin \vec{E}$, which proves (22). □

**Case 3 ($\ell \geq 6$).** By the argument of Case 1, $x_{\max}$ belongs to a directed 4-cycle $\vec{C}_4$. We first observe that $x_{\max}$ does not belong to a directed 3-cycle $\vec{C}_3$. Indeed[1],

[1]This statement holds for all integers $\ell \geq 3$ outside of $\ell = 5$, and can be proven by inducting on $\ell = \lfloor \ell/2 \rfloor + \lceil \ell/2 \rceil$.

(26) *every integer* $\ell \geq 6$ *can be expressed as* $\ell = 3i + 4j$ *for some integers* $i, j \geq 0$,

and so the inclusion of $x_{\max}$ along both a directed 3-cycle $\vec{C}_3$ and a directed 4-cycle $\vec{C}_4$ would place $x_{\max}$ in a closed directed $\ell$-walk in $\vec{G}$, contradicting our hypothesis. We next observe that a longest directed path $\vec{P} = (y_1, \dots, y_k)$ in $N^+_{\vec{G}}(x_{\max})$ satisfies $k = \Omega(n)$. Indeed, $|N^+_{\vec{G}}(y_k) \cap N^+_{\vec{G}}(x_{\max})| \leq k - 2$ holds by the optimal length of $\vec{P}$, and so

$$\begin{aligned}(27)\quad & \left|N^+_{\vec{G}}(y_k) \cup N^+_{\vec{G}}(x_{\max})\right| \\ &= \deg^+_{\vec{G}}(y_k) + \deg^+_{\vec{G}}(x_{\max}) - \left|N^+_{\vec{G}}(y_k) \cap N^+_{\vec{G}}(x_{\max})\right| \geq 2\delta^+\big(\vec{G}\big) - k.\end{aligned}$$

Since $x_{\max}$ belongs to no directed 3-cycles $\vec{C}_3$,

$$\begin{aligned}(28)\quad & N^+_{\vec{G}}(y_k) \cap N^-_{\vec{G}}(x_{\max}) = \emptyset = N^+_{\vec{G}}(x_{\max}) \cap N^-_{\vec{G}}(x_{\max}) \\ &\implies N^+_{\vec{G}}(y_k) \cup N^+_{\vec{G}}(x_{\max}) \subseteq V \setminus N^-_{\vec{G}}(x_{\max}) \\ &\implies \left|N^+_{\vec{G}}(y_k) \cup N^+_{\vec{G}}(x_{\max})\right| \leq n - \deg^-_{\vec{G}}(x_{\max}) = n - \Delta^-\big(\vec{G}\big).\end{aligned}$$

Then $k = \Omega(n)$ follows comparing (27) and (28):

$$\begin{aligned}k \geq 2\delta^+\big(\vec{G}\big) + \Delta^-\big(\vec{G}\big) - n &\overset{(20)}{>} 3\delta^+\big(\vec{G}\big) + \tfrac{1}{4}\beta^3 n - n \\ &\overset{(7)}{\geq} n\left(\tfrac{1}{4}\beta^3 - 3\alpha\right) \overset{(15)}{=} n\left(\tfrac{1}{4}\beta^3 - \tfrac{1}{32}\beta^6\right) \geq \beta^3 n/8.\end{aligned}$$

To conclude Case 3, set $R = N^-_{\vec{G}}(x_{\max})$ and $S = N^+_{\vec{G}}(y_k)$, which we observed above are disjoint. Then (21) guarantees a path $(s, v, r)$ with $s \in S$ and $r \in R$, whence

$$\big(x_{\max}, y_{k-\ell+5}, y_{k-\ell+6}, \dots, y_k, s, v, r, x_{\max}\big)$$

is a closed directed $\ell$-walk, contradicting our hypothesis. In other words, (20) must be false when $\ell \geq 6$. which proves Claim 3.3. □

## 4. Proof of Proposition 3.2

In this section, we prove Proposition 3.2, where we will use several auxiliary facts. The first fact is taken from Corollary 1.5 in [6].

**Fact 4.1.** *Fix an integer $\ell \geq 4$. Let $\vec{G} = (V, \vec{E})$ be a large $n$-vertex oriented graph which contains no directed triangle, but which satisfies $\delta_0(\vec{G}) \geq (0.3025)n$. Then $\vec{G}$ admits a directed $\ell$-cycle $\vec{C}_\ell$.*

Our remaining facts are independent of the context of proving Proposition 3.2, and are therefore verified in Section 5.

**Fact 4.2.** *Fix an integer $\ell \geq 4$ and an $\varepsilon \in (0, 1/(11)]$. Let $\vec{G} = (V, \vec{E})$ be an oriented graph on $n \geq n_0(\ell, \varepsilon)$ many vertices which admits no closed directed $\ell$-walk, but which satisfies $\delta_0(\vec{G}) \geq ((1/3) - \varepsilon)n$. Let $(U_0, U_1)$ be a pair of subsets $U_0, U_1 \subseteq V$ satisfying the following conditions:*

*(i)* $|U_0|, |U_1| \geq \delta_0(\vec{G})$;
*(ii)* $|U_0 \cap U_1| \leq ((1/3) - 21\varepsilon)n$;
*(iii)* *$\vec{G}$ admits no directed paths $u_0 \to v \to u_1$, where $u_0 \in U_0$ and $u_1 \in U_1$.*

*Then, there exist independent sets $I_0 \subseteq U_0 \setminus U_1$ and $I_1 \subseteq U_1 \setminus U_0$ with sizes*

$$|I_0| \geq |U_0 \setminus U_1| - 7\varepsilon n \geq 20\varepsilon n \quad \text{and} \quad |I_1| \geq |U_1 \setminus U_0| - 7\varepsilon n \geq 20\varepsilon n. \tag{29}$$

**Remark 4.3.** In many applications of Fact 4.2, the pair $(U_0, U_1)$ will satisfy $U_0 \cap U_1 = \emptyset$. □

**Fact 4.4.** *Fix an integer $\ell \geq 4$ and an $\varepsilon \in (0, 1/(54))$. Let $\vec{G} = (V, \vec{E})$ be an oriented graph on $n \geq n_0(\ell, \varepsilon)$ many vertices which admits no closed directed $\ell$-walk, but which satisfies $\delta_0(\vec{G}) \geq ((1/3) - \varepsilon)n$. Let $(x, y, z, x)$ be a directed 3-cycle $\vec{C}_3$ in $\vec{G}$, and assume that neither $x$ nor $y$ belongs to a directed 4-cycle $\vec{C}_4$. Then, $|N^-_{\vec{G}}(x) \cap N^+_{\vec{G}}(y)| \geq ((1/3) - 18\varepsilon)n$.*

We now prove Proposition 3.2, and distinguish whether or not $\ell = 5$.

### 4.1. Proof of Proposition 3.2 when $\ell \neq 5$

Fix $\lambda_0 > 0$. Define the promised constant

$$\beta = \beta(\lambda_0) = \min\left\{\tfrac{1}{21}\lambda_0,\ \tfrac{1}{3} - 0.3025,\ \tfrac{1}{55}\right\}. \tag{30}$$

Fix an integer $\ell \geq 4$, where $\ell \neq 5$. Let $\vec{H} = (V, \vec{E})$ be an $m$-vertex oriented graph, where $m \geq m_0(\lambda_0, \beta, \ell)$ is assumed to be sufficiently large whenever needed. Assume that $\vec{H}$ admits no closed directed $\ell$-walk but satisfies $\delta_0(\vec{H}) \geq ((1/3) - \beta)m$. We prove that $\vec{H}$ is $\lambda_0$-extremal.

The central observation of the proof is that $\vec{H}$ admits directed triangles, since otherwise with

$$\delta_0(\vec{H}) \geq \left(\tfrac{1}{3} - \beta\right) m \overset{(30)}{\geq} (0.3025)m$$

Fact 4.1 would guarantee a directed $\ell$-cycle $\vec{C}_\ell$ in $\vec{H}$, contradicting our hypothesis. Thus, fix a directed 3-cycle $(v_0, v_1, v_2, v_0)$ in $\vec{H}$. Our observation in (26) guarantees that no vertex $v_i \in \{v_0, v_1, v_2\}$ can belong to a directed 4-cycle $\vec{C}_4$ lest $\vec{H}$ admits a closed directed $\ell$-walk. For fixed $i \in \mathbb{Z}_3$, we define

$$U_i = N^-_{\vec{H}}(v_i) \cap N^+_{\vec{H}}(v_{i+1}). \tag{31}$$

Then $U_0$, $U_1$, and $U_2$ are pairwise disjoint because $\vec{H}$ is an oriented graph. By our choice of $\beta < 1/(54)$ in (30), and by no $v_j \in \{v_0, v_1, v_2\}$ belonging to a directed 4-cycle $\vec{C}_4$, Fact 4.4 guarantees that

$$|U_i| = \left|N^-_{\vec{H}}(v_i) \cap N^+_{\vec{H}}(v_{i+1})\right| \geq \left(\tfrac{1}{3} - 18\beta\right) n. \tag{32}$$

We claim that each $u_i \in U_i$ satisfies

$$\left|U_{i+1} \cap N^+_{\vec{H}}(u_i)\right| \geq \left(\tfrac{1}{3} - 45\beta\right) n. \tag{33}$$

If true, any partition $V = V_0 \cup V_1 \cup V_2$, where $U_j \subseteq V_j$ for each $j \in \mathbb{Z}_3$, is $\lambda_0$-extremal since

$$\begin{aligned} e_{\vec{H}}(V_i, V_{i+1}) \geq e_{\vec{H}}(U_i, U_{i+1}) &= \sum_{u_i \in U_i} \left|U_{i+1} \cap N^+_{\vec{H}}(u_i)\right| \overset{(33)}{\geq} |U_i| \left(\tfrac{1}{3} - 45\beta\right) n \\ &\overset{(32)}{\geq} \left(\tfrac{1}{3} - 45\beta\right)\left(\tfrac{1}{3} - 18\beta\right) n^2 \geq \left(\tfrac{1}{9} - 21\beta\right) n^2 \overset{(30)}{\geq} \left(\tfrac{1}{9} - \lambda_0\right) n^2. \end{aligned}$$

To prove (33), fix $i \in \mathbb{Z}_3$, and w.l.o.g. assume $i = 0$. Fix $u_0 \in U_0 = N^-_{\vec{H}}(v_0) \cap N^+_{\vec{H}}(v_1)$. Then $(v_1, u_0, v_0, v_1)$ is a directed 3-cycle $\vec{C}_3$, and so (26) gives that $u_0$ can belong to no directed 4-cycle $\vec{C}_4$. As such, Fact 4.4 (applied to $(v_1, u_0, v_0, v_1)$) guarantees that

$$\left|N^-_{\vec{H}}(v_1) \cap N^+_{\vec{H}}(u_0)\right| \geq \left(\tfrac{1}{3} - 18\beta\right) n, \tag{34}$$

which isn't yet (33), but it will be very close. With an error we can control, we shall 'replace' $N^-_{\vec{H}}(v_1)$ in (34) with $U_1 \subseteq N^-_{\vec{H}}(v_1)$ from (31). We claim this error will be small if

$$\deg^-_{\vec{H}}(v_1) \le \left(\tfrac{1}{3} + 9\beta\right)n. \tag{35}$$

Indeed, if (35) holds, then we would have

$$\begin{aligned} \left|N^-_{\vec{H}}(v_1) \setminus U_1\right| &\overset{(31)}{=} \deg^-_{\vec{H}}(v_1) - |U_1| \\ &\overset{(32)}{\le} \deg^-_{\vec{H}}(v_1) - \left(\tfrac{1}{3} - 18\beta\right) n \overset{(35)}{\le} 27\beta n, \end{aligned} \tag{36}$$

and so comparing (34) with (36) yields

$$\left|U_1 \cap N^+_{\vec{H}}(u_0)\right| + 27\beta n \overset{(36)}{\ge} \left|N^+_{\vec{H}}(u_0) \cap N^-_{\vec{H}}(v_1)\right| \overset{(34)}{\ge} \left(\tfrac{1}{3} - 18\beta\right) n,$$

which gives (33). It thus remains to prove that (35) holds.

To prove (35), we will apply Fact 4.2 to the pair $(N^-_{\vec{H}}(v_1), N^+_{\vec{H}}(v_1))$. Note that the hypotheses (i)–(iii) of Fact 4.2 are met by $(N^-_{\vec{H}}(v_1), N^+_{\vec{H}}(v_1))$ since $|N^-_{\vec{H}}(v_1)|$, $|N^+_{\vec{H}}(v_1)| \ge \delta_0(\vec{H})$, since $|N^-_{\vec{H}}(v_1) \cap N^+_{\vec{H}}(v_1)| = 0$, and since there are no paths $u^+ \to v \to u^-$ with $u^+ \in N^+_{\vec{H}}(v_1)$ and $u^- \in N^-_{\vec{H}}(v_1)$ lest $(v_1, u^+, v, u^-, v_1)$ is a directed 4-cycle containing $v_1$. Fact 4.2 guarantees an independent set $I_{v_1} \subseteq N^-_{\vec{H}}(v_1)$ of size

$$\begin{aligned} |I_{v_1}| &\ge \deg^-_{\vec{H}}(v_1) - 7\beta n \ge \delta_0\big(\vec{H}\big) - 7\beta n \\ &\ge \left(\tfrac{1}{3} - \beta\right) n - 7\beta n \ge \left(\tfrac{1}{3} - 8\beta\right) n \overset{(30)}{>} 0, \end{aligned} \tag{37}$$

so fix $w_1 \in I_{v_1}$. Now, $N^+_{\vec{H}}(w_1) \cup N^-_{\vec{H}}(w_1) \cup I_{v_1} \subseteq V$ is a pairwise disjoint union, in which case

$$\begin{aligned} n \ge \deg^+_{\vec{H}}(w_1) + \deg^-_{\vec{H}}(w_1) + |I_{v_1}| &\overset{(37)}{\ge} \deg^+_{\vec{H}}(w_1) + \deg^-_{\vec{H}}(w_1) + \deg^-_{\vec{H}}(v_1) - 7\beta n \\ &\ge 2\delta_0\big(\vec{H}\big) + \deg^-_{\vec{H}}(v_1) - 7\beta n \\ &\ge 2\left(\tfrac{1}{3} - \beta\right) n + \deg^-_{\vec{H}}(v_1) - 7\beta n = \deg^-_{\vec{H}}(v_1) + \left(\tfrac{2}{3} - 9\beta\right) n, \end{aligned}$$

from which (35) now follows.

### 4.2. Proof of Proposition 3.2 when $\ell = 5$

To prove Proposition 3.2 when $\ell = 5$, we use Facts 4.1–4.4 together with the following two additional facts (which are also proven in Section 5).

**Fact 4.5.** *Fix an integer $\ell \geq 4$ and an $\varepsilon > 0$. Let $\vec{G} = (V, \vec{E})$ be an oriented graph on $n \geq n_0(\ell, \varepsilon)$ many vertices which admits no closed directed $\ell$-walk. Then, $\delta^+(\vec{G}) \leq ((1/3) + \varepsilon)n$ and $\delta^-(\vec{G}) \leq ((1/3) + \varepsilon)n$.*

**Fact 4.6.** *For all $\lambda > 0$, there exists $\varepsilon = \varepsilon(\lambda) > 0$ so that every oriented graph $\vec{G} = (V, \vec{E})$ on $n \geq n_0(\lambda, \varepsilon)$ many vertices with $\delta_0(\vec{G}) \geq ((1/3) - \varepsilon)n$ will be $\lambda$-extremal, provided $\vec{G}$ has either:*

1. *a partition $V = V_0 \cup V_1 \cup V_2$ with $|V_1|, |V_2| \geq ((1/3) - \varepsilon)n$ and $e_{\vec{G}}(V_1)$, $e_{\vec{G}}(V_2), e_{\vec{G}}(V_2, V_1) \leq \varepsilon n^2$,*
2. *or no transitive triangles.*

Now, let $\lambda_0 > 0$ be given. Let

$$\varepsilon_{\text{Fct. 4.6}} = \varepsilon_{\text{Fct. 4.6}}(\lambda = \lambda_0) > 0 \tag{38}$$

be the constant guaranteed by Fact 4.6. We define the promised constant

$$\beta = \tfrac{1}{109}\varepsilon_{\text{Fct. 4.6}}. \tag{39}$$

Let $\vec{H} = (V, \vec{E})$ be an $m$-vertex oriented graph, where in all that follows we assume $m \geq m_0(\lambda_0, \varepsilon_{\text{Fct. 4.6}}, \beta)$ is sufficiently large. Assume that $\vec{H}$ admits no closed directed 5-walks, i.e., directed 5-cycles $\vec{C}_5$, but which satisfies $\delta_0(\vec{H}) \geq ((1/3) - \beta)m$. We prove that $\vec{H}$ is $\lambda_0$-extremal.

For sake of argument, we assume that $\vec{H}$ admits some transitive triangles, as otherwise by our choice of $\beta$ and $\varepsilon_{\text{Fct. 4.6}}$ in (38) and (39), Conclusion (2) of Fact 4.6 would give that $\vec{H}$ is $\lambda_0$-extremal. For the remainder of the proof, we fix a transitive triangle $(x, y), (x, z), (y, z) \in \vec{E}$. Let $I = I_{x,z} = N^-_{\vec{H}}(x) \cap N^+_{\vec{H}}(z)$, which is an independent set lest $(a, b) \in \vec{E} \cap (I \times I)$ gives the directed 5-cycle $(x, y, z, a, b, x)$. Our first main observation is that $I$ is 'large'.

**Claim 4.7.**

$$|I| \geq \left(\tfrac{1}{3} - 21\beta\right) n. \tag{40}$$

*Proof of Claim 4.7.* Assume for contradiction that (40) fails to hold. We will apply Fact 4.2 to the pair $(N^-_{\vec{H}}(x), N^+_{\vec{H}}(z))$. Note that the hypotheses (i)–(iii) are met by $(N^-_{\vec{H}}(x), N^+_{\vec{H}}(z))$ since $|N^-_{\vec{H}}(x)|, |N^+_{\vec{H}}(z))| \geq \delta_0(\vec{H})$, since $|N^-_{\vec{H}}(x) \cap N^+_{\vec{H}}(z))| \leq ((1/3) - 21\beta)n$ on account that (40) failed, and since there are no paths $u^+ \to v \to u^-$ with $u^+ \in N^+_{\vec{H}}(z)$ and $u^- \in N^-_{\vec{H}}(x)$ lest $(x, z, u^+, v, u^-, x)$ is a directed 5-cycle $\vec{C}_5$. Fact 4.2 guarantees disjoint independent sets $I_x \subseteq N^-_{\vec{H}}(x) \setminus N^+_{\vec{H}}(z)$ and $I_z \subseteq N^+_{\vec{H}}(z) \setminus N^-_{\vec{H}}(x)$ (disjoint also from $I$) with sizes

$$\begin{aligned} |I_x| &\geq \left|N^-_{\vec{H}}(x) \setminus N^+_{\vec{H}}(z)\right| - 7\beta n = \deg^-_{\vec{H}}(x) - |I| - 7\beta n \geq 20\beta n \quad \text{and} \\ |I_z| &\geq \left|N^+_{\vec{H}}(z) \setminus N^-_{\vec{H}}(x)\right| - 7\beta n = \deg^+_{\vec{H}}(z) - |I| - 7\beta n \geq 20\beta n. \end{aligned} \tag{41}$$

Fix $a_x \in I_x$ and $b_z \in I_z$. One may check that

$$\begin{aligned} N^-_{\vec{H}}(a_x) \cap N^+_{\vec{H}}(b_z) &= N^-_{\vec{H}}(a_x) \cap I = N^+_{\vec{H}}(b_z) \cap I \\ &= N^-_{\vec{H}}(a_x) \cap I_z = N^+_{\vec{H}}(b_z) \cap I_x = \emptyset. \end{aligned}$$

Thus, together with the independence of $I$, $I_x$, and $I_z$, we have that $I \cup I_x \cup I_y \cup N^-(a_x) \cup N^+(b_z) \subseteq V$ is a pairwise disjoint union, and so

$$\begin{aligned} n &\geq |I| + |I_x| + |I_z| + \deg^-_{\vec{H}}(a_x) + \deg^+_{\vec{H}}(b_z) \\ &\overset{(41)}{\geq} \deg^-_{\vec{H}}(x) + \deg^+_{\vec{H}}(z) - |I| + \deg^-_{\vec{H}}(a_x) + \deg^+_{\vec{H}}(b_z) - 14\beta n \\ &\geq 4\delta_0(\vec{H}) - |I| - 14\beta n \\ &\overset{\text{hyp}}{\geq} 4\left(\tfrac{1}{3} - \beta\right)n - |I| - 14\beta n = n - |I| + \left(\tfrac{1}{3} - 18\beta\right)n, \end{aligned}$$

from which $|I| \geq ((1/3) - 18\beta)n$ follows, and contradicts our assumption that (40) failed to hold. □

Continuing the proof of Proposition 3.2, we attempt to meet Condition (1) of Fact 4.6 to $\vec{H}$ with $V_1 = I$ and with $V_2$ which we now define. For the remainder of the proof, fix $v \in I$ and take $V_2 = I_v$ to be an independent set which is as large as possible subject to either $I_v \subseteq N^+_{\vec{H}}(v)$ or $I_v \subseteq N^-_{\vec{H}}(v)$. In the former case, $0 = e_{\vec{H}}(V_1) = e_{\vec{H}}(V_2) = e_{\vec{H}}(V_2, V_1)$ since each of $V_1 = I$ and $V_2 = I_v$ is independent and since

$$a \in N^+_{\vec{H}}(v) \text{ forbids } b \in N^+_{\vec{H}}(a) \cap I, \tag{42}$$

lest $(x, z, v, a, b, x)$ is a directed 5-cycle $\vec{C}_5$ in $\vec{H}$. In the latter case, $0 = e_{\vec{H}}(V_1) = e_{\vec{H}}(V_2) = e_{\vec{H}}(V_1, V_2)$, where the last equality holds by $a \in N^-_{\vec{G}}(v)$ forbidding $b \in N^-_{\vec{H}}(a) \cap I$ lest $(x, z, b, a, v, x)$ is a directed 5-cycle $\vec{C}_5$. In either case, we make the following claim.

**Claim 4.8.**

$$|V_2| = \big|I_v\big| \geq \big(\tfrac{1}{3} - 24\beta\big)\, n. \tag{43}$$

If Claim 4.8 holds, then together with (40) and the considerations above, the partition $V = V_0 \cup V_1 \cup V_2$, where $V_0 = V \setminus (V_1 \cup V_2)$, meets the hypotheses of Fact 4.6. By our choice of $\varepsilon_{\text{Fct. 4.6}}$ and $\beta$ in (38) and (39), Fact 4.6 guarantees that $V = V_0 \cup V_1 \cup V_2$ is a $\lambda_0$-extremal partition of $\vec{H}$. Thus, the proof of Proposition 3.2 when $\ell = 5$ will be complete upon proving Claim 4.8.

*Proof of Claim 4.8.* Assume for contradiction that the $\vec{H}$-subgraphs $\vec{H}[N^+_{\vec{H}}(v)]$ and $\vec{H}[N^-_{\vec{H}}(v)]$ induced respectively on $N^+_{\vec{H}}(v)$ and $N^-_{\vec{H}}(v)$ satisfy

$$\alpha\big(\vec{H}\big[N^+_{\vec{H}}(v)\big]\big) < \big(\tfrac{1}{3} - 24\beta\big)\, n \quad \text{and} \quad \alpha\big(\vec{H}\big[N^-_{\vec{H}}(v)\big]\big) < \big(\tfrac{1}{3} - 24\beta\big)\, n, \tag{44}$$

where $\alpha(\cdot)$ denotes the independence number. Since

$$\big|N^+_{\vec{H}}(v)\big| = \deg^+_{\vec{H}}(v) \geq \delta_0\big(\vec{H}\big) \geq \big(\tfrac{1}{3} - \beta\big)\, n > \big(\tfrac{1}{3} - 24\beta\big)\, n > \alpha\big(\vec{H}\big[N^+_{\vec{H}}(v)\big]\big),$$

$\vec{H}[N^+_{\vec{H}}(v)]$ admits edges $(a,b) \in \vec{E}$. We fix one such and will observe that

$$N^+_{\vec{H}}(b) \cap N^+_{\vec{H}}(v) \neq \emptyset. \tag{45}$$

Indeed, assuming otherwise the set $N^+_{\vec{H}}(b) \cap N^-_{\vec{H}}(v)$ satisfies

$$\big|N^+_{\vec{H}}(b) \cap N^-_{\vec{H}}(v)\big| = \big|N^+_{\vec{H}}(b)\big| + \big|N^-_{\vec{H}}(v)\big| - \big|N^+_{\vec{H}}(b) \cup N^-_{\vec{H}}(v)\big|. \tag{46}$$

From $\emptyset = N^+_{\vec{H}}(b) \cap N^+_{\vec{H}}(v) = N^+_{\vec{H}}(b) \cap I$ (cf. (42) and (45)) we infer $N^+_{\vec{H}}(b) \subseteq V \setminus (N^+_{\vec{H}}(v) \cup I)$, and from $\emptyset = N^-_{\vec{H}}(v) \cap N^+_{\vec{H}}(v) = N^-_{\vec{H}}(v) \cap I$ (recall that $I$ is independent and $v \in I$) we infer $N^+_{\vec{H}}(b) \cup N^-_{\vec{H}}(v) \subseteq V \setminus (N^+_{\vec{H}}(v) \cup I)$, where $N^+_{\vec{H}}(v) \cup I$ is a disjoint union by the independence of $I$ and because $v \in I$. Thus,

$$\begin{aligned}
&\big|N^+_{\vec{H}}(b) \cap N^-_{\vec{H}}(v)\big| \overset{(46)}{\geq} \deg^+_{\vec{H}}(b) + \deg^-_{\vec{H}}(v) - \big(n - \deg^+_{\vec{H}}(v) - |I|\big) \\
&\geq 3\delta_0\big(\vec{H}\big) - n + |I| \geq 3\,\big(\tfrac{1}{3} - \beta\big)\, n - n + |I| \overset{(40)}{\geq} \big(\tfrac{1}{3} - 24\beta\big)\, n \overset{(44)}{>} \alpha\big(\vec{H}\big[N^-_{\vec{H}}(v)\big]\big).
\end{aligned}$$

Consequently, there exists an edge $(c,d) \in \vec{E}$ with $c, d \in N^+_{\vec{H}}(b) \cap N^-_{\vec{H}}(v)$, in which case $(v,a,b,c,d,v)$ is a directed 5-cycle $\vec{C}_5$ in $\vec{H}$. This proves (45).

