

On the asymptotic behavior of the q -analog of Kostant's partition function

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Kostant's partition function counts the number of distinct ways to express a weight of a classical Lie algebra \mathfrak{g} as a sum of positive roots of \mathfrak{g} . We refer to each of these expressions as decompositions of a weight. Our main result considers an infinite family of weights, irrespective of Lie type, for which we establish a closed formula for the q -analog of Kostant's partition function and then prove that the (normalized) distribution of the number of positive roots in the decomposition of any of these weights converges to a Gaussian distribution as the rank of the Lie algebra goes to infinity. We also extend these results to the highest root of the classical Lie algebras and we end our analysis with some directions for future research.

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1. Introduction

A classical problem in analytic number theory is to determine the behavior of certain distributions associated to the decompositions of positive integers, as sums of positive integers. For example, define the Fibonacci numbers as $F_n = F_{n-1} + F_{n-2}$, whenever $n \geq 3$ and $F_1 = 1$, $F_2 = 2$. Then Zeckendorf's Theorem [21] states that the positive integers can be uniquely expressed as a sum of nonconsecutive Fibonacci numbers and such an expression is called a decomposition. Lekkerkerker [19] later established that if $m \in [F_n, F_{n+1}]$, then the number of summands needed in the decomposition of m is asymptotic to $\left(\frac{1}{2}\left(1 - \frac{1}{\sqrt{5}}\right)\right)n$ as $n \rightarrow \infty$. From this work, it was found that the distribution of the number of summands in decompositions of positive integers actually converges to a Gaussian [18]. These results have been extended to numerous other sequences of integers which allow unique decompositions of the positive integers as a sum of elements in the sequence [6, 5, 10, 9, 7, 2].

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In our work, we bring the tools of analytic number theory to the study of vector partitions, a more general setting as these decompositions are not unique. In particular, we study Kostant's partition function which counts the number of ways of expressing a weight (vector) of a simple Lie algebra \mathfrak{g} as a linear combination of the positive roots of \mathfrak{g} (a finite set of vectors). As is standard in analytic number theory, we refer to such expressions as decompositions.

We recall that Lusztig [20] defined the q -analog of Kostant's partition function [17] as the polynomial valued function

$$\wp_q(\xi) = c_0 + c_1q + c_2q^2 + \cdots + c_kq^k$$

where c_i denotes the number of ways the weight ξ can be expressed as a sum of i positive roots. Hence, evaluating $\wp_q(\xi)|_{q=1}$ yields the total number of decompositions of the weight ξ as a sum of positive roots. However, the q -analog gives us more detailed information as it keeps track of the number of positive roots used in the decompositions and plays a key role in our analysis.

Our first main result considers an infinite family of weights of a classical Lie algebra of rank r , irrespective of Lie type, for which we establish a closed formula for the q -analog of Kostant's partition function and then prove that the (normalized) distribution of the number of positive roots in the decomposition of these weights converges to a Gaussian distribution as $r \rightarrow \infty$.

Theorem 1. Let \mathfrak{g} be a classical Lie algebra of rank r , with $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ a set of simple roots of \mathfrak{g} . If

$$\lambda = \left(\sum_{i=1}^r \alpha_i \right) + \sum_{i \in I} c_i \alpha_i$$

where $I = \{i_1, i_2, \dots, i_\ell\}$ is a set of nonconsecutive integers satisfying $1 < i_1 < i_2 < \cdots < i_\ell < r - 2$, and $c_{i_1}, c_{i_2}, \dots, c_{i_\ell} \in \mathbb{Z}_{>0}$, then

$$\wp_q(\lambda) = q^{m+1}(1+q)^{r-1-2\ell}(2+2q+q^2)^\ell.$$

Moreover, if Y_r is the random variable denoting the total number of positive roots used in the decompositions of λ , normalize Y_r to $Y'_r = (Y_r - \mu_r) / \sigma_r$ where μ_r and σ_r^2 are the mean and variance of Y_r , respectively, then

$$\mu_r = \frac{r+1}{2} - \frac{1}{5}\ell + \sum_{i=1}^{\ell} c_i \quad \text{and} \quad \sigma_r^2 = \frac{r-1}{4} + \frac{3}{50}\ell,$$

and Y_r' converges in distribution to the standard normal distribution as $r \rightarrow \infty$ for every classical Lie algebra \mathfrak{g} of rank r .

In Lie type A , note that when $c_i = 0$ for all $1 \leq i \leq r$, then the weight λ defined in Theorem 1 is the highest root. This motivates extending Theorem 1 by considering the case when λ is the highest root of a classical Lie algebra. Harris, Insko, and Omar gave closed formulas for the value of the q -analog of Kostant's partition function on the highest root of a classical Lie algebra in [13]. Our second main result follows.

Theorem 2. Let $\mathcal{P}_{\mathfrak{g}_r}(q)$ denote $\wp_q(\tilde{\alpha})$ for a classical Lie algebra \mathfrak{g} of rank r . Then $\mathcal{P}_{\mathfrak{g}_r}(q)$ is asymptotically Gaussian with mean and variance given by:

Type A_r ($r \geq 1$):

$$\mu_r = \frac{r+1}{2} \quad \text{and} \quad \sigma_r^2 = \frac{r-1}{4}$$

Type B_r ($r \geq 2$):

$$\begin{aligned} \mu_r = & \frac{(5 - \sqrt{5} + (25 - 13\sqrt{5})r)(5 - \sqrt{5})^r}{5 \left[(5 - 3\sqrt{5})(5 - \sqrt{5})^r + (5 + 3\sqrt{5})(5 + \sqrt{5})^r \right]} \\ & + \frac{(5 + \sqrt{5} + (25 + 13\sqrt{5})r)(5 + \sqrt{5})^r}{5 \left[(5 - 3\sqrt{5})(5 - \sqrt{5})^r + (5 + 3\sqrt{5})(5 + \sqrt{5})^r \right]} \end{aligned}$$

and

$$\begin{aligned} \sigma_r^2 = & \frac{20^{r+1}r^2}{-5 \left[(5 - \sqrt{3})(5 - \sqrt{5})^r + (5 + 3\sqrt{5})(5 + \sqrt{5})^r \right]^2} \\ & + \frac{\left[26(3\sqrt{5} - 7)(5 - \sqrt{5})^{2r} - 26(3\sqrt{5} + 7)(5 + \sqrt{5})^{2r} + 36 \cdot \frac{20^{r+1}}{5} \right] r}{-5 \left[(5 - \sqrt{3})(5 - \sqrt{5})^r + (5 + 3\sqrt{5})(5 + \sqrt{5})^r \right]^2} \\ & + \frac{2(73 - 25\sqrt{5})(5 - \sqrt{5})^{2r} + 2(73 + 25\sqrt{5})(5 + \sqrt{5})^{2r} - 63 \cdot \frac{20^{r+1}}{5}}{-5 \left[(5 - \sqrt{3})(5 - \sqrt{5})^r + (5 + 3\sqrt{5})(5 + \sqrt{5})^r \right]^2} \end{aligned}$$

