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The weighted spectrum of the universal cover and an Alon-Boppana result for the normalized Laplacian*

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We provide a lower bound for the spectral radius of the universal cover of irregular graphs in the presence of symmetric edge weights. We use this bound to derive an Alon-Boppana type bound for the second eigenvalue of the normalized Laplacian.

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1. Introduction

Let $G = (V, E)$ be a simple, connected, n vertex graph and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of its adjacency matrix. Letting $\lambda(G) = \lambda_2$ one of the versions of the famous Alon-Boppana theorem states that

Alon-Boppana Theorem ([14])**.** *For any sequence of simple connected d-regular graphs G_i with increasing diameter, we have that $\liminf_{i\to\infty} \lambda(G_i) \geq 2\sqrt{d-1}$. Furthermore, for any particular d-regular graph with two edges of distance at least $2k+2$, $\lambda(G) \geq 2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{k+1}$.*

Since the spectrum of the adjacency matrix of a regular graph is closely related to the expansion properties of the graph (see [1], for example), the Alon-Boppana result may be thought of as upper bound on the quality of the expansion in a d-regular graph. Recently, Friedman has confirmed a conjecture of Alon and shown that for any $\epsilon > 0$ a sufficiently large d-regular random graph has $\lambda(G) \leq 2\sqrt{d-1} + \epsilon$ with high probability, and thus may be thought of as extremal with respect to $\lambda(\cdot)$ [4, 5].

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Given the wide ranging practical and theoretical applications of expanders (see [8]), it is natural to consider what an analogous statement of the Alon-Boppana theorem would be for irregular graphs. To that end, we say a graph has r-robust average degree d if for every vertex v, $G[V \backslash B_r(v)]$ has average degree at least d, where $G[S]$ is the graph induced by S and $B_r(v)$ consists of all vertices at distance most r from v. Now, with this definition Hoory generalized the Alon-Boppana results as follows.

Theorem 1 ([7])**.** *Let G_i be a sequence of graphs such that G_i has r_i-robust average degree $d \geq 2$. If $r_i \to \infty$, then $\liminf_{i\to\infty} \max\{|\lambda_2(G_i)|, |\lambda_n(G_i)|\} \geq 2\sqrt{d-1}$.*

Jiang has recently improved this result [9] by relaxing the requirements on the r-robust average degree and improving the rate at which the bound converges to $2\sqrt{d-1}$

Theorem 2 ([9])**.** *Let G be a sequence of graphs such that G has r-robust average degree $d \geq 1$, then*

$$\lambda(G) \geq 2\sqrt{d-1}\cos\left(\frac{\pi}{r+1}\right).$$

At this point is also worth mentioning Mohar's recent work on a multipartite generalization of the Alon-Boppana theorem [12] which provides a clean description of the spectrum of the t-partite regular graph in terms of the spectrum of a $t \times t$ matrix. By introducing the concept of a sub-universal cover Mohar is able to lift this result to general results about graphs that are not necessarily multipartite, for instance:

Theorem 3 ([12])**.** *Let $d_1 \leq d_2 \leq d$ be positive integers, and let $\mathcal{G}^d_{d_1,d_2}$ be the set of all graphs whose maximum vertex degree is at most d and whose vertex set is the union of (not necessarily disjoint) subsets U_1, U_2, such that every vertex in U_i has at least d_i neighbors in U_{3-i} for $i = 1, 2$. For every $\epsilon > 0$, every n-vertex graph $G \in \mathcal{G}^d_{d_1,d_2}$ has $\Omega_\epsilon(n)$ eigenvalues larger than $\sqrt{d_1 - 1} + \sqrt{d_2 - 1} - \epsilon$.*

We note that the results of Mohar are stronger than Theorem 3, in that they imply a family of bounds along the same lines that are significantly harder state compactly.

It is relatively easy to find examples where the bound given by Mohar is an improvement on the bound given by Hoory and Jiang, for example, consider the graphs G on $4n$ vertices where A, A', B, are n vertex independent sets and C is a n vertex 8-regular graph. The vertices in B are connected to

A, A', and C by $(10,10)$, $(3,3)$, and $(2,2)$-regular bipartite graphs, and A' and C are connected by a $(7,7)$-regular graph. These graphs have a average degree 11, and so asymptotically Theorem 1 gives an asymptotic lower-bound of $2\sqrt{10}$. Letting $U_1 = A \cup A' \cup C$ and $U_2 = B \cup C$, gives two sets where every vertex in U_1 has degree 10 into U_2 and every vertex in U_2 has degree 15 into U_1. Thus Theorem 3 gives a bound of $\sqrt{9} + \sqrt{14} > 2\sqrt{10}$. We note that this improvement comes at a cost as there is not an immediately obvious way to quickly verify that the conditions of Theorem 3, whereas it is clear that in a class of graphs of bounded degree (or even families of graphs where the maximum degree grows sufficiently slowly with n) the conditions of Theorem 1 are satisfied asymptotically.

We note that by passing to the irregular case the tight relationship with the expansion of the graph is lost, and so neither Hoory's nor Mohar's generalizations can not be thought of as a bound on the expansion properties of a family of graphs with specified properties. In order to maintain the connection between expansion and the spectrum of an irregular graph, we consider instead the normalized Laplacian $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$ where D is the diagonal matrix of degrees. Note that if a graph is regular, then the spectrum of the normalized Laplacian can be found by applying an affine transformation of the spectrum of the adjacency matrix. However, if the graph is irregular the structure of the two spectra can differ significantly.

We make the standard observations that all of the eigenvalues of $\mathcal{L}$ are in $[0,2]$ and that there is an eigenvector with eigenvalue 0, namely $\sqrt{\frac{\deg(v_i)}{\mathrm{Vol}(G)}}$, where $\mathrm{Vol}(G) = \sum_{i=1}^{n} \deg(v_i)$. Thus for a given graph G define $\lambda^{\mathcal{L}}(G)$ as the second smallest eigenvalue of the normalized Laplacian. It is well known that $\lambda^{\mathcal{L}}(G)$ is tightly connected with expansion and algorithmic properties of G (see [3] for an overview of such results).