Now, define

$$C = \left\{ c \in N^+_{\vec{H}}(v) : \exists \text{ a directed path on 3 vertices contained in } \vec{H}[N^+_{\vec{H}}(v)] \text{ that ends in } c \right\}. \tag{47}$$

Note that (45) implies that $C$ is non-empty. By this definition, every element $c \in C$ satisfies

$$N^+_{\vec{H}}(c) \cap N^+_{\vec{H}}(v) = N^+_{\vec{H}}(c) \cap C. \tag{48}$$

Since the $\vec{H}$-subgraph $\vec{H}[C]$ induced on $C$ admits no directed 5-cycles $\vec{C}_5$, Fact 4.5 guarantees the existence of a vertex $c_0 \in C \subseteq N^+_{\vec{H}}(v)$ (cf. (47)) so that

$$\begin{aligned} \left|N^+_{\vec{H}}(c_0) \cap N^+_{\vec{H}}(v)\right| &\stackrel{(48)}{=} \left|N^+_{\vec{H}}(c_0) \cap C\right| \\ &\leq \left\{ \begin{array}{cc} \left(\frac{1}{3} + \beta\right) |C| & \text{if } |C| = \Omega(1) \\ |C| & \text{else} \end{array} \right\} \leq \left(\frac{1}{3} + \beta\right) \left|N^+_{\vec{H}}(v)\right|. \end{aligned} \tag{49}$$

Consider now $N^+_{\vec{H}}(c_0) \cap N^-_{\vec{H}}(v) = (N^+_{\vec{H}}(c_0) \setminus C) \cap N^-_{\vec{H}}(v)$, where $C \subseteq N^+_{\vec{H}}(v)$ from (47) but where $N^+_{\vec{H}}(v) \cap N^-_{\vec{H}}(v) = \emptyset$. Then

$$\begin{aligned} \left|N^+_{\vec{H}}(c_0) \cap N^-_{\vec{H}}(v)\right| &= \left|\left(N^+_{\vec{H}}(c_0) \setminus C\right) \cap N^-_{\vec{H}}(v)\right| \\ &= \left|\left(N^+_{\vec{H}}(c_0) \setminus C\right)\right| + \left|N^-_{\vec{H}}(v)\right| - \left|\left(N^+_{\vec{H}}(c_0) \setminus C\right) \cup N^-_{\vec{H}}(v)\right| \\ &\stackrel{(49)}{\geq} \deg^+_{\vec{H}}(c_0) - \left(\tfrac{1}{3} + \beta\right) \deg^+_{\vec{H}}(v) + \deg^-_{\vec{H}}(v) - \left|\left(N^+_{\vec{H}}(c_0) \setminus C\right) \cup N^-_{\vec{H}}(v)\right| \\ &\stackrel{(48)}{=} \deg^+_{\vec{H}}(c_0) - \left(\tfrac{1}{3} + \beta\right) \deg^+_{\vec{H}}(v) + \deg^-_{\vec{H}}(v) - \left|\left(N^+_{\vec{H}}(c_0) \setminus N^+_{\vec{H}}(v)\right) \cup N^-_{\vec{H}}(v)\right|. \end{aligned}$$

Using (42) and the independence of $I$, the last union resides in $V \setminus (N^+_{\vec{H}}(v) \cup I)$, and so

$$\begin{aligned} &\left|N^+_{\vec{H}}(c_0) \cap N^-_{\vec{H}}(v)\right| \\ &\geq \deg^+_{\vec{H}}(c_0) - \left(\tfrac{1}{3} + \beta\right) \deg^+_{\vec{H}}(v) + \deg^-_{\vec{H}}(v) - \left(n - \deg^+_{\vec{H}}(v) - |I|\right) \\ &= \deg^+_{\vec{H}}(c_0) + \deg^-_{\vec{H}}(v) + \left(\tfrac{2}{3} - \beta\right) \deg^+_{\vec{H}}(v) + |I| - n \\ &\geq \left(\tfrac{8}{3} - \beta\right) \delta_0(\vec{H}) + |I| - n \geq \left(\tfrac{8}{3} - \beta\right) \left(\tfrac{1}{3} - \beta\right) n + |I| - n \\ &\stackrel{(40)}{\geq} \left(\tfrac{8}{3} - \beta\right) \left(\tfrac{1}{3} - \beta\right) n + \left(\tfrac{1}{3} - 21\beta\right) n - n \geq \left(\tfrac{2}{9} - 24\beta\right) n, \end{aligned} \tag{50}$$

which is positive by (39). Now, (47) and (50) render a directed path $a \to b \to c_0 \to d$ where $a, b, c_0 \in N^+_{\vec{H}}(v)$ and $d \in N^+_{\vec{H}}(c_0) \cap N^-_{\vec{H}}(v)$, in which case $(v, a, b, c_0, d, v)$ is a directed 5-cycle in $\vec{H}$. Thus, our assumption in (44) is incorrect, which completes the proof of Claim 4.8. $\square$

## 5. Proofs of Facts 4.2–4.6

The easiest proof here is that of Fact 4.5, which we give immediately. Fix an integer $\ell \geq 4$ and fix $\varepsilon > 0$. Let $\vec{G} = (V, \vec{E})$ be an oriented graph on $n \geq n_0(\ell, \varepsilon)$ many vertices which admits no closed directed $\ell$-walk. The latter conclusion of Fact 4.5 follows from the former by reversing the orientations on $\vec{E}$. If the former fails, then Proposition 3.1 ensures a large $m$-vertex subgraph $\vec{H} \subseteq \vec{G}$ satisfying

$$\delta_0(\vec{H}) \geq \left(\tfrac{\delta^+(\vec{G})}{n} - \tfrac{\varepsilon}{2}\right)m \geq \left(\tfrac{1}{3} + \tfrac{\varepsilon}{2}\right)m \geq \tfrac{m+1}{3},$$

and so Theorem 1.5 guarantees a directed $\ell$-cycle $\vec{C}_\ell$ in $\vec{H}$, and hence in $\vec{G}$.

### Proof of Fact 4.2

Fix an integer $\ell \geq 4$ and an $\varepsilon \in (0, 1/(11)]$. Let $\vec{G} = (V, \vec{E})$ be an oriented graph on $n \geq n_0(\ell, \varepsilon)$ many vertices which admits no closed directed $\ell$-walk, but which satisfies $\delta_0(\vec{G}) \geq ((1/3) - \varepsilon)n$. Let $(U_0, U_1)$ be a pair of subsets satisfying (i)–(iii) in the hypotheses of Fact 4.2. We prove that there exist independent sets $I_0 \subseteq U_0 \setminus U_1$ and $I_1 \subseteq U_1 \setminus U_0$ satisfying (29). To that end, define

$$T_0 = \left\{u_0 \in U_0 \setminus U_1 : N^-_{\vec{G}}(u_0) \cap (U_0 \setminus U_1) \neq \emptyset\right\} \quad \text{and}$$
$$T_1 = \left\{u_1 \in U_1 \setminus U_0 : N^+_{\vec{G}}(u_1) \cap (U_1 \setminus U_0) \neq \emptyset\right\}.$$

For fixed $j \in \mathbb{Z}_2$, the set $I_j = U_j \setminus (U_{j+1} \cup T_j)$ is independent and of size $|I_j| = |U_j \setminus U_{j+1}| - |T_j|$, so to prove (29) we will prove $|T_j| \leq 7\varepsilon n$. In particular, our argument will show that $|T_0| \geq \varepsilon n$ and $|T_1| \geq \varepsilon n$ can't both hold, and that $|T_j| < \varepsilon n$ implies $|T_{j+1}| \leq 7\varepsilon n$. It remains to verify these details.

Write $U = U_0 \cup U_1$, and define

$$(51) \quad S_0 = N^+(T_0) \setminus U \quad \text{and} \quad S_1 = N^-(T_1) \setminus U, \qquad \text{where} \quad S_0 \cap S_1 \overset{\text{(iii)}}{=} \emptyset.$$

For $j \in \mathbb{Z}_2$, we will verify the implications

$$|T_j| \geq \varepsilon n \quad \Longrightarrow \quad |S_j| \geq \delta_0(\vec{G}) - \left(\tfrac{1}{3}+\varepsilon\right)|T_j| \quad \Longrightarrow \quad |T_{j+1}| < \varepsilon n. \tag{52}$$

Indeed, $\vec{G}[T_j]$ is a large oriented graph with no closed directed $\ell$-walks, so Fact 4.5 guarantees $t_j \in T_j$:

$$|N^+_{\vec{G}}(t_0) \cap T_0| \leq \left(\tfrac{1}{3}+\varepsilon\right)|T_0| \quad \text{and} \quad |N^-_{\vec{G}}(t_1) \cap T_1| \leq \left(\tfrac{1}{3}+\varepsilon\right)|T_1|. \tag{53}$$

By definition of $T_j$, there exist $u_0 \in U_0 \setminus U_1$ and $u_1 \in U_1 \setminus U_0$ with $(u_0, t_0), (t_1, u_1) \in \vec{E}$, where $N^+(t_0) \cap T_0 = N^+(t_0) \cap (U_0 \setminus U_1)$ and $N^-(t_1) \cap T_1 = N^-(t_1) \cap (U_1 \setminus U_0)$ hold. Moreover, $u \in N^+_{\vec{G}}(t_0) \cap U_1$ is impossible lest $(u_0, t_0, u)$ violates (iii), and $N^-_{\vec{G}}(t_1) \cap U_0 \neq \emptyset$ is similarly impossible. Altogether,

$$\begin{aligned}
&\deg^+_{\vec{G}}(t_0) = \left|N^+_{\vec{G}}(t_0) \cap U\right| + \left|N^+_{\vec{G}}(t_0) \setminus U\right| \\
&= \left|N^+_{\vec{G}}(t_0) \cap (U_0 \setminus U_1)\right| + \left|N^+_{\vec{G}}(t_0) \setminus U\right| = \left|N^+_{\vec{G}}(t_0) \cap T_0\right| + \left|N^+_{\vec{G}}(t_0) \setminus U\right| \\
&\qquad\qquad \overset{(53)}{\leq} \left(\tfrac{1}{3}+\varepsilon\right)|T_0| + \left|N^+_{\vec{G}}(t_0) \setminus U\right| \overset{(51)}{\leq} \left(\tfrac{1}{3}+\varepsilon\right)|T_0| + |S_0|,
\end{aligned}$$

and so the former implication of (52) holds with $j = 0$. Similarly,

$$\begin{aligned}
\deg^-_{\vec{G}}(t_1) &= \left|N^-_{\vec{G}}(t_1) \cap U\right| + \left|N^-_{\vec{G}}(t_1) \setminus U\right| \\
&\leq \left|N^-_{\vec{G}}(t_1) \cap T_1\right| + |S_1| \leq \left(\tfrac{1}{3}+\varepsilon\right)|T_1| + |S_1|,
\end{aligned}$$

and so the former implication of (52) holds with $j = 1$. Finally, if both $|T_0|, |T_1| \geq \varepsilon n$, then

$$\begin{aligned}
n &\overset{(51)}{\geq} |U| + |S_0| + |S_1| \\
&\overset{(52)}{\geq} |U| + 2\delta_0(\vec{G}) - \left(\tfrac{1}{3}+\varepsilon\right)(|T_0| + |T_1|) \\
&\geq |U| + 2\delta_0(\vec{G}) - \left(\tfrac{1}{3}+\varepsilon\right)|U_0 \triangle U_1| \\
&= \left(\tfrac{2}{3}-\varepsilon\right)(|U_0| + |U_1|) + 2\delta_0(\vec{G}) - \left(\tfrac{1}{3}-2\varepsilon\right)|U_0 \cap U_1| \\
&\overset{\text{(i)}}{\geq} 2\delta_0(\vec{G})\left(\tfrac{5}{3}-\varepsilon\right) - \left(\tfrac{1}{3}-2\varepsilon\right)|U_0 \cap U_1| \\
&\overset{\text{(ii)}}{\geq} 2\delta_0(\vec{G})\left(\tfrac{5}{3}-\varepsilon\right) - \left(\tfrac{1}{3}-2\varepsilon\right)\left(\tfrac{1}{3}-21\varepsilon\right)n \\
&\geq 2\delta_0(\vec{G})\left(\tfrac{5}{3}-\varepsilon\right) - \left(\tfrac{1}{9}-\tfrac{23}{3}\varepsilon+42\varepsilon^2\right)n \\
&\geq 2\left(\tfrac{1}{3}-\varepsilon\right)\left(\tfrac{5}{3}-\varepsilon\right)n - \left(\tfrac{1}{9}-\tfrac{23}{3}\varepsilon+42\varepsilon^2\right)n,
\end{aligned}$$

from which $\varepsilon \geq 11/(120)$ follows and contradicts the hypothesis $\varepsilon \leq 1/(11)$. This proves (52).

By (52), it suffices to assume for fixed $j \in \mathbb{Z}_2$ that $|T_j| \geq \varepsilon n$, and then to prove that $|T_j| \leq 7\varepsilon n$. To that end, we find a vertex $z_{j+1} \in U_{j+1} \setminus T_{j+1}$ where,

($a$) when $j = 0$, the vertex $z_1 \in U_1 \setminus T_1$ has no in-neighbors from $U_1 \setminus T_1$;
($b$) when $j = 1$, the vertex $z_0 \in U_0 \setminus T_0$ has no out-neighbors in $U_0 \setminus T_0$.

We start by fixing $v_{j+1} \in I_{j+1} = U_{j+1} \setminus (U_j \cup T_{j+1})$, which is possible by

$$\begin{aligned} |I_{j+1}| = \big|U_{j+1} \setminus (U_j \cup T_{j+1})\big| &= |U_{j+1}| - |U_0 \cap U_1| - |T_{j+1}| \\ &\overset{\text{(i)}}{\geq} \delta_0\big(\vec{G}\big) - |U_0 \cap U_1| - |T_{j+1}| \overset{\text{(ii)}}{\geq} \delta_0\big(\vec{G}\big) - \big(\tfrac{1}{3} - 21\varepsilon\big)\, n - |T_{j+1}| \\ &\overset{(52)}{\geq} \delta_0\big(\vec{G}\big) - \big(\tfrac{1}{3} - 21\varepsilon\big)\, n - \varepsilon n \geq 19\varepsilon n. \end{aligned}$$

Consider ($a$) above ($j = 0$). If $v_1$ has an in-neighbor $w_1 \in U_1 \setminus T_1$, then $w_1 \in U_0 \cap U_1$ because $I_1$ is independent. If $w_1$ has an in-neighbor $x_1 \in U_1 \setminus T_1$, then $x_1 \in I_1$ lest we violate (iii). If $x_1$ has an in-neighbor $y_1 \in U_1 \setminus T_1$, then $y_1 \in U_0 \cap U_1$ because $I_1$ is independent, but now we have violated (iii). Thus, some $z_1 \in \{v_1, w_1, x_1, y_1\}$ has no in-neighbor within $U_1 \setminus T_1$. Purely symmetric arguments establish ($b$).

We use ($a$) and ($b$) above to conclude the proof of Fact 4.2, where we first consider $j = 0$. The sets $U_1 \setminus T_1$, $S_0$, and $T_0$ are pairwise disjoint by construction, and the set $Z_1 = N^-_{\vec{G}}(z_1)$ is disjoint from $U_1 \setminus T_1$ by ($a$) and is disjoint from each of $S_0$ and $T_0$ by (iii). When $j = 1$, the sets $Z_0 = N^+(z_0)$, $U_0 \setminus T_0$, $S_1$, and $T_1$ are similarly pairwise disjoint. Thus, for whichever $j \in \mathbb{Z}_2$ satisfies $|T_j| \geq \varepsilon n$, we have

$$\begin{aligned} n \geq |Z_{j+1}| + |U_{j+1} \setminus T_{j+1}| + |S_j| + |T_j| &\geq \delta_0\big(\vec{G}\big) + |U_{j+1}| - |T_{j+1}| + |S_j| + |T_j| \\ &\overset{(52)}{>} 2\delta_0\big(\vec{G}\big) + |U_{j+1}| - \varepsilon n + \big(\tfrac{2}{3} - \varepsilon\big)\, |T_j| \\ &\overset{\text{(i)}}{\geq} 3\delta_0\big(\vec{G}\big) - \varepsilon n + \big(\tfrac{2}{3} - \varepsilon\big)\, |T_j| \geq n - 4\varepsilon n + \big(\tfrac{2}{3} - \varepsilon\big)\, |T_j|, \end{aligned}$$

from which $|T_j| \leq (132/(19))\varepsilon n < 7\varepsilon n$ follows from $\varepsilon \in (0, 1/(11)]$.

## Proof of Fact 4.4

Fix an integer $\ell \geq 4$ and an $\varepsilon \in (0, 1/(54))$. Let $\vec{G} = (V, \vec{E})$ be an oriented graph on $n \geq n_0(\ell, \varepsilon)$ many vertices which admits no closed directed $\ell$-walk, but which satisfies $\delta_0(\vec{G}) \geq ((1/3)-\varepsilon)n$. Let $(x,y,z,x)$ be a directed 3-cycle $\vec{C}_3$ in $\vec{G}$, and assume that neither $x$ nor $y$ belongs to a directed 4-cycle $\vec{C}_4$. Assume, on the contrary, that

$$\left|N^-_{\vec{G}}(x) \cap N^+_{\vec{G}}(y)\right| < \left(\tfrac{1}{3} - 18\varepsilon\right) n. \tag{54}$$

We will show that our assumption in (54) implies

$$N^+_{\vec{G}}(x) \cap N^-_{\vec{G}}(y) \neq \emptyset, \tag{55}$$

in which case an element $v \in N^+_{\vec{G}}(x) \cap N^-_{\vec{G}}(y)$ would result in the directed 4-cycle $(x,v,y,z,x)$ containing both $x$ and $y$. We now establish the details for (55).

First, we apply Fact 4.2 to each of the pairs $(X_0,X_1)$ and $(Y_0,Y_1)$, where

$$X_0 = N^+_{\vec{G}}(x), \qquad X_1 = N^-_{\vec{G}}(x), \qquad Y_0 = N^+_{\vec{G}}(y), \qquad Y_1 = N^-_{\vec{G}}(y).$$

Note that the hypotheses of Fact 4.2 are met since $\vec{G}$ admits no closed directed $\ell$-walks but satisfies $\delta_0(\vec{G}) \geq ((1/3)-\varepsilon)n$ for $0 < \varepsilon < 1/(54) < 1/(11)$, and where e.g. $(X_0,X_1)$ satisfies the hypotheses (i)–(iii) of Fact 4.2 since $|X_0|, |X_1| \geq \delta_0(\vec{G})$, since $|X_0 \cap X_1| = 0$, and since a directed path $x_0 \to v \to x_1$ with $x_0 \in X_0$ and $x_1 \in X_1$ would give a directed 4-cycle $\vec{C}_4$ containing $x$. Fact 4.2 guarantees independent sets $I_x \subseteq X_1 \setminus X_0 = X_1 = N^-_{\vec{G}}(x)$ and $I_y \subseteq Y_0 \backslash Y_1 = Y_0 = N^+_{\vec{G}}(y)$ of respective sizes $|I_x| \geq |N^-_{\vec{G}}(x)| - 7\varepsilon n$ and $|I_y| \geq |N^+_{\vec{G}}(y)| - 7\varepsilon n$. Note that $|I_x \cap I_y|$ is bounded by (54), we but claim that $I_x \cap I_y = \emptyset$. Indeed, a vertex $v \in I_x \cap I_y$ must have its neighborhood $N_{\vec{G}}(v) = N^-_{\vec{G}}(v) \cup N^+_{\vec{G}}(v)$ outside of $I_x \cup I_y$, and so

$$\begin{aligned}
\left|N^-_{\vec{G}}(x) \cap N^+_{\vec{G}}(y)\right| &\geq |I_x \cap I_y| = |I_x| + |I_y| - |I_x \cup I_y| \\
&\geq |I_x| + |I_y| + \left|N^+_{\vec{G}}(v)\right| + \left|N^-_{\vec{G}}(v)\right| - n \\
&\overset{\text{Fct. 4.2}}{\geq} \left|N^-_{\vec{G}}(x)\right| - 7\varepsilon n + \left|N^+_{\vec{G}}(y)\right| - 7\varepsilon n + \left|N^+_{\vec{G}}(v)\right| + \left|N^-_{\vec{G}}(v)\right| - n \\
&\geq 4\delta_0\big(\vec{G}\big) - 14\varepsilon n - n \geq 4\left(\tfrac{1}{3} - \varepsilon\right) n - 14\varepsilon - n = \left(\tfrac{1}{3} - 18\varepsilon\right) n
\end{aligned}$$

contradicts (54).

Second, we claim that every $a_x \in I_x$ and $b_y \in I_y$ satisfy

$$N^-_{\vec{G}}(a_x) \cap N^+_{\vec{G}}(b_y) \neq \emptyset. \tag{56}$$

Indeed, $(b_y, a_x) \notin \vec{E}$ lest $(x, y, b_y, a_x, x)$ is a directed 4-cycle containing both $x$ and $y$. Thus, $V \setminus (I_x \cup I_y)$ contains each of $N^-_{\vec{G}}(a_x)$ and $N^+_{\vec{G}}(b_y)$, and hence their union. As such,

$$(57)\quad \left|N^-_{\vec{G}}(a_x) \cap N^+_{\vec{G}}(b_y)\right| = \left|N^-_{\vec{G}}(a_x)\right| + \left|N^+_{\vec{G}}(b_y)\right| - \left|N^-_{\vec{G}}(a_x) \cup N^+_{\vec{G}}(b_y)\right|$$
$$\geq \left|N^-_{\vec{G}}(a_x)\right| + \left|N^+_{\vec{G}}(b_y)\right| + |I_x \cup I_y| - n = \left|N^-_{\vec{G}}(a_x)\right| + \left|N^+_{\vec{G}}(b_y)\right| + |I_x| + |I_y| - n$$
$$\overset{\text{Fct. 4.2}}{\geq} \left|N^-_{\vec{G}}(a_x)\right| + \left|N^+_{\vec{G}}(b_y)\right| + \left|N^-_{\vec{G}}(x)\right| + \left|N^+_{\vec{G}}(y)\right| - 14\varepsilon n - n$$
$$\geq 4\delta_0\big(\vec{G}\big) - 14\varepsilon n - n \geq 4\left(\tfrac{1}{3} - \varepsilon\right) n - 14\varepsilon n - n = \left(\tfrac{1}{3} - 18\varepsilon\right) n > 0.$$

Third and finally, we observe that $N^+_{\vec{G}}(x) \cap I_y = \emptyset = N^-_{\vec{G}}(y) \cap I_x$. Indeed, and for example, any $b_y \in N^+_{\vec{G}}(x) \cap I_y$ and $a_x \in I_x$ beget $c_{xy} \in N^-_{\vec{G}}(a_x) \cap N^+_{\vec{G}}(b_y)$ by (56), in which case $(x, b_y, c_{xy}, a_x, x)$ would be a directed 4-cycle containing $x$. Now, $V \setminus (I_x \cup I_y)$ contains each of $N^+_{\vec{G}}(x)$ and $N^-_{\vec{G}}(y)$, and so

$$\left|N^+_{\vec{G}}(x) \cap N^-_{\vec{G}}(y)\right| \geq \left|N^+_{\vec{G}}(x)\right| + \left|N^-_{\vec{G}}(y)\right| + |I_x \cup I_y| - n,$$

where calculations identical to (57) establish (55).

## Proof of Fact 4.6

Let $\lambda > 0$ be given. The promised constant $\varepsilon = \varepsilon(\lambda) > 0$ will be defined in context. Let $\vec{G} = (V, \vec{E})$ be an oriented graph on $n \geq n_0(\lambda, \varepsilon)$ many vertices which satisfies $\delta_0(\vec{G}) \geq ((1/3) - \varepsilon)n$. We show that $\vec{G}$ is $\lambda$-extremal when Conditions (1) or (2) hold, which we handle separately.

For Condition (1), it suffices to take $\varepsilon \in (0, \lambda/7]$. Let $V = V_0 \cup V_1 \cup V_2$ be a partition satisfying $|V_1|, |V_2| \geq ((1/3) - \varepsilon)n$ and $e_{\vec{G}}(V_1), e_{\vec{G}}(V_2), e_{\vec{G}}(V_2, V_1) \leq \varepsilon n^2$. We bound each of $e_{\vec{G}}(V_2, V_0)$, $e_{\vec{G}}(V_0, V_1)$, and $e_{\vec{G}}(V_1, V_2)$ suitably from below. First, our hypotheses give

$$(58)\quad e_{\vec{G}}(V_2, V_0) \geq \Big(\sum_{v_2 \in V_2} \deg^+_{\vec{G}}(v_2)\Big) - e_{\vec{G}}(V_2) - e_{\vec{G}}(V_2, V_1)$$
$$\geq \left(\tfrac{1}{3} - \varepsilon\right)^2 n^2 - 2\varepsilon n^2 \geq \left(\tfrac{1}{9} - \lambda\right) n^2,$$

where we used $3\varepsilon \leq \lambda$. Second, and similarly,

$$e_{\vec{G}}(V_0, V_1) \geq \Big(\sum_{v_1 \in V_1} \deg^-_{\vec{G}}(v_1)\Big) - e_{\vec{G}}(V_1) - e_{\vec{G}}(V_2, V_1)$$

$$\geq \left(\tfrac{1}{3}-\varepsilon\right)^2 n^2 - 2\varepsilon n^2 \geq \left(\tfrac{1}{9}-\lambda\right) n^2,$$

where we again used $3\varepsilon \leq \lambda$. Note that, since $\vec{G}$ is oriented, our hypotheses and (58) give

$$e_{\vec{G}}(V_0, V_2) \leq |V_0||V_2| - e_{\vec{G}}(V_2, V_0) \leq \left(\tfrac{n-|V_1|}{2}\right)^2 - e_{\vec{G}}(V_2, V_0) \leq 5\varepsilon n^2. \tag{59}$$

Third, our hypotheses and (59) give

$$e_{\vec{G}}(V_1, V_2) \geq \Big(\sum_{v_2 \in V} \deg^-_{\vec{G}}(v_2)\Big) - e_{\vec{G}}(V_2) - e_{\vec{G}}(V_0, V_2)$$
$$\geq \left(\tfrac{1}{3}-\varepsilon\right)^2 n^2 - 6\varepsilon n^2 \geq \left(\tfrac{1}{9}-\lambda\right) n^2,$$

where we used $7\varepsilon \leq \lambda$.