Type C_r ($r \geq 3$):

$$\mu_r = \frac{((1 - \sqrt{5}) + (7 - \sqrt{5})r)(5 - \sqrt{5})^r}{10 \left((5 - \sqrt{5})^r + (5 + \sqrt{5})^r \right)}$$

$$+ \frac{((1 + \sqrt{5}) + (7 + \sqrt{5})r)(5 + \sqrt{5})^r}{10 \left((5 - \sqrt{5})^r + (5 + \sqrt{5})^r \right)}$$

and

$$\begin{aligned} \sigma_r^2 &= \frac{\frac{20^{r+1}}{4}r^2 + \left[13 \left((5 - \sqrt{5})^{2r} + (5 + \sqrt{5})^{2r} \right) + 9 \cdot \frac{20^{r+1}}{5} \right] r}{25 \left((5 - \sqrt{5})^r + (5 + \sqrt{5})^r \right)^2} \\ &+ \frac{(-21 + 4\sqrt{5})(5 + \sqrt{5})^{2r} - (21 + 4\sqrt{5})(5 - \sqrt{5})^{2r} - 37 \cdot 20^r}{25 \left((5 - \sqrt{5})^r + (5 + \sqrt{5})^r \right)^2} \end{aligned}$$

Type D_r ($r \geq 4$):

$$\begin{aligned} \mu_r &= \frac{(15 - \sqrt{5} + r(-5 + 7\sqrt{5}))(5 - \sqrt{5})^r}{10\sqrt{5} \left((5 - \sqrt{5})^r - (5 + \sqrt{5})^r \right)} \\ &+ \frac{(15 + \sqrt{5} - r(5 + 7\sqrt{5}))(5 + \sqrt{5})^r}{10\sqrt{5} \left((5 - \sqrt{5})^r - (5 + \sqrt{5})^r \right)} \end{aligned}$$

and

$$\begin{aligned} \sigma_r^2 &= \frac{\frac{20^{r+1}}{4}r^2 - \left[13 \left((5 + \sqrt{5})^{2r} + (5 - \sqrt{5})^{2r} \right) + \frac{20^{r+1}}{5} \right] r}{-25 \left[(5 + \sqrt{5})^r - (5 - \sqrt{5})^r \right]^2} \\ &+ \frac{(34 - 3\sqrt{5})(5 + \sqrt{5})^{2r} + (34 + 3\sqrt{5})(5 - \sqrt{5})^{2r} - 23 \cdot 20^r}{-25 \left[(5 + \sqrt{5})^r - (5 - \sqrt{5})^r \right]^2}. \end{aligned}$$

This paper is organized as follows. Section 3 contains the proof of Theorem 1. We give two proofs of Theorem 2: the first in Section 4 uses the formulas of Harris, Insko, and Omar [13] along with Bender's Theorem [3] and moment generating functions, while the second uses the classical approach via moment generating functions and formulas of Harris, Insko, and Omar [13] only. This second proof is lengthy; hence we present it in Appendix A. We end with Section 5 where we present directions for further study.

2. Background

We begin by recalling the positive roots of each Lie type.

Type A_r : If $r \geq 1$, the set of simple roots is given by $\Delta = \{\alpha_1, \dots, \alpha_r\}$, and the set of positive roots is given by

$$\Phi^+ = \Delta \cup \{\alpha_i + \alpha_{i+1} + \dots + \alpha_j : 1 \leq i < j \leq r\}.$$

The highest root is given by $\tilde{\alpha} = \alpha_1 + \dots + \alpha_r$.

Type B_r : If $r \geq 2$, the set of simple roots is given by $\Delta = \{\alpha_1, \dots, \alpha_r\}$ and the set of positive roots is given by

$$\begin{aligned} \Phi^+ = \Delta \cup \{ & \alpha_i + \dots + \alpha_j : 1 \leq i < j \leq r\} \\ & \cup \{\alpha_i + \dots + \alpha_{j-1} + 2\alpha_j + \dots + 2\alpha_r : 1 \leq i < j \leq r\}, \end{aligned}$$

where $\tilde{\alpha}_{B_r} = \alpha_1 + 2\alpha_2 + \dots + 2\alpha_r$ is the highest root.

Type C_r : If $r \geq 3$, the set of simple roots is given by $\Delta = \{\alpha_1, \dots, \alpha_r\}$ and the set of positive roots is given by

$$\begin{aligned} \Phi^+ = \Delta \cup \{ & \alpha_i + \dots + \alpha_{j-1} + 2\alpha_j + \dots + 2\alpha_{r-1} + \alpha_r : \\ & 1 \leq i < j \leq r-1\} \\ & \cup \{\tilde{\alpha}_{C_r}\} \cup \{\alpha_i + \dots + \alpha_j : 1 \leq i < j \leq r\}, \end{aligned}$$

where $\tilde{\alpha}_{C_r} = 2\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{r-1} + \alpha_r$ is the highest root.

Type D_r : If $r \geq 4$, the set of simple roots is given by $\Delta = \{\alpha_1, \dots, \alpha_r\}$ and the set of positive roots is given by

$$\begin{aligned} \Phi^+ = \Delta \cup \{ & \alpha_i + \dots + \alpha_{j-1} : 1 \leq i < j \leq r\} \\ & \cup \{\alpha_i + \dots + \alpha_{r-2} + \alpha_r : 1 \leq i \leq r-2\} \\ & \cup \{\alpha_i + \dots + \alpha_{j-1} + 2\alpha_j + \dots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r : \\ & 1 \leq i < j \leq r-2\}, \end{aligned}$$

where $\tilde{\alpha}_{D_r} = \alpha_1 + 2\alpha_2 + \dots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$ is the highest root.

Let the random variable Y_r denote the total number of positive roots used in the decompositions of the highest root of a chosen classical Lie algebra of rank r as sums of positive roots, and let $p_{r,k}$ denote the number of decompositions of the highest root as a sum of exactly k positive roots. We use the following result in our analysis.

Proposition 1 ([8], Propositions 4.7, 4.8). Let $F(x, y) = \sum_{r,k \geq 0} p_{r,k} x^r y^k$ be the generating function of $p_{r,k}$, and let $g_r(y) = \sum_{k=0}^r p_{r,k} y^k$ be the coefficient of x^r in $F(x, y)$. Then the mean of Y_r is

$$\mu_r = \frac{g'_r(1)}{g_r(1)}$$

and the variance of Y_r is

$$\sigma_r^2 = \frac{\frac{d}{dy}(y g'_r(y))|_{y=1}}{g_r(1)} - \mu_r^2.$$

3. Proof of Theorem 1

We begin with the following type A result.

Proposition 2. Let $r \geq 3$. If $I \subset \{2, \dots, r - 1\}$ is a set of nonconsecutive integers and $\beta = \sum_{i=1}^r \alpha_i + \sum_{i \in I} \alpha_i$ is a weight of the Lie algebra of type A_r , then

$$\wp_q(\beta) = q^{|I|+1} (1 + q)^{r-1-2|I|} (2 + 2q + q^2)^{|I|}.$$

Proof. If $I = \emptyset$, then the result follows from the fact that $\wp_q(\sum_{i=1}^r \alpha_i) = q(1 + q)^{r-1}$ (see [12]).

Suppose the formula holds for any indexing set with cardinality n , which is a subset of $\{2, \dots, r - 1\}$ consisting of nonconsecutive integers. Consider I where $|I| = n + 1$ and $j = \max(I)$. Let $I' = I - \{j\}$ and note $\max(I') \leq j - 2$. First consider the case where the additional α_j appears as a simple root in a decomposition of β . The number of such decompositions of β is

$$(1) \quad q \cdot q^{n+1} (1 + q)^{r-1-2n} (2 + 2q + q^2)^n = q^{n+2} (1 + q)^{r-1-2n} (2 + 2q + q^2)^n,$$

where the factor of q on the left hand side accounts for the α_j appearing as a simple root, and, by the induction hypothesis, the remaining factors are associated with taking I' as the indexing set.