In the context of the normalized Laplacian, the Alon-Boppana theorem says that for a d-regular graph G, $\lambda^{\mathcal{L}}(G) \le 1 - 2\frac{\sqrt{d-1}}{d} + o(1)$. Thus, one might naturally conjecture that in the case of the normalized Laplacian of irregular graphs, this would generalize to $\lambda^{\mathcal{L}}(G) \le 1 - \frac{2\sqrt{d-1}}{d} + o(1)$ where d is the average degree of G. However, as we will show in Section 3 there exists a fixed $\epsilon > 0$ and an infinite family of graphs $\mathcal{G}$ with common average degree d and increasing diameters, such that $\lambda^{\mathcal{L}}(G) - \left(1 - \frac{2\sqrt{d-1}}{d}\right) > \epsilon$ for all $G \in \mathcal{G}$. Thus, our main result is that if G_i is a sequence of graphs with average degree at least 2 and increasing "robustness" with respect to the average degree d, then there is a constant δ, dependent on the degree sequence, such that $\lambda^{\mathcal{L}}(G_i) \le 1 - \frac{2\sqrt{d-1}}{\delta} + o(1)$. We note that if the G_i's are regular then this bound will agree exactly with the Alon-Boppana result, however for irregular graphs this yields a higher upper bound than the natural conjecture.

2. Alon-Boppana for the normalized Laplacian

Rather than attack the normalized Laplacian directly, we adapt the work of Hoory [7] and provide a lower bound for the spectral radius of the universal cover graph with appropriate weights. Recall that the universal cover of a connected graph G is the unique (up to isomorphism) infinite graph that is a universal cover of G as a topological space. This graph may be constructed explicitly by fixing a root vertex r and considering all non-backtracking walks in G starting at r. Two such walks are adjacent if one can be extend to the other by a single step in the walk.

Noting that a weight function $w\colon E \to \mathbb{R}^+$ on G lifts in a natural way to a weight function $\tilde{w}$ on the universal cover $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, for any $v \in \mathcal{V}$ we define $t_{2k}^{(w)}(v)$ as the total weight of all closed walks of length $2k$ from v to itself in $\mathcal{G}$, where the weight of a walk is the product of all the edge weights in the walk. By well known results (see [13]) the weighted spectral radius of $\mathcal{G}$ is $\rho_w(\mathcal{G}) = \limsup_{k\to\infty} \sqrt[2k]{t_{2k}^{(w)}(v)}$ for any $v \in \mathcal{V}$.

Theorem 4. *Let $G = (V, E)$ be any graph with minimum degree 2 and let $f\colon V \to \mathbb{R}^+$. Define a weight function $w\colon E \to \mathbb{R}^+$ by $w(u, v) = f(u)f(v)$, then the weighted spectral radius of the universal cover $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is at least $2\prod_{v\in V}\left(f(v)^2\sqrt{(\deg(v)-1)}\right)^{\frac{\deg(v)}{\mathrm{Vol}(G)}}$.*

Proof. In order to provide a lower bound on the spectral radius of $\mathcal{G}$, we will show that $t_{2k}^{(w)}(v)$ is bounded below by the expected weight of certain class of random walks on G. Specifically, we will consider sampling from what we term *stack based walks* uniformly at random. A stack based walk starts at a vertex v with some neighbor of v, say v', at the top of a stack. If the stack based walk is currently at vertex v with v' (a neighbor of v) on the top of the stack, on a *forward step* some neighbor v'' of v is chosen such that $v' \neq v''$ and the walk moves to vertex v'' and v is pushed onto the stack. If a *backwards step* is taken, v' is popped off the stack and the walk moves to v'. Note that at no point in the progress of a stack based walk is the stack allowed to be empty, in particular, when the stack has a single element a backwards step is forbidden. We will refer to a stack based walk that returns to its initial state (including the state of the stack) as a *closed* stack based walk. We will denote by $\Omega_{2k}(v, v')$ the set of closed stack based walks of length $2k$ that start at the vertex v and with v' in the stack.

Notice that the number of elements in a stack during a closed stack based walk is a Dyck path with 1 as the baseline. Thus letting T_{2k} be the collection of Dyck paths of length $2k$ (with baseline 1), we can further parameterize $\Omega_{2k}(v, v')$ by $\tau \in T_{2k}$. That is, $\Omega_{2k,\tau}(v, v')$ is those walks in $\Omega_{2k}(v, v')$ where the size of the stack agrees with τ at every step. It is obvious that for every $\omega \in \Omega_{2k}(v, v')$ there is a unique $\tau \in T_{2k}$ such that $\omega \in \Omega_{2k,\tau}(v, v')$.

Now any stack based walk $\omega \in \Omega_{2k}(v, v')$ can be immediately lifted to a closed walk of length $2k$ in $\mathcal{G}$, and further, the Dyck path associated with the ω corresponds to the current distance from the initial vertex v. Thus we have that $t_{2k}^{(w)}(v) > w(\Omega_{2k}(v, v'))$, where $w(S)$ is the sum of the weights of all walks in S. The strict inequality comes from the fact that all walks whose first step is to vertex v' are counted in $t_{2k}^{(w)}(v)$ but missing from $\Omega_{2k}(v, v')$. Furthermore, by averaging over the choice of v', we have that $t_{2k}^{(w)}(v) \geq \sum_{v'\sim v} \frac{1}{\deg(v)} w(\Omega_{2k}(v, v') = \sum_{\tau\in T_{2k}} \sum_{v'\sim v} \frac{1}{\deg(v)} w(\Omega_{2k,\tau}(v, v'))$. As a consequence of this observation, and the independence of the spectral radius from the choice of v, we have that

$$\begin{aligned}\rho_w(\mathcal{G}) &= \limsup_k \sqrt[2k]{\sum_v \frac{\deg(v)}{\mathrm{Vol}(G)} t_{2k}^{(w)}(v)} \\ &= \limsup_k \sqrt[2k]{\sum_v \frac{\deg(v)}{\mathrm{Vol}(G)} \sum_{\tau\in T_{2k}} \sum_{v'\sim v} \frac{1}{\deg(v)} w(\Omega_{2k,\tau}(v, v'))}.\end{aligned}$$

For simplicity of notation we will let

$$S_{2k}^{(w)} = \sum_v \frac{\deg(v)}{\mathrm{Vol}(G)} \sum_{\tau\in T_{2k}} \sum_{v'\sim v} \frac{1}{\deg(v)} w(\Omega_{2k,\tau}(v, v')).$$