For Condition (2), we consider a suitably small $\gamma \in (0, \lambda/3]$ in context, and we take $\varepsilon = \varepsilon(\gamma) > 0$ according to an application of the Erdős-Stone theorem [4], discussed below. Assume that $\vec{G} = (V, \vec{E})$ has no transitive triangles. Then the underlying graph $G = (V, E)$ (obtained by removing orientations on arcs) is $K_4$-free. By our hypothesis,

$$\big|\vec{E}\big| = \sum_{v \in V} \deg^+_{\vec{G}}(v) \geq n\delta_0\big(\vec{G}\big) \geq \left(\tfrac{1}{3}-\varepsilon\right) n^2,$$

and so altogether the underlying graph $G = (V, E)$ is $K_4$-free with $|E| \geq ((1/3) - \varepsilon)n^2$ many edges. As such, the Erdős-Stone theorem [4] guarantees a partition $V = V_0 \cup V_1 \cup V_2$, where $|V_0| \leq |V_1| \leq |V_2| \leq |V_0| + 1$, and where each $0 \leq i < j \leq 2$ satisfies

$$|E(G[V_i, V_j])| \geq \left(\tfrac{1}{9}-\gamma\right) n^2. \tag{60}$$

Then[2], the 3-partite graph

$$G[V_0, V_1] \cup G[V_1, V_2] \cup G[V_2, V_0] \tag{61}$$
$$\text{admits at least } ((1/(27)) - \mu)n^3 \text{ many triangles } K_3,$$

where $\mu = \mu(\gamma) \to 0$ as $\gamma \to 0$. Since $\vec{G}$ has no transitive triangles, every triangle of $G$ corresponds to a directed 3-cycle $\vec{C}_3$ in $\vec{G}$. Among other conclusions, we will show that almost all of the triangles of (61) are

[2] See, for example, the proof of Fact 2.5.

commonly oriented, in one of the following two senses. We say that a directed 3-cycle $\vec{C}_3$ of $\vec{G}$ is *positively oriented* when all of its arcs are among $(V_0 \times V_1) \cup (V_1 \times V_2) \cup (V_2 \times V_0)$, and we say that it is *negatively oriented* when all of its arcs are among $(V_0 \times V_2) \cup (V_2 \times V_1) \cup (V_1 \times V_0)$.

We average (61) over, say $V_0$, to obtain a vertex $\bar{v}_0 \in V_0$ belonging to at least $((1/9) - 2\mu)n^2$ many directed 3-cycles $\vec{C}_3$ of $\vec{G}$. At least half of these directed 3-cycles are commonly oriented, so w.l.o.g. assume that at least half are positively oriented. Then

$$\begin{aligned}(62)\quad & \bar{v}_0 \in V_0 \text{ belongs to at least } ((1/9) - 2\mu)n^2 \\ &\qquad \text{many directed 3-cycles } \vec{C}_3 \text{ of } \vec{G}, \text{ and in particular} \\ &\qquad\quad \bar{v}_0 \in V_0 \text{ belongs to at least } ((1/(18)) - \mu)n^2 \\ &\qquad\qquad\qquad \text{many positively oriented 3-cycles } \vec{C}_3 \text{ of } \vec{G}.\end{aligned}$$

From (62), we will prove that

$$e_{\vec{G}}(V_1, V_2) \geq \left(\tfrac{1}{9} - \lambda\right) n^2 \tag{63}$$

follows. Indeed, recall that $K[A, B]$ is the complete bipartite graph with parts $A$ and $B$ and that $\vec{G}$ has no transitive triangles, so each of

$$K\left[N^+_{\vec{G}}(\bar{v}_0) \cap V_1, N^+_{\vec{G}}(\bar{v}_0) \cap V_2\right] \qquad \text{and} \qquad K\left[N^-_{\vec{G}}(\bar{v}_0) \cap V_1, N^-_{\vec{G}}(\bar{v}_0) \cap V_2\right]$$

is edge-disjoint from $E$. Consequently, (60) gives that each has size at most

$$\begin{aligned}&|V_1||V_2| - \left(\tfrac{1}{9} - \gamma\right) n^2 \leq \left(\tfrac{n-|V_0|}{2}\right)^2 - \left(\tfrac{1}{9} - \gamma\right) n^2 \leq 2\gamma n^2 \\ &\quad\Longrightarrow \qquad \left|N^+_{\vec{G}}(\bar{v}_0) \cap V_1\right| \leq \sqrt{2\gamma} n \quad \text{or} \quad \left|N^+_{\vec{G}}(\bar{v}_0) \cap V_2\right| \leq \sqrt{2\gamma} n, \\ &\qquad\qquad \text{and} \qquad \left|N^-_{\vec{G}}(\bar{v}_0) \cap V_1\right| \leq \sqrt{2\gamma} n \quad \text{or} \quad \left|N^-_{\vec{G}}(\bar{v}_0) \cap V_2\right| \leq \sqrt{2\gamma} n.\end{aligned}$$

By (62), it must be that both $\left|N^+_{\vec{G}}(\bar{v}_0) \cap V_2\right| \leq \sqrt{2\gamma} n$ and $|N^-_{\vec{G}}(\bar{v}_0) \cap V_1| \leq \sqrt{2\gamma} n$ hold. As such, $\bar{v}_0 \in V_0$ belongs to at most $2\gamma n^2$ many negatively oriented triangles, and so (62) may be updated to say that $\bar{v}_0 \in V_0$ belongs to at least $((1/9) - 2\mu - 2\gamma)n^2$ many positively oriented triangles. As such,

$$e_{\vec{G}}(V_1, V_2) \geq \left(\tfrac{1}{9} - 2\mu - 2\gamma\right)n^2 \geq \left(\tfrac{1}{9} - \lambda\right)n^2$$

holds by taking $2\mu + 2\gamma \leq \lambda$, and renders (63).

The argument above shows that, for each $i \in \mathbb{Z}_3$,

$(a_i)$ either $e_{\vec{G}}(V_i, V_{i+1}) \geq ((1/9) - \lambda)n^2$,
$(b_i)$ or $e_{\vec{G}}(V_{i+1}, V_i) \geq ((1/9) - \lambda)n^2$.

These outcomes must be consistent across $i \in \mathbb{Z}_3$, which is to say that either $(a_0)$, $(a_1)$, and $(a_2)$ all hold, or $(b_0)$, $(b_1)$, and $(b_2)$ all hold. Indeed, assuming otherwise $\vec{G}$ would have $\Omega(n^3)$ many transitive triangles, contradicting our hypothesis. This proves that $\vec{G}$ is $\lambda$-extremal, as desired.

## 6. Proof of Lemma 2.9 – Part 1: strategy and coarse structure

It suffices to take the promised constant $\lambda_0 > 0$ as

$$\lambda_0 = \left(\tfrac{1}{45{,}000}\right)^4. \tag{64}$$

Now, fix $0 < \lambda \leq \lambda_0$ and fix an integer $\ell \geq 4$ which is not divisible by three. In all that follows, we take the integer $n_0 = n_0(\lambda_0, \lambda, \ell)$ to be sufficiently large whenever needed. The proof of Lemma 2.9 is fairly technical, so we begin by outlining some of its strategy.

### 6.1. Initial strategy

Let $(G, c)$ be a $\lambda$-extremal edge-colored graph on $n \geq n_0$ many vertices. Recall that the hypotheses in Statements (1) and (2) of Lemma 2.9 assume

$$\delta^c(G) \geq \begin{cases} (n+5)/3 & \text{in Statement (1),} \\ (n+4)/3 & \text{in Statement (2).} \end{cases} \tag{65}$$

If $(G, c)$ admits a rainbow $\ell$-cycle $C_\ell$, then the conclusions of Lemma 2.9 hold, so

(66) *we assume throughout this proof that* $(G, c)$ *admits no rainbow* $\ell$*-cycles* $C_\ell$.

Moreover,

(67) *we assume throughout this proof that* $(G, c)$ *is edge-minimal w.r.t. satisfying both* (65) *and* (66).

Observe, for example, that (67) implies that $(G, c)$ admits no monochromatic paths $P$ or cycles $C$ on three or more edges, lest removing an internal edge

$\{x, y\} \in E$ from $P$ or $C$ lowers neither $\deg_G^c(x)$ nor $\deg_G^c(y)$. Finally, for both cases of (65), we set $m = \lfloor n/3 \rfloor$, where

$$\delta^c(G) \geq \tfrac{n+4}{3} \quad \Longrightarrow \quad \delta^c(G) \geq \left\lfloor \tfrac{n}{3} \right\rfloor + 2 = m + 2. \tag{68}$$

Since we assume in Lemma 2.9 that $(G, c)$ is $\lambda$-extremal, fix a $\lambda$-extremal partition $V = V(G) = V_0 \cup V_1 \cup V_2$ of $(G, c)$ (recall Definition 2.4). Our first main goal in proving Lemma 2.9 is to infer from (67) that $(G, c)$ enjoys *nearly canonical* structure on $V_0 \cup V_1 \cup V_2$, in the following sense. Let $H = K[V_0, V_1, V_2]$ be the complete 3-partite graph with vertex partition $V_0 \cup V_1 \cup V_2$, and consider an edge-coloring $\kappa$ on $H$ where, for each $i \in \mathbb{Z}_3$ and for each $v_i \in V_i$,

- $(a)$ $\kappa$ assigns all distinct colors to the edges $\{v_i, v_{i+1}\} \in E$, where $v_{i+1} \in V_{i+1}$;
- $(b)$ $\kappa$ assigns a common color to all the edges $\{v_i, v_{i-1}\} \in E$, where $v_{i-1} \in V_{i-1}$.

We say that any such $(H, \kappa)$ is *canonical* w.r.t. $V_0 \cup V_1 \cup V_2$. (For example, a canonical edge-colored graph $(H, \kappa)$ was used in Case 1 of Section 1.1, where $\kappa = c_+$ was defined in (2).) In the immediate sequel, we use (67) to prove that $(G, c)$ is *nearly canonical* w.r.t. $V_0 \cup V_1 \cup V_2$, in the sense that for each $i \in \mathbb{Z}_3$, *almost all* $v_i \in V_i$ admit distinctly colored edges to *almost all* $v_{i+1} \in V_{i+1}$, and *almost all* $v_i \in V_i$ admit commonly colored edges to *almost all* $v_{i-1} \in V_{i-1}$. We now make these details precise.

### 6.2. $(G, c)$ is nearly canonical: getting started

Definition 2.4 ensures that each $i \in \mathbb{Z}_3$ satisfies

$$|V_i| = \left(\tfrac{1}{3} \pm 3\sqrt{\lambda}\right) n. \tag{69}$$

Indeed, $\sum_{i \in \mathbb{Z}_3} |E(G[V_i, V_{i+1}])| \geq (1/3 - 3\lambda) n^2$, so $\binom{V_0}{2} \cup \binom{V_1}{2} \cup \binom{V_2}{2}$ consists of at most $((1/6) + 3\lambda) n^2$ many pairs. Set $|V_i| = ((1/3) + e_i) n$, $i \in \mathbb{Z}_3$, so that $e_0 + e_1 + e_2 = 0$. Then $\binom{V_0}{2} \cup \binom{V_1}{2} \cup \binom{V_2}{2}$ has size

$$(1 + o(1)) \left( \tfrac{n^2}{6} + \tfrac{n^2}{2} \left( e_0^2 + e_1^2 + e_2^2 \right) \right),$$

and this is too large when $\max\{|e_0|, |e_1|, |e_2|\} > \sqrt{6\lambda}$.

Next, fix $i \in \mathbb{Z}_3$. We shall say that a vertex $v_i \in V_i$ is an *$i$-good* vertex if

$$\deg_G^c(v_i, V_{i+1}) \geq |V_{i+1}| - \lambda^{1/4} n \quad \text{and} \quad \deg_G(v_i, V_{i-1}) \geq |V_{i-1}| - \lambda^{1/4} n, \tag{70}$$

where as usual $\deg_G(v_i, V_{i-1})$ denotes the number of neighbors of $v_i$ in $V_{i-1}$, and where here $\deg_G^c(v_i, V_{i+1})$ denotes the number of colors seen on the edges of $v_i$ to $V_{i+1}$. Then (70) says an $i$-good vertex $v_i$ admits distinctly colored edges to all but $\lambda^{1/4}n$ many vertices $v_{i+1} \in V_{i+1}$, and it admits edges of varying colors to all but $\lambda^{1/4}n$ many vertices $v_{i-1} \in V_{i-1}$. Let $V_i^{\text{good}}$ denote the set of $i$-good vertices $v_i \in V_i$. Using (69) and Definition 2.4, it is easy to show that

$$\left|V_i^{\text{good}}\right| \geq |V_i| - 24\lambda^{1/4}n. \tag{71}$$

With $i \in \mathbb{Z}_3$ still fixed, we shall say that a vertex $v_i \in V_i \setminus V_i^{\text{good}}$ is an *$i$-bad vertex*. We write $V_i^{\text{bad}} = V_i \setminus V_i^{\text{good}}$ for the set of $i$-bad vertices, and we write $V^{\text{bad}} = V_0^{\text{bad}} \cup V_1^{\text{bad}} \cup V_2^{\text{bad}}$ for the set of *bad vertices*. Then bad vertices total at most $72\lambda^{1/4}n$ by (71).

We now alter the partition $V = V_0 \cup V_1 \cup V_2$ to $V = U_0 \cup U_1 \cup U_2$, as follows. For each $i$-good vertex $v_i \in V_i^{\text{good}}$, we put $v_i \in U_i$. For each bad vertex $v \in V^{\text{bad}}$, let $j_v \in \mathbb{Z}_3$ achieve

$$\begin{aligned}\deg_G^c\left(v, V_{j_v}^{\text{good}}\right) \\ = \max\Big\{ \deg_G^c\left(v, V_0^{\text{good}}\right), \deg_G^c\left(v, V_1^{\text{good}}\right), \deg_G^c\left(v, V_2^{\text{good}}\right)\Big\}.\end{aligned} \tag{72}$$

We then put $v \in U_{j_v-1}$. Then $U_i$ consists of $V_i^{\text{good}}$ together with those bad vertices $v \in V^{\text{bad}}$ satisfying

$$\deg_G^c\left(v, V_{i+1}^{\text{good}}\right) \geq \max\Big\{ \deg_G^c\left(v, V_i^{\text{good}}\right), \deg_G^c\left(v, V_{i-1}^{\text{good}}\right)\Big\}. \tag{73}$$

We write $U_i^{\text{good}} = U_i \cap V_i^{\text{good}} = V_i^{\text{good}}$, and we continue to call these vertices *good*. We write $U_i^{\text{bad}} = U_i \cap V^{\text{bad}}$, and we continue to call these vertices *bad*. Then (69)–(71) give:

$$\begin{gathered} |U_i| = \left(\tfrac{1}{3} \pm 75\lambda^{1/4}\right)n, \quad \left|U_i^{\text{good}}\right| = \left(\tfrac{1}{3} \pm 75\lambda^{1/4}\right)n, \\ \left|V^{\text{bad}}\right| = \left|U_0^{\text{bad}}\right| + \left|U_1^{\text{bad}}\right| + \left|U_2^{\text{bad}}\right| \leq 72\lambda^{1/4}n, \\ \forall\, i \in \mathbb{Z}_3,\ \forall\, u \in U_i^{\text{good}},\ \deg_G^c(u, U_{i+1}) \geq |V_{i+1}| - 73\lambda^{1/4}n \geq |U_{i+1}| - 145\lambda^{1/4}n, \\ \forall\, i \in \mathbb{Z}_3,\ \forall\, u \in U_i^{\text{good}},\ \deg_G^c\left(u, U_{i+1}^{\text{good}}\right) \geq \left(\tfrac{1}{3} - 76\lambda^{1/4}\right)n, \\ \forall\, i \in \mathbb{Z}_3,\ \forall\, u \in U_i^{\text{good}},\ \deg_G(u, U_{i-1}) \geq |V_{i-1}| - 73\lambda^{1/4}n \geq |U_{i-1}| - 145\lambda^{1/4}n, \\ \forall\, i \in \mathbb{Z}_3,\ \forall\, u \in U_i^{\text{good}},\ \deg_G\left(u, U_{i-1}^{\text{good}}\right) \geq \left(\tfrac{1}{3} - 76\lambda^{1/4}\right)n, \end{gathered} \tag{74}$$

$$\forall\, i \in \mathbb{Z}_3,\ \forall\, u \in U_i^{\mathrm{bad}},\ \deg_G^c(u, U_{i+1}) \overset{(72)}{\geq} \tfrac{1}{3}\delta^c(G) - 72\lambda^{1/4} n \overset{(68)}{\geq} \left(\tfrac{1}{9} - 72\lambda^{1/4}\right)n.$$

Henceforth, the initial partition $V = V_0 \cup V_1 \cup V_2$ is largely usurped by $V = U_0 \cup U_1 \cup U_2$.

### 6.3. $(G, c)$ is nearly canonical: a next step

The inequalities in (74) show that $|U_0|, |U_1|, |U_2|$ are nearly balanced, and that $G[U_0, U_1, U_3]$ differs from the complete 3-partite graph $K[U_0, U_1, U_2]$ on few edges. The inequalities in (74) also show that $(G, c)$ deviates very little from property $(a)$ of Section 6.1, in that good vertices $u_i \in U_i^{\mathrm{good}}$ (which are pervasive) have distinctly colored edges to nearly all $u_{i+1} \in U_{i+1}$. We now show that $(G, c)$ deviates little from the corresponding property $(b)$. For that, we first show that good vertices $u_i \in U_i^{\mathrm{good}}$ are incident to few colors $c(\{u_i, u_{i-1}\})$, where $u_{i-1} \in U_{i-1}^{\mathrm{good}}$.

**Fact 6.1.** *For each $i \in \mathbb{Z}_3$ and for each $u_i \in U_i^{\mathrm{good}}$, we have $\deg_G^c(u_i, U_{i-1}^{\mathrm{good}}) \leq 240\lambda^{1/4} n$.*

*Proof of Fact 6.1.* Assume for contradiction that Fact 6.1 is false for some index $i \in \mathbb{Z}_3$ and vertex $u_i \in U_i^{\mathrm{good}}$, and w.l.o.g. assume $i = 2$. We will first determine a set $T_1^{\mathrm{good}} \subseteq U_1^{\mathrm{good}}$ so that the fixed good vertex $u_2 \in U_2^{\mathrm{good}}$ satisfies

$$\text{(75)}\quad \deg_G^c\left(u_2, T_1^{\mathrm{good}}\right) > 160\lambda^{1/4} n \text{ and}$$
$$\text{every path } (u_2, u_1, v) \text{ in } G \text{ where } u_1 \in T_1^{\mathrm{good}} \text{ is rainbow.}$$

Note that $T_1^{\mathrm{good}}$ will be an eventual 'target space'. To prove (75), we distinguish $\deg_G^c(u_2)$.

**Case 1 ($\deg_G^c(u_2) \geq (n + 10)/3$).** Set $T_1^{\mathrm{good}} = U_1^{\mathrm{good}}$, where our contrary assumption gives

$$\text{(76)}\qquad \deg_G^c(u_2, T_1^{\mathrm{good}}) = \deg_G^c(u_2, U_1^{\mathrm{good}}) > 240\lambda^{1/4} n > 160\lambda^{1/4},$$

as desired. Now, every path $(u_2, u_1, v)$ with $u_1 \in T_1^{\mathrm{good}} = U_1^{\mathrm{good}}$ is rainbow lest the hypothesis of Case 1 gives that removing the edge $\{u_2, u_1\} \in E$ from $G$ contradicts (67). $\square$

**Case 2 ($\deg^c_G(u_2) < (n+10)/3$).** Set

$$T_1^{\text{good}} = \big\{u_1 \in N_G(u_2, U_1^{\text{good}}) : \exists\, u_0 \in N_G(u_2, U_0) \\ \text{where } c(\{u_2, u_0\}) = c(\{u_2, u_1\})\big\}.$$

Observe that

$$\deg^c_G(u_2) \ge \deg^c_G(u_2, U_0) + \deg^c_G\left(u_2, U_1^{\text{good}}\right) - \deg^c_G\left(u_2, T_1^{\text{good}}\right),$$

and so

$$\begin{aligned}\deg^c_G\left(u_2, T_1^{\text{good}}\right) &\ge \deg^c_G(u_2, U_0) + \deg^c_G\left(u_2, U_1^{\text{good}}\right) - \deg^c_G(u_2)\\ &\overset{(74)}{\ge} \left(\tfrac{1}{3} - 76\lambda^{1/4}\right)n + \deg^c_G\left(u_2, U_1^{\text{good}}\right) - \deg^c_G(u_2)\\ &\overset{(76)}{>} \left(\tfrac{1}{3} - 76\lambda^{1/4}\right)n + 240\lambda^{1/4}n - \deg^c_G(u_2)\\ &\overset{\text{Case 2}}{>} \left(\tfrac{1}{3} - 76\lambda^{1/4}\right)n + 240\lambda^{1/4}n - \tfrac{n+10}{3} \ge 160\lambda^{1/4}n.\end{aligned}$$

If $(u_2, u_1, v)$ is a monochromatic path with $u_1 \in T_1^{\text{good}}$, then there exists $u_0 \in U_0$ where $(u_0, u_2, u_1, v)$ is monochromatic. Whether or not $v = u_0$, removing the edge $\{u_2, u_1\} \in E$ from $G$ contradicts (67). $\square$

We now use (75) to complete the proof of Fact 6.1. Fix an arbitrary vertex $u_1 \in T_1^{\text{good}}$ (cf. (75)), and fix an arbitrary vertex $u_0 \in N_G(u_1, U_0^{\text{good}})$ (cf. (74)). By (75), the path $(u_2, u_1, u_0)$ is rainbow. We now distinguish the cases $\ell \equiv 1, 2 \pmod 3$.

**Case A ($\ell \equiv 1 \pmod 3$).** Using (74) and $n \ge n_0(\ell)$ sufficiently large, we can easily extend the rainbow path $(u_2, u_1, u_0)$ to a rainbow path $R_{\ell-4} = (u_2, u_1, u_0, v_1, v_2, \dots, v_{\ell-4})$ on $\ell - 1$ vertices, where for each $1 \le j \le \ell - 4$, we may choose $v_j \in U_J^{\text{good}}$ for $J \equiv j \pmod 3$. By (75), $u_2 \in U_2^{\text{good}}$ sees $160\lambda^{1/4}n$ colors into $T_1^{\text{good}}$, and at most $\ell - 1$ of them were used on $R_{\ell-4}$. Similarly, (74) gives that the vertex $v_{\ell-4} \in U_0^{\text{good}}$ (recall $\ell \equiv 1 \pmod 3$) sees $((1/3) - 76\lambda^{1/4})n$ colors into $U_1^{\text{good}}$, and at most $\ell - 1$ of them were used on $R_{\ell-4}$. Then for some $w_1 \in N_G(u_2, T_1^{\text{good}}) \cap N_G(v_{\ell-4}, U_1^{\text{good}})$ both paths $(w_1, u_2, u_1, u_0, v_1, \dots, v_{\ell-4})$ and $(u_2, u_1, u_0, v_1, \dots, v_{\ell-4}, w_1)$ are rainbow since

$$160\lambda^{1/4}n - (\ell-1) + \left(\tfrac{1}{3} - 76\lambda^{1/4}\right)n - (\ell-1) \ge \left(\tfrac{1}{3} + 84\lambda^{1/4}\right)n - O(1) \overset{(74)}{>} |U_1|.$$

Since $w_1 \in T_1^{\text{good}}$, the path $(u_2, w_1, v_{\ell-4})$ is rainbow by (75), and so the $\ell$-cycle $(u_2, u_1, u_0, v_1, \dots, v_{\ell-4}, w_1, u_2)$ is rainbow. $\square$

**Case B ($\ell \equiv 2 \pmod 3$).** The rainbow path $(u_2, u_1)$ may be extended to a rainbow path $\hat{R}_{\ell-2} = (u_2, u_1, v_2, \dots, v_{\ell-2})$ on $\ell - 1$ vertices, where for each $2 \leq j \leq \ell - 2$, we may choose $v_j \in U_J^{\text{good}}$ for $J \equiv j \pmod 3$. Identically to the above, we may extend the rainbow path $\hat{R}_{\ell-2}$ to a rainbow $\ell$-cycle $C_\ell$. $\square$

### 6.4. $(G, c)$ is nearly canonical: finale

We now show that, for a fixed $u_i \in U_i^{\text{good}}$, edges $\{u_i, u_{i-1}\} \in E$, where $u_{i-1} \in U_{i-1}^{\text{good}}$, are dominated by a single color.

**Proposition 6.2.** *For each $i \in \mathbb{Z}_3$ and for each $u_i \in U_i^{\text{good}}$, there exists a color $c_{u_i}$ from $c$ where all but $241\lambda^{1/4}n$ many vertices $u_{i-1} \in U_{i-1}^{\text{good}}$ satisfy $\{u_i, u_{i-1}\} \in E$ and $c(\{u_i, u_{i-1}\}) = c_{u_i}$. Together with (74), all but $313\lambda^{1/4}n$ vertices $u_{i-1} \in U_{i-1}$ satisfy $\{u_i, u_{i-1}\} \in E$ and $c(\{u_i, u_{i-1}\}) = c_{u_i}$.*

For the proof and use of Proposition 6.2, we establish some notation. Fix $i \in \mathbb{Z}_3$ and fix $u_i \in U_i$. On the edges $E_G(u_i, U_{i-1})$ between $u_i$ and $U_{i-1}$, let $c_{u_i}$ be a most frequent color, which we call the *primary* color of $E_G(u_i, U_{i-1})$. Edges of $E_G(u_i, U_{i-1})$ colored by $c_{u_i}$ are called *typical* edges, and edges of $E_G(u_i, U_{i-1})$ colored otherwise are called *special* edges. We write $N_G^{\text{typ}}(u_i, U_{i-1})$ for the set of $u_{i-1} \in U_{i-1}$ where $\{u_i, u_{i-1}\} \in E$ is a typical edge, and we write $N_G^{\text{spec}}(u_i, U_{i-1})$ for the set of $u_{i-1} \in U_{i-1}$ where $\{u_i, u_{i-1}\} \in E$ is a special edge. We write

$$(77) \quad \deg_G^{\text{typ}}(u_i, U_{i-1}) = \left|N_G^{\text{typ}}(u_i, U_{i-1})\right| \quad \text{and} \quad \deg_G^{\text{spec}}(u_i, U_{i-1}) = \left|N_G^{\text{spec}}(u_i, U_{i-1})\right|.$$

*Proof of Proposition 6.2.* Assume for contradiction that Proposition 6.2 is false for some index $i \in \mathbb{Z}_3$ and vertex $u_i \in U_i^{\text{good}}$, and w.l.o.g. assume $i = 2$. Then, the fixed vertex $u_2 \in U_2^{\text{good}}$ satisfies

$$(78) \quad \deg_G^{\text{spec}}\left(u_2, U_1^{\text{good}}\right) \geq 241\lambda^{1/4}n \quad \text{while} \quad \deg_G^{c}\left(u_2, U_1^{\text{good}}\right) \overset{\text{Fact 6.1}}{\leq} 240\lambda^{1/4}n.$$

We will produce a contradiction similar to that for Fact 6.1, where we will use (78) to construct a rainbow $\ell$-cycle $C_\ell$ in $(G, c)$, which will contradict (66). We again distinguish the cases $\ell \equiv 1, 2 \pmod 3$.