Next, consider the case where α_j does not appear as a simple root in a decomposition of β . We treat the roots $\alpha' = \alpha_{j-1} + \alpha_j$ and $\alpha'' = \alpha_j + \alpha_{j+1}$ as quasi-simple roots. In other words, they cannot be separated for this count of the decompositions. Thus it suffices to find the number of ways to write $\sum_{i=1}^{j-2} \alpha_i + \sum_{i \in I'} \alpha_i + (\alpha')$ and $\sum_{i=j+2}^r \alpha_i + (\alpha'')$ as sums of positive roots, and take the product of the results.

For $\sum_{i=1}^{j-2} \alpha_i + \sum_{i \in I'} \alpha_i + (\alpha')$, we use the inductive hypothesis (acting like in type A_{j-1} and treating α' as α_{j-1}) to get

$$(2) \quad q^{n+1}(1+q)^{j-1-1-2n}(2+2q+q^2)^n.$$

For $\sum_{i=j+2}^r \alpha_i + (\alpha'')$, we use the base case (acting like in type A_{r-j} and treating α'' as α_{j+1}) to get

$$(3) \quad q(1+q)^{r-j-1}.$$

Then, the number of ways that α_j does not appear as a simple root in the decompositions of β is obtained by taking the product of Equations (2) and (3) which yields

$$(4) \quad q^{n+2}(1+q)^{r-3-2n}(2+2q+q^2)^n.$$

Thus the number of decompositions of β must account for the cases where α_j appears as a simple root and where it does not appear as a simple root. This is given by taking the sum of Equations (1) and (4) which yields

$$\begin{aligned} \wp_q(\beta) &= q^{n+2}(1+q)^{r-1-2n}(2+2q+q^2)^n + q^{n+2}(1+q)^{r-3-2n}(2+2q+q^2)^n \\ &= q^{n+2}(1+q)^{r-3-2n}(2+2q+q^2)^n [(1+q)^2 + 1] \\ &= q^{n+2}(1+q)^{r-3-2n}(2+2q+q^2)^{n+1} \\ &= q^{(n+1)+1}(1+q)^{r-1-2(n+1)}(2+2q+q^2)^{n+1} \end{aligned}$$

as desired. □

We now give a more general result.

Proposition 3. Let $r \geq 3$ and let $I = \{i_1, i_2, \dots, i_\ell\}$ be a set of nonconsecutive integers satisfying $1 < i_1 < i_2 < \dots < i_\ell < r$, and $c_{i_1}, c_{i_2}, \dots, c_{i_\ell} \in \mathbb{Z}_{>0}$. If

$$\xi = \sum_{i=1}^r \alpha_i + \sum_{j=1}^{\ell} c_{i_j} \alpha_{i_j}$$

is a weight of the Lie algebra of type A_r , then

$$\wp_q(\xi) = q^{m+1}(1+q)^{r-1-2\ell}(2+2q+q^2)^\ell$$

where $m = \sum_{j=1}^{\ell} c_{i_j}$.

Proof. The result follows from taking the formula in Proposition 2 and multiplying by $q^{\sum_{i=1}^{\ell}(c_i-1)}$, which accounts for the additional simple roots that we must use to decompose ξ than what we needed to decompose β . Hence

$$\begin{aligned} \wp_q(\xi) &= q^{\sum_{i=1}^{\ell}(c_i-1)} \cdot q^{\ell+1}(1+q)^{r-1-2\ell}(2+2q+q^2)^\ell \\ &= q^{m-\ell} \cdot q^{\ell+1}(1+q)^{r-1-2\ell}(2+2q+q^2)^\ell \\ &= q^{m+1}(1+q)^{r-1-2\ell}(2+2q+q^2)^\ell. \end{aligned} \quad \square$$

By further restricting the set I we can give a general result for all Lie types.

Proposition 4. Let \mathfrak{g} be a classical simple Lie algebra of rank $r \geq 5$. Let $I = \{i_1, i_2, \dots, i_\ell\}$ be a set of nonconsecutive integers satisfying $1 < i_1 < i_2 < \dots < i_\ell < r - 2$, and $c_{i_1}, c_{i_2}, \dots, c_{i_\ell} \in \mathbb{Z}_{>0}$. If

$$\lambda = \sum_{i=1}^r \alpha_i + \sum_{j=1}^{\ell} c_{i_j} \alpha_{i_j}$$

is a weight of \mathfrak{g} , then

$$\wp_q(\lambda) = q^{m+1}(1+q)^{r-1-2\ell}(2+2q+q^2)^\ell$$

where $m = \sum_{j=1}^{\ell} c_{i_j}$.

Proof. The result follows from Proposition 3 and the fact that under this restriction on the index set I , the only positive roots one can use in decompositions of λ are of type A_r . □

We define $\wp(\lambda) = \wp_q(\lambda)|_{q=1}$. Setting $q = 1$ in Proposition 4 establishes the following result.

Corollary 1. Let I and λ be defined as in Proposition 4. Then

$$\wp(\lambda) = 2^{r-1} \left(\frac{5}{4}\right)^\ell = 2^{r-1-2\ell} 5^\ell.$$

Proposition 5. Let λ be defined as in Proposition 4. If Y_r denotes the random variable for the total number of positive roots used in the decompositions of λ , then the mean and variance of Y_r are given by

$$\mu_r = \frac{r+1}{2} - \frac{1}{5}\ell + m \quad \text{and} \quad \sigma_r^2 = \frac{r-1}{4} + \frac{3}{50}\ell,$$

respectively.

Proof. The result follows from Proposition 1 and Corollary 1. □

Proposition 6. Let μ_r and σ_r^2 be defined as in Proposition 5 and λ be defined as in Proposition 4. Then the random variable $Y'_r = (Y_r - \mu_r) / \sigma_r$ converges in distribution to the standard normal distribution as $r \rightarrow \infty$.

Proof. Let $M_{Y'_r}$ be the moment generating function of Y'_r . In this proof, we establish that the moment generating function of Y'_r converges to that of the standard normal distribution, which is $e^{t^2/2}$.

By Proposition 4, we let $g_r(y) = y^{m+1}(1+y)^{r-1-2\ell}(2+2y+y^2)^\ell$; hence

$$\begin{aligned} \log[g_r(e^n)] &= \log[(e^n)^{m+1}(1+e^n)^{r-1-2\ell}(2+2e^n+e^{2n})^\ell] \\ &= (m+1)\log(e^n) + (r-1-2\ell)\log(1+e^n) + \ell\log(2+2e^n+e^{2n}) \\ (5) \quad &= (m+1)\log(1+n+n^2/2) + (r-1-2\ell)\log(2+n+n^2/2) \\ &\quad + \ell\log(5+4n+3n^2) + O(n^3). \end{aligned}$$

Using Taylor's series expansion for $\log(x)$ we have

$$(6) \quad \log(1+n+n^2/2) = n + O(n^3)$$

$$(7) \quad \log(2+n+n^2/2) = \log(2) + \frac{1}{2}n + \frac{1}{8}n^2 + O(n^3)$$

$$(8) \quad \log(5+4n+3n^2) = \log(5) + \frac{4}{5}n - \frac{1}{50}n^2 + O(n^3).$$

Substituting Equations (6), (7), and (8), into Equation (5) yields

$$(9) \quad \log[g_r(e^n)] = n^2 \left(\frac{r-1}{8} - \frac{27\ell}{100} \right) + n \left(\frac{r+1}{2} - \frac{1}{5}\ell + m \right)$$

$$(10) \quad + \ell \log \left(\frac{5}{4} \right) + (r-1)\log(2) + O(n^3).$$

By Corollary 1 we know

$$(11) \quad \log[g_r(1)] = \log \left[2^{r-1} \left(\frac{5}{4} \right)^\ell \right] = \ell \log \left(\frac{5}{4} \right) + (r-1)\log(2).$$

Hence, substituting Equations (9) and (11), $\mu_r = \frac{r+1}{2} - \frac{1}{5}\ell + m$, $\sigma_r = \sqrt{\frac{r-1}{4} + \frac{3}{50}\ell}$, and $n = \frac{t}{\sigma_r}$ yields

$$\log(M_{Y'_r}(t)) = \log[g_r(e^n)] - \log[g_r(1)] - \frac{t\mu_r}{\sigma_r}$$

$$(12) \quad = \left(\frac{25 + 54\ell - 25r}{50 - 12\ell - 50r} \right) t^2 + O \left(\left(\frac{t}{\sqrt{\frac{r-1}{4} + \frac{3}{50}\ell}} \right)^3 \right).$$

Taking the limit of Equation (12), as $r \rightarrow \infty$, we have that $\log(M_{Y'_r}(t))$ converges to $\frac{1}{2}t^2$. Thus Y'_r converges to the standard normal distribution as $r \rightarrow \infty$. □

Theorem 1 follows directly from Proposition 4 and Proposition 6.