In order to complete the proof it will suffice to place a lower bound on $S_{2k}^{(w)}$. To that end, we will define a probability measure on elements of $\Omega_{2k,\tau}(v, v')$, specifically the uniform random walk measure. That is, for every forward step in the stack based walk we choose uniformly from the set of all allowable neighbors, that is, the neighbors that are not the top element of the stack. Specifically, consider a fixed walk $\omega \in \Omega_{2k,\tau}(v, v')$ and let $(v_1, u_1), \ldots, (v_k, u_k)$ be the forward edges of the walk w. Define $p(\omega) = \prod_{i=1}^k \frac{1}{\deg(v_i)-1}$, which is the probability of choosing the walk ω at random, given that the vertex v' is not the first step. Thus we may rewrite

$$S_{2k}^{(w)} = \sum_{\tau\in T_{2k}} \sum_{v\in V} \sum_{v'\sim v} \sum_{\omega\in\Omega_{2k,\tau}(v,v')} \frac{p(\omega)}{\mathrm{Vol}(G)} \frac{w(\omega)}{p(\omega)}$$

and observe that $\sum_{\omega\in\Omega_{2k,\tau}(v,v')} p(\omega) = 1$ and thus

$$\sum_{v\in V}\sum_{v'\sim v}\sum_{\omega\in\Omega_{2k,\tau}(v,v')} \frac{p(\omega)}{\mathrm{Vol}(G)} = 1$$

and hence, by the (weighted) arithmetic-geometric mean inequality

$$S_{2k}^{(w)} \geq \sum_{\tau\in T_{2k}} \prod_{v\in V(G)} \prod_{v'\sim v} \prod_{\omega\in\Omega_{2k,\tau}(v,v')} \left(\frac{w(\omega)}{p(\omega)}\right)^{\frac{p(\omega)}{\mathrm{Vol}(G)}}.$$

Since every forward edges is traversed in the backtracking direction as well, the weight of the walk $w(\omega) = \prod_{i=1}^{k}(f(v_i)f(u_i))^2$. Thus we have that $\frac{w(\omega)}{p(\omega)} = \prod_{i=1}^{k}(\deg(v_i)-1)f(v_i)^2f(u_i)^2$. Hence, it suffices to understand, for any ordered edge (v,v'), the number of times (weighted by $\frac{p(\omega)}{\mathrm{Vol}(G)}$) any stack based walk governed by τ crosses (v,v') on a forward step. To that end, for any ordered edge (v,v') let $\delta_{v,v'}(\omega)$ be the number of times the walk ω goes from v to v' on a forward step and let $\Omega_{2k,\tau} = \bigcup_{v\in V}\bigcup_{v'\sim v}\Omega_{2k,\tau}(v,v')$. We note that by the stack based description we have that $\Omega_{2k,\tau}(v,v')\cap\Omega_{2k,\tau}(v,v'') = \emptyset$ if $v' \neq v''$ as the initial stack differs even for the same walk in G. Hence we have that

$$\begin{aligned} S_{2k}^{(w)} &\geq \sum_{\tau\in T_{2k}} \prod_{v\in V} \prod_{v'\sim v} \prod_{\omega\in\Omega_{2k,\tau}(v,v')} \left(\frac{w(\omega)}{p(\omega)}\right)^{\frac{p(\omega)}{\mathrm{Vol}(G)}} \\ &= \sum_{\tau\in T_{2k}} \prod_{v\in V} \prod_{v'\sim v} \left((\deg(v)-1)f(v)^2f(v')^2\right)^{\frac{1}{\mathrm{Vol}(G)}\sum_{\omega\in\Omega_{2k,\tau}} p(\omega)\delta_{v,v'}(\omega)}. \end{aligned}$$

Thus we are interested in

$$\frac{1}{\mathrm{Vol}(G)}\sum_{\omega\in\Omega_{2k,\tau}} p(\omega)\delta_{v,v'}(\omega),$$

for any ordered edge (v,v') and all τ and k. But this is just the expected number of times a random stack based walk crosses (v,v') on a forward step given an initial state chosen uniformly at random. It is easy to see that uniform distribution on directed edges is stationary for the random stack based walk, and thus

$$\frac{1}{\mathrm{Vol}(G)}\sum_{\omega\in\Omega_{2k,\tau}} p(\omega)\delta_{v,v'}(\omega) = \frac{k}{\mathrm{Vol}(G)}.$$

Furthermore, we have that

$$S_{2k}^{(w)} \geq \sum_{\tau \in T_{2k}} \prod_{v \in V(G)} \prod_{v' \sim v} \left(f(v)^2 f(v')^2 (\deg(v)-1)\right)^{\frac{k}{\mathrm{Vol}(G)}}$$
$$= |T_{2k}| \prod_{v \in V(G)} \left((\deg(v)-1) f(v)^4\right)^{\frac{\deg(v)k}{\mathrm{Vol}(G)}},$$

which proves the result as

$$\rho_w(\tilde{G}) \geq \limsup_{k \to \infty} \sqrt[2k]{S_{2k}^{(w)}} \geq 2 \prod_{v \in V(G)} \left(f(v)^2 \sqrt{(\deg(v)-1)}\right)^{\frac{\deg(v)}{\mathrm{Vol}(G)}}$$

□

Corollary 5. *For any graph G with minimum degree at least 2 with weight function $w(u,v) = (\deg(v)\deg(u))^{-1/2}$, the weighted spectral radius of the universal cover is at least $2\sqrt{\prod_{v \in V} \left(\frac{\deg(v)-1}{\deg(v)^2}\right)^{\frac{\deg(v)}{\mathrm{Vol}(G)}}}$.*

Following the notation of Chung, Lu, and Vu [2] we denote the average degree of a graph by d and the *second order average degree* of a graph $G = (V, E)$ by $\tilde{d} = \frac{\sum_{v \in V} \deg(v)^2}{\mathrm{Vol}(G)} = \frac{\sum_{v \in V} \deg(v)^2}{d|V|}$. Using this notation, we can reformulate the bound in Corollary 5 into a more natural one in terms of global statistics of G. Specifically, since $(x-1)^x$ is log-convex for $x \geq 2$, we have $\prod_{v \in V} (\deg(v)-1)^{\frac{\deg(v)}{\mathrm{Vol}(G)}} \geq (d-1)^{\frac{dn}{\mathrm{Vol}(G)}} = d-1$. Additionally, by the arithmetic-geometric mean inequality,

$$\prod_{v \in V} \deg(v)^{\frac{\deg(v)}{\mathrm{Vol}(G)}} \leq \frac{\sum_{v \in V(G)} \deg(v)^2}{\mathrm{Vol}(G)} = \tilde{d}.$$

Building on this observation we have the following natural extension of Corollary 5.