**Case 1 ($\ell \equiv 1 \pmod 3$).** The inequalities in (78) together imply that there exist neighbors $u_1 \neq v_1 \in N_G(u_2, U_1^{\text{good}})$ for which $c(\{u_2, u_1\}) = c(\{u_2, v_1\})$ differs from the primary color $c_{u_2}$. For simplicity, let $c_{u_2}$ be *blue* and let $c(\{u_2, u_1\}) = c(\{u_2, v_1\})$ be *red*. Using (74), fix $u_0 \neq v_0 \in N_G(u_1, U_0^{\text{good}}) \cap N_G(v_1, U_0^{\text{good}})$. Since $(G, c)$ admits no monochromatic paths on four vertices, none of the edges of the four-cycle $(u_1, u_0, v_1, v_0)$ can be red, and not all of them can be blue. W.l.o.g., assume $\{u_1, u_0\}$ is colored *yellow* so that $(u_2, u_1, u_0)$ is a red-yellow path which avoids the primary color blue for $u_2$.

Similarly to the proof of Fact 6.1, we will extend $(u_2, u_1, u_0)$ to a rainbow $\ell$-cycle $C_\ell$, which will contradict (66). Consider the following set which will be an eventual 'target space':

$$T_1(u_2) = \Big\{ t_1 \in N_G\big(u_2, U_1^{\text{good}}\big) : \\ c(\{u_2, t_1\}) \text{ is neither red nor yellow } \Big\} \subseteq U_1^{\text{good}}.$$

Since blue is the primary color for $u_2$, some edges $\{u_2, t_1\}$ with $t_1 \in T_1(u_2)$ are colored blue. Now, among the colors blue, red, and yellow, neither red nor yellow are primary, so at most a $2/3$ portion of neighbors $v_1 \in N_G(u_2, U_1^{\text{good}})$ have red or yellow edges with $u_2$. Thus,

$$|T_1(u_2)| \geq \tfrac{1}{3} \deg_G\big(u_2, U_1^{\text{good}}\big) \overset{(74)}{\geq} \tfrac{1}{3}\big(\tfrac{1}{3} - 76\lambda^{1/4}\big)n \overset{(64)}{\geq} \tfrac{n}{10}, \\ \text{while } u_2 \text{ sees at most } 240\lambda^{1/4}n \text{ colors into } T_1(u_2) \text{ (cf. Fact 6.1).} \tag{79}$$

Let $C(u_2)$ be the set of colors used on edges between $u_2$ and $T_1(u_2)$. As we did for Fact 6.1, we extend the rainbow path $(u_2, u_1, u_0)$ to a rainbow path $R_{\ell-4} = (u_2, u_1, u_0, w_1, w_2, \dots, w_{\ell-4})$ on $\ell - 1$ vertices, where for each $1 \leq j \leq \ell - 4$, we may choose $w_j \in U_J^{\text{good}}$ for $J \equiv j \pmod 3$, but where this time we avoid the $|C(u_2)| \leq 240\lambda^{1/4}n$ many colors of $C(u_2)$, which we may do on account of (74). Since $w_{\ell-4} \in U_0^{\text{good}}$ (recall $\ell \equiv 1 \pmod 3$), (74) gives that $\deg_G^c(w_{\ell-4}, U_1^{\text{good}}) \geq |U_1^{\text{good}}| - 145\lambda^{1/4}n$, so from $T_1(u_2) \subseteq U_1^{\text{good}}$,

$$\deg_G^c(w_{\ell-4}, T_1(u_2)) \geq |T_1(u_2)| - 145\lambda^{1/4}n \overset{(79)}{\geq} \tfrac{n}{10} - 145\lambda^{1/4}n \\ \overset{(64)}{>} 240\lambda^{1/4}n + \ell - 1 \overset{(79)}{\geq} |C(u_2)| + \ell - 1. \tag{80}$$

Thus, we may choose a neighbor $t_1 \in N_G(w_{\ell-4}) \cap T_1(u_2)$ where $c(\{w_{\ell-4}, t_1\}) \notin C(u_2)$ differs from any color used on $R_{\ell-4}$. Now, $(u_2, u_1, u_0, w_1, w_2, \ldots, w_{\ell-4}, t_1)$ is a rainbow $\ell$-cycle $C_\ell$ in $(G, c)$ (where $c(\{u_2, t_1\}) \in C(u_2)$ but where $C(u_2)$ was used nowhere else on $C_\ell$), which contradicts (66). □

**Case 2 ($\ell \equiv 2 \pmod 3$).** The proof is analogous to that above, where we may simplify the preamble of Case 1. Here, fix a single neighbor $u_1 \in N_G(u_2, U_1^{\text{good}})$ where $c(\{u_2, u_1\})$ (which we assume is red) differs from the primary color blue for $u_2$. We will extend the rainbow path $(u_2, u_1)$ to a rainbow $\ell$-cycle $C_\ell$, which will contradict (66). To do so, this time we define

$$T_1(u_2) = \bigl\{t_1 \in N_G\bigl(u_2, U_1^{\text{good}}\bigr) : \ c(\{u_2, t_1\}) \text{ is not red}\bigr\},$$

and again we define $C(u_2)$ to be the set of colors on edges between $u_2$ and $T_1(u_2)$. Since red is not the primary color of $u_2$, at most half the neighbors $v_1 \in N_G(u_2, U_1^{\text{good}})$ have a red edge with $u_2$, and so the final conclusions of (79) hold. On account of (74), we may extend the rainbow path $(u_2, u_1)$ to a rainbow path $\hat{R}_{\ell-2} = (u_2, u_1, v_2, \ldots, v_{\ell-2})$ on $\ell-1$ vertices, where for each $2 \le j \le \ell-2$, we may choose $v_j \in U_J^{\text{good}}$ for $J \equiv j \pmod 3$, and where again we may avoid the $|C(u_2)| \le 240\lambda^{1/4} n$ many colors of $C(u_2)$. The inequality in (80) holds for the vertex $v_{\ell-2} \in U_0^{\text{good}}$ (recall $\ell \equiv 2 \pmod 3$), so we may choose $t_1 \in N_G(v_{\ell-2}) \cap T_1(u_2)$ where $c(\{v_{\ell-2}, t_1\}) \notin C(u_2)$ differs from any color used on $\hat{R}_{\ell-2}$. Now, $(u_2, u_1, v_2, \ldots, v_{\ell-2}, t_1)$ is a rainbow $\ell$-cycle $C_\ell$ in $(G, c)$, which contradicts (66). □

We conclude the nearly canonical structure of $(G, c)$ by noting that, for each $i \in \mathbb{Z}_3$, distinct good vertices $u_i \neq v_i \in U_i^{\text{good}}$ admit distinct primary colors.

**Corollary 6.3.** *For each $i \in \mathbb{Z}_3$ and for each $u_i \neq v_i \in U_i^{\text{good}}$, the primary colors $c_{u_i}$ and $c_{v_i}$ differ.*

*Proof of Corollary 6.3.* Fix $i \in \mathbb{Z}_3$ and fix $u_i \neq v_i \in U_i^{\text{good}}$. Then

$$\begin{aligned} \bigl|N_G^{\text{typ}}(u_i, U_{i-1}) \cap N_G^{\text{typ}}(v_i, U_{i-1})\bigr| &\overset{\text{Prop. 6.2}}{\ge} |U_{i-1}| - 626\lambda^{1/4} \\ &\overset{(74)}{\ge} \bigl(\tfrac{1}{3} - 701\lambda^{1/4}\bigr)n \overset{(64)}{\ge} 2. \end{aligned}$$

If $c_{u_i} = c_{v_i}$, then any pair from the set above renders a monochromatic 4-cycle, contradicting (67). □

# 7. Proof of Lemma 2.9 – Part 2: strong cycles and the case $\ell \equiv 2 \pmod 3$

Continuing from the previous section, we now prepare to prove Lemma 2.9 when $\ell \equiv 2 \pmod 3$. The central tools of this proof are important observations on so-called *strong cycles* in the nearly canonical edge-colored graph $(G, c)$. Many of these observations will also be important later when we prove the case $\ell \equiv 1 \pmod 3$ of Lemma 2.9.

## 7.1. Strong cycles

We say that a cycle $C_k = (u_1, \dots, u_k)$ (with prescribed vertex $u_1$) is a *strong* cycle if there exists $i \in \mathbb{Z}_3$ so that $u_1 \in U_i^{\mathrm{good}}$ and $u_k \in N_G^{\mathrm{typ}}(u_1, U_{i-1})$. We determine conditions under which rainbow or properly colored paths can be extended to strong rainbow or strong properly colored cycles.

**Proposition 7.1.** *Fix integers $1 \le k < K \le \ell$ and fix $i, j \in \mathbb{Z}_3$ for which $K - k \equiv (i-1) - j \pmod 3$. Let $P$ be a $(u_i, u_j)$-path on $k$ vertices linking $u_i \in U_i^{\mathrm{good}}$ and $u_j \in U_j$. The following statements hold:*

1. *If $P$ is rainbow and $c_{u_i}$-free, then $P$ may be extended to a strong rainbow $K$-cycle $C_K$;*
2. *If $P$ is properly colored and its $u_i$-edge is not $c_{u_i}$-colored, then $P$ may be extended to a strong properly colored $K$-cycle $C_K$;*
3. *When $K \equiv k \pmod 3$ and $(G, c)$ admits a strong rainbow $k$-cycle $C_k$, then $(G, c)$ also admits a strong rainbow $K$-cycle $C_K$;*
4. *When $K \equiv k \pmod 3$ and $(G, c)$ admits a strong properly colored $k$-cycle $C_k$, then $(G, c)$ also admits a strong properly colored $K$-cycle $C_K$.*

*Proof of Proposition 7.1.* Let integers $1 \le k < K \le \ell$ and elements $i, j \in \mathbb{Z}_3$ be given satisfying $K - k \equiv (i-1) - j \pmod 3$, and let $P = (u_i, \dots, u_j)$ be a $(u_i, u_j)$-path on $k$ vertices linking $u_i \in U_i^{\mathrm{good}}$ and $u_j \in U_j$. To prove Statement (1), assume that $P = R$ is rainbow and $c_{u_i}$-free. Similarly to the proofs of Fact 6.1 and Proposition 6.2, we will extend $R$ to a $c_{u_i}$-free rainbow path $\tilde{R}_{K-k-1} = (u_i, \dots, u_j, v_{j+1}, \dots, v_{j+K-k-1})$ on $K-1$ vertices, where for each $j+1 \le h \le j+K-k-1$, we may choose $v_h \in U_H^{\mathrm{good}}$ for $H \equiv h \pmod 3$. We begin with the first step, where it is not guaranteed in our hypothesis that $u_j \in U_j$ is a good vertex. If $u_j \in U_j^{\mathrm{bad}}$, then

$$(81) \quad \deg^c_G(u_j, U_{j+1}) \overset{(74)}{\geq} \left(\tfrac{1}{9} - 72\lambda^{1/4}\right)n \overset{(74)}{\Longrightarrow} \deg^c_G\left(u_j, U^{\text{good}}_{j+1}\right) \overset{(74)}{\geq} \left(\tfrac{1}{9} - 144\lambda^{1/4}\right)n.$$

Thus, we may select $v_{j+1} \in N_G(u_j, U^{\text{good}}_{j+1})$ for $\tilde{R}_{K-k-1}$ while avoiding $c_{u_i}$ and the colors of $R$. If $u_j \in U^{\text{good}}_j$ is a good vertex, then the neighborhood $N_G(u_j, U^{\text{good}}_{j+1})$ is larger still (cf. (74)), and again we may select $v_{j+1}$ for $\tilde{R}_{K-k-1}$ as described above. We select all remaining vertices $v_h$ for $\tilde{R}_{K-k-1}$, where $j+2 \leq h \leq j+K-k-1$, in a similar fashion. By our hypothesis $K - k \equiv (i-1) - j \pmod 3$, the terminal vertex $v_{j+K-k-1} \in U^{\text{good}}_{i-2}$ while the initial vertex $u_i \in U^{\text{good}}_i$. Comparing Proposition 6.2 and (74), we see

$$(82) \quad d^{\text{typ}}_G(u_i, U_{i-1}) + d^c_G(v_{j+K-k-1}, U_{i-1}) - |U_{i-1}| \geq |U_{i-1}| - 386\lambda^{1/4}n \overset{(74)}{\geq} \left(\tfrac{1}{3} - 461\lambda^{1/4}\right)n$$

and so we may select a vertex $u_{i-1}$ from $N^{\text{typ}}_G(u_i, U_{i-1}) \cap N_G(v_{j+K-k-1}, U_{i-1})$ whose adjacency with $v_{j+K-k-1}$ avoids $c_{u_i}$ and the colors of $\tilde{R}_{K-k-1}$. Since $c(\{u_i, u_{i-1}\}) = c_{u_i}$ is the primary color of $u_i$, which hasn't yet been used, $C_K = (u_i, \ldots, u_j, v_{j+1}, \ldots, v_{j+K-k-1}, u_{i-1})$ is a strong rainbow $K$-cycle.

The proof of Statement (2) is absolutely the same as that of Statement (1). In particular, for the properly colored $k$-vertex path $P = (u_i, \ldots, u_j)$ linking $u_i \in U^{\text{good}}_i$ and $u_j \in U_j$ whose $u_i$-edge is not $c_{u_i}$-colored, the proof above allows the segment $(u_j, v_{j+1}, \ldots, v_{j+K-k-1}, u_{i-1})$ of $\tilde{R}_{K-k-1}$ to be rainbow, $c_{u_i}$-free, and to be free of the colors from $P$. Thus, $C_K = (u_i, \ldots, u_j, v_{j+1}, \ldots, v_{j+K-k-1}, u_{i-1})$ is a strong properly colored $K$-cycle.

Statements (3) and (4) now follow immediately from Statements (1) and (2). Indeed, let $C_k = (u_1, \ldots, u_k)$ be a strong rainbow or properly colored $k$-cycle where $u_1 \in U^{\text{good}}_i$ and $u_k \in N^{\text{typ}}(u_1, U_{i-1})$ for some $i \in \mathbb{Z}_3$. Ignoring the edge $\{u_1, u_k\}$, the path $P_k = (u_1, \ldots, u_k)$ is rainbow or properly colored, where $u_k \in U_{i-1}$ assumes $j = i-1$. Taking $K \equiv k+(i-1)-(i-1) \equiv k \pmod 3$ and $K \leq \ell$, Statements (1) or (2) extend $P_k$ to a strong rainbow or properly colored $K$-cycle $C_K$. □

It will be convenient to have the following corollary of Proposition 7.1 in the case $\ell \equiv 2 \pmod 3$.

**Corollary 7.2.** *Let $\ell \equiv 2 \pmod 3$ and fix $i \in \mathbb{Z}_3$. The following statements hold:*

1. *Each* $u_i \in U_i^{\text{good}}$ *satisfies* $N_G^{\text{spec}}(u_i, U_{i-1}) = \emptyset$;
2. *Let* $R = (u_i, v, w_i)$ *be a rainbow path with* $u_i \in U_i^{\text{good}}$ *and* $w_i \in U_i$. *Then* $c_{u_i}$ *appears on* $R$. *In particular,* $c(\{u_i, v\}) = c_{u_i}$ *or* $(G, c)$ *admits a properly colored* $\ell$*-cycle* $C_\ell$;
3. *Let* $R = (u_i, v, u_{i-1})$ *be a rainbow path with* $u_i \in U_i^{\text{good}}$ *and* $u_{i-1} \in U_{i-1}^{\text{good}}$. *Then* $c_{u_i} = c_{u_{i-1}}$ *or* $c_{u_i}$ *or* $c_{u_{i-1}}$ *appears on* $R$. *As well, if* $c(\{u_i, v\}) \neq c_{u_i}$ *and* $c(\{u_{i-1}, v\}) \neq c_{u_{i-1}}$, *then* $(G, c)$ *admits a properly colored* $\ell$*-cycle* $C_\ell$;
4. *Let* $\ell \neq 5$, *and let* $u_i, v_i \in U_i^{\text{good}}$ *and* $w_i, x_i \in U_i$ *span disjoint edges* $\{u_i, w_i\}, \{v_i, x_i\} \in E(G)$. *Then* $c_{u_i}$, $c_{v_i}$, $c(\{u_i, w_i\})$, *and* $c(\{v_i, x_i\})$ *can't all be distinct.*
5. *Let* $\ell \neq 5$, *and let* $T_i \subseteq U_i^{\text{good}}$ *be a set with the property that for all* $u_i \in T_i$, *there exist* $v_i \neq w_i \in N_G(u_i, U_i)$ *so that* $c(\{u_i, v_i\})$, $c(\{u_i, w_i\})$, *and* $c_{u_i}$ *are all distinct. Then* $|T_i| \leq 5$.

*Proof of Corollary 7.2.* Let $\ell \equiv 2 \pmod 3$ and fix $i \in \mathbb{Z}_3$. For Statement (1), fix $u_i \in U_i^{\text{good}}$. If $u_{i-1} \in N_G^{\text{spec}}(u_i, U_{i-1})$, then $\{u_i, u_{i-1}\}$ is a $c_{u_i}$-free rainbow path which Proposition 7.1 guarantees can be extended to a strong rainbow $\ell$-cycle $C_\ell$ (by setting $j = i - 1$ and $k = 2$, and with $\ell \equiv 2 \pmod 3$), which contradicts (66).

For Statement (2), let $R = (u_i, v, w_i)$ be a rainbow path with $u_i \in U_i^{\text{good}}$ and $w_i \in U_i$. If $R$ is $c_{u_i}$-free, then Proposition 7.1 guarantees that $R$ can be extended to a strong rainbow $\ell$-cycle $C_\ell$ (by setting $j = i$ and $k = 3$, and with $\ell \equiv 2 \pmod 3$), which again contradicts (66). In particular, if $c(\{u_i, v\}) \neq c_{u_i}$, then Proposition 7.1 guarantees that $R$ can be extended to a strong properly colored $\ell$-cycle $C_\ell$.

For Statement (3), let $R = (u_i, v, u_{i-1})$ be a rainbow path with $u_i \in U_i^{\text{good}}$ and $u_{i-1} \in U_{i-1}^{\text{good}}$. Assume for contradiction that $R$ avoids both $c_{u_i} \neq c_{u_{i-1}}$. Since $u_{i-1} \in U_{i-1}^{\text{good}}$ is a good vertex, Proposition 6.2 guarantees a vertex $u_{i-2} \in N_G(u_{i-1}, U_{i-2}^{\text{good}})$ distinct from $v$ for which $c(\{u_{i-1}, u_{i-2}\}) = c_{u_{i-1}}$. Then the path $S = (u_i, v, u_{i-1}, u_{i-2})$ is rainbow (because $R$ is rainbow and avoids $c_{u_{i-1}}$), and the path $S$ avoids $c_{u_i}$ (because $R$ does and because $c_{u_i} \neq c_{u_{i-1}}$). As such, Proposition 7.1 guarantees that $S$ can be extended to a strong rainbow $\ell$-cycle $C_\ell$ (by setting $j = i - 2$ and $k = 4$, and with $\ell \equiv 2 \pmod 3$), which contradicts (66). In particular, assume $c(\{u_i, v\}) \neq c_{u_i}$ and $c(\{u_{i-1}, v\}) \neq c_{u_{i-1}}$. Then $S = (u_i, v, u_{i-1}, u_{i-2})$ is proper (because $R$ is rainbow and $c(\{u_{i-1}, v\}) \neq c_{u_{i-1}}$). Since $c(\{u_i, v\}) \neq c_{u_i}$, Proposition 7.1 guarantees that $S$ can be extended to a strong properly colored $\ell$-cycle $C_\ell$.

For Statement (4), let $\ell \neq 5$, and let $u_i, v_i \in U_i^{\text{good}}$ and $w_i, x_i \in U_i$ span disjoint edges $\{u_i, w_i\}, \{v_i, x_i\} \in E(G)$. Assume, on the contrary, that

$C = \{c_{u_i}, c_{v_i}, c(\{u_i, w_i\}), c(\{v_i, x_i\})\}$ is a set of four distinct colors. Fix any $u_{i+1} \in N_G(w_i, U_{i+1}^{\text{good}})$ where the edge $\{w_i, u_{i+1}\} \in E(G)$ is $C$-free (which is possible by the argument in (81)). Now, fix any

$$u_{i-1} \in N_G(u_{i+1}, U_{i-1}) \cap N_G^{\text{typ}}(v_i, U_{i-1})$$

where the edge $\{u_{i-1}, u_{i+1}\} \in E(G)$ is $(C \cup c(\{w_i, u_{i+1}\}))$-free (which is possible by the argument in (82)). Now, $(u_i, w_i, u_{i+1}, u_{i-1}, v_i, x_i)$ is a rainbow path avoiding $c_{u_i}$, which Proposition 7.1 guarantees can be extended to a strong rainbow $\ell$-cycle $C_\ell$ (by setting $j = i$ and $k = 6$, and with $\ell \equiv 2 \pmod 3$), which again contradicts (66).

For Statement (5), let $T_i \subseteq U_i^{\text{good}}$ be a set with the property so described, but assume for contradiction that $|T_i| \geq 6$. Fix $u_i \in T_i$, where we take $c_{u_i}$ to be *blue*, and let $v_i \neq w_i \in N_G(u_i, U_i)$ be guaranteed by the definition of $T_i$, where we take $c(\{u_i, v_i\})$ to be *red* and $c(\{u_i, w_i\})$ to be *yellow*. Since $|T_i| \geq 6$, there exists $x_i \in T_i \setminus \{u_i, v_i, w_i\}$ where $c_{x_i}$ is neither red nor yellow. Since $x_i \neq u_i$, Corollary 6.3 guarantees that $c_{x_i} \neq c_{u_i}$ can't be blue, so we take $c_{x_i}$ to be *green*. Let $y_i \neq z_i \in N_G(x_i, U_i)$ be guaranteed by the definition of $T_i$. We now distinguish the extent to which $\{u_i, v_i, w_i\}$ and $\{x_i, y_i, z_i\}$ overlap.

**Case 1 ($\{u_i, v_i, w_i\} \cap \{x_i, y_i, z_i\} = \emptyset$).** If $c(\{x_i, y_i\})$ is yellow, then $\{u_i, v_i\}$ and $\{x_i, y_i\}$ violate Statement (4) above. Similarly, if $c(\{x_i, y_i\})$ is red, then $\{u_i, w_i\}$ and $\{x_i, y_i\}$ violate the same. Assume neither $\{x_i, y_i\}$ nor $\{x_i, z_i\}$ is red or yellow, where the definition of $T_i$ ensures neither is green. At most one of these pairs can be blue, so assume $\{x_i, y_i\}$ is neither red, yellow, green, nor blue. Now, $\{u_i, v_i\}$ and $\{x_i, y_i\}$ violate Statement (4) above. $\square$

**Case 2 ($u_i \in \{y_i, z_i\}$).** Assume w.l.o.g. that $u_i = z_i$. If $c(\{u_i, x_i\})$ is yellow, then $(x_i, u_i, v_i)$ violates Statement (2) above. If $c(\{u_i, x_i\})$ is not yellow, then it is also not green by the definition of $T_i$, and so $(x_i, u_i, w_i)$ violates the same Statement (2). $\square$

**Remark.** Since $x_i \in U_i^{\text{good}} \setminus \{u_i, v_i, w_i\}$, we do not have the case $x_i \in \{v_i, w_i\}$. $\square$

**Case 3 ($u_i \notin \{y_i, z_i\}$; $\{v_i, w_i\} \cap \{y_i, z_i\} \neq \emptyset$).** Assume w.l.o.g. that $w_i = y_i$. If $c(\{x_i, y_i\})$ is yellow, then $\{u_i, v_i\}$ and $\{x_i, y_i\}$ violate Statement (4) above. If $c(\{x_i, y_i\})$ is red, then $(u_i, w_i = y_i, x_i)$ violates Statement (2) above. If $c(\{x_i, y_i\})$ is blue, then $(x_i, y_i = w_i, u_i)$ violates Statement (2) above. Otherwise, $c(\{x_i, y_i\})$ isn't green by the definition of $T_i$, so $\{u_i, v_i\}$ and $\{x_i, y_i\}$ violate Statement (4) above. $\square$

### 7.2. Proof of Lemma 2.9: Statement (1) when $\ell \equiv 2 \pmod 3$

Let $\ell \equiv 2 \pmod 3$, where $\ell \neq 5$. The hypothesis of Statement (1) of Lemma 2.9 gives that $\delta^c(G) \geq (n+5)/3$. Assume w.l.o.g. that

$$(83) \quad |U_2| \leq |U_1| \leq |U_0|, \qquad \text{in which case} \qquad |U_2| \leq \left\lfloor \tfrac{n}{3} \right\rfloor \leq \left\lceil \tfrac{n}{3} \right\rceil \leq |U_0|.$$

In the immediate sequel, we motivate the main approach of the proof.