4. Proof of Theorem 2

In type A_r we know

$$(13) \quad \wp_q(\tilde{\alpha}) = q(1+q)^{r-1} = q \sum_{i=0}^{r-1} \binom{r-1}{i} q^i = \sum_{k=1}^r \binom{r-1}{k-1} q^k.$$

It is well-known that the binomial distribution converges to a standard normal, see [1] for four distinct proofs of this result.

Let $\mathcal{P}_{\mathfrak{g}_r}(q)$ denote $\wp_q(\tilde{\alpha})$ for a classical Lie algebra \mathfrak{g} of rank r . Harris, Insko, and Omar gave closed formulas for the generating functions for the q -analog of Kostant’s partition function for Lie algebras of Type B , C , and D in [13], which we restate below for ease of reference.

Theorem (Generating Functions [13]). The closed formulas for the generating functions $\sum_{r \geq 1} \mathcal{P}_{B_r}(q)x^r$, $\sum_{r \geq 1} \mathcal{P}_{C_r}(q)x^r$, and $\sum_{r \geq 4} \mathcal{P}_{D_r}(q)x^r$, are given by

$$(14) \quad \sum_{r \geq 1} \mathcal{P}_{B_r}(q)x^r = \frac{qx + (-q - q^2)x^2 + q^2x^3}{1 - (2 + 2q + q^2)x + (1 + 2q + q^2 + q^3)x^2},$$

$$(15) \quad \sum_{r \geq 1} \mathcal{P}_{C_r}(q)x^r = \frac{qx + (-q - q^2)x^2}{1 - (2 + 2q + q^2)x + (1 + 2q + q^2 + q^3)x^2},$$

$$(16) \quad \sum_{r \geq 4} \mathcal{P}_{D_r}(q)x^r = \frac{(q + 4q^2 + 6q^3 + 3q^4 + q^5)x^4}{1 - (2 + 2q + q^2)x + (1 + 2q + q^2 + q^3)x^2} - \frac{(q + 4q^2 + 6q^3 + 5q^4 + 3q^5 + q^6)x^5}{1 - (2 + 2q + q^2)x + (1 + 2q + q^2 + q^3)x^2}.$$

We now use Bender’s Theorem (Theorem 1 from [3]) to show that each of these generating functions have asymptotically Gaussian coefficients. For reference, we restate a special case of Bender’s Theorem

Theorem (Bender's Theorem [3]). Suppose $f(z, w) = g(z, w)/P(z, w)^{m+1}$. If

- (i) $P(z, w)$ is a polynomial in z with coefficients continuous in w ,
- (ii) $P(z, 1)$ has a simple root at r and all other roots have larger absolute value,
- (iii) $g(z, w)$ is analytic for w near 1 and $z < r + \epsilon$,
- (iv) $g(r, 1) \neq 0$,

then the coefficients of $f(z, w)$ are asymptotically normal.

In order to match the notation in [3], we define $f_{\mathfrak{g}}(x, q) = \sum_{r \geq r_{\mathfrak{g}}} \mathcal{P}_{\mathfrak{g}}(q)x^r$, where

$$r_{\mathfrak{g}} = \begin{cases} 1, & \mathfrak{g} = B_r, C_r \\ 4, & \mathfrak{g} = D_r \end{cases}.$$

Since these generating functions are rational functions in q and x , we need only check conditions (ii) and (iv) of Bender's Theorem for each Lie type.

Corollary 2. Let $\sum_{r \geq 1} \mathcal{P}_{B_r}(q)x^r$ be the generating function for the q -analog of Kostant's partition function for Lie algebras of Type B . Then the coefficients of this generating function are asymptotically Gaussian.

Proof. Let $f_{B_r}(x, q) = g_{B_r}(x, q)/P(x, q) = \sum_{r \geq 1} \mathcal{P}_{B_r}(q)x^r$ as given in (14). Then the roots of $P(x, 1) = 1 - 5x + 5x^2$ are $\frac{1}{10}(5 - \sqrt{5})$ and $\frac{1}{10}(5 + \sqrt{5})$. Hence, condition (ii) of Bender's Theorem 1 is satisfied. And we have $g_{B_r}(\frac{1}{10}(5 - \sqrt{5}), 1) \neq 0$. Hence, condition (iv) of Bender's Theorem is satisfied. \square

Notice that the denominators for the generating functions for Types C and D are the same as for Type B . Hence, condition (ii) of Bender's Theorem will be satisfied for all three Lie algebra types. So, for Types C and D , we need only check condition (iv) of Bender's Theorem.

Corollary 3. Let $\sum_{r \geq 1} \mathcal{P}_{C_r}(q)x^r$ be the generating function for the q -analog of Kostant's partition function for Lie algebras of Type C . Then the coefficients of this generating function are asymptotically Gaussian.

Proof. Let $f_{C_r}(x, q) = g_{C_r}(x, q)/P(x, q) = \sum_{r \geq 1} \mathcal{P}_{C_r}(q)x^r$ as given in (15). Since $g_{C_r}(\frac{1}{10}(5 - \sqrt{5}), 1) \neq 0$, condition (iv) of Bender's Theorem is satisfied. \square

Corollary 4. Let $\sum_{r \geq 4} \mathcal{P}_{D_r}(q)x^r$ be the generating function for the q -analog of Kostant's partition function for Lie algebras of Type D . Then the coefficients of this generating function are asymptotically Gaussian.

Proof. Let $f_{D_r}(x, q) = g_{D_r}(x, q)/P(x, q) = \sum_{r \geq 1} \mathcal{P}_{D_r}(q)x^r$ as given in (16). Since $g_{D_r}(\frac{1}{10}(5 - \sqrt{5}), 1) \neq 0$, condition (iv) of Bender’s Theorem is satisfied. \square

We now compute the means and variances for these generating functions. Harris, Insko, and Omar also gave closed formulas for the value of the q -analog of Kostant’s partition function on the highest root of a classical Lie algebra in [13], which we restate below for ease of reference.