Corollary 6. *If $G = (V, E)$ is a graph with average degree $d \geq 2$ and $w(u,v) = (\deg(u)\deg(v))^{-1/2}$, then the spectral radius of the universal cover is at least $2\sqrt{d-1}/\tilde{d}$, where $\tilde{d} = \frac{\sum_{v \in V} \deg(v)^2}{\mathrm{Vol}(G)}$ is the second order average degree.*

Proof. Since the average degree of G is at least 2 and removing a degree one vertex can not decrease the average degree, G has a non-empty 2-core, $G' = (V', E')$. Letting $\mathcal{G}$ denote the universal cover of G and $\mathcal{G}'$ denote the universal cover of G', we have have that

$$\rho_w(\mathcal{G}) \geq \rho_w(\mathcal{G}') \geq 2\sqrt{\prod_{v' \in V'} \left(\frac{\deg_{G'}(v') - 1}{\deg_G(v')^2} \right)^{\frac{\deg_{G'}(v')}{\mathrm{Vol}(G')}}}$$

by adapting the proof of Theorem 4. Specifically, note that the first inequality comes from the limiting of the closed walks to those entirely within G' while preserving the weight of all those walks. Now since G' has minimum degree at least 2 by definition and deleting degree one vertices in G can not decrease the average degree,

$$\prod_{v' \in V'} (\deg_{G'}(v') - 1)^{\frac{\deg_{G'}(v')}{\mathrm{Vol}(G')}} \geq d' - 1 \geq d - 1.$$

We observe that if $y \geq 2x$ and $\alpha \in (0, 1]$, then $\alpha^{\frac{x}{y}} \geq \alpha^{\frac{x+1}{y+2}}$. Thus by sequentially adding the vertices deleted to reach the 2-core, we have

$$\prod_{v' \in V'} \deg_G(v')^{-\frac{\deg_{G'}(v')}{\mathrm{Vol}(G')}} \geq \prod_{v \in V} \deg_G(v)^{-\frac{\deg_{G'}(v)}{\mathrm{Vol}(G')}} \geq \prod_{v \in} \deg_G(v)^{-\frac{\deg_G(v)}{\mathrm{Vol}(G)}} \geq \frac{1}{\tilde{d}}.$$

Combining these observations gives the desired result. □

Let $B_r(v)$ be the set of vertices are distance at most r from v. If G is a connected graph, let f_P be the unit principle eigenvector of the normalized Laplacian. We will say that a graph has *normalized Laplacian eigenradius* r if for every vertex v, $\sum_{u \in B_r(v)} f_P(u)^2 \leq \frac{1}{2}$ and there is some vertex v' such that $\sum_{u \in B_{r+1}(v)} f_P(u)^2 > \frac{1}{2}$. Using this notation we have the following analogue of the Alon-Boppana theorem.

Theorem 7. *If $G = (V, E)$ is a connected graph with normalized Laplacian eigenradius at least $2k + 1 \geq 3$, average degree $d \geq 2$, then $\lambda^{\mathcal{L}}(G) \leq 1 - {}^{2\sqrt{d-1}}/_{\tilde{d}} \left(1 - \frac{3\ln(k+1)}{4k}(1 + o(1))\right)$, where $\tilde{d}$ is the second order average degree $\frac{\sum_{v \in V} \deg(v)^2}{\mathrm{Vol}(G)}$.*

Proof. Let f_P be the principle unit eigenvector of $\mathcal{L}$, let $M = I - \mathcal{L}$, and let $w(u, v) = (\deg(u)\deg(v))^{-1/2}$. Now by Corollary 6, if $\mathcal{G}$ is the universal cover of G, then

$$\rho_w(\mathcal{G}) \geq 2 \prod_{v \in V} \left(\frac{\sqrt{\deg(v) - 1}}{\deg(v)} \right)^{\frac{\deg(v)}{\mathrm{Vol}(G)}}.$$

As there is a natural weight preserving injection from closed walks in $\mathcal{G}$ to walks in G, there exists vertex v in G such that the total weight of all closed walks of length $2k$ is at least $C_k\left(\frac{(d-1)^k}{\tilde{d}^{2k}}\right)$, where C_k is the k^{th} Catalan number.

Let $R = V \backslash B_r(v)$, that is, the set of vertices of distance at least $r+1$ from v. Let f_R be the projection of f_P onto the coordinates of R, we note that $\|f_R\|_2^2 \geq \frac{1}{2}$, by the definition of eigenradius. Letting $\mathbb{1}_v$ be the indicator vector for the vertex v, define $f = \|f_R\|_2 \mathbb{1}_v - \frac{f_P^T\mathbb{1}_v}{\|f_R\|_2} f_R$. We first note that

$$f_P^T f = \|f_R\|_2 f_P^T \mathbb{1}_v - \frac{f_P^T\mathbb{1}_v}{\|f_R\|_2} f_P^T f_R = 0.$$

Now consider the Raleigh quotient for f:

$$\begin{aligned}
\frac{f^T M^{2k} f}{\|f\|_2^2} &= \frac{\|f_R\|_2^2 \mathbb{1}_v^T M^{2k}\mathbb{1}_v - 2f_P^T\mathbb{1}_v\mathbb{1}_v M^{2k} f_R + \frac{(f_P^T\mathbb{1}_v)^2}{\|f_R\|_2^2} f_R^T M^{2k} f_R}{\|f_R\|_2^2 + (f_P^T\mathbb{1}_v)^2} \\
&= \frac{\|f_R\|_2^2 \mathbb{1}_v^T M^{2k}\mathbb{1}_v - 2f_P^T\mathbb{1}_v \sum_u f_R(u)\mathbb{1}_v^T M^{2k}\mathbb{1}_u + \frac{(f_P^T\mathbb{1}_v)^2}{\|f_R\|_2^2} f_R^T M^{2k} f_R}{\|f_R\|_2^2 + (f_P^T\mathbb{1}_v)^2} \\
&= \frac{\|f_R\|_2^2 \mathbb{1}_v^T M^{2k}\mathbb{1}_v + \frac{(f_P^T\mathbb{1}_v)^2}{\|f_R\|_2^2} f_R^T M^{2k} f_R}{\|f_R\|_2^2 + (f_P^T\mathbb{1}_v)^2} \\
&\geq \frac{\|f_R\|_2^2 \mathbb{1}_v^T M^{2k}\mathbb{1}_v}{\|f_R\|_2^2 + (f_P^T\mathbb{1}_v)^2} \\
&\geq \frac{1}{2}\mathbb{1}_v^T M^{2k}\mathbb{1}_v \\
&\geq \frac{1}{2} C_k\left(\frac{(d-1)^k}{\tilde{d}^{2k}}\right),
\end{aligned}$$