**7.2.1. Main idea of proof** We shall make repeated use of Statement (5) of Corollary 7.2, for which we establish the following notation. Fix $i \in \mathbb{Z}_3$ and $u_i \in U_i^{\rm good}$, and define

$$(84) \qquad c^{\rm spec}(u_i, U_i) = \left\{ c(\{u_i, v_i\}) \neq c_{u_i} : v_i \in N_G(u_i, U_i) \right\}$$

for the set of special (non-primary) colors on edges $\{u_i, v_i\} \in E_G(u_i, U_i)$ incident to $u_i$ in $U_i$. By Statement (1) of Corollary 7.2, all edges $\{u_i, u_{i-1}\} \in E_G(u_i, U_{i-1})$ are colored by $c_{u_i}$, and so

$$(85) \quad \begin{aligned} \left|c^{\rm spec}(u_i, U_i)\right| &\geq \deg_G^c(u_i) - \deg_G^c(u_i, U_{i-1}) - \deg_G^c(u_i, U_{i+1}) \\ &= \deg_G^c(u_i) - 1 - \deg_G^c(u_i, U_{i+1}) \geq \delta^c(G) - 1 - |U_{i+1}| \\ &\geq \tfrac{n+5}{3} - 1 - |U_{i+1}| = \tfrac{n+2}{3} - |U_{i+1}|. \end{aligned}$$

In particular, when $i = 1 \in \mathbb{Z}_3$, we infer that every $u_1 \in U_1^{\rm good}$ satisfies

$$(86) \qquad \left|c^{\rm spec}(u_1, U_1)\right| \geq \tfrac{n+2}{3} - |U_2| \overset{(83)}{\geq} \tfrac{n+2}{3} - \left\lfloor \tfrac{n}{3} \right\rfloor.$$

As such, if $n \equiv 2 \pmod 3$, then (86) gives $|c^{\rm spec}(u_1, U_1)| \geq 2$ for every $u_1 \in U_1^{\rm good}$, and so $T_1 = U_1^{\rm good}$ readily contradicts Statement (5) of Corollary 7.2 (because $|U_1^{\rm good}|$ from (74) is much too large). Similarly, if $|U_2| \leq \lfloor n/3 \rfloor - 1$, then (86) gives $|c^{\rm spec}(u_1, U_1)| \geq 2$ for every $u_1 \in U_1^{\rm good}$, giving the same contradiction. The main idea of the current proof exploits a similar theme to the instances $n \equiv 2 \pmod 3$ or $|U_2| \leq \lfloor n/3 \rfloor - 1$, which we announce as our goal:

(87) *we seek to determine a large set $T_i \subseteq U_i^{\rm good}$, for some $i \in \mathbb{Z}_3$, where every $u_i \in T_i$ satisfies $|c^{\rm spec}(u_i, U_i)| \geq 2$.*

When so, we contradict Statement (5) of Corollary 7.2.

**7.2.2. Supporting details** From the discussion above, it suffices to consider the case $n \not\equiv 2 \pmod 3$ and $|U_2| = \lfloor n/3 \rfloor$. As such, $|U_2| = |U_1| = \lfloor n/3 \rfloor$ and $|U_0| = \lceil n/3 \rceil$. Now, for $u_1 \in U_1^{\text{good}}$, we define

$$S(u_1) = \{v_1 \in N_G(u_1, U_1) : c(\{u_1, v_1\}) \neq c_{u_1}\}.$$

We refine the partition $U_1 = U_1^{\text{good}} \cup U_1^{\text{bad}}$ from (74) by subdividing $U_1^{\text{good}}$ into

$$A_1 = \{u_1 \in U_1^{\text{good}} : S(u_1) \cap U_1^{\text{bad}} \neq \emptyset\} \quad \text{and} \quad B_1 = U_1^{\text{good}} \setminus A_1. \tag{88}$$

We will observe the following fact.

**Fact 7.3.** *Every $u_1 \in B_1$ satisfies $S(u_1) \subseteq A_1$.*

*Proof of Fact 7.3.* Fix $u_1 \in B_1$, but assume for contradiction that there exists $v_1 \in S(u_1) \cap B_1$. Since both $u_1 \neq v_1 \in U_1^{\text{good}}$ are good vertices, Corollary 6.3 guarantees that $c_{u_1} \neq c_{v_1}$, where we will take $c_{u_1}$ to be *red* and $c_{v_1}$ to be *blue*. From $v_1 \in S(u_1)$, we infer that $c(\{u_1, v_1\})$ is not $c_{u_1} =$ red. We distinguish two cases.

**Case 1 ($c(\{u_1, v_1\}) \neq c_{v_1}$).** Here, we will take $c(\{u_1, v_1\})$ to be *yellow*. Proposition 6.2 and Corollary 6.3 guarantee a vertex $u_0 \in N_G^{\text{typ}}(v_1, U_0^{\text{good}})$ so that $c(\{u_0, v_1\}) = c_{v_1}$ is blue but $c_{u_0}$ is neither red, blue, nor yellow. We take $c_{u_0}$ to be *green*. Now, $R = (u_1, v_1, u_0)$ is a rainbow path where $u_1 \in U_1^{\text{good}}$, where $u_0 \in U_0^{\text{good}}$, but where neither $c_{u_1} \neq c_{u_0}$ (red nor green) appear on $R$, which contradicts Statement (3) of Corollary 7.2. □

**Case 2 ($c(\{u_1, v_1\}) = c_{v_1}$).** From (86), we infer that $|S(v_1)| \geq 1$, where $u_1 \notin S(v_1)$ on account that $c(\{u_1, v_1\}) = c_{v_1}$ is blue. From $v_1 \in B_1$, we infer that $S(v_1) \cap U_1^{\text{bad}} = \emptyset$, and so there exists $u_1 \neq w_1 \in S(v_1) \subseteq U_1^{\text{good}}$. From $w_1 \in S(v_1)$, we infer that $c(\{v_1, w_1\}) \neq c_{v_1}$ is not blue. So the path $(u_1, v_1, w_1)$ is rainbow, and Statement (2) of Corollary 7.2 implies that $c(\{v_1, w_1\})$ is $c_{u_1}$ (red). Since $u_1, v_1, w_1 \in U_1^{\text{good}}$ are good and distinct, Corollary 6.3 guarantees that the primary colors $c_{u_1}$ (red), $c_{v_1}$ (blue), and $c_{w_1}$ are distinct, where we take $c_{w_1}$ to be *green*. Since $w_1 \in U_1^{\text{good}}$ is good, Proposition 6.2 and Corollary 6.3 guarantee a vertex $u_0 \in N_G^{\text{typ}}(w_1, U_0^{\text{good}})$ so that $c_{u_0}$ is neither $c_{v_1}$ (blue), $c(\{u_0, w_1\})$ (green), nor $c(\{v_1, w_1\})$ (red). Now, $(v_1, w_1, u_0)$ contradicts Statement (3) of Corollary 7.2. □

Fact 7.3 admits the following corollary.

**Corollary 7.4.** *There exist distinct $u_1, v_1, w_1 \in U_1^{\mathrm{good}}$ satisfying $S(u_1) \cap S(v_1) \cap S(w_1) \neq \emptyset$.*

*Proof of Corollary 7.4.* Define the auxiliary directed graph $\vec{\Gamma} = (U_1, \vec{E})$ by the rule that for each $(u_1, v_1) \in U_1 \times U_1$, we put $(u_1, v_1) \in \vec{E}$ if, and only if, $v_1 \in S(u_1)$. In this notation, $S(u_1) = N^+_{\vec{\Gamma}}(u_1)$. We now distinguish two cases.

**Case 1 ($|B_1| > 2|A_1|$).** For the bipartition $A_1 \cup B_1$ (cf. (88)), we infer

$$\sum_{a_1 \in A_1} \left|N^-_{\vec{\Gamma}}(a_1) \cap B_1\right| = \sum_{b_1 \in B_1} \left|N^+_{\vec{\Gamma}}(b_1) \cap A_1\right|$$
$$= \sum_{b_1 \in B_1} |S(b_1) \cap A_1| \overset{\text{Fct. } 7.3}{=} \sum_{b_1 \in B_1} |S(b_1)| \overset{(86)}{\geq} |B_1| > 2|A_1|.$$

By averaging, there exists $\bar{a}_1 \in A_1$ which satisfies $|N^-_{\vec{\Gamma}}(\bar{a}_1) \cap B_1| \geq 3$, so let $b_1, b_1', b_1'' \in N^-_{\vec{\Gamma}}(\bar{a}_1)$. Then

$$\bar{a}_1 \in N^+_{\vec{\Gamma}}(b_1) \cap N^+_{\vec{\Gamma}}\big(b_1'\big) \cap N^+_{\vec{\Gamma}}\big(b_1''\big) = S(b_1) \cap S\big(b_1'\big) \cap S\big(b_1''\big),$$

and so $S(b_1) \cap S(b_1') \cap S(b_1'') \neq \emptyset$. □

**Case 2 ($|B_1| \leq 2|A_1|$).** For the bipartition $A_1 \cup U_1^{\mathrm{bad}}$ (recall $A_1 \subseteq U_1^{\mathrm{good}}$), we infer

$$(89) \quad \sum_{u_1 \in U_1^{\mathrm{bad}}} \left|N^-_{\vec{\Gamma}}(u_1) \cap A_1\right| = \sum_{a_1 \in A_1} \left|N^+_{\vec{\Gamma}}(a_1) \cap U_1^{\mathrm{bad}}\right|$$
$$= \sum_{a_1 \in A_1} \left|S(a_1) \cap U_1^{\mathrm{bad}}\right| \overset{\text{def}}{\geq} |A_1|,$$

where we used the definition of $A_1$ from (88). Moreover, from the bipartition $U_1^{\mathrm{good}} = A_1 \cup B_1$, we infer

$$(90) \quad 3|A_1| \geq |A_1| + |B_1| = \left|U_1^{\mathrm{good}}\right| \overset{(74)}{\geq} \left(\tfrac{1}{3} - 75\lambda^{1/4}\right)n$$
$$\overset{(64)}{\geq} 648\lambda^{1/4}n \overset{(74)}{\geq} 9\left|U_1^{\mathrm{bad}}\right| \qquad \Longrightarrow \qquad |A_1| \geq 3\left|U_1^{\mathrm{bad}}\right|.$$

Combining (89) and (90) yields $\sum_{u_1 \in U_1^{\mathrm{bad}}} |N^-_{\vec{\Gamma}}(u_1) \cap A_1| \geq 3|U_1^{\mathrm{bad}}|$, and so an average vertex $\bar{u}_1 \in U_1^{\mathrm{bad}}$ satisfies $|N^-_{\vec{\Gamma}}(\bar{u}_1) \cap A_1| \geq 3$. Let $a_1, a_1', a_1'' \in$

$N^-_{\vec{\Gamma}}(\bar{u}_1)$, in which case

$$\bar{u}_1 \in N^+_{\Gamma}(a_1) \cap N^+_{\Gamma}\left(a_1'\right) \cap N^+_{\Gamma}\left(a_1''\right) = S(a_1) \cap S\left(a_1'\right) \cap S\left(a_1''\right),$$

and so $S(a_1) \cap S(a_1') \cap S(a_1'') \neq \emptyset$. □

For the remainder of the proof, we fix distinct $u_1, v_1, w_1 \in U_1^{\text{good}}$ guaranteed by Corollary 7.4. We also fix an element $x_1 \in S(u_1) \cap S(v_1) \cap S(w_1)$. We garner the following useful corollary.

**Corollary 7.5.** *The coloring $c$ is constant on the edges $E_G(x_1, U_1)$.*

*Proof of Corollary 7.5.* We first show that

$$c(\{u_1, x_1\}) = c(\{v_1, x_1\}) = c(\{w_1, x_1\}). \tag{91}$$

For that, since $u_1, v_1, w_1 \in U_1^{\text{good}}$ are distinct good vertices, Corollary 6.3 guarantees that $c_{u_1}$, $c_{v_1}$, and $c_{w_1}$ are distinct, so we take $c_{u_1}$ to be *red*, $c_{v_1}$ to be *blue*, and $c_{w_1}$ to be *yellow*. Assume, on the contrary, that $c(\{u_1, x_1\}) \neq c(\{v_1, x_1\})$. Then $(u_1, x_1, v_1)$ is a rainbow $U_1$-path where $u_1 \in U_1^{\text{good}}$ is a good vertex, so Statement (2) of Corollary 7.2 guarantees that $c(\{x_1, v_1\})$ is $c_{u_1} =$ red. Applying the same argument to $(v_1, x_1, u_1)$, we infer that $c(\{u_1, x_1\})$ is $c_{v_1} =$ blue. Now, $c_{w_1} =$ yellow appears on neither $(w_1, x_1, u_1)$ nor $(w_1, x_1, v_1)$ (since $x_1 \in S(w_1)$ guarantees that $c(\{w_1, x_1\})$ is not $c_{w_1} =$ yellow). Since $w_1 \in U_1^{\text{good}}$ is a good vertex, Statement (2) of Corollary 7.2 guarantees that both $(w_1, x_1, u_1)$ and $(w_1, x_1, v_1)$ are monochromatic, and so $c(\{w_1, x_1\})$ is both red and blue, a contradiction.

Corollary 7.5 now easily follows from (91), where we take that common color to be *green*. By the argument above, any edge $\{x_1, y_1\} \in E_G(x_1, U_1)$ that isn't colored green must be colored each of red, blue, and yellow, which isn't possible. □

**7.2.3. Finale** We return to our goal in (87). Let $u_1, v_1, w_1 \in U_1^{\text{good}}$ and $x_1 \in S(u_1) \cap S(v_1) \cap S(w_1)$ be fixed from the previous subsection, where all of $E_G(x_1, U_1)$ is colored *green*, which is the only color from before which we now need to reference. Then $E_G(x_1, U_1 \cup U_2)$ admits at most $|U_2| + 1$ colors, the set of which we call $C = C(x_1, U_1, U_2)$. As such, the number of non-$C$ colors on $E_G(x_1, U_0)$ is at least

$$\begin{aligned}\deg^c_G(x_1) - \deg^c_G(x_1, U_1) - \deg^c_G(x_1, U_2) &\geq \delta^c(G) - 1 - |U_2| \\ &\geq \tfrac{n+5}{3} - 1 - |U_2| = \tfrac{n+2}{3} - |U_2| \overset{(83)}{\geq} \tfrac{n+2}{3} - \left\lfloor \tfrac{n}{3} \right\rfloor,\end{aligned}$$

which is positive. Fix $u_0 \in N_G(x_1, U_0)$ where $c(\{u_0, x_1\}) \notin C$. In particular, $c(\{u_0, x_1\})$ is not green, and we take $c(\{u_0, x_1\})$ to be *purple*. (It won't matter if $c(\{u_0, x_1\})$ appeared in the previous subsection, so long as $c(\{u_0, x_1\})$ is not green.) Define

(92)
$$T_0 = \{v_0 \in U_0^{\text{good}} : v_0 \neq u_0, \, c_{v_0} \neq c(\{u_0, x_1\}) = \text{purple}, \, c_{v_0} \neq \text{green}\},$$

where Corollary 6.3 guarantees

$$|T_0| \geq |U_0^{\text{good}}| - 3 \overset{(74)}{\geq} \left(\tfrac{1}{3} - 75\lambda^{1/4}\right)n - 3 = \Omega(n). \tag{93}$$

We make the following critical observation.

**Observation 7.6.** *An edge $\{v_0, x_1\} \in E_G(x_1, T_0)$ must be colored $c_{v_0}$.*

*Proof of Observation 7.6.* For a fixed $\{v_0, x_1\} \in E_G(x_1, T_0)$, we distinguish two cases.

**Case 1 ($c(\{v_0, x_1\}) \neq c(\{u_0, x_1\}) =$ purple).** Here, $(v_0, x_1, u_0)$ is a rainbow path where $v_0 \in T_0 \subseteq U_0^{\text{good}}$ is a good vertex. Statement (2) of Corollary 7.2 guarantees that $c_{v_0}$ appears on $(v_0, x_1, u_0)$, and since $c_{v_0} \neq c(\{u_0, x_1\}) =$ purple holds by the definition of $T_0$, we must have $c(\{v_0, x_1\}) = c_{v_0}$. □

**Case 2 ($c(\{v_0, x_1\}) = c(\{u_0, x_1\}) =$ purple).** Among the fixed distinct vertices $u_1, v_1, w_1 \in U_1^{\text{good}}$ above, Corollary 6.3 guarantees that at most one of the distinct colors $c_{u_1}$, $c_{v_1}$, $c_{w_1}$ can equal $c(\{u_0, x_1\}) = c(\{v_0, x_1\}) =$ purple, and at most one of $c_{u_1}$, $c_{v_1}$, $c_{w_1}$ can equal $c_{v_0}$. Assume w.l.o.g. that

$$c_{v_0} \neq c_{u_1} \neq c(\{u_0, x_1\}) = c(\{v_0, x_1\}) = \text{purple}.$$

Now, the path $(u_1, x_1, v_0)$ is a green-purple rainbow path where $u_1 \in U_1^{\text{good}}$ and $v_0 \in T_0 \subseteq U_0^{\text{good}}$ are good vertices satisfying $c_{u_1} \neq c_{v_0}$. Statement (3) guarantees that one of $c_{v_0} \neq c_{u_1}$ appears on $(u_1, x_1, v_0)$, but neither do. Indeed, $c_{v_0}$ is neither green nor purple by the definition of $T_0$ (cf. (92)), and $c_{u_1}$ is not green by $x_1 \in S(u_1)$ and it is not purple by our choice above. □

We now conclude the proof of Statement (1) of Lemma 2.9 when $\ell \equiv 2 \pmod 3$. Fix a vertex $v_0 \in T_0$. By combining Statement (1) of Corollary 7.2 with Observation 7.6, we conclude that all edges $E_G(v_0, U_2 \cup \{x_1\})$ are colored the single primary color $c_{v_0}$. However distinctly the edges $E_G(v_0, U_1 \setminus \{x_1\})$ are colored, the edges $E_G(v_0, U_1 \cup U_2)$ are colored with at most $1 + (|U_1| - 1) = |U_1| = \lfloor n/3 \rfloor$ many colors, one of which is the primary color $c_{v_0}$. (Recall that it suffices to consider the case $|U_2| = |U_1| = \lfloor n/3 \rfloor$.) All remaining colors incident to $v_0$ are special and are applied to $E_G(v_0, U_0)$, the number of which is precisely given by the parameter $|c^{\text{spec}}(v_0, U_0)|$ from (84). Altogether, we conclude

$$\begin{aligned} (94) \quad \left|c^{\text{spec}}(v_0, U_0)\right| &\geq \deg^c_G(v_0, U_0) - |c(E_G(v_0, U_1 \cup U_2)| \\ &\geq \delta^c(G) - \left\lfloor \tfrac{n}{3} \right\rfloor \geq \tfrac{n+5}{3} - \left\lfloor \tfrac{n}{3} \right\rfloor \geq \tfrac{5}{3}, \end{aligned}$$

and therefore $|c^{\text{spec}}(v_0, U_0)| \geq 2$. Now, (93) and (94) together contradict Statement (5) of Corollary 7.2.

### 7.3. Proof of Lemma 2.9: Statement (2) when $\ell \equiv 2 \pmod 3$

Let $\ell \equiv 2 \pmod 3$. The hypothesis of Statement (2) of Lemma 2.9 gives that $\delta^c(G) \geq (n+4)/3$ (cf. (129)). We again assume w.l.o.g. that (83) holds, and we want to conclude that $(G, c)$ admits a properly colored $\ell$-cycle $C_\ell$.

(95) *We assume, on the contrary, that* $(G, c)$ *does not admit a properly colored* $\ell$*-cycle* $C_\ell$*.*

We will show that our assumption in (95) guarantees vertices $x_1 \in U_1^{\text{good}}$ and $y_1, z_1 \in U_1$, where $(x_1, y_1, z_1)$ is a rainbow $U_1$-path satisfying $c(\{x_1, y_1\}) \neq c_{x_1}$. Then (95) contradicts Statement (2) of Corollary 7.2.

We begin our work with an observation. Fix an auxiliary vertex $u_0 \in U_0^{\text{good}}$, where Statement (1) of Corollary 7.2 guarantees that $E_G(u_0, U_2)$ is colored only with $c_{u_0}$. We observe that

(96) $\qquad E_G(u_0, U_0)$ is also colored only with $c_{u_0}$.

To see (96), suppose $v_0 \in N_G(u_0, U_0)$ admits $c(\{u_0, v_0\}) \neq c_{u_0}$. Let $u_1 \in N_G(v_0, U_1^{\text{good}})$ have color $c(\{v_0, u_1\}) \neq c(\{u_0, v_0\})$, where we used $\deg^c_G(v_0, U_1^{\text{good}}) \geq ((1/9) - 144\lambda^{1/4})n$ implicit in (74). Statement (1) of Corollary 7.2 guarantees that $c(\{v_0, u_1\}) = c_{u_1}$. As such, the number $|c^{\text{spec}}(u_1, U_1)|$ of special colors incident to $u_1$ in $U_1$ satisfies

$$\begin{aligned}(97)\quad \left|c^{\mathrm{spec}}(u_1,U_1)\right| &\geq \deg^c_G(u_1) - \deg^c_G(u_1,U_0) - \deg^c_G(u_1,U_2) \\ &\geq \delta^c(G) - \deg^c_G(u_1,U_0) - \deg^c_G(u_1,U_2) = \delta^c(G) - 1 - \deg^c_G(u_1,U_2) \\ &\geq \tfrac{n+4}{3} - 1 - |U_2| \overset{(83)}{\geq} \tfrac{n+1}{3} - \left\lfloor \tfrac{n}{3} \right\rfloor \geq \tfrac{1}{3},\end{aligned}$$

so fix $v_1 \in N_G(u_1,U_1)$ where $c(\{u_1,v_1\}) \neq c_{u_1}$. Now, $P = (u_0,v_0,u_1,v_1)$ is a properly colored path where $c(\{u_0,v_0\}) \neq c_{u_0}$. Proposition 7.1 guarantees (with $i=0$, $j=1$, $k=4$, and $\ell \equiv 2 \pmod 3$) that $P$ can be extended to a strong rainbow $\ell$-cycle $C_\ell$, contradicting (95). This proves (96).

We choose the first promised vertex $x_1 \in U_1^{\mathrm{good}}$ arbitrarily, where the auxiliary vertex $u_0 \in U_0^{\mathrm{good}}$ above is still fixed. To choose the second promised vertex $y_1 \in U_1$, define

$$(98)\qquad \begin{aligned} A_{u_0} &= \{u_1 \in N_G(u_0,U_1) : c(\{u_0,u_1\}) \neq c_{u_0}\} \text{ and} \\ B_{x_1} &= \{v_1 \in N_G(x_1,U_1) : c(\{x_1,v_1\}) \neq c_{x_1}\}.\end{aligned}$$

(The set $B_{x_1}$ is the same as $S(x_1)$ from the previous subsection.) Then $A_{u_0} \cup B_{x_1} \subseteq U_1$, and so

$$(99)\qquad |A_{u_0} \cap B_{x_1}| = |A_{u_0}| + |B_{x_1}| - |A_{u_0} \cup B_{x_1}| \geq |A_{u_0}| + |B_{x_1}| - |U_1|.$$

From our observation above (cf. (96)), all of $E_G(u_0, U_0 \cup U_2)$ is colored with $c_{u_0}$, and therefore $|A_{u_0}| \geq \deg^c_G(u_0) - 1$. Since $x_1 \in U_1^{\mathrm{good}}$ is a good vertex, Statement (1) of Corollary 7.2 guarantees that all of $E_G(x_1,U_0)$ is colored with $c_{x_1}$, and therefore

$$|B_{x_1}| \geq \deg^c_G(x_1) - \deg^c_G(x_1,U_0) - \deg^c_G(x_1,U_2) \geq \deg^c_G(x_1) - 1 - |U_2|.$$

Returning to (99), we conclude

$$\begin{aligned}|A_{u_0} \cap B_{x_1}| &\geq |A_{u_0}| + |B_{x_1}| - |U_1| \geq \deg^c_G(u_0) + \deg^c_G(x_1) - 2 - |U_1| - |U_2| \\ &= \deg^c_G(u_0) + \deg^c_G(x_1) - 2 - (n - |U_0|) \geq 2\delta^c(G) - 2 - n + |U_0| \\ &\geq 2\left(\tfrac{n+4}{3}\right) - 2 - n + |U_0| = \tfrac{2n+2}{3} - n + |U_0| \overset{(83)}{\geq} \tfrac{2n+2}{3} - n + \left\lceil \tfrac{n}{3} \right\rceil \geq \tfrac{2}{3}.\end{aligned}$$

Fix $y_1 \in A_{u_0} \cap B_{x_1}$ arbitrarily.

To choose the third promised vertex $z_1 \in U_1$, we make a couple observations. First, we observe that the path $(x_1, y_1, u_0)$ must be monochromatic.

Indeed, since $y_1 \in A_{u_0} \cap B_{x_1}$, we infer from (98) that $c(\{u_0, y_1\}) \neq c_{u_0}$ and $c(\{x_1, y_1\}) \neq c_{x_1}$. Thus, if $(x_1, y_1, u_0)$ were rainbow, then Statement (3) of Corollary 7.2 would guarantee that $(G, c)$ admits a properly colored $\ell$-cycle $C_\ell$, contradicting (95). Henceforth, we take $c(\{u_0, y_1\}) = c(\{x_1, y_1\})$ to be *blue*. Second, we observe that

$$\text{(100)} \qquad \text{all of } E_G(y_1, U_0) \text{ is colored by } c(\{u_0, y_1\}) = c(\{x_1, y_1\}) = \text{blue}.$$

Indeed, suppose $\{v_0, y_1\} \in E_G(y_1, U_0)$ admitted $c(\{v_0, y_1\}) \neq c(\{u_0, y_1\}) =$ blue. Then $(u_0, y_1, v_0)$ is a rainbow path with $c(\{u_0, y_1\}) \neq c_{u_0}$ (because $y_1 \in A_{u_0}$ from (98)). Statement (2) of Corollary 7.2 then guarantees that $(G, c)$ admits a properly colored $\ell$-cycle $C_\ell$, again contradicting (95). Now, all of $E_G(y_1, U_0 \cup \{x_1\})$ is colored blue, and so the number of non-blue colors of $E_G(y_1, U_1)$ is at least

$$\begin{aligned}\deg_G^c(y_1) - \deg_G^c(y_1, U_0) - \deg_G^c(y_1, U_2) &= \deg_G^c(y_1) - 1 - \deg_G^c(y_1, U_2) \\ &\geq \delta^c(G) - 1 - |U_2| \geq \tfrac{n+4}{3} - 1 - |U_2| \overset{(83)}{\geq} \tfrac{n+1}{3} - \left\lfloor \tfrac{n}{3} \right\rfloor \geq \tfrac{1}{3}.\end{aligned}$$

Fix any $z_1 \in N_G(y_1, U_1)$ for which $c(\{y_1, z_1\})$ is not blue. Since $c(\{x_1, y_1\})$ is blue, we infer that $(x_1, y_1, z_1)$ is a rainbow path where $c_{x_1} \neq c(\{x_1, y_1\}) =$ blue is guaranteed by $y_1 \in B_{x_1}$ from (98). Thus, Statement (2) of Corollary 7.2 guarantees from the rainbow $U_1$-path $(x_1, y_1, z_1)$ (where $c_{x_1} \neq c(\{x_1, y_1\}) =$ blue) that $(G, c)$ admits a properly colored $\ell$-cycle $C_\ell$, again contradicting (95).