Corollary 5 (Explicit formulas [13]). Let $\tilde{\alpha}$ denote the highest root of a Lie algebra. Let $\mathcal{P}_{A_r}(q)$, $\mathcal{P}_{B_r}(q)$, $\mathcal{P}_{C_r}(q)$, and $\mathcal{P}_{D_r}(q)$ denote $\wp_q(\tilde{\alpha})$, in the Lie algebras of type A_r , B_r , C_r , and D_r , respectively. Then explicit formulas for the value of the q -analog of Kostant’s partition function on the highest root of the classical Lie algebras are as follow:

(17)

$$\text{Type } A_r \ (r \geq 1) : \mathcal{P}_{A_r}(q) = q(1+q)^{r-1},$$

(18)

$$\text{Type } B_r \ (r \geq 2) : \mathcal{P}_{B_r}(q) = b_+(q) \cdot (\beta_+(q))^{r-2} + b_-(q) \cdot (\beta_-(q))^{r-2},$$

(19)

$$\text{Type } C_r \ (r \geq 1) : \mathcal{P}_{C_r}(q) = c_+(q) \cdot (\beta_+(q))^{r-1} + c_-(q) \cdot (\beta_-(q))^{r-1},$$

(20)

$$\text{Type } D_r \ (r \geq 4) : \mathcal{P}_{D_r}(q) = d_+(q) \cdot (\beta_+(q))^{r-4} + d_-(q) \cdot (\beta_-(q))^{r-4},$$

where

$$\beta_{\pm}(q) = \frac{(q^2 + 2q + 2) \pm q\sqrt{q^2 + 4}}{2}$$

and

$$b_{\pm}(q) = \frac{(q^5 + q^4 + 5q^3 + 4q^2 + 4q) \pm (q^4 + q^3 + 3q^2 + 2q)\sqrt{q^2 + 4}}{2(q^2 + 4)},$$

$$c_{\pm}(q) = \frac{(q^3 + 4q) \pm q^2\sqrt{q^2 + 4}}{2(q^2 + 4)},$$

$$d_{\pm}(q) = \frac{q^7 + 3q^6 + 10q^5 + 16q^4 + 25q^3 + 16q^2 + 4q}{2(q^2 + 4)} \pm \frac{(q^6 + 3q^5 + 8q^4 + 12q^3 + 9q^2 + 2q)\sqrt{q^2 + 4}}{2(q^2 + 4)}.$$

For $1 \leq k \leq r$, let $p_{r,k}$ denote the number of ways to write $\tilde{\alpha}$ as a sum of exactly k positive roots in type A_r . Hence $p_{r,k}$ is obtained by extracting the coefficient of q^k in $\mathcal{P}_{\mathfrak{g}}(q)$ and from Equation (13) we know that $p_{r,k} = \binom{r-1}{k-1}$ ways to write $\tilde{\alpha}$ as a sum of exactly k positive roots. We note that $p_{r,0} = 0$ for all r , and $p_{0,k} = 0$ for all k .

Proposition 7. Let $F(x, y) = \sum_{r \geq 0} \sum_{k \geq 0} p_{r,k} x^r y^k$ be the generating function for the coefficients $p_{r,k}$ in type A_r . Then

$$(21) \quad F(x, y) = \frac{xy}{1 - x - xy}.$$

Proof. The result follows from the bivariate generating function of the binomial coefficients

$$\sum_{r \geq 0} \sum_{k \geq 0} \binom{r}{k} x^r y^k = \frac{1}{1 - y - yx}. \quad \square$$

Proposition 8. For $r \geq 1$, the mean and variance of Y_{A_r} are given by $\mu_r = \frac{r+1}{2}$ and $\sigma_r^2 = \frac{r-1}{4}$, respectively.

Proof. By Equation (21) and use of the geometric sum formula, we note

$$F(x, y) = xy \left(\frac{1}{1 - (1+y)x} \right) = \sum_{m=0}^{\infty} y(1+y)^m x^{m+1}.$$

Hence $g_r(y) = y(1+y)^{r-1}$. Now observe that by Proposition 1

$$\mu_r = \frac{g'_r(1)}{g_r(1)} = \frac{y(r-1)(1+y)^{r-2} + (1+y)^{r-1} |_{y=1}}{2^{r-1}} = \frac{r+1}{2}.$$

The variance follows from a similar calculation. □

Proposition 9. For $r \geq 2$, the mean and variance of Y_{B_r} are given by

$$\begin{aligned} \mu_r = & \frac{(5 - \sqrt{5} + (25 - 13\sqrt{5})r)(5 - \sqrt{5})^r}{5[(5 - 3\sqrt{5})(5 - \sqrt{5})^r + (5 + 3\sqrt{5})(5 + \sqrt{5})^r]} \\ & + \frac{(5 + \sqrt{5} + (25 + 13\sqrt{5})r)(5 + \sqrt{5})^r}{5[(5 - 3\sqrt{5})(5 - \sqrt{5})^r + (5 + 3\sqrt{5})(5 + \sqrt{5})^r]} \end{aligned}$$

and

$$\begin{aligned} \sigma_r^2 = & \frac{20^{r+1}r^2}{-5[(5-\sqrt{3})(5-\sqrt{5})^r + (5+3\sqrt{5})(5+\sqrt{5})^r]^2} \\ & + \frac{[26(3\sqrt{5}-7)(5-\sqrt{5})^{2r} - 26(3\sqrt{5}+7)(5+\sqrt{5})^{2r} + 36 \cdot \frac{20^{r+1}}{5}]r}{-5[(5-\sqrt{3})(5-\sqrt{5})^r + (5+3\sqrt{5})(5+\sqrt{5})^r]^2} \\ & + \frac{[2(73-25\sqrt{5})(5-\sqrt{5})^{2r} + 2(73+25\sqrt{5})(5+\sqrt{5})^{2r} - 63 \cdot \frac{20^{r+1}}{5}]}{-5[(5-\sqrt{3})(5-\sqrt{5})^r + (5+3\sqrt{5})(5+\sqrt{5})^r]^2}, \end{aligned}$$

respectively.

Proof. Applying the result in Proposition 1 to Equation (18) yields the desired result, albeit after some straightforward, but lengthy calculations. \square

Proposition 10. For $r \geq 3$, the mean and variance of Y_{C_r} are given by

$$\begin{aligned} \mu_r = & \frac{((1-\sqrt{5}) + (7-\sqrt{5})r)(5-\sqrt{5})^r}{10((5-\sqrt{5})^r + (5+\sqrt{5})^r)} \\ & + \frac{((1+\sqrt{5}) + (7+\sqrt{5})r)(5+\sqrt{5})^r}{10((5-\sqrt{5})^r + (5+\sqrt{5})^r)} \end{aligned}$$

and

$$\begin{aligned} \sigma_r^2 = & \frac{\frac{20^{r+1}}{4}r^2 + [13((5-\sqrt{5})^{2r} + (5+\sqrt{5})^{2r}) + 9 \cdot \frac{20^{r+1}}{5}]r}{25((5-\sqrt{5})^r + (5+\sqrt{5})^r)^2} \\ & + \frac{(-21+4\sqrt{5})(5+\sqrt{5})^{2r} - (21+4\sqrt{5})(5-\sqrt{5})^{2r} - 37 \cdot 20^r}{25((5-\sqrt{5})^r + (5+\sqrt{5})^r)^2}, \end{aligned}$$

respectively.

Proof. Applying the result in Proposition 1 to Equation (19) yields the desired result, albeit after some straightforward, but lengthy, calculations. \square

Proposition 11. The mean and variance of Y_{D_r} are given by

$$\mu_r = \frac{(15-\sqrt{5} + r(-5+7\sqrt{5}))(5-\sqrt{5})^r}{10\sqrt{5}((5-\sqrt{5})^r - (5+\sqrt{5})^r)}$$

$$+ \frac{(15 + \sqrt{5} - r(5 + 7\sqrt{5})) (5 + \sqrt{5})^r}{10\sqrt{5} \left((5 - \sqrt{5})^r - (5 + \sqrt{5})^r \right)}$$

and

$$\begin{aligned} \sigma_r^2 = & \frac{\frac{20^{r+1}}{4}r^2 - \left[13 \left((5 + \sqrt{5})^{2r} + (5 - \sqrt{5})^{2r} \right) + \frac{20^{r+1}}{5} \right] r}{-25 \left[(5 + \sqrt{5})^r - (5 - \sqrt{5})^r \right]^2} \\ & + \frac{(34 - 3\sqrt{5}) (5 + \sqrt{5})^{2r} + (34 + 3\sqrt{5}) (5 - \sqrt{5})^{2r} - 23 \cdot 20^r}{-25 \left[(5 + \sqrt{5})^r - (5 - \sqrt{5})^r \right]^2}, \end{aligned}$$

respectively.