where the third equality comes from the fact that $\mathbb{1}_v^T M^{2k}\mathbb{1}_u = 0$ for all $u \in R$, the second inequality from the definition of eigenradius, and the final inequality by the choice of v. As a consequence the spectral norm of M is at least

$$\left(\frac{1}{2}C_k\right)^{\frac{1}{2k}} \frac{\sqrt{d-1}}{\tilde{d}} \geq 2\frac{\sqrt{d-1}}{\tilde{d}}\left(1 - \frac{3\ln(k+1)}{4k}(1+o(1))\right)$$

yielding the desired bound on $\lambda^{\mathcal{L}}(G)$. □

It is worth noting that the $o(1)$ term can be bounded by $\frac{\ln(4\pi)+\frac{18k+1}{72k^2+3k}}{3\ln(k+1)}$ and so is at most $\frac{3}{2}$ for all k. We note as well that this result can also be rephrased in the r-robust average degree framework of Hoory except that within that framework d is a lower bound on the r-robust average degree and $\tilde{d}$ is an upper bound on the r-robust second order average degree. It is also worth noting that the sole contribution of the $\frac{1}{2}$ in the definition of normalized Laplacian eigenradius is the leading term in this inequality, and thus it can be replaced by any arbitrary constant ϵ. In fact, it suffices for ϵ to tend towards zero sufficiently slowly with respect to the normalized Laplacian eigenradius r, that is, it suffices for $\epsilon^{\frac{1}{r-1}} = \epsilon^{\frac{1}{2k}} \to 1$.

Corollary 8. *Let G_n be a sequence graphs with average degrees $d_n \geq 2$, second order average degrees $\tilde{d}_n$, and maximum degrees Δ_n, satisfying that $\log(\Delta_n) \in o(\log(\mathrm{Vol}(G_n)))$. If $\lim_{n\to\infty} 1 - 2\sqrt{d_n-1}/\tilde{d}_n = L$, then we have that $\limsup_{n\to\infty} \lambda^{\mathcal{L}}(G_n) \leq L$.*

Proof. We first observe that if the maximum degree of G_n is Δ_n, then for any vertex v, $|B_r(v)| \leq \Delta_n(\Delta_n-1)^{r-1}$ and thus $\mathrm{Vol}(B_r(v)) \leq \Delta_n^2(\Delta_n-1)^{r-1}$. Thus, in order for $\mathrm{Vol}(B_r(v)) \geq \frac{1}{2}\mathrm{Vol}(G_n)$, it must be the case that $r \geq 1 + \frac{-\ln(2\Delta_n^2)+\ln(\mathrm{Vol}(G_n))}{\ln(\Delta_n-1)}$. As $\ln(\Delta_n) \in o(\ln(\mathrm{Vol}(G_n)))$, this implies that the normalized Laplacian eigenradius diverges to infinity with n. In combination with Theorem 7, this gives the desired result. $\square$

We note that in general the natural conjectured bound on $\lambda^{\mathcal{L}}(G)$ extending Alon-Boppana result is $1 - 2\sqrt{d-1}/d$ which is in general smaller than $1-2\sqrt{d-1}/\tilde{d}$. In the following section, we show that this separation is essential by providing a class of graphs such that $\lambda^{\mathcal{L}}(G) \geq 1 - 2\sqrt{d-1}/d + \epsilon$ for some fixed positive ϵ.

3. Regular graphs are not extremal

We first observe that there is a trivial obstruction to regular graphs being extremal with respect to $\lambda^{\mathcal{L}}$. Specifically, if G_n is a sequence of d-regular, n-vertex graphs which are sufficiently close to Ramanujan, then the graphs G_n' formed by adding a dominating vertex have average degree approaching $d+2$, while $\limsup_{n\to\infty} \lambda^{\mathcal{L}}(G_n') = 1 - \frac{2\sqrt{d-1}}{d+1} > 1 - \frac{2\sqrt{d+1}}{d+2}$. However, all the graphs G_n' have diameter 2, in contrast to the proof of Nilli which uses the diameter to control the error term [14]. Thus, one might suppose that it suffices to impose a growing diameter condition to recover the natural generalization of Alon-Boppana. However, in this section we will provide a means of constructing an infinite family of graphs $\{G_i\}$, with common average degree d and common maximum degree (and hence increasing diameter), such that $\liminf_{i\to\infty} \lambda^{\mathcal{L}}(G) \geq 1 - \frac{2\sqrt{d-1}}{d} + \epsilon$ for some fixed $\epsilon > 0$. To this end, given graphs H_1 on n_1 vertices, H_2 on n_2 vertices, and B a bipartite graph on (n_1, n_2) vertices, we define $G(H_1, H_2, B)$ to be any of the graphs formed by gluing the vertices of H_1 and H_2 to the appropriate side of the bipartition of B.

Lemma 9. *If H_1 is an n vertex d_1-regular graph, H_2 is a rn vertex d_2-regular graph, and B is a (n, rn) vertex (rk, k)-regular bipartite graph, then $G = G(H_1, H_2, B)$ is such that*

$$\max\left\{\omega, \frac{\lambda(G_1)}{d_1 + rk}, \frac{\lambda(G_2)}{d_2 + k}\right\} \leq 1 - \lambda^{\mathcal{L}}(G) \leq \max\{\omega, \rho\}$$

where

$$\rho = \begin{cases} \frac{\frac{1}{4}\lambda(B)^2 - \lambda(G_1)\lambda(G_2)}{\xi\lambda(B) - (d_2+k)\lambda(G_1) - (d_1+rk)\lambda(G_2)} & \frac{\lambda(G_1)}{d_1+rk} + \frac{\lambda(G_2)}{d_2+k} < \frac{\lambda(B)}{\xi} \\ \max\left\{\frac{\lambda(G_1)}{d_1+rk}, \frac{\lambda(G_2)}{d_2+k}\right\} & \frac{\lambda(G_1)}{d_1+rk} + \frac{\lambda(G_2)}{d_2+k} \geq \frac{\lambda(B)}{\xi} \end{cases},$$