## 8. Proof of Lemma 2.9 – Part 3: strong or short cycles and the case $\ell \equiv 1 \pmod 3$

Continuing from the previous sections, we prove Lemma 2.9 in the case $\ell \equiv 1 \pmod 3$. For this, we will need a number of supporting details, where we begin by establishing an analogue of Corollary 7.2 for $\ell \equiv 1 \pmod 3$ (another corollary of Proposition 7.1). We use the following terminology and notation. For a fixed $j \in \mathbb{Z}_3$, recall the set $U_j = U_j^{\text{good}} \cup U_j^{\text{bad}}$ (cf. (74)). We shall say that a vertex $u_j \in U_j^{\text{bad}}$ is an *internal (bad) vertex* if $\deg_G^c(u_j, U_j) \geq 3$, and that $u_j \in U_j^{\text{bad}}$ is an *external (bad) vertex* otherwise. We then define

$$\text{(101)} \qquad \begin{aligned} I_j^{\text{bad}} &= \left\{ u_j \in U_j^{\text{bad}} : \ \deg_G^c(u_j, U_j) \geq 3 \right\} \qquad \text{and} \\ E_j^{\text{bad}} &= \left\{ u_j \in U_j^{\text{bad}} : \ \deg_G^c(u_j, U_j) \leq 2 \right\}. \end{aligned}$$

**Corollary 8.1.** *Let* $\ell \equiv 1 \pmod 3$. *Fix an index* $j \in \mathbb{Z}_3$, *a vertex* $u_j \in U_j^{\text{good}}$, *and an edge* $\{u_j, v\} \in E$.

1. *If* $v \in U_j$ *or* $v \in I_{j+1}^{\text{bad}}$, *then* $c(\{u_j, v\}) = c_{u_j}$ *is the primary color of* $u_j$.
2. *The edges* $E_G(u_j, U_{j-1})$ *admit at least* $\deg_G^c(u_j) - 1 - |U_{j+1} \setminus I_{j+1}^{\text{bad}}|$ *non-*$c_{u_j}$ *colors.*
3. *If* $v \in U_{j-1}^{\text{bad}}$ *and* $c(\{u_j, v\}) \neq c_{u_j}$, *then* $\deg_G^c(v, U_j) \geq ((1/6) - 37\lambda^{1/4})n$.
4. *If* $v \in E_{j-1}^{\text{bad}}$ *and* $c(\{u_j, v\}) \neq c_{u_j}$, *then* $\deg_G^c(v, U_j) \geq \deg_G^c(v) - 3$.
5. *If* $v \in U_{j-1}^{\text{good}}$ *and* $c(\{u_j, v\}) \neq c_{u_j}$, *then* $\deg_G^c(v, U_j \setminus I_j^{\text{bad}}) \geq \deg_G^c(v) - 1$.

*Proof of Corollary 8.1.* Let $\ell \equiv 1 \pmod 3$. Fix an index $j \in \mathbb{Z}_3$, and w.l.o.g. let $j = 0 \in \mathbb{Z}_3$. Fix a good vertex $u_0 \in U_0^{\text{good}}$, and fix an edge $\{u_0, v\} \in E$.

For Statement (1), assume first that $v \in U_0$. If $c(\{u_0, v\}) \neq c_{u_0}$, then $P = (u_0, v)$ is a $c_{u_0}$-free rainbow path which Proposition 7.1 extends to a strong rainbow $\ell$-cycle $C_\ell$ (using $i = j = 0$, $k = 2$, and $K = \ell \equiv 1 \pmod 3$), contradicting (66). Assume next that $v \in I_1^{\text{bad}}$, in which case $v$ sees at least three colors in $U_1$. If $c(\{u_0, v\}) \neq c_{u_0}$, then $v$ admits a neighbor $w_1 \in U_1$ where $c(\{v, w_1\})$ is neither $c_{u_0}$ nor $c(\{u_0, v\})$. Now, $P = (u_0, v, w_1)$ is a $c_{u_0}$-free rainbow path which Proposition 7.1 extends to a strong rainbow $\ell$-cycle $C_\ell$ (using $i = 0$, $j = 1$, $k = 3$, and $K = \ell \equiv 1 \pmod 3$), again contradicting (66).

Statement (2) is an easy consequence of Statement (1). For $u_0 \in U_0^{\text{good}}$ satisfies

$$\deg_G^c(u_0) = \deg_G^c(u_0, U_0) + \deg_G^c(u_0, U_1) + \deg_G^c(u_0, U_2), \tag{102}$$

where Statement (1) guarantees that all edges of $E_G(u_0, U_0 \cup I_1^{\text{bad}})$ are colored $c_{u_0}$. However colors are assigned to $E_G(u_0, U_1 \setminus I_1^{\text{bad}})$, at least $\deg_G^c(u_0) - 1 - |U_1 \setminus I_1^{\text{bad}}|$ many non-$c_{u_0}$ colors remain, and these must occur on the edges of $E_G(u_0, U_2)$.

For Statement (3), we prepare an observation used multiple times below. For an edge $\{u_0, u_2\} \in E_G(u_0, U_2)$ satisfying $c(\{u_0, u_2\}) \neq c_{u_0}$, we observe that

(103) *every edge* $\{u_2, u_1\} \in E_G(u_2, U_1)$ *must be colored either* $c(\{u_0, u_2\})$ *or* $c_{u_0}$,

lest $(u_0, u_2, u_1)$ is a $c_{u_0}$-free rainbow path which Proposition 7.1 extends to a strong rainbow $\ell$-cycle $C_\ell$ (using $i = 0$, $j = 1$, $k = 3$, and $K = \ell \equiv 1 \pmod 3$), contradicting (66). Now, as in Statement (3), assume $v \in U_2^{\text{bad}}$

where $c(\{u_0, v\}) \neq c_{u_0}$. Then (103) gives $\deg_G^c(v, U_1^{\text{good}}) \leq 2$, where (72) and (73) add that $v \in U_2^{\text{bad}}$ satisfies

$$\begin{aligned}&\deg_G^c\left(v, U_0^{\text{good}}\right)\\&\qquad\geq \max\left\{\deg_G^c\left(v, U_1^{\text{good}}\right), \deg_G^c\left(v, U_2^{\text{good}}\right)\right\} \geq \deg_G^c\left(v, U_2^{\text{good}}\right).\end{aligned}$$

Since $V = U_1^{\text{good}} \cup U_2^{\text{good}} \cup U_3^{\text{good}} \cup V^{\text{bad}}$ is a partition,

$$\begin{aligned}\deg_G^c(v) &= \deg_G^c\left(v, U_0^{\text{good}}\right) + \deg_G^c\left(v, U_1^{\text{good}}\right)\\&\quad + \deg_G^c\left(v, U_2^{\text{good}}\right) + \deg_G^c\left(v, V^{\text{bad}}\right)\\&\leq \deg_G^c\left(v, U_0^{\text{good}}\right) + \deg_G^c\left(v, U_1^{\text{good}}\right)\\&\quad + \deg_G^c\left(v, U_2^{\text{good}}\right) + \left|V^{\text{bad}}\right|\\&\overset{(103)}{\leq} \deg_G^c\left(v, U_0^{\text{good}}\right) + 2 + \deg_G^c\left(v, U_2^{\text{good}}\right) + \left|V^{\text{bad}}\right|\\&\overset{(73)}{\leq} 2\deg_G^c\left(v, U_0^{\text{good}}\right) + 2 + \left|V^{\text{bad}}\right|\\&\overset{(74)}{\leq} 2\deg_G^c\left(v, U_0^{\text{good}}\right) + 2 + 72\lambda^{1/4}n\\&\leq 2\deg_G^c\left(v, U_0^{\text{good}}\right) + 73\lambda^{1/4}n.\end{aligned} \tag{104}$$

Thus, we conclude Statement (3) from

$$\begin{aligned}\deg_G^c(v, U_0) &\geq \deg_G^c\left(v, U_0^{\text{good}}\right) \overset{(104)}{\geq} \tfrac{1}{2}\left(\deg_G^c(v) - 73\lambda^{1/4}n\right)\\&\geq \tfrac{1}{2}\left(\delta^c(G) - 73\lambda^{1/4}n\right) \geq \tfrac{1}{2}\left(\tfrac{n+4}{3} - 73\lambda^{1/4}n\right) \geq \left(\tfrac{1}{6} - 37\lambda^{1/4}\right)n.\end{aligned}$$

For Statement (4), assume that $v \in E_2^{\text{bad}}$ and that $c(\{u_0, v\}) \neq c_{u_0}$. As before with (102), we have

$$\deg_G^c(v) = \deg_G^c(v, U_0) + \deg_G^c(v, U_1) + \deg_G^c(v, U_2).$$

By the definition of $E_2^{\text{bad}}$ in (101), we have $\deg_G^c(v, U_2) \leq 2$. Moreover, (103) gives $\deg_G(v, U_1) \leq 2$, where these colors can only be $c(\{u_0, v\})$ and $c_{u_0}$. Since $c(\{u_0, v\})$ is used on $E_G(v, U_0)$, Statement (4) follows.

For Statement (5), assume that $v \in U_2^{\text{good}}$ and that $c(\{u_0, v\}) \neq c_{u_0}$. As before with (102), we have

$$\deg_G^c(v) = \deg_G^c(v, U_0) + \deg_G^c(v, U_1) + \deg_G^c(v, U_2).$$

Statement (1) ensures that all edges of $E_G(v, U_2 \cup I_0^{\text{bad}})$ are assigned the primary color $c_v$. Moreover, (103) gives all edges of $E_G(v, U_1)$ are assigned $c(\{u_0, v\})$ and $c_{u_0}$, which must include $c_v$. Now, all non-$\{c_{u_0}, c_v\}$ colors incident to $v$ must be on the edges $E_G(v, U_0 \setminus I_0^{\mathrm{bad}})$, where $c(\{u_0, v\})$ is one such color used. In either case of $c_v \in \{c_{u_0}, c(\{u_0, v\})\}$, only $c_{u_0}$ is possibly not used on $E_G(v, U_0 \setminus I_0^{\mathrm{bad}})$. $\square$

## 8.1. On 4-cycles $C_4$ in $(G, c)$ when $\ell \equiv 1 \pmod 3$

Since $4 \equiv 1 \pmod 3$, which is the modular case of Lemma 2.9 we seek to prove, we study 4-cycles $C_4$ in $(G, c)$ from the point of view of Proposition 7.1 and Corollary 8.1. We begin with the following notation and terminology. For $j \in \mathbb{Z}_3$ and $u_j \in U_j$, recall from (77) that an edge $\{u_j, u_{j-1}\} \in E_G(u_j, U_{j-1})$ is said to be *typical* when $c(\{u_j, u_{j-1}\}) = c_{u_j}$ is the primary color of $u_j$, and is said to be *special* otherwise. In the reverse of (77), we write

$$\begin{aligned}
N_G^{\mathrm{typ}}\big(u_{j-1}, U_j^{\mathrm{good}}\big) &= \Big\{u_j \in N_G\big(u_{j-1}, U_j^{\mathrm{good}}\big) : c(\{u_{j-1}, u_j\}) = c_{u_j}\Big\}, \\
N_G^{\mathrm{spec}}\big(u_{j-1}, U_j^{\mathrm{good}}\big) &= \Big\{u_j \in N_G\big(u_{j-1}, U_j^{\mathrm{good}}\big) : c(\{u_{j-1}, u_j\}) \neq c_{u_j}\Big\}, \\
\deg_G^{\mathrm{typ}}\big(u_{j-1}, U_j^{\mathrm{good}}\big) &= \Big|N_G^{\mathrm{typ}}\big(u_{j-1}, U_j^{\mathrm{good}}\big)\Big|, \qquad \text{and} \\
\deg_G^{\mathrm{spec}}\big(u_{j-1}, U_j^{\mathrm{good}}\big) &= \Big|N_G^{\mathrm{spec}}\big(u_{j-1}, U_j^{\mathrm{good}}\big)\Big|.
\end{aligned}$$

We proceed with an initial observation.

**Observation 8.2.** *Let $\ell \equiv 1 \pmod 3$. Fix $j \in \mathbb{Z}_3$, $u_{j-1} \in U_{j-1}$, and $u_j \neq v_j \in N_G^{\mathrm{spec}}(u_{j-1}, U_j^{\mathrm{good}})$. Then $c(\{u_{j-1}, u_j\})$, $c(\{u_{j-1}, v_j\})$, $c_{u_j}$, and $c_{v_j}$ can't all be distinct.*

*Proof of Observation 8.2.* Let $\ell \equiv 1 \pmod 3$. Fix $j \in \mathbb{Z}_3$, and w.l.o.g. let $j = 0$. Fix $u_2 \in U_2$ and fix $u_0 \neq v_0 \in N_G^{\mathrm{spec}}(u_2, U_0^{\mathrm{good}})$. We apply Proposition 6.2 and Corollary 6.3 to each of $u_0 \neq v_0 \in U_0^{\mathrm{good}}$ to determine at least

$$\big|U_2^{\mathrm{good}}\big| - 482\lambda^{1/4} n \overset{(74)}{\geq} \left(\frac{1}{3} - 557\lambda^{1/4}\right) n \overset{(64)}{>} 0$$

many vertices $w_2 \in U_2^{\mathrm{good}}$ for which $\{u_0, w_2\}, \{v_0, w_2\} \in E$ and

$$c(\{u_0, w_2\}) = c_{u_0} \neq c_{v_0} = c(\{v_0, w_2\}).$$

If Observation 8.2 is false, then $(u_0, w_2, v_0, u_2)$ is a strong rainbow 4-cycle which Statement (3) of Proposition 7.1 extends to a strong rainbow $\ell$-cycle $C_\ell$, contradicting (66). $\square$

We continue with a second observation, which is a corollary of the one above.

**Corollary 8.3.** *Let $\ell \equiv 1 \pmod 3$. Fix $j \in \mathbb{Z}_3$ and $u_{j-1} \in U_{j-1}$. If $\deg_G^{\mathrm{spec}}(u_{j-1}, U_j^{\mathrm{good}}) \geq 4$, then all but at most one of the edges $\{u_{j-1}, u_j\} \in E$, where $u_j \in N_G^{\mathrm{spec}}(u_{j-1}, U_j^{\mathrm{good}})$ are monochromatic. Thus,*

$$\deg_G^{\mathrm{typ}}\left(u_{j-1}, U_j^{\mathrm{good}}\right) \geq \deg_G^{c}\left(u_{j-1}, U_j^{\mathrm{good}}\right) - 3,$$

*where in particular*

$$\deg_G^{\mathrm{typ}}\left(u_{j-1}, U_j^{\mathrm{good}}\right) \geq \begin{cases} \left(\frac{1}{6} - 110\lambda^{1/4}\right)n & \text{if } \deg_G^{\mathrm{spec}}(u_{j-1}, U_j^{\mathrm{good}}) \geq 1, \\ \left(\frac{1}{9} - 144\lambda^{1/4}\right)n & \text{otherwise.} \end{cases}$$

*Proof of Corollary 8.3.* Let $\ell \equiv 1 \pmod 3$. Fix $j \in \mathbb{Z}_3$ and w.l.o.g. assume $j = 0$. Fix $u_2 \in U_2$ and assume that $\deg_G^{\mathrm{spec}}(u_2, U_0^{\mathrm{good}}) \geq 4$. For sake of argument,

(105) *we assume the special edges $\{u_2, u_0\} \in E$, where $u_0 \in N_G^{\mathrm{spec}}(u_2, U_0^{\mathrm{good}})$, are not entirely monochromatic.*

We consider two cases.

**Case 1 ($\exists\ u_0 \neq v_0 \in N_G^{\mathrm{spec}}(u_2, U_0^{\mathrm{good}})$: $c(\{u_2, u_0\}) = c(\{u_2, v_0\})$).** Fix $u_0 \neq v_0 \in N_G^{\mathrm{spec}}(u_2, U_0^{\mathrm{good}})$ with $c(\{u_2, u_0\}) = c(\{u_2, v_0\})$. Using (105), we infer the existence of an edge $\{u_2, w_0\} \in E$, where $w_0 \in N_G^{\mathrm{spec}}(u_2, U_0^{\mathrm{good}})$, for which $c(\{u_2, w_0\}) \neq c(\{u_2, u_0\}) = c(\{u_2, v_0\})$. We claim that

$$c(\{u_2, u_0\}) = c(\{u_2, v_0\}) = c_{w_0}. \tag{106}$$

Indeed, pivoting $\{u_2, u_0\}$ against $\{u_2, w_0\}$, Observation 8.2 ensures $c(\{u_2, u_0\}) = c_{w_0}$ or $c(\{u_2, w_0\}) = c_{u_0}$. In the former case, (106) holds by the hypothesis of Case 1. In the latter case, we pivot $\{u_2, v_0\}$ against $\{u_2, w_0\}$, where Observation 8.2 gives $c(\{u_2, v_0\}) = c_{w_0}$ or $c(\{u_2, w_0\}) = c_{v_0}$. If $c(\{u_2, w_0\}) = c_{u_0}$, then Corollary 6.3 gives $c(\{u_2, w_0\}) \neq c_{v_0}$, and so $c(\{u_2, v_0\}) = c_{w_0}$ and again (106) holds.

If the first conclusion of Corollary 8.3 does not hold, we ignore the edge $\{u_2, w_0\}$ to infer the existence of an edge $\{u_2, x_0\} \in E$, where $x_0 \in N_G^{\mathrm{spec}}(u_2, U_0^{\mathrm{good}})$, for which $c(\{u_2, x_0\}) \neq c(\{u_2, u_0\})$. By (106),

$$c_{x_0} = c(\{u_2, u_0\}) = c(\{u_2, v_0\}) = c_{w_0},$$

and $x_0 \neq w_0 \in U_0^{\mathrm{good}}$ contradicts Corollary 6.3. $\square$

**Case 2 ($\forall\ u_0 \neq v_0 \in N_G^{\rm spec}(u_2, U_0^{\rm good})$, $c(\{u_2, u_0\}) \neq c(\{u_2, v_0\})$).** Since $\deg_G^{\rm spec}(u_2, U_0^{\rm good}) \geq 4$, fix distinct $u_0, v_0, w_0, x_0 \in N_G^{\rm spec}(u_2, U_0^{\rm good})$. Using Observation 8.2, we take w.l.o.g. $c(\{u_2, u_0\}) = c_{v_0}$. Observation 8.2 then ensures that $c(\{u_2, w_0\}) = c_{u_0}$ (since $c_{v_0} \neq c_{w_0}$ from Corollary 6.3) and $c(\{u_2, x_0\}) = c_{u_0}$, which contradicts the hypothesis of Case 2. □

The remaining assertions of Corollary 8.3 are now easy to establish. When $\deg_G^{\rm spec}(u_2, U_0^{\rm good}) = 0$, all edges of $E_G(u_2, U_0^{\rm good})$ are typical, and so

$$\begin{aligned}\deg_G^{\rm typ}\left(u_2, U_0^{\rm good}\right) &\geq \deg_G^c\left(u_2, U_0^{\rm good}\right) \\ &\overset{(74)}{\geq} \left(\tfrac{1}{9} - 72\lambda^{1/4}\right)n - 72\lambda^{1/4}n = \left(\tfrac{1}{9} - 144\lambda^{1/4}\right)n.\end{aligned}$$

When $\deg_G^{\rm spec}(u_2, U_0^{\rm good}) \geq 1$, the first assertion of Corollary 8.3 guarantees that edges $\{u_2, u_0\} \in E$ with $u_0 \in N_G^{\rm spec}(u_2, U_0^{\rm good})$ are colored with at most three colors. Thus,

$$\begin{aligned}\deg_G^{\rm typ}\left(u_2, U_0^{\rm good}\right) \geq \deg_G^c\left(u_2, U_0^{\rm good}\right) - 3 &\overset{\text{Cor. 8.1}}{\geq} \left(\tfrac{1}{6} - 37\lambda^{1/4}\right)n - 72\lambda^{1/4}n - 3 \\ &= \left(\tfrac{1}{6} - 109\lambda^{1/4}\right)n - 3 \geq \left(\tfrac{1}{6} - 110\lambda^{1/4}\right)n,\end{aligned}$$

as promised. □

**Definition 8.4** ($j$-special)**.** A 4-cycle $(u_j, u_{j-1}, v_j, v_{j-1})$ is $j$*-special* for some $j \in \mathbb{Z}_3$ if $u_j, v_j \in U_j^{\rm good}$, $u_{j-1}, v_{j-1} \in U_{j-1}$, $\{u_j, u_{j-1}\}, \{v_j, v_{j-1}\} \in E$ are typical, and $\{u_j, v_{j-1}\}, \{v_j, u_{j-1}\} \in E$ are special.

**Observation 8.5.** *Let $\ell \equiv 1 \pmod 3$, and fix $j \in \mathbb{Z}_3$. A $j$-special 4-cycle $(u_j, u_{j-1}, v_j, v_{j-1})$ receives precisely three colors, where in particular $c(\{u_j, v_{j-1}\}) = c(\{v_j, u_{j-1}\})$.*

*Proof of Observation 8.5.* Let $\ell \equiv 1 \pmod 3$. Fix $j \in \mathbb{Z}_3$, and w.l.o.g. let $j = 0$. Fix a 0-special 4-cycle $(u_0, u_2, v_0, v_2)$. By its definition, we infer $c(\{u_2, u_0\}) = c_{u_0}$ and $c(\{v_2, v_0\}) = c_{v_0}$ are primary, which Corollary 6.3 ensures are distinct. By its definition, $c(\{u_0, v_2\}) \neq c_{u_0}$ and $c(\{v_0, u_2\}) \neq c_{v_0}$ are special. Observe that $c(\{u_0, v_2\}) \neq c_{v_0}$ since otherwise Proposition 6.2 guarantees that $G - \{v_0, v_2\}$ (here denoting edge-removal) contradicts (67). Similarly, $c(\{v_0, u_2\}) \neq c_{u_0}$. Thus, if $c(\{u_0, v_2\}) \neq c(\{v_0, u_2\})$, then $(u_0, u_2, v_0, v_2)$ is a strong rainbow 4-cycle which Statement (3) of Proposition 7.1 extends to a strong rainbow $\ell$-cycle $C_\ell$, contradicting (66). □

We conclude this subsection with a corollary of the preceding observation.

**Corollary 8.6.** *Let $\ell \equiv 1 \pmod 3$. Fix $j \in \mathbb{Z}_3$, $u_{j-1} \neq v_{j-1} \in U_{j-1}$, and a color $\alpha$ from $c$. Set*

$$A = A_\alpha(u_{j-1}) = \left\{u_j \in N_G^{\mathrm{spec}}(u_{j-1}, U_j) : c(\{u_j, u_{j-1}\}) = \alpha\right\} \quad \textit{and}$$
$$B = B_\alpha(v_{j-1}) = \left\{v_j \in N_G^{\mathrm{spec}}(v_{j-1}, U_j) : c(\{v_j, v_{j-1}\}) \neq \alpha\right\}.$$

*Then $|A \cup B| < ((1/6) + 258\lambda^{1/4})n$.*

*Proof of Corollary 8.6.* Let $\ell \equiv 1 \pmod 3$. Fix $j \in \mathbb{Z}_3$, and w.l.o.g. let $j = 0$. Fix $u_2 \neq v_2 \in U_2$ and fix a color $\alpha$ of $c$. Let $A = A(u_2)$ and $B = B(v_2)$ be defined as above, but assume for contradiction that

$$|A \cup B| \geq \left(\tfrac{1}{6} + 258\lambda^{1/4}\right)n. \tag{107}$$

We will use (107) to guarantee distinct vertices

$$N_G^{\mathrm{typ}}\left(u_2, U_0^{\mathrm{good}}\right) \cap (A \cup B) \ni u_0 \neq v_0 \in N_G^{\mathrm{typ}}\left(v_2, U_0^{\mathrm{good}}\right) \cap (A \cup B). \tag{108}$$

If (108) holds, then it will conclude our proof, because $(u_2, u_0, v_2, v_0)$ would be a rainbow 0-special 4-cycle, contradicting Observation 8.5. To see this, we first note from (108) that $u_0 \neq v_0 \in U_0^{\mathrm{good}}$ are good vertices, where $u_0 \in N_G^{\mathrm{typ}}(u_2, U_0^{\mathrm{good}})$ guarantees $c(\{u_0, u_2\}) = c_{u_0}$ and $v_0 \in N_G^{\mathrm{typ}}(v_2, U_0^{\mathrm{good}})$ guarantees $c(\{v_0, v_2\}) = c_{v_0}$, and where $c_{u_0} \neq c_{v_0}$ is guaranteed by Corollary 6.3. Since $u_0 \in A \cup B$ happens only from $u_0 \in B \setminus A$ (because $c(\{u_0, u_2\}) = c_{u_0}$ is not special for $u_0$), we infer $c(\{u_0, v_2\}) \neq \alpha$ is some non-$\alpha$ special color for $u_0$. Since $v_0 \in A \cup B$ happens only from $v_0 \in A \setminus B$ (because $c(\{v_0, v_2\}) = c_{v_0}$ is not special for $v_0$), we infer $c(\{v_0, u_2\}) = \alpha$ is special for $v_0$. (Note: the existence of vertices $u_0$ and $v_0$ in (108) implies they are necessarily distinct.) Thus, $(u_2, u_0, v_2, v_0)$ is a rainbow 0-special 4-cycle, as claimed.