Proof. Applying the result in Proposition 1 to Equation (20) yields the desired result, albeit after some straightforward, but lengthy, calculations. \square

5. Future work

In this section we provide some directions for future study.

1. Recall that for a fixed r , we let $p_{r,k}$ denote the number of ways to express the highest root as a sum of exactly k positive roots. In our work in type A_r the sequence $p_{r,0}, p_{r,1}, \dots, p_{r,r}$ is unimodal as it is a sequence of binomial coefficients. In other Lie types (for fixed r) we ask: Do the coefficients $p_{r,k}$ also form a unimodal sequence?
2. We established that the distribution of the number of positive roots used in the decompositions of certain weights, including the highest root, of the classical Lie algebras converges to a Gaussian. One potential problem would be to give a uniform proof of the second main result in this paper.
3. Additionally, we provide some computational experiments for $2\tilde{\alpha}$ and 2ρ where 2ρ is the sum of all positive roots of the classical simple Lie algebras. The results are provided in Figure 1. In light of these experiments, we pose the following: Give necessary and sufficient conditions on the weight μ of a Lie algebra of rank r , such that the (normalized) distribution of the number of positive roots used in the decompositions of μ as a sum of positive roots converges to a Gaussian distribution as $r \rightarrow \infty$.

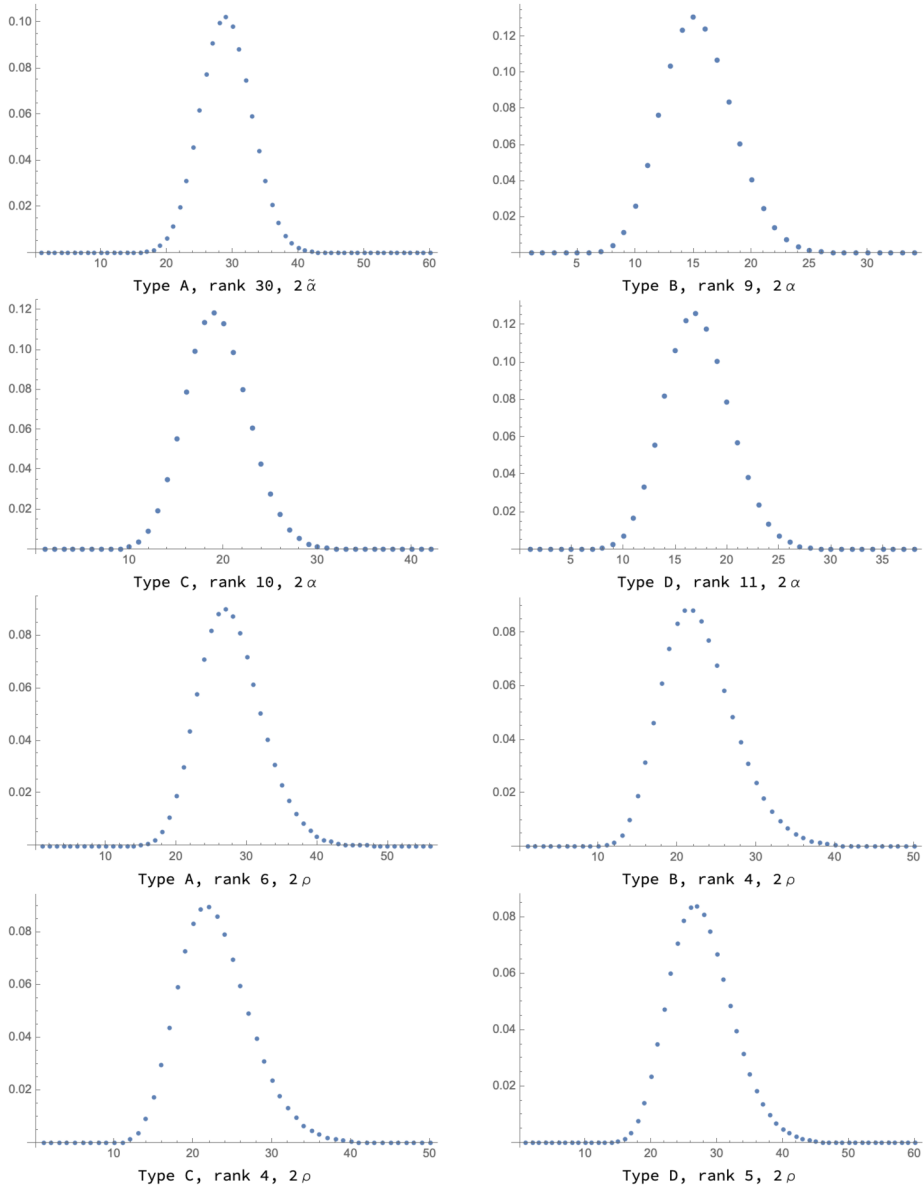


Figure 1: Distributions of the number of positive roots used in the decompositions of $2\tilde{\alpha}$ and 2ρ in the classical simple Lie algebras.

4. Although our experimentations seem to support an affirmative solution to the above problem, we would also welcome an example of a weight μ whose associated distribution does not converge to a Gaussian distribution as $r \rightarrow \infty$. Hence, determining for what weight such a distribution converges to a Gaussian remains an open problem worthy of study.

Appendix A. Alternate Proof of Theorem 2

To prove Theorem 2, we proceed via a case-by-case analysis and establish that the moment generating function of $Y'_{g,r}$ converges to that of the standard normal, which is $e^{t^2/2}$.

A.1. Type A

First we prove Theorem 2 for the Lie algebra of type A_r .

Theorem 2 (Type A_r). *Let μ_r and σ_r^2 be defined as in Proposition 8. Then the random variable $Y'_r = (Y_r - \mu_r) / \sigma_r^2$ converges to the standard Gaussian distribution as $r \rightarrow \infty$.*

Proof. Recall $g_r(y) = y(1+y)^{r-1}$, hence

$$\begin{aligned}
 \log[g_r(e^n)] &= \log[e^n(1+e^n)^{r-1}] \\
 &= \log(e^n) + (r-1)\log(1+e^n) \\
 (22) \quad &= \log\left(1+n+\frac{n^2}{2}\right) + (r-1)\log\left(2+n+\frac{n^2}{2}\right) + O(n^3).
 \end{aligned}$$

Using Taylor's series expansion for $\log(x)$ we have

$$\begin{aligned}
 \log\left(1+n+\frac{n^2}{2}\right) &= \log(1) + \frac{1}{1}\left(1+n+\frac{n^2}{2}-1\right) \\
 &\quad - \frac{1}{1}\frac{\left(1+n+\frac{n^2}{2}-1\right)^2}{2} + O(n^3) \\
 &= n + \frac{n^2}{2} - \frac{1}{2}\left(n+\frac{n^2}{2}\right)^2 + O(n^3) \\
 &= n + \frac{n^2}{2} - \frac{1}{2}\left(n^2+n^3+\frac{n^4}{4}\right) + O(n^3) \\
 (23) \quad &= n + O(n^3)
 \end{aligned}$$

and

$$\begin{aligned}
 \log\left(2 + n + \frac{n^2}{2}\right) &= \log(2) + \frac{1}{2}\left(2 + n + \frac{n^2}{2} - 2\right) \\
 &\quad - \frac{1}{4}\frac{\left(2 + n + \frac{n^2}{2} - 2\right)^2}{2} + O(n^3) \\
 &= \log(2) + \frac{1}{2}\left(n + \frac{n^2}{2}\right) - \frac{1}{8}\left(n + \frac{n^2}{2}\right)^2 + O(n^3) \\
 (24) \qquad &= \log(2) + \frac{1}{2}n + \frac{1}{8}n^2 + O(n^3).
 \end{aligned}$$