$\xi = \sqrt{(d_1 + rk)(d_2 + k)}$, and

$$\omega = \frac{1}{d}\frac{r}{r+1}\left(\frac{(d_2 + k)d_1}{d_1 + rk} - 2k + \frac{(d_1 + rk)d_2}{(d_2 + k)r^2}\right).$$

Proof. Rather than dealing directly with the normalized Laplacian, $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$, we will again deal with the matrix $M = D^{-1/2}AD^{-1/2}$. Now the largest eigenvalue of M has value one (corresponding to the zero eigenvalue of $\mathcal{L}$) and has eigenvector $D^{1/2}\mathbb{1}$. For convenience of notation, let $\mathbb{1}_t$ be an appropriately sized vector whose first t entries are 1 and remaining entries are zero, and similarly let $\mathbb{1}'_s$ be an appropriately sized vector whose last s entries are one and the remaining entries are zero. We fix an ordering of vertices of G so that the n side of the bipartition appears first and thus the primary eigenvector of M is $\sqrt{d_1 + rk}\mathbb{1}_n + \sqrt{d_2 + k}\mathbb{1}'_{rn}$.

Let σ be the unit vector

$$\sqrt{\frac{(d_2 + k)r}{\operatorname{Vol}(G)}}\mathbb{1}_n - \sqrt{\frac{d_1 + rk}{r\operatorname{Vol}(G)}}\mathbb{1}'_{rn}.$$

Now any unit vector v orthogonal to the first eigenspace of M, can be written in the form $\alpha f + \beta g + \gamma\sigma$ where $\alpha^2 + \beta^2 + \gamma^2 = 1$, $\|f\| = \|g\| = 1$,

$f^T\sigma = g^T\sigma = f^Tg = 0$, and f is only non-zero on the first n entries and g is only non-zero on the last rn entries. Thus to understand v^TMv it suffices to understand f^TMf, g^TMg, f^TMg, $f^TM\sigma$, $g^TM\sigma$, and $\sigma^TM\sigma$. The lower bound comes immediately from considering f^TMf, g^TMg, and $\sigma^TM\sigma$ and the value of $\sigma^TM\sigma$ which we calculate later.

It is easy to see that $f^TMf \leq \frac{\lambda(H_1)}{d_1+rk}$ and $g^TMg \leq \frac{\lambda(H_2)}{d_2+k}$. Noting that $M\sigma = \eta\mathbb{1}_n + \zeta\mathbb{1}'_{rn}$ for some η and ζ where

$$\begin{aligned}
\eta &= \frac{d_1}{d_1+rk}\sqrt{\frac{(d_2+k)r}{\mathrm{Vol}(G)}} - \frac{rk}{\xi}\sqrt{\frac{d_1+rk}{r\,\mathrm{Vol}(G)}} \\
\zeta &= \frac{k}{\xi}\sqrt{\frac{(d_2+k)r}{\mathrm{Vol}(G)}} - \frac{d_2}{d_2+k}\sqrt{\frac{d_1+rk}{r\,\mathrm{Vol}(G)}},
\end{aligned}$$

it is clear that $f^TM\sigma = g^TM\sigma = 0$. Furthermore, we have that

$$\begin{aligned}
\sigma^TM\sigma &= \eta n\sqrt{\frac{(d_2+k)r}{\mathrm{Vol}(G)}} - \zeta rn\sqrt{\frac{d_1+rk}{r\,\mathrm{Vol}(G)}} \\
&= \frac{d_2+k}{d_1+rk}\frac{d_1rn}{\mathrm{Vol}(G)} - \frac{krn}{\mathrm{Vol}(G)} - \frac{krn}{\mathrm{Vol}(G)} + \frac{d_1+rk}{d_2+k}\frac{d_2n}{r\,\mathrm{Vol}(G)} \\
&= \frac{1}{d}\frac{r}{r+1}\left(\frac{(d_2+k)d_1}{d_1+rk} - 2k + \frac{(d_1+rk)d_2}{(d_2+k)r^2}\right)
\end{aligned}$$

We now consider f^TMg. If we let $u = f+g$, then $f^TMg + g^TMf = \frac{1}{\xi}u^TA_Bu$, where A_B is the adjacency matrix for the graph B. Furthermore, the space of vector u that can be formed in this manner spans a $(r+1)n-2$ dimensional subspace. Further, the orthogonal complement of this spaces is spanned by $\sqrt{rk}\mathbb{1}_n + \sqrt{k}\mathbb{1}'_{rn}$ and $\sqrt{rk}\mathbb{1}_n - \sqrt{k}\mathbb{1}'_{rn}$. But since $\sqrt{rk}\mathbb{1}_n + \sqrt{k}\mathbb{1}'_{rn}$ is the principle eigenvector for A_B and

$$\left(\sqrt{rk}\mathbb{1}_n - \sqrt{k}\mathbb{1}'_{rn}\right)^T A_B\left(\sqrt{rk}\mathbb{1}_n - \sqrt{k}\mathbb{1}'_{rn}\right) = -2\sqrt{k}rkn < 0,$$

we have that $u^TA_Bu \leq \lambda(B)$. Thus $f^TMg \leq \frac{\lambda(B)}{2\xi}$.