To prove (108) (from (107)), define

$$A^{\mathrm{good}} = A \cap U_0^{\mathrm{good}} \qquad \text{and} \qquad B^{\mathrm{good}} = B \cap U_0^{\mathrm{good}}.$$

Then

$$\left|A^{\mathrm{good}} \cup B^{\mathrm{good}}\right| \overset{(74)}{\geq} |A \cup B| - 72\lambda^{1/4}n \overset{(107)}{\geq} \left(\tfrac{1}{6} + 186\lambda^{1/4}\right)n, \tag{109}$$

and so one of $|A^{\mathrm{good}}|$ or $|B^{\mathrm{good}}|$ is large. Our proof will ultimately show that both are large, and so we begin by assuming the former is non-empty. We therefore infer that $\deg_G^{\mathrm{spec}}(u_2, U_0^{\mathrm{good}}) \geq |A^{\mathrm{good}}| > 0$, and so Corollary 8.3

gives

$$\deg_G^{\mathrm{typ}}\left(u_2, U_0^{\mathrm{good}}\right) \geq \left(\tfrac{1}{6} - 110\lambda^{1/4}\right)n. \tag{110}$$

If $N_G^{\mathrm{typ}}(u_2, U_0^{\mathrm{good}})$ and $A \cup B$ were disjoint, we would have

$$\begin{aligned}|U_0| &\geq \deg_G^{\mathrm{typ}}\left(u_2, U_0^{\mathrm{good}}\right) + |A \cup B| \overset{(110)}{\geq} \left(\tfrac{1}{6} - 110\lambda^{1/4}\right)n + |A \cup B| \\ &\overset{(107)}{\geq} \left(\tfrac{1}{6} - 110\lambda^{1/4}\right)n + \left(\tfrac{1}{6} + 258\lambda^{1/4}\right)n = \left(\tfrac{1}{3} + 148\lambda^{1/4}\right)n \overset{(74)}{>} |U_0|,\end{aligned} \tag{111}$$

a contradiction. This guarantees the existence of the vertex $u_0$ in (108).

To guarantee the existence of the vertex $v_0$ in (108), we argue similarly. For that, $N_G^{\mathrm{typ}}(u_2, U_0^{\mathrm{good}})$ and $A^{\mathrm{good}}$ are disjoint subsets of $U_0^{\mathrm{good}}$, and so

$$\begin{aligned}\left|A^{\mathrm{good}}\right| + \left(\tfrac{1}{6} - 110\lambda^{1/4}\right)n &\overset{(110)}{\leq} \left|A^{\mathrm{good}}\right| + \deg_G^{\mathrm{typ}}\left(u_2, U_0^{\mathrm{good}}\right) \\ &\leq \left|U_0^{\mathrm{good}}\right| \overset{(74)}{\leq} \left(\tfrac{1}{3} + 75\lambda^{1/4}\right)n \implies \left|A^{\mathrm{good}}\right| \leq \left(\tfrac{1}{6} + 185\lambda^{1/4}\right)n.\end{aligned} \tag{112}$$

Thus,

$$\begin{aligned}\deg_G^{\mathrm{spec}}\left(v_2, U_0^{\mathrm{good}}\right) &\geq \left|B^{\mathrm{good}} \setminus A^{\mathrm{good}}\right| = \left|A^{\mathrm{good}} \cup B^{\mathrm{good}}\right| - \left|A^{\mathrm{good}}\right| \\ &\overset{(109)}{\geq} \left(\tfrac{1}{6} + 186\lambda^{1/4}\right)n - \left|A^{\mathrm{good}}\right| \overset{(112)}{\geq} \lambda^{1/4}n > 0,\end{aligned}$$

and so Corollary 8.3 gives

$$\deg_G^{\mathrm{typ}}\left(v_2, U_0^{\mathrm{good}}\right) \geq \left(\tfrac{1}{6} - 110\lambda^{1/4}\right)n.$$

We now proceed identically to before with (110)–(111). □

### 8.2. Amenable elements of $\mathbb{Z}_3$

Recall the partition $V(G) = U_0 \cup U_1 \cup U_2$ of $(G, c)$ from (74). For each $j \in \mathbb{Z}_3$, recall the (so-called *internal* and *external*) sets of bad vertices

$$I_j^{\mathrm{bad}} = \{u_j \in U_j^{\mathrm{bad}} : \deg_G^c(u_j, U_j) \geq 3\} \qquad \text{and}$$
$$E_j^{\mathrm{bad}} = \{u_j \in U_j^{\mathrm{bad}} : \deg_G^c(u_j, U_j) \leq 2\}$$

from (101). In particular, $U_j^{\mathrm{bad}} = I_j^{\mathrm{bad}} \cup E_j^{\mathrm{bad}}$ is a partition, and so

$$U_j = U_j^{\mathrm{good}} \cup U_j^{\mathrm{bad}} = U_j^{\mathrm{good}} \cup I_j^{\mathrm{bad}} \cup E_j^{\mathrm{bad}} \tag{113}$$

are partitions. We set

$$\hat{U}_j = U_j^{\text{good}} \cup E_j^{\text{bad}} \qquad \text{and} \qquad \Delta_j = m - \big|\hat{U}_j\big|, \tag{114}$$

where $m = \lfloor n/3 \rfloor$ from (68). The following observation follows by elementary means (independent from $\ell \pmod 3$), and plays an important role in our proof of Lemma 2.9.

**Observation 8.7.** *There exists $j \in \mathbb{Z}_3$ so that*

1. $\Delta_j \geq 0$;
2. $|I_{j+1}^{\text{bad}}| \leq 2\Delta_j$;
3. $|U_{j+2}| \leq m + 2\Delta_j + 2$.

We say that an element $j \in \mathbb{Z}_3$ satisfying Conclusions (1)–(3) of Observation 8.7 is *amenable.*

*Proof of Observation 8.7.* Without loss of generality, let

$$\Delta_0 = \max\{\Delta_0, \Delta_1, \Delta_2\}. \tag{115}$$

Conclusion (1) now holds with $j = 0$ (cf. 115), lest $\Delta_1, \Delta_2 \leq \Delta_0 \leq -1$ and $m = \lfloor n/3 \rfloor \geq (n-2)/3$ give

$$3m - \big|\hat{U}_0\big| - \big|\hat{U}_1\big| - \big|\hat{U}_2\big| \overset{(114)}{=} \Delta_0 + \Delta_1 + \Delta_2 \leq -3$$
$$\implies \qquad 3m+3 \leq \big|\hat{U}_0\big| + \big|\hat{U}_1\big| + \big|\hat{U}_2\big| \overset{(113),(114)}{\leq} |U_0|+|U_1|+|U_2| = n \leq 3m+2,$$

a contradiction. Conclusion (3) also holds with $j = 0$ (cf. 115), since

$$|U_2| = n - |U_0| - |U_1| \overset{(113),(114)}{=} n - \big|I_0^{\text{bad}}\big| - \big|\hat{U}_0\big| - \big|I_1^{\text{bad}}\big| - \big|\hat{U}_1\big| \leq n - \big|\hat{U}_0\big| - \big|\hat{U}_1\big|$$
$$\leq m+2+\Big(m-\big|\hat{U}_0\big|\Big)+\Big(m-\big|\hat{U}_1\big|\Big) \overset{(114)}{=} m+2+\Delta_0+\Delta_1 \overset{(115)}{\leq} m+2\Delta_0+2.$$

For sake of argument, we assume that Conclusion (2) fails with $j = 0$ (cf. 115):

$$\big|I_1^{\text{bad}}\big| \geq 2\Delta_0 + 1. \tag{116}$$

Observe that

$$\big|I_0^{\text{bad}}\big| + \big|I_1^{\text{bad}}\big| \leq \big|I_0^{\text{bad}}\big| + \big|I_1^{\text{bad}}\big| + \big|I_2^{\text{bad}}\big| \tag{117}$$

$$\overset{(113),(114)}{=} n - |\hat{U}_0| - |\hat{U}_1| - |\hat{U}_2|$$

$$\leq 2 + \left(m - |\hat{U}_0|\right) + \left(m - |\hat{U}_1|\right) + \left(m - |\hat{U}_2|\right) \overset{(114)}{=} 2 + \Delta_0 + \Delta_1 + \Delta_2,$$

$$\implies \qquad 0 \leq |I_0^{\mathrm{bad}}| \leq 2 + \Delta_0 + \Delta_1 + \Delta_2 - |I_1^{\mathrm{bad}}| \overset{(116)}{\leq} 1 + \Delta_1 + \Delta_2 - \Delta_0.$$

We claim that, when $\Delta_1 \leq -1$ holds, $j = 2$ satisfies Observation 8.7. Indeed,

$$0 \leq |I_0^{\mathrm{bad}}| \overset{(117)}{\leq} 1 + \Delta_1 + \Delta_2 - \Delta_0 \leq 0$$

follows from (115) and our current assumption $\Delta_1 \leq -1$. Necessarily then, $I_0^{\mathrm{bad}} = \emptyset$, $\Delta_1 = -1$, and $\Delta_2 = \Delta_0 \geq 0$ also achieves the maximum in (115). In particular, $j = 2$ satisfies Conclusion (2) of Observation 8.7 by $|I_0^{\mathrm{bad}}| = 0 \leq 2\Delta_2$, and it satisfies Conclusions (1) and (3) of Observation 8.7 by $\Delta_2 = \max\{\Delta_0, \Delta_1, \Delta_2\}$ (see our work between (115) and (116)).

For sake of argument, we assume that

$$\Delta_1 \geq 0. \tag{118}$$

Note that (118) says Conclusion (1) holds with $j = 1$. Conclusion (3) also holds with $j = 1$, since

$$|U_0| \overset{(113),(114)}{=} |\hat{U}_0| + |I_0^{\mathrm{bad}}| \overset{(117)}{\leq} |\hat{U}_0| + 1 + \Delta_1 + \Delta_2 - \Delta_0$$

$$\overset{(115)}{\leq} |\hat{U}_0| + 1 + \Delta_1 \overset{(114)}{=} m - \Delta_0 + 1 + \Delta_1 \overset{(115)}{\leq} m + 1 + \Delta_1 \overset{(118)}{\leq} m + 2\Delta_1 + 2,$$

where we used $\Delta_0 \geq 0$. For sake of argument, we assume Conclusion (2) fails with $j = 1$:

$$|I_2^{\mathrm{bad}}| \geq 2\Delta_1 + 1. \tag{119}$$

We conclude that Observation 8.7 holds with $j = 2$. For that, observe that

$$|I_0^{\mathrm{bad}}| + 2\Delta_0 + 2\Delta_1 + 2 \overset{(116),(119)}{\leq} |I_0^{\mathrm{bad}}| + |I_1^{\mathrm{bad}}| + |I_2^{\mathrm{bad}}| \overset{(117)}{\leq} 2 + \Delta_0 + \Delta_1 + \Delta_2$$

$$\implies \qquad 0 \leq |I_0^{\mathrm{bad}}| \leq \Delta_2 - \Delta_0 - \Delta_1 \overset{(115)}{\leq} -\Delta_1 \overset{(118)}{\leq} 0.$$

Thus, $I_0^{\mathrm{bad}} = \emptyset$, $\Delta_1 = 0$, and $\Delta_2 = \Delta_0$ is the maximum from (115). As $\Delta_2 = \Delta_0$ is the maximum from (115), we already observed that Conclusion (1) holds, i.e., $\Delta_2 \geq 0$, and that Conclusion (3) holds, i.e., $|U_1| \leq m + 2\Delta_2 + 2$. Since $\Delta_1 = 0$ and $I_0^{\mathrm{bad}} = \emptyset$, Conclusion (2) also holds, i.e., $|I_0^{\mathrm{bad}}| = 0 \leq 2\Delta_2$, which completes the proof of Observation 8.7. $\square$

Observation 8.7 guarantees at least one *amenable* element $j \in \mathbb{Z}_3$, i.e., one where Conclusions (1)–(3) of Observation 8.7 hold. We next consider properties of an amenable $j \in \mathbb{Z}_3$ when $\ell \equiv 1 \pmod 3$.

**Fact 8.8.** *Let $\ell \equiv 1 \pmod 3$. Let $j \in \mathbb{Z}_3$ be amenable, and fix $u_{j+1} \in U_{j+1}$. Then*

$$\deg_G^{\mathrm{spec}}(u_{j+1}, U_{j+2}^{\mathrm{good}}) \leq \begin{cases} 2\Delta_j + 5 & \text{if } u_{j+1} \in \hat{U}_{j+1} \text{ (cf. (114))}, \\ n/(10) & \text{if } u_{j+1} \in I_{j+1}^{\mathrm{bad}} \text{ (cf. (101) and (114))}. \end{cases}$$

*Proof of Fact 8.8.* Let $\ell \equiv 1 \pmod 3$. Fix an amenable $j \in \mathbb{Z}_3$, and w.l.o.g. let $j = 0$. Note first that

$$\deg_G^c(v, U_2) \geq \deg_G^c(v) - 3 \qquad \text{for every } v \in \hat{U}_1 \text{ with } \deg_G^{\mathrm{spec}}(v, U_2^{\mathrm{good}}) > 0, \tag{120}$$

since $u_2 \in N_G^{\mathrm{spec}}(v, U_2^{\mathrm{good}})$ gives $u_2 \in U_2^{\mathrm{good}}$ and $c(\{u_2, v\}) \neq c_{u_2}$, and $v \in \hat{U}_1 = U_1^{\mathrm{good}} \cup E_1^{\mathrm{bad}}$ allows us to apply Statement (4) or (5) of Corollary 8.1.

Fix a vertex $u_1 \in U_1$, and first let $u_1 \in \hat{U}_1$. Assume, on the contrary, that

$$\deg_G^{\mathrm{spec}}(u_1, U_2^{\mathrm{good}}) \geq 2\Delta_0 + 6 \overset{\text{Obs. 8.7}}{\geq} 6. \tag{121}$$

Now, the edges $E_G(u_1, U_2)$ consist of $\deg_G^c(u_1, U_2)$ many distinctly colored edges together with some number of edges of repeated colors. By Corollary 8.3, the special edges $E_G^{\mathrm{spec}}(u_1, U_2^{\mathrm{good}})$, i.e., those of the form $\{u_1, u_2\} \in E$ for $u_2 \in N_G^{\mathrm{spec}}(u_1, U_2^{\mathrm{good}})$, come in at most two colors, so we have

$$\begin{aligned} |U_2| &\geq \deg_G(u_1, U_2) \geq \deg_G^c(u_1, U_2) + \deg_G^{\mathrm{spec}}\left(u_1, U_2^{\mathrm{good}}\right) - 2 \\ &\overset{(120)}{\geq} \deg_G^c(u_1) + \deg_G^{\mathrm{spec}}\left(u_1, U_2^{\mathrm{good}}\right) - 5 \overset{(121)}{\geq} \deg_G^c(u_1) + 2\Delta_0 + 1 \overset{(68)}{\geq} m + 2\Delta_0 + 3, \end{aligned}$$

which contradicts Conclusion (3) of Observation 8.7.

Now let $u_1 \in I_1^{\mathrm{bad}}$. Assume, on the contrary, that

$$\deg_G^{\mathrm{spec}}\left(u_1, U_2^{\mathrm{good}}\right) > \tfrac{n}{10}. \tag{122}$$

By Corollary 8.3, all but one of the edges $E_G^{\mathrm{spec}}(u_1, U_2^{\mathrm{good}})$ are monochromatic, and in some color $\alpha$ of $c$. To prepare an upcoming application of Corollary 8.6, we define

$$(123)\quad A_2 = A_2(u_1) = \left\{ v_2 \in N_G^{\mathrm{spec}}(u_1, U_2^{\mathrm{good}}) : c(\{u_1, v_2\}) = \alpha \right\},$$
$$\text{where} \qquad |A_2| \overset{\text{Cor. 8.3}}{\geq} \deg_G^{\mathrm{spec}}(u_1, U_2^{\mathrm{good}}) - 1 \overset{(122)}{>} \tfrac{n}{10} - 1 \geq \tfrac{n}{11}.$$

For Corollary 8.6, we will identify a set $B_2 \subseteq U_2^{\mathrm{good}}$ corresponding to $A_2$ above[3]. For that, we first consider the following superset $\overline{B}_2 \supseteq B_2$, from which we will later extract $B_2$:

$$(124)\quad \overline{B}_2 = N_G^{\mathrm{typ}}(u_1, U_2^{\mathrm{good}}), \qquad \text{where} \qquad |\overline{B}_2| \overset{\text{Cor. 8.3}}{\geq} \left(\tfrac{1}{6} - 110\lambda^{1/4}\right)n.$$

We claim that for fixed $u_2 \in \overline{B}_2$,

$$(125)\qquad \deg_G^{\mathrm{typ}}(v_1, A_2) > 0 \text{ for every } v_1 \in N_G^{\mathrm{spec}}(u_2, \hat{U}_1),$$

since then $\deg_G^{\mathrm{spec}}(v_1, U_2^{\mathrm{good}}) > 0$ (with $u_2 \in N_G^{\mathrm{spec}}(v_1, U_2^{\mathrm{good}})$) and inclusion-exclusion gives

$$\begin{aligned}
&U_2^{\mathrm{good}} \supseteq A_2 \cup N_G^{\mathrm{typ}}(v_1, U_2^{\mathrm{good}}) \\
&\quad\Longrightarrow \quad |U_2^{\mathrm{good}}| \geq |A_2| + \deg_G^{\mathrm{typ}}(v_1, U_2^{\mathrm{good}}) - \deg_G^{\mathrm{typ}}(v_1, A_2) \\
&\quad\Longrightarrow \quad \deg_G^{\mathrm{typ}}(v_1, A_2) \geq |A_2| + \deg_G^{\mathrm{typ}}(v_1, U_2^{\mathrm{good}}) - |U_2^{\mathrm{good}}| \\
&\qquad\qquad \overset{(123)}{>} \tfrac{n}{11} + \deg_G^{\mathrm{typ}}(v_1, U_2^{\mathrm{good}}) - |U_2^{\mathrm{good}}| \\
&\qquad\qquad \overset{(74)}{\geq} \tfrac{n}{11} + \deg_G^{\mathrm{typ}}(v_1, U_2^{\mathrm{good}}) - \left(\tfrac{1}{3} + 75\lambda^{1/4}\right)n \\
&\qquad\qquad \overset{\text{Cor. 8.3}}{\geq} \tfrac{n}{11} + \deg_G^{c}(v_1, U_2^{\mathrm{good}}) - 3 - \left(\tfrac{1}{3} + 75\lambda^{1/4}\right)n \\
&\qquad\qquad \overset{(74)}{\geq} \tfrac{n}{11} + \deg_G^{c}(v_1, U_2) - 3 - \left(\tfrac{1}{3} + 147\lambda^{1/4}\right)n \\
&\overset{(120)}{\geq} \tfrac{n}{11} + \deg_G^{c}(v_1) - 6 - \left(\tfrac{1}{3} + 147\lambda^{1/4}\right)n \geq \tfrac{n}{11} + \delta_c(G) - 6 - \left(\tfrac{1}{3} + 147\lambda^{1/4}\right)n \\
&\qquad\qquad\qquad \overset{(65)}{\geq} \tfrac{n}{11} - 6 - 147\lambda^{1/4}n \geq \tfrac{n}{11} - 148\lambda^{1/4}n \overset{(64)}{>} 0.
\end{aligned}$$

[3]Strictly speaking, the set $B_2$ we will define below will be a subset of that in the hypothesis of Corollary 8.6.

We can now show that

$$c(\{u_2, v_1\}) = \alpha \text{ for every } v_1 \in N_G^{\mathrm{spec}}(u_2, \hat{U}_1). \tag{126}$$

Indeed, by (125), there exists $v_2 \in N_G^{\mathrm{typ}}(v_1, A_2)$. Note that $v_2 \in A_2$ implies that $\{u_1, v_2\}$ is a special edge with color $\alpha$. Furthemore, both $u_2 \in \overline{B}_2$ and $v_2 \in A_2$ are good vertices, both $\{u_2, u_1\}$ and $\{v_2, v_1\}$ are typical edges, and both $\{u_2, v_1\}$ and $\{v_2, u_1\}$ are special edges. Thus, the 4-cycle $(u_1, u_2, v_1, v_2)$ is 2-special (cf. Definition 8.4), so Observation 8.5 guarantees that $c(\{u_2, v_1\}) = c(\{v_2, u_1\}) = \alpha$.

To extract the desired subset $B_2 \subseteq \overline{B}_2$ from (124), we double-count

$$Z = \Big\{\{u_2, v_1\} \in E : \, u_2 \in \overline{B}_2, \, v_1 \in N_G^{\mathrm{spec}}(u_2, I_1^{\mathrm{bad}}), \text{ and } c(\{v_1, u_2\}) \neq \alpha\Big\}.$$

For each $u_2 \in \overline{B}_2$, Statement (2) of Corollary 8.1 ensures that $E_G(u_2, U_1)$ admits at least

$$\deg_G^c(u_2) - 1 - \big|U_0 \setminus I_0^{\mathrm{bad}}\big| \overset{(114)}{=} \deg_G^c(u_2) - 1 - \big|\hat{U}_0\big|$$

many special colors for $u_2$. By (126), every special edge $\{u_2, v_1\} \in E_G(u_2, \hat{U}_1)$ is colored $c(\{u_2, v_1\}) = \alpha$, which is forbidden in $Z$. Thus,

$$\begin{aligned} |Z| \geq \sum_{u_2 \in \overline{B}_2} \Big(\deg_G^c(u_2) - 2 - \big|\hat{U}_0\big|\Big) &\overset{(68)}{\geq} \sum_{u_2 \in \overline{B}_2} \Big(m - \big|\hat{U}_0\big|\Big) \\ &\overset{(114)}{=} \sum_{u_2 \in \overline{B}_2} \Delta_0 = \Delta_0 \big|\overline{B}_2\big| \overset{\text{Obs. } 8.7}{\geq} \tfrac{1}{2}\big|I_1^{\mathrm{bad}}\big|\big|\overline{B}_2\big|. \end{aligned}$$

Averaging $|Z|$ over $I_1^{\mathrm{bad}}$, we infer the existence of a vertex $v_1 \in I_1^{\mathrm{bad}}$ where

$$B_2 = B_2(v_1) = \big\{u_2 \in N_G^{\mathrm{spec}}\big(v_1, \overline{B}_2\big) : \, c(\{u_2, v_1\}) \neq \alpha\big\} \tag{127}$$

satisfies

$$|B_2| \geq \tfrac{1}{2}\big|\overline{B}_2\big| \overset{(124)}{\geq} \tfrac{1}{2}\big(\tfrac{1}{6} - 110\lambda^{1/4}\big)n. \tag{128}$$

Consider the sets $A_2 = A_2(u_1)$ and $B_2 = B_2(v_1)$ from (123) and (127). Since $B_2 \subseteq \overline{B}_2$ where $A_2 \cap \overline{B}_2 = \emptyset$ from (124), we infer that $A_2 \cup B_2$ is a disjoint union of size

$$|A_2 \cup B_2| = |A_2| + |B_2| \overset{(123)}{\geq} \tfrac{n}{11} + |B_2| \overset{(128)}{\geq} \tfrac{n}{11} + \tfrac{1}{2}\big(\tfrac{1}{6} - 110\lambda^{1/4}\big)n$$
$$= \big(\tfrac{23}{132} - 55\lambda^{1/4}\big)n \overset{(64)}{>} \big(\tfrac{1}{6} + 258\lambda^{1/4}\big)n,$$

which contradicts Corollary 8.6, and concludes the proof of Fact 8.8. □

Fact 8.8 admits the following easy but useful corollary.

**Corollary 8.9.** *Let $\ell \equiv 1 \pmod 3$, and fix an amenable element $j \in \mathbb{Z}_3$. Fix an integer $\Delta \geq \max\{1, \Delta_j\}$, and fix $W_{j+1} \subseteq U_{j+1}$ of size $|W_{j+1}| < n/(100)$. Then*

$$\Big|E_G^{\mathrm{spec}}\big(W_{j+1}, U_{j+2}^{\mathrm{good}}\big)\Big| \leq \tfrac{3}{10}\Delta n,$$

*where $E_G^{\mathrm{spec}}(W_{j+1}, U_{j+2}^{\mathrm{good}})$ includes all $\{w_{j+1}, u_{j+2}\} \in E$ with $w_{j+1} \in W_{j+1}$ and $u_{j+2} \in N_G^{\mathrm{spec}}(w_{j+1}, U_{j+2}^{\mathrm{good}})$.*

*Proof of Corollary 8.9.* Let $\ell \equiv 1 \pmod 3$. Fix an amenable element $j \in \mathbb{Z}_3$, and w.l.o.g. let $j = 0$. Fix an integer $\Delta \geq \max\{1, \Delta_0\}$, and fix $W_1 \subseteq U_1$ of size $|W_1| < n/(100)$. Then

$$\Big|E_G^{\mathrm{spec}}\big(W_1, U_2^{\mathrm{good}}\big)\Big| = \sum_{w_1 \in W_1} \deg_G^{\mathrm{spec}}\big(w_1, U_2^{\mathrm{good}}\big)$$
$$= \sum_{w_1 \in W_1 \cap \hat{U}_1} \deg_G^{\mathrm{spec}}\big(w_1, U_2^{\mathrm{good}}\big) + \sum_{w_1 \in W_1 \cap I_1^{\mathrm{bad}}} \deg_G^{\mathrm{spec}}\big(w_1, U_2^{\mathrm{good}}\big)$$
$$\overset{\text{Fct. } 8.8}{\leq} \big|W_1 \cap \hat{U}_1\big|(2\Delta_0 + 5) + \big|W_1 \cap I_1^{\mathrm{bad}}\big|\tfrac{n}{10}$$
$$\leq |W_1|(2\Delta_0 + 5) + \big|I_1^{\mathrm{bad}}\big|\tfrac{n}{10}$$
$$\overset{\text{Obs. } 8.7}{\leq} |W_1|(2\Delta_0 + 5) + \tfrac{n}{5}\Delta_0 \leq \tfrac{n}{100}(2\Delta + 5) + \tfrac{n}{5}\Delta,$$

and the quantity above is at most $27\Delta n/100$. □

### 8.3. Proof of Lemma 2.9 in the case $\ell \equiv 1 \pmod 3$

We now prove Lemma 2.9 in the case $\ell \equiv 1 \pmod 3$. For this case, recall from (65) that the hypotheses of Lemma 2.9 assume

$$\delta^c(G) \geq \begin{cases} (n+5)/3 & \text{in Statement (1) with } \ell \equiv 1 \pmod 3, \\ (n+4)/3 & \text{in Statement (2) with } \ell \equiv 1 \pmod 3. \end{cases} \tag{129}$$

We begin our work with Statement (2).