Substituting Equations (23) and (24) into Equation (22) yields

$$\begin{aligned}
 \log[g_r(e^n)] &= [n + O(n^3)] + (r-1)\left[\log(2) + \frac{1}{2}n + \frac{1}{8}n^2 + O(n^3)\right] + O(n^3) \\
 &= n + (r-1)\log(2) + \frac{1}{2}n(r-1) + \frac{1}{8}n^2(r-1) + O(n^3) \\
 (25) \qquad &= n + \frac{1}{2}n(r-1) + \frac{1}{8}n^2(r-1) + O(n^3) + \log(g_r(1)).
 \end{aligned}$$

Recall

$$(26) \qquad \log(M_{Y'_r}(t)) = \log[g_r(e^n)] - \log[g_r(1)] - \frac{t\mu_r}{\sigma_r}$$

where $\mu_r = \frac{r+1}{2}$, $\sigma_r = \sqrt{\frac{r-1}{4}}$, and $n = \frac{t}{\sigma_r} = \frac{t}{\sqrt{\frac{r-1}{4}}} = \frac{2t}{\sqrt{r-1}}$. Substituting Equation (25) and $n = \frac{2t}{\sqrt{r-1}}$ into Equation (26) yields

$$\begin{aligned}
 \log[M_{Y'_r}(t)] &= \frac{2t}{\sqrt{r-1}} + \frac{t}{\sqrt{r-1}}(r-1) + \frac{1}{8}\left(\frac{2t}{\sqrt{r-1}}\right)^2(r-1) \\
 &\quad - \frac{t\left(\frac{r+1}{2}\right)}{\sqrt{\frac{r-1}{4}}} + O\left(\left(\frac{2t}{\sqrt{r-1}}\right)^3\right) \\
 (27) \qquad &= \frac{1}{2}t^2 + O\left(\left(\frac{2t}{\sqrt{r-1}}\right)^3\right).
 \end{aligned}$$

Taking the limit of Equation (27) as $r \rightarrow \infty$, we have that $\log(M_{Y'_r}(t))$ converges to $\frac{1}{2}t^2$. Thus Y'_r converges to the standard normal distribution as $r \rightarrow \infty$. \square

A.2. Technical results for other Lie types

For Types B , C , and D , we also show that $\log \left[M_{Y'_{\mathfrak{g},r}(t)} \right]$ converges to $\frac{1}{2}t^2$ as $r \rightarrow \infty$, thus proving that $Y'_{\mathfrak{g},r}$ converges to the standard normal distribution as $r \rightarrow \infty$. In this section, we work through simplifying the equation for $\log \left[M_{Y'_{\mathfrak{g},r}(t)} \right]$, as many terms overlap for the various Lie types.

Recall from Corollary 5 that $\wp_q(\tilde{\alpha}) = \mathfrak{g}_+(q) \cdot (\beta_+(q))^{r-i_{\mathfrak{g}}} + \mathfrak{g}_-(q) \cdot (\beta_-(q))^{r-i_{\mathfrak{g}}}$, where $\mathfrak{g}_{\pm} \in \{b_{\pm}, c_{\pm}, d_{\pm}\}$ and

$$i_{\mathfrak{g}} = \begin{cases} 2, & \mathfrak{g} = B_r \\ 1, & \mathfrak{g} = C_r \\ 4, & \mathfrak{g} = D_r \end{cases} .$$

Hence, if we let the random variable $Y_{\mathfrak{g},r}$ denote the total number of positive roots used in the decompositions of the highest root of the Lie algebra of type \mathfrak{g} as sums of positive roots, we can write $g_r(y) = \wp_q(\tilde{\alpha})|_{q=y}$ and $\log \left[M_{Y'_{\mathfrak{g},r}(t)} \right] = \log[g_r(e^n)] - \log[g_r(1)] - \frac{t\mu_{\mathfrak{g},r}}{\sigma_{\mathfrak{g},r}}$. Let

$$(28) \quad M = \mathfrak{g}_+(e^n) \cdot (\beta_+(e^n))^{r-i_{\mathfrak{g}}} ,$$

$$(29) \quad A = \mathfrak{g}_-(e^n) \cdot (\beta_-(e^n))^{r-i_{\mathfrak{g}}} , \quad \text{and}$$

$$(30) \quad S = A/M = \frac{\mathfrak{g}_-(e^n)}{\mathfrak{g}_+(e^n)} \left(\frac{\beta_-(e^n)}{\beta_+(e^n)} \right)^{r-i_{\mathfrak{g}}} .$$

Then

$$(31) \quad \log \left[M_{Y'_{\mathfrak{g},r}(t)} \right] = \log[M] + \log[1 + S] - \log[g_r(1)] - \frac{t\mu_{\mathfrak{g},r}}{\sigma_{\mathfrak{g},r}} .$$

We first evaluate

$$\beta_{\pm}(e^n) = \frac{(e^{2n} + 2e^n + 2) \pm e^n \sqrt{e^{2n} + 4}}{2} .$$

Then, using Taylor expansion of $y = e^x$ about $x = 0$, we replace e^n with $1 + n + \frac{1}{2}n^2 + O(n^3)$ and obtain

$$\begin{aligned} \beta_{\pm}(e^n) &= \frac{1}{2}(5 + 4n + 3n^2 + O(n^3)) \\ &\quad \pm \frac{1}{2}(1 + n + \frac{1}{2}n^2 + O(n^3))\sqrt{5 + 2n + 2n^2 + O(n^3)} . \end{aligned}$$

By Taylor expanding $y = \sqrt{x}$ about $x = 5$ and then replacing $x = 5 + 2n + 2n^2$ and simplifying, we get

$$(32) \quad \beta_+(e^n) = \frac{5}{2} + \frac{\sqrt{5}}{2} + \left(2 + \frac{3}{\sqrt{5}}\right)n + \left(\frac{3}{2} + \frac{11}{5\sqrt{5}}\right)n^2 + O(n^3) \quad \text{and}$$

$$(33) \quad \beta_-(e^n) = \frac{5}{2} - \frac{\sqrt{5}}{2} + \left(2 - \frac{3}{\sqrt{5}}\right)n + \left(\frac{3}{2} - \frac{11}{5\sqrt{5}}\right)n^2 + O(n^3).$$

Using Equation (32), we can rewrite

$$\begin{aligned} \log[M] &= \log[\mathfrak{g}_+(e^n)] \\ &+ (r - i_{\mathfrak{g}}) \log \left[\frac{5}{2} + \frac{\sqrt{5}}{2} + \left(2 + \frac{3}{\sqrt{5}}\right)n + \left(\frac{3}{2} + \frac{11}{5\sqrt{5}}\right)n^2 + O(n^3) \right]. \end{aligned}$$