Now since $f^TM\sigma = g^TM\sigma$, the second largest eigenvalue of M occurs either when $\gamma^2 = 0$ or when $\alpha^2+\beta^2 = 0$. Thus, optimizing for choice of α when $\gamma = 0$, gives the result. □

Corollary 10. *If H_1 is an n vertex d_1-regular Ramanujan graph, H_2 is a rn vertex d_2-regular Ramanujan graph, B is a (n, rn) vertex (rk, k)-regular* *bipartite Ramanujan graph, $3d_2 > k+4$, and r is sufficiently large, then $G = G(H_1, H_2, B)$ is such that $\lambda^{\mathcal{L}}(G) = 1 - \frac{2\sqrt{d_2-1}}{d_2+k}$.*

Proof. We note that for large enough r, we have that $\omega < 0$. Additionally, $\frac{2\sqrt{d_1-1}}{d_1+rk} \to 0$ and $\frac{\sqrt{k-1}+\sqrt{rk-1}}{\sqrt{(d_1+rk)(d_2+k)}} \to \frac{1}{\sqrt{d_2+k}}$ as $r \to \infty$. Now since $3d_2 > k+4$, it follows that $\frac{2\sqrt{d_2-1}}{d_2+k} > \frac{1}{\sqrt{d_2+k}}$ and by Lemma 9 the result follows. □

In fact, it suffices that graphs H_1, H_2, and B be sufficiently close to Ramanujan. That is, it suffices for H_1, H_2, and B to be such that $\lambda(H_1) \leq 2\sqrt{d_1-1} + o(1)$, $\lambda(H_2) \leq 2\sqrt{d_2-1} + o(1)$, and $\lambda(B) \leq \sqrt{k-1} + \sqrt{rk-1} + o(1)$ where the $o(1)$ is in terms of n. We will refer to sequences of graphs satisfying these conditions as *nearly Ramanujan.*

Theorem 11. *For any fixed choice of integers $d_1 \geq 3, d_2 \geq 8$, there is an infinite family of graphs $\{G_i\}_{i\in I}$ with common average degree d, such that $\lambda^{\mathcal{L}}(G_i) \geq 1 - \frac{2\sqrt{d-1}}{d} + \epsilon$ for some fixed $\epsilon > 0$.*

Proof. First we observe that since $\frac{(d_2+3)d_1}{d_1+3r} + \frac{(d_1+3r)d_2}{(d_2+3)r^2} \to 0$ as $r \to \infty$ and there is a choice of r so that $\frac{(d_2+3)d_1}{d_1+3r} + \frac{(d_1+3r)d_2}{(d_2+3)r^2} < 6$ and $d_1 < (r+1)(d_2+6)$. Now let $\{H_n\}$ be a sequence of d_1-regular nearly Ramanujan graphs on n vertices, and let $\{\hat{H}_n\}$ be a sequence of d_2-regular nearly Ramanujan graphs on rn vertices, and let $\{B_n\}$ be a sequence of random $(3r, 3)$-regular bipartite graphs on (n, rn) vertices. We note that by the work of Friedman [5], the classes $\{H_n\}$ and $\{\hat{H}_n\}$ exist. Define $G_n = G(H_n, \hat{H}_n, B_n)$. Now the average degree for each G_n is $\frac{r}{r+1}(d_2+6) + \frac{d_1}{r+1} < d_2+6$ by the choice of r. Furthermore, the choice of r and the observation that $\frac{2\sqrt{d_2-1}}{d_2+3} > \frac{3\sqrt{r}}{\sqrt{(d_1+3r)(d_2+3)}}$, together with Corollary 10, gives that $1 - \lambda^{\mathcal{L}}(G_n) = \frac{2\sqrt{d_2-1}+o(1)}{d_2+3}$. Since $\frac{2\sqrt{x-1}}{x}$ is a decreasing function for $x \geq 2$ and $d \leq d_2+6$, it suffices to show that there is an $\epsilon > 0$ such that $\frac{2\sqrt{d_2-1}}{d_2+3} < \frac{2\sqrt{d_2+5}}{d_2+6}$. Rearranging, this is equivalent to $\frac{d_2+6}{d_2+3} < \sqrt{\frac{d_2+5}{d_2-1}}$. Since both sides are positive, it suffices to show that $1 + \frac{6}{d_2+3} + \frac{9}{(d_2+3)^2} < 1 + \frac{6}{d_2-1}$. Alternatively we may show that $6d_2^2 + 21d_2 - 27 = (6(d_2+3)+9)(d_2-1) < 6(d_2+3)^2 = 6d_2^2 + 36d_2 + 54$, which clearly holds. Thus there is an $\epsilon > 0$ such that for a sufficiently large n, $\lambda^{\mathcal{L}}(G_n) \geq 1 - \frac{2\sqrt{d-1}}{d} + \epsilon$. □

It is worth noting that the this construction could be extended to larger class of degrees if the existence of a larger class of nearly Ramanujan biregular bipartite graphs were known. Although it is clear that by subdivision any k-regular nearly Ramanujan graph gives rise to a $(2, k)$-regular nearly Ramanujan bipartite graph [6] and Li and Solé have provided a construction of a limited class of biregular bipartite graphs based generalized n-gons [10], neither of these constructions yields a sufficient diversity of bipartite near Ramanujan graphs to meaningfully expand the range of degrees chosen. Since the initial submission of this work, Marcus, Spielman, and Srivastava have provided a construction of (c, d)-biregular bipartite Ramanujan graphs for all $c, d \geq 3$ via 2-lifts of the complete bipartite graph [11].

4. Spectral bounds for the normalized Laplacian of bipartite graphs

In order to deal with bipartite graphs, we need the following result which appears in [7]. First, let T_{2k} be the collection of length $2k$ Dyck paths and for $\tau \in T_{2k}$ let $\mathrm{odd}(\tau)$ be the number of positive steps on τ starting from an odd height. Similarly, define $\mathrm{even}(\tau)$ and note that $\mathrm{odd}(\tau) + \mathrm{even}(\tau) = k$.

Lemma 12. *For any positive constants $a, b > 0$,*

$$\lim_{k\to\infty} \sqrt[2k]{\sum_{\tau\in T_{2k}} a^{\mathrm{even}(\tau)} b^{\mathrm{odd}(\tau)}} = \sqrt{a} + \sqrt{b}.$$

For the sake of completeness we provide this alternative proof.