**8.3.1. Statement (2) of Lemma 2.9** Statement (2) of Lemma 2.9 seeks to conclude that $(G, c)$ admits a properly colored $\ell$-cycle $C_\ell$. To prove this, we proceed by fixing an amenable element $j \in \mathbb{Z}_3$ from Observation 8.7. We first claim that

$$E_G^{\rm spec}(U_{j+1}^{\rm good}, U_{j+2}^{\rm good}) \neq \emptyset. \tag{130}$$

To prove (130), we consider the identity

$$\big|E_G^{\rm spec}\big(U_{j+1}, U_{j+2}^{\rm good}\big)\big| = \big|E_G^{\rm spec}\big(U_{j+1}^{\rm bad}, U_{j+2}^{\rm good}\big)\big| + \big|E_G^{\rm spec}\big(U_{j+1}^{\rm good}, U_{j+2}^{\rm good}\big)\big|. \tag{131}$$

We will use Corollary 8.9 to bound $|E_G^{\rm spec}(U_{j+1}^{\rm bad}, U_{j+2}^{\rm good})|$, and we will use Corollary 8.1 to bound $|E_G^{\rm spec}(U_{j+1}, U_{j+2}^{\rm good})|$. First, in the context of Corollary 8.9, we set $\Delta = 1 + \Delta_j \geq \max\{1, \Delta_j\}$, where we used $\Delta_j \geq 0$ from the amenability of $j \in \mathbb{Z}_3$. We also set $W_{j+1} = U_{j+1}^{\rm bad}$, where

$$|W_{j+1}| = \big|U_{j+1}^{\rm bad}\big| \overset{(74)}{\leq} 72\lambda^{1/4} n \overset{(64)}{<} \tfrac{n}{100}.$$

Consequently, Corollary 8.9 guarantees

$$\big|E_G^{\rm spec}\big(U_{j+1}^{\rm bad}, U_{j+2}^{\rm good}\big)\big| \leq \tfrac{3}{10}\Delta n. \tag{132}$$

Second, Statement (2) of Corollary 8.1 guarantees that every $u_{j+2} \in U_{j+2}^{\rm good}$ satisfies

$$\begin{aligned} \big|E_G^{\rm spec}(u_{j+2}, U_{j+1})\big| &\geq \deg_G^c(u_{j+2}) - 1 - \big|U_j \setminus I_j^{\rm bad}\big| \\ &\overset{(114)}{=} \deg_G^c(u_{j+2}) - 1 - \big|\hat{U}_j\big| \overset{(68)}{\geq} m + 1 - \big|\hat{U}_j\big| \overset{(114)}{=} 1 + \Delta_j = \Delta, \end{aligned} \tag{133}$$

and so

$$\big|E_G^{\rm spec}\big(U_{j+1}, U_{j+2}^{\rm good}\big)\big| = \sum_{u_{j+2} \in U_{j+2}^{\rm good}} \deg_G^{\rm spec}(u_{j+2}, U_{j+1}) \geq \Delta \big|U_{j+2}^{\rm good}\big|. \tag{134}$$

Applying (132) and (134) to (131) yields

$$\Delta\big|U_{j+2}^{\rm good}\big| \leq \big|E_G^{\rm spec}\big(U_{j+1}, U_{j+2}^{\rm good}\big)\big| \leq \tfrac{3}{10}\Delta n + \big|E_G^{\rm spec}\big(U_{j+1}^{\rm good}, U_{j+2}^{\rm good}\big)\big|,$$

and so

$$\begin{aligned}\big|E_G^{\text{spec}}\big(U_{j+1}^{\text{good}}, U_{j+2}^{\text{good}}\big)\big| &\geq \Delta\big(\big|U_{j+2}^{\text{good}}\big| - \tfrac{3}{10}n\big)\\ &\overset{(74)}{\geq} \Delta n\big(\tfrac{1}{3} - 75\lambda^{1/4} - \tfrac{3}{10}\big) > \Delta n\big(\tfrac{3}{100} - 75\lambda^{1/4}\big) \overset{(64)}{>} 0,\end{aligned}$$

where we used $\Delta = 1 + \Delta_j \geq 1$ from the amenability of $j \in \mathbb{Z}_3$. This proves (130).

To prove Statement (2) of Lemma 2.9, fix an edge $\{u_{j+1}, u_{j+2}\} \in E_G^{\text{spec}}(U_{j+1}^{\text{good}}, U_{j+2}^{\text{good}})$ from (130), where $u_{j+1} \in U_{j+1}^{\text{good}}$ and $u_{j+2} \in U_{j+2}^{\text{good}}$. We claim that

$$E_G^{\text{spec}}\Big(N_G^{\text{typ}}(u_{j+2}, U_{j+1}), N_G^{\text{typ}}\big(u_{j+1}, U_{j+2}^{\text{good}}\big)\Big) \neq \emptyset. \tag{135}$$

If (135) holds, then it concludes our proof, as follows. Fix $\{v_{j+1}, v_{j+2}\} \in E$ of (135), where $v_{j+1} \in N_G^{\text{typ}}(u_{j+2}, U_{j+1})$ and $v_{j+2} \in N_G^{\text{typ}}(u_{j+1}, U_{j+2}^{\text{good}})$. We first observe that $(u_{j+2}, v_{j+1}, v_{j+2}, u_{j+1})$ is a $(j+2)$-special 4-cycle (cf. Definition 8.4). Indeed, $u_{j+2} \in U_{j+2}^{\text{good}}$ is good from (130) and $v_{j+2} \in U_{j+2}^{\text{good}}$ is good from (135). The edge $\{u_{j+2}, v_{j+1}\} \in E$ is typical because $v_{j+1} \in N_G^{\text{typ}}(u_{j+2}, U_{j+1})$ from (135), and the edge $\{v_{j+2}, u_{j+1}\} \in E$ is typical because $v_{j+2} \in N_G^{\text{typ}}(u_{j+1}, U_{j+2}^{\text{good}})$ from (135). The edge $\{u_{j+2}, u_{j+1}\} \in E$ is special from from (130), and the edge $\{v_{j+2}, v_{j+1}\} \in E$ is special from (135). Since the 4-cycle $(u_{j+2}, v_{j+1}, v_{j+2}, u_{j+1})$ is $(j+2)$-special, Observation 8.5 guarantees that its edges receive precisely 3-colors, where the special edges $(\{u_{j+2}, u_{j+1}\})$ and $\{v_{j+2}, v_{j+1}\}$ match in color and the typical edges $\{u_{j+2}, v_{j+1}\}$ and $\{v_{j+2}, u_{j+1}\}$ do not. Thus, $(u_{j+2}, v_{j+1}, v_{j+2}, u_{j+1})$ is a properly colored 4-cycle. Equivalently, $(u_{j+2}, u_{j+1}, v_{j+2}, v_{j+1})$ is a strong properly colored 4-cycle which Proposition 7.1 extends to a strong properly colored $\ell$-cycle $C_\ell$, as promised by Statement (2) of Lemma 2.9.

To prove (135), we proceed similarly to (130), and begin by considering the identity

$$\begin{aligned}\big|E_G^{\text{spec}}\big(U_{j+1}, N_G^{\text{typ}}\big(u_{j+1}, U_{j+2}^{\text{good}}\big)\big)\big| &\\ = \big|E_G^{\text{spec}}\big(U_{j+1} \setminus N_G^{\text{typ}}(u_{j+2}, U_{j+1}), N_G^{\text{typ}}\big(u_{j+1}, U_{j+2}^{\text{good}}\big)\big)\big| &\\ + \big|E_G^{\text{spec}}\big(N_G^{\text{typ}}(u_{j+2}, U_{j+1}), N_G^{\text{typ}}\big(u_{j+1}, U_{j+2}^{\text{good}}\big)\big)\big|&.\end{aligned} \tag{136}$$

As before, Corollary 8.9 will bound the first summand of (136), and Corollary 8.1 will bound the left hand side of (136). First, we again set $\Delta = 1 +$

$\Delta_j \geq 0$, but we now set $W_{j+1} = U_{j+1} \setminus N_G^{\mathrm{typ}}(u_{j+2}, U_{j+1})$. Since $u_{j+2} \in U_{j+2}^{\mathrm{good}}$ is a good vertex, Proposition 6.2 guarantees

$$|W_{j+1}| = \left|U_{j+1} \setminus N_G^{\mathrm{typ}}(u_{j+2}, U_{j+1})\right| \leq 313\lambda^{1/4} n \overset{(64)}{<} \tfrac{n}{100}.$$

Consequently, Corollary 8.9 guarantees

$$\begin{aligned}(137)\quad & \left|E_G^{\mathrm{spec}}\big(U_{j+1} \setminus N_G^{\mathrm{typ}}(u_{j+2}, U_{j+1}), N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\big)\right| \\ & \qquad \leq \left|E_G^{\mathrm{spec}}\big(U_{j+1} \setminus N_G^{\mathrm{typ}}(u_{j+2}, U_{j+1}), U_{j+2}^{\mathrm{good}}\big)\right| \leq \tfrac{3}{10}\Delta n.\end{aligned}$$

Second, and identically to (133) and (134),

$$\begin{aligned}(138)\quad & \left|E_G^{\mathrm{spec}}\big(U_{j+1}, N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\big)\right| \\ & = \sum_{v_{j+2} \in N_G^{\mathrm{typ}}(u_{j+1}, U_{j+2}^{\mathrm{good}})} \deg_G^{\mathrm{spec}}(v_{j+2}, U_{j+1}) \geq \Delta \left|N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\right|.\end{aligned}$$

Applying (137) and (138) to (136) yields

$$\begin{aligned}(139)\quad & \left|E_G^{\mathrm{spec}}\big(N_G^{\mathrm{typ}}(u_{j+2}, U_{j+1}), N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\big)\right| \\ & \geq \Delta\Big(\left|N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\right| - \tfrac{3}{10} n\Big) \overset{\text{Cor. } 8.3}{\geq} \Delta\Big(\deg_G^c\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big) - 3 - \tfrac{3}{10} n\Big) \\ & \overset{(74)}{\geq} \Delta\Big(\big(\tfrac{1}{3} - 76\lambda^{1/4}\big)n - 3 - \tfrac{3}{10} n\Big) > \Delta n\big(\tfrac{3}{100} - 76\lambda^{1/4}\big) \overset{(64)}{>} 0,\end{aligned}$$

where we used that $\Delta = 1 + \Delta_j \geq 1$ and that $n$ is sufficiently large. This proves (135), and completes the proof of Statement (2) of Lemma 2.9.

**8.3.2. Statement (1) of Lemma 2.9** Statement (1) of Lemma 2.9 assumes that $\delta^c(G) \geq (n+5)/3$ and seeks to conclude that $(G, c)$ admits[4] a rainbow $\ell$-cycle $C_\ell$. The argument here is similar to that of the previous subsection, where in fact we build upon that same argument. For that, note that $\delta^c(G) \geq (n+5)/3 \geq (n+4)/3$ allows all conclusions of the previous subsection to hold for the amenable element $j \in \mathbb{Z}_3$. As before, let

[4]Throughout our proof, we have assumed in (66) that $(G, c)$ avoids rainbow $\ell$-cycles $C_\ell$. Finding one now shows that our assumption (66) is flawed.

$\{u_{j+1}, u_{j+2}\} \in E_G^{\mathrm{spec}}(U_{j+1}^{\mathrm{good}}, U_{j+2}^{\mathrm{good}})$ be fixed. We first observe that

$$\Delta_{j+2} \leq -1, \tag{140}$$

since

$$\begin{aligned} \left|\hat{U}_{j+2}\right| &\geq \deg_G^c\left(u_{j+1}, \hat{U}_{j+2}\right) \\ &= \deg_G^c\left(u_{j+1}, U_{j+2} \setminus I_{j+2}^{\mathrm{bad}}\right) \overset{\text{Cor. 8.1}}{\geq} \deg_G^c(u_{j+1}) - 1 \overset{(68)}{\geq} m+1 \\ &\implies \qquad -1 \geq m - \left|\hat{U}_{j+2}\right| \overset{(114)}{=} \Delta_{j+2}. \end{aligned}$$

Second, we observe that

$$n \equiv 2 \pmod 3 \qquad \text{or} \qquad \Delta_j \geq 1. \tag{141}$$

To argue (141),

(142) *we assume, on the contrary, that* $n \not\equiv 2 \pmod 3$ *and* $\Delta_j = 0$.

From (142), we will conclude that $j+1 \in \mathbb{Z}_3$ is also amenable, whence (140) also holds for $j+1 \in \mathbb{Z}_3$, in which case $\Delta_{j+1+2} = \Delta_j \leq -1$ contradicts $\Delta_j = 0$ of (142). To see that $j+1 \in \mathbb{Z}_3$ is amenable, we note from (114) that

$$\left|\hat{U}_j\right| \overset{(142)}{=} m, \quad \text{and} \qquad \left|\hat{U}_{j+2}\right| \overset{(140)}{\geq} m+1 \qquad \implies \qquad \left|\hat{U}_{j+1}\right| \leq m, \tag{143}$$

lest $3m+2 \leq |\hat{U}_0| + |\hat{U}_1| + |\hat{U}_2| \leq n$ contradicts (142) (recall $m = \lfloor n/3 \rfloor$ from (68)). Thus,

$$\Delta_{j+1} \overset{(114)}{=} m - \left|\hat{U}_{j+1}\right| \geq 0 \tag{144}$$

satisfies the first condition of amenability in Observation 8.7. Moreover,

$$\begin{aligned} \left|I_j^{\mathrm{bad}}\right|, \left|I_{j+2}^{\mathrm{bad}}\right| &\leq \sum_{k \in \mathbb{Z}_3} \left|I_k^{\mathrm{bad}}\right| \overset{(113),(114)}{=} n - \sum_{k \in \mathbb{Z}_3} \left|\hat{U}_k\right| \\ &\overset{(142)}{\leq} 1 + \sum_{k \in \mathbb{Z}_3} \left(m - \left|\hat{U}_k\right|\right) \overset{(114)}{=} 1 + \sum_{k \in \mathbb{Z}_3} \Delta_k = 1 + \Delta_j + \Delta_{j+1} + \Delta_{j+2} \\ &\overset{(140)}{\leq} \Delta_j + \Delta_{j+1} \overset{(142)}{=} \Delta_{j+1} \overset{(144)}{\leq} 2\Delta_{j+1}, \end{aligned} \tag{145}$$

and so $|I_{j+2}^{\mathrm{bad}}| \leq 2\Delta_{j+1}$ satisfies the second condition of amenability in Observation 8.7. Finally,

$$|U_{j+3}| = |U_j| \overset{(113),(114)}{=} \big|\hat{U}_j\big| + \big|I_j^{\mathrm{bad}}\big| \overset{(142)}{=} m + \big|I_j^{\mathrm{bad}}\big| \overset{(145)}{\leq} m + 2\Delta_{j+1} \leq m + 2\Delta_{j+1} + 2,$$

and so $|U_{j+3}| \leq m + 2\Delta_{j+1} + 2$ satisfies the third condition of amenability in Observation 8.7.

The remainder of our proof for Statement (1) of Lemma 2.9 splits into the two cases of (141). For these, recall that $\{u_{j+1}, u_{j+2}\} \in E_G^{\mathrm{spec}}(U_{j+1}^{\mathrm{good}}, U_{j+2}^{\mathrm{good}})$ was fixed at the start of this proof, where we now set $c(\{u_{j+1}, u_{j+2}\}) = \alpha$ for $\alpha \neq c_{u_{j+2}}$ on account that $\{u_{j+1}, u_{j+2}\}$ is a special edge.

**Case 1 ($n \equiv 2$ (mod 3)).** We revisit (135) by confirming that

$$\begin{aligned} &\emptyset \neq E_{G,\neg\alpha}^{\mathrm{spec}}\big(N_G^{\mathrm{typ}}(u_{j+2}, U_{j+1}), N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\big) \\ &\quad = \big\{\{v_{j+1}, v_{j+2}\} \in \\ &\quad E_G^{\mathrm{spec}}\big(N_G^{\mathrm{typ}}(u_{j+2}, U_{j+1}), N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\big) : c(\{v_{j+1}, v_{j+2}\}) \neq \alpha\big\}, \end{aligned} \tag{146}$$

where the set above consists of those edges of (135) which are not colored $\alpha$. If true, then any $\{v_{j+1}, v_{j+2}\} \in E$ of (146) gives a strong rainbow 4-cycle $(u_{j+2}, u_{j+1}, v_{j+2}, v_{j+1})$ (recall Observation 8.5) which Proposition 7.1 extends to a strong rainbow $\ell$-cycle $C_\ell$, as promised by Statement (1) of Lemma 2.9. To see (146), we replay the details of (136)–(139) with the added hypothesis $n = 3m + 2$. We again have

$$\begin{aligned} &\big|E_{G,\neg\alpha}^{\mathrm{spec}}\big(U_{j+1}, N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\big)\big| \\ &\quad = \big|E_{G,\neg\alpha}^{\mathrm{spec}}\big(U_{j+1} \setminus N_G^{\mathrm{typ}}(u_{j+2}, U_{j+1}), N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\big)\big| \\ &\qquad + \big|E_{G,\neg\alpha}^{\mathrm{spec}}\big(N_G^{\mathrm{typ}}(u_{j+2}, U_{j+1}), N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\big)\big|, \end{aligned} \tag{147}$$

where the left hand side and first summand of (147) are defined analogously to (146). Setting $\Delta = \Delta_j + 1 \geq \max\{\Delta_j, 1\}$, we clearly have

$$\begin{aligned} &\big|E_{G,\neg\alpha}^{\mathrm{spec}}\big(U_{j+1} \setminus N_G^{\mathrm{typ}}(u_{j+2}, U_{j+1}), N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\big)\big| \\ &\quad \leq \big|E_G^{\mathrm{spec}}\big(U_{j+1} \setminus N_G^{\mathrm{typ}}(u_{j+2}, U_{j+1}), N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\big)\big| \overset{(137)}{\leq} \tfrac{3}{10}\Delta n. \end{aligned} \tag{148}$$

Moreover, $\delta^c(G) \geq (n+5)/3$ for $n \equiv 2 \pmod 3$ ensures $\delta^c(G) \geq m+3$, and so Statement (2) of Corollary 8.1 guarantees that every $v_{j+2} \in U_{j+2}^{\mathrm{good}}$ satisfies

$$(149)\quad \left|E_G^{\mathrm{spec}}(v_{j+2}, U_{j+1})\right| \geq \deg_G^c(v_{j+2}) - 1 - \left|U_j \setminus I_j^{\mathrm{bad}}\right| \overset{(114)}{=} \deg_G^c(v_{j+2}) - 1 - \left|\hat{U}_j\right| \geq m + 2 - \left|\hat{U}_j\right| \overset{(114)}{=} 2 + \Delta_j = 1 + \Delta.$$

Consequently, those edges $E_{G,\neg\alpha}^{\mathrm{spec}}(v_{j+2}, U_{j+1})$ above not colored $\alpha$ satisfy $|E_{G,\neg\alpha}^{\mathrm{spec}}(v_{j+2}, U_{j+1})| \geq \Delta$, and so similarly to (138), we have

$$(150)\quad \left|E_{G,\neg\alpha}^{\mathrm{spec}}\big(N_G^{\mathrm{typ}}(u_{j+2}, U_{j+1}), N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\big)\right| \geq \Delta \left|N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\right|.$$

Similarly to (139), we apply (148) and (150) to (147) to infer

$$(151)\quad \left|E_{G,\neg\alpha}^{\mathrm{spec}}\big(N_G^{\mathrm{typ}}(u_{j+2}, U_{j+1}), N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\big)\right| \geq \Delta\Big(\left|N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\right| - \tfrac{3}{10}n\Big) \overset{(139)}{>} 0,$$

which proves (146).

**Case 2 ($\Delta_j \geq 1$).** We again confirm (146), in which case any $\{v_{j+1}, v_{j+2}\} \in E(G)$ of (146) gives a strong rainbow 4-cycle $(u_{j+2}, u_{j+1}, v_{j+2}, v_{j+1})$ which Proposition 7.1 extends to a strong rainbow $\ell$-cycle $C_\ell$, as promised by Statement (1) of Lemma 2.9. We again replay the details of (147)–(151), only this time we set $\Delta = \Delta_j \geq 1$ (as opposed to before, when $\Delta = 1 + \Delta_j$). Then (148) is updated to say that

$$(152)\quad \left|E_{G,\neg\alpha}^{\mathrm{spec}}\big(U_{j+1} \setminus N_G^{\mathrm{typ}}(u_{j+2}, U_{j+1}), N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\big)\right| \leq \left|E_G^{\mathrm{spec}}\big(U_{j+1} \setminus N_G^{\mathrm{typ}}(u_{j+2}, U_{j+1}), N_G^{\mathrm{typ}}\big(u_{j+1}, U_{j+2}^{\mathrm{good}}\big)\big)\right| \overset{(137)}{\leq} \tfrac{3}{10}\Delta_j n,$$

while (149) is updated to say that each $v_{j+2} \in U_{j+2}^{\mathrm{good}}$ satisfies

$$\left|E_G^{\mathrm{spec}}(v_{j+2}, U_{j+1})\right| \geq \deg_G^c(v_{j+2}) - 1 - \left|\hat{U}_j\right| \overset{(68)}{\geq} m + 1 - \left|\hat{U}_j\right| = 1 + \Delta_j.$$

Consequently, (150) is updated to say

$$(153)\quad \left|E^{\text{spec}}_{G,\neg\alpha}\big(N^{\text{typ}}_G(u_{j+2},U_{j+1}),N^{\text{typ}}_G(u_{j+1},U^{\text{good}}_{j+2})\big)\right| \geq \Delta_j\left|N^{\text{typ}}_G(u_{j+1},U^{\text{good}}_{j+2})\right|.$$

We apply (152) and (153) to (147) to infer

$$\left|E^{\text{spec}}_{G,\neg\alpha}\big(N^{\text{typ}}_G(u_{j+2},U_{j+1}),N^{\text{typ}}_G(u_{j+1},U^{\text{good}}_{j+2})\big)\right| \geq \Delta_j\big(\left|N^{\text{typ}}_G(u_{j+1},U^{\text{good}}_{j+2})\right|-\tfrac{3}{10}n\big) \overset{(139)}{>} 0,$$

where we used $\Delta_j \geq 1$ from Case 2. This confirms (146), and concludes our proof of Lemma 2.9.

## Appendix: Proof-sketch for case 3 from the introduction

Recall the partition $V = U_0 \cup U_1 \cup U_2 \cup \{x\} \cup \{y\}$ from Case 3 of Section 1.1. Let $\vec{G}_1 = (V, \vec{E}_1)$ be the oriented graph whereby $(u,v) \in \vec{E}_1$ if, and only if, $(u,v)$ is an element of one of the following sets:

$$U_0\times U_1,\quad U_1\times U_2,\quad U_2\times U_0,\quad (U_0\cup U_2)\times\{x\},\quad \{x\}\times(\{y\}\cup U_1),\quad \{y\}\times U_2.$$

Let $\vec{G}_2 = (V, \vec{E}_2)$ satisfy $(u,v) \in \vec{E}_2$ if, and only if, $(u,v)$ is an element of one of the following sets:

$$U_0\times U_1,\quad U_1\times U_2,\quad U_2\times U_0,\quad \{x\}\times U_1,\quad U_1\times\{y\},\quad \{y\}\times(\{x\}\cup U_2).$$

One can see that neither $\vec{G}_1 = (V, \vec{E}_1)$ nor $\vec{G}_2 = (V, \vec{E}_2)$ admit directed $\ell$-cycles $\vec{C}_\ell$ when $\ell \equiv 2 \pmod 3$. Recalling $(\hat{G}, \hat{c})$ from (5), construct $\hat{G}_1 \subseteq \hat{G}$ by removing all edges between $U_1$ and $y$, and set $\hat{c}_1 = \hat{c}|_{E(\hat{G}_1)}$. Construct $\hat{G}_2 \subseteq \hat{G}$ by removing all edges between $U_0 \cup U_2$ and $x$, and set $\hat{c}_2 = \hat{c}|_{E(\hat{G}_2)}$. Then $E(\hat{G}) = E(\hat{G}_1) \cup E(\hat{G}_2)$, and $(\hat{G}_i, \hat{c}_i)$, $i = 1, 2$, is isomorphic to the edge-colored graph determined by $\vec{G}_i$. Therefore, a rainbow $\ell$-cycle of $(\hat{G}, \hat{c})$ can coincide entirely with neither $\hat{G}_1$ nor $\hat{G}_2$, and must admit an edge from $E(\hat{G}_1) \setminus E(\hat{G}_2)$ and an edge from $E(\hat{G}_2) \setminus E(\hat{G}_1)$. But this is impossible, because (5) ensures that every edge in the symmetric difference $E(\hat{G}_1) \triangle E(\hat{G}_2)$ is assigned the color $\star$.

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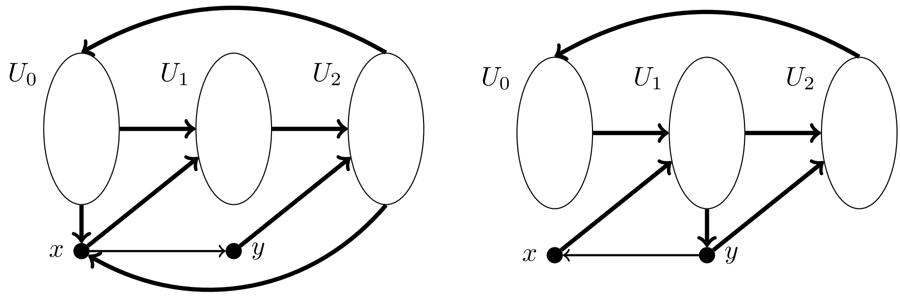


(a) The oriented graph $\vec{G}_1 = (V, \vec{E}_1)$.

(b) The oriented graph $\vec{G}_2 = (V, \vec{E}_2)$.

Figure 1: Neither $\vec{G}_1$ nor $\vec{G}_2$ admit directed $\ell$-cycles when $\ell \equiv 2 \pmod 3$.

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