Proof. Let $C_k^{(a,b)} = \sum_{\tau\in T_{2k}} a^{\mathrm{even}(\tau)} b^{\mathrm{odd}(\tau)}$ and let $C(a, b, x) = \sum_{k=0}^{\infty} C_k^{(a,b)} x^k$. Making the standard observation that for any Dyck path of length $2k$ which first returns to height 0 at step $2t$, the subpath from step 1 to step $2t - 1$ is also a Dyck path, we have that $C_{k+1}^{(a,b)} = \sum_{i=0}^{k} aC_i^{(b,a)} C_{k-i}^{(a,b)}$, where the interchange in (a, b) occurs because the sub-Dyck path starts at an odd value. Thus we have that $C(a, b, x) = 1 + axC(b, a, x)C(a, b, x)$ and $C(b, a, x) = 1 + bxC(a, b, x)C(b, a, x)$. Letting $C^*(\{a, b\}, x) = C(a, b, x)C(b, a, x)$ and combining these relationships we get $C^* = 1 + (a + b)xC^* + abx^2C^{*2}$. Thus

$$C^*(\{a, b\}, x) = \frac{1 - (a + b)x - \sqrt{(1 - (a + b)x)^2 - 4abx^2}}{2abx^2},$$

where the negative square root is chosen to eliminate the pole at $x = 0$. Thus

$$C(a, b, x) = 1 + ax\frac{1 - (a + b)x - \sqrt{(1 - (a + b)x)^2 - 4abx^2}}{2abx^2}$$

$$= \frac{1 + (b-a)x - \sqrt{1 - 2(a+b)x + (b-a)^2 x^2}}{2bx}.$$

Note that this has poles at $x = \frac{1}{(\sqrt{a}+\sqrt{b})^2}$ and $x = \frac{1}{(\sqrt{a}-\sqrt{b})^2}$ (if $a \neq b$), and thus $C_k^{(a,b)} \sim \left(\sqrt{a}+\sqrt{b}\right)^{2k}$ as desired. □

With this lemma in hand, we now have the following normalized Laplacian analogue of the bounds on the spectral radius of the universal cover of irregular bipartite graphs [7].

Theorem 13. *For any bipartite graph $B = (L, R, E)$ with minimum degree at least 2 and weight function $w(u,v) = f(u)f(v)$, the weighted spectral radius of the universal cover is at least*

$$\left(\sqrt{d_L - 1} + \sqrt{d_R - 1}\right)\sqrt{\prod_{v \in L \cup R} f(v)^{\frac{2\deg(v)}{|E|}}},$$

where d_L and d_R are the average degrees of the L and R sides of the partition, respectively.

Proof. We will consider the same class of walks as in the proof of Theorem 4, except the starting vertex will be restricted to vertices in L. Specifically, using the same notation, we have

$$\begin{aligned}
\rho_w(\tilde{B}) &\geq \limsup_{k\to\infty} \sqrt[2k]{\sum_{v \in L} \frac{\deg(v)}{|E|} \sum_{v' \sim v} \frac{1}{\deg(v)} \sum_{\tau \in T_{2k}} w(\Omega_{v,v',\tau,2k})} \\
&= \limsup_{k\to\infty} \sqrt[2k]{\frac{1}{|E|} \sum_{v \in L} \sum_{v' \sim v} \sum_{\tau \in T_{2k}} w(\Omega_{v,v',\tau,2k})}.
\end{aligned}$$

Thus we consider

$$\sum_{\tau \in T_{2k}} \sum_{v \in L} \sum_{v' \sim v} \sum_{\omega \in \Omega_{v,v',\tau,2k}} \frac{w(\omega)}{|E|} \frac{p(\omega)}{p(\omega)} \geq \sum_{\tau \in T_{2k}} \prod_{v \in L} \prod_{v' \sim v} \prod_{\omega \in \Omega_{v,v',\tau,2k}} \left(\frac{w(\omega)}{p(\omega)}\right)^{\frac{p(\omega)}{|E|}}.$$

Now since $\frac{w(\omega)}{p(\omega)} = \prod_{i=1}^{k} (\deg(v_i) - 1) f(v_i)^2 f(u_i)^2$ it suffices to understand for any edge how many times the ordered edge (v, v') is crossed in a non-backtracking walk by a forward step (weighted by $\frac{p(\omega)}{|E|}$). That is, we are

interested in
$$\sum_{u\in L}\sum_{u'\sim u}\sum_{\omega\in\Omega_{u,u',\tau,2k}}\delta_{v,v'}(\omega).$$
If $v\in L$, this is $\mathrm{even}(\tau)$ while if $v\in R$, this is $\mathrm{odd}(\tau)$. Thus
$$\begin{aligned}\sum_{v\in L}\sum_{v'\sim v}\sum_{\omega\in\Omega_{v,v',\tau,2k}}\frac{w(\omega)}{|E|}\frac{p(\omega)}{p(\omega)} &= \prod_{v\in L}(\deg(v)-1)^{\frac{\deg(v)\,\mathrm{even}(\tau)}{|E|}}\\ &\quad\times\prod_{v\in R}(\deg(v)-1)^{\frac{\deg(v)\,\mathrm{odd}(\tau)}{|E|}}\\ &\quad\times\prod_{v\in L\cup R}f(v)^{\frac{4\deg(v)k}{|E|}}\\ &\geq (d_L-1)^{\ \mathrm{odd}(\tau)}(d_R-1)^{\ \mathrm{even}(\tau)}\\ &\quad\times\prod_{v\in L\cup R}f(v)^{\frac{2\deg(v)(\mathrm{even}(\tau)+\mathrm{odd}(\tau))}{|E|}}.\end{aligned}$$

Thus $\rho_w(\tilde{B})\geq\left(\sqrt{d_L-1}+\sqrt{d_R-1}\right)\sqrt{\prod_{v\in L\cup R}f(v)^{\frac{2\deg(v)}{|E|}}}$. □

Applying the weighting from the normalized Laplacian, we have the following result.

Corollary 14. *For any bipartite graph $B=(L,R,E)$ with minimum degree at least 2 and weight function $w(u,v)=\frac{1}{\sqrt{\deg(v)\deg(u)}}$, the weighted spectral radius of the universal cover is at least*
$$\left(\sqrt{d_L-1}+\sqrt{d_R-1}\right)\left(\tilde{d}_L\tilde{d}_R\right)^{-1/2},$$
where d_L and d_R are the average degrees of the sides and $\tilde{d}_L=\frac{\sum_{v\in L}\deg(v)^2}{\sum_{v\in L}\deg(v)}$ and $\tilde{d}_R=\frac{\sum_{v\in R}\deg(v)^2}{\sum_{v\in R}\deg(v)}$ are the second order average degrees of the sides.

It is worth noting that the term $\left(\tilde{d}_L\tilde{d}_R\right)^{-1/2}$ can be replaced by $\left(\hat{d}_L\hat{d}_R\right)^{-1/2}$ where $\hat{d}_L$ and $\hat{d}_R$ are the averages of the $3/2$ powers of the degrees in L and R, respectively. It is also worth noting that unlike Theorem 4, this theorem can not be extended to the case where the average degree is at least 2, as the average degree on each side of the partition could decreased by deleting a vertex of degree 1.

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