Now we Taylor expand $y = \log(x)$ about $x = \frac{5}{2} + \frac{\sqrt{5}}{2}$ and replace $x = \beta_+(e^n)$ as in Equation (32) to get

$$(34) \quad \begin{aligned} \log[M] &= \log[\mathfrak{g}_+(e^n)] \\ &+ (r - i_{\mathfrak{g}}) \left[\frac{(13 + 5\sqrt{5})n}{5(3 + \sqrt{5})} + \frac{13n^2}{50} + \log \left(\frac{1}{2}(5 + \sqrt{5}) \right) + O(n^3) \right]. \end{aligned}$$

To simplify Equation (30), we denote $N = \left(\frac{\beta_-(e^n)}{\beta_+(e^n)}\right)^{r-i_{\mathfrak{g}}}$, which we can simplify by Taylor expanding $y = 1/x$ about $x = \frac{5}{2} + \frac{\sqrt{5}}{2}$ and then replacing $x = \beta_+(e^n)$ as in Equation (32) and multiplying this Taylor expansion to Equation (33). We obtain

$$\begin{aligned} N &= \left(\frac{1}{(5 + \sqrt{5})^3} \right)^{r-i_{\mathfrak{g}}} \\ &\quad \left[(20(5 + \sqrt{5})) + n(-20(1 + \sqrt{5}) + 2(5 + \sqrt{5})n) \right]^{r-i_{\mathfrak{g}}}. \end{aligned}$$

Using the Binomial Theorem, we have

$$(35) \quad \begin{aligned} N &= \left(\frac{1}{(5 + \sqrt{5})^3} \right)^{r-i_{\mathfrak{g}}} \left[(20(5 + \sqrt{5}))^{r-i_{\mathfrak{g}}} \right. \\ &\quad + (r - i_{\mathfrak{g}}) (20(5 + \sqrt{5}))^{r-i_{\mathfrak{g}}-1} \left(n(-20(1 + \sqrt{5}) + 2(5 + \sqrt{5})n) \right) \\ &\quad \left. + \frac{(r - i_{\mathfrak{g}})(r - i_{\mathfrak{g}} - 1)}{2} (20(5 + \sqrt{5}))^{r-i_{\mathfrak{g}}-2} \right] \end{aligned}$$

$$\left. \left(n \left(-20(1 + \sqrt{5}) + 2(5 + \sqrt{5})n \right) \right)^2 + O(n^3) \right].$$

In order to completely describe Equation (31), we need to work through each Lie algebra type separately.

A.2.1. Type C Next we prove Theorem 2 for the Lie algebra of type C_r .

Theorem 2 (Type C_r). *Let μ_r and σ_r^2 be defined as in Proposition 11. Then the random variable $Y'_{C_r} = (Y_{C_r} - \mu_r) / \sigma_r^2$ converges to the standard Gaussian distribution as $r \rightarrow \infty$.*

Proof. Recall

$$(36) \quad c_{\pm}(q) = \frac{(q^3 + 4q) \pm q^2 \sqrt{q^2 + 4}}{2(q^2 + 4)}.$$

Replacing e^n with $1 + n + \frac{1}{2}n^2 + O(n^3)$ gives

$$c_{\pm}(e^n) = \frac{\left(5 + 7n + \frac{13n^2}{2} + O(n^3) \right)}{10 + 4n + 4n^2 + O(n^3)} \pm \frac{\left(1 + 2n + 2n^2 + O(n^3) \right) \sqrt{5 + 2n + 2n^2 + O(n^3)}}{10 + 4n + 4n^2 + O(n^3)}.$$

Now, Taylor expanding $y = \sqrt{x}$ about $x = 5$ and $z = 1/x$ about $x = 10$, and replacing $x = 5 + 2n + 2n^2$ and $x = 10 + 4n + 4n^2$, respectively, gives

$$(37) \quad \begin{aligned} c_+(e^n) &= \left(\left(5 + 7n + \frac{13n^2}{2} \right) + (1 + 2n + 2n^2) \left(\sqrt{5} + \frac{n}{\sqrt{5}} + \frac{9n^2}{10\sqrt{5}} \right) \right) \\ &\quad \left(\frac{1}{10} - \frac{n}{25} - \frac{3n^2}{125} \right) + O(n^3) \\ &= \frac{1}{500} \left(50(5 + \sqrt{5}) + 10(25 + 9\sqrt{5})n + (125 + 73\sqrt{5})n^2 \right) + O(n^3). \end{aligned}$$

Similarly,

$$(38) \quad c_-(e^n) = \frac{1}{500} \left(50(5 - \sqrt{5}) + 10(25 - 9\sqrt{5})n + (125 - 73\sqrt{5})n^2 \right) + O(n^3).$$

Taylor expanding $y = 1/x$ about $x = \frac{1}{500}(50(5 + \sqrt{5}))$, replacing $x = C_+(e^n)$ as in Equation (37), and multiplying the resulting expression to Equation (38) gives

$$(39) \quad \frac{c_-(e^n)}{c_+(e^n)} = \frac{1}{(5 + \sqrt{5})^3} \left[20(5 + \sqrt{5}) - 40(1 + \sqrt{5})n - 8(-3 + \sqrt{5})n^2 \right] + O(n^3).$$

Taylor expanding $y = \log(x)$ about $x = \frac{1}{500}(50(5 + \sqrt{5}))$ and replacing $x = C_+(e^n)$ as in Equation (37) gives

$$(40) \quad \log(c_+(e^n)) = \frac{1}{25(3 + \sqrt{5})} \left[(85 + 35\sqrt{5})n + 2(-1 + \sqrt{5})n^2 \right] - \log(2) - \log(5) + \log(5 + \sqrt{5}) + O(n^3).$$

Now, we can substitute Equation (40) into Equation (34), where $i_g = 1$, to get

$$(41) \quad \log[M] = \frac{(4 + 2\sqrt{5} + 13r + 5\sqrt{5}r)n}{5(3 + \sqrt{5})} + \frac{(-43 - 9\sqrt{5} + 13(3 + \sqrt{5})r)n^2}{50(3 + \sqrt{5})} - r \log(2) - \log(5) + r \log(5 + \sqrt{5}) + O(n^3).$$

Substituting Equations (39) and (35) (with $i_g = 1$) into Equation (30), we get an equation for $1 + S$:

$$(42) \quad 1 + S = 1 + 4^{1+r}5^{r-1/2}(5 + \sqrt{5})^{-3-2r} \left[50(2 + \sqrt{5}) - 10(5 + 2\sqrt{5})n(1 + r) + n^2(-30 - 11\sqrt{5} + 10(2 + \sqrt{5})r + 5(2 + \sqrt{5})r^2) \right] + O(n^3).$$

Next, we can Taylor expand $y = \log(x)$ about $x = 1 + 4^{1+r}5^{r-1/2}(5 + \sqrt{5})^{-3-2r}50(2 + \sqrt{5})$ and replace $x = 1 + S$ as in Equation (42) to obtain

$$(43) \quad \log(1 + S) = \log \left[1 + 20^r(5 + \sqrt{5})^{-2r} \right] - \frac{2^{2r-1}5^{r-2}n}{(20^r + (5 + \sqrt{5})^{2r})^2} \left[10\sqrt{5}(20^r + (5 + \sqrt{5})^{2r})(1 + r) + n \left(2^{3+2r}5^{1/2+r} + (5 + \sqrt{5})^{2r}(-5 + 8\sqrt{5}) \right) \right]$$

$$-10(5 + \sqrt{5})^{2r}r - 5(5 + \sqrt{5})^{2r}r^2 \Big] + O(n^3).$$

Given Equations (41) and (43), $g_r(1) = \frac{1}{5} \left(\frac{5+\sqrt{5}}{2}\right)^r + \frac{1}{5} \left(\frac{5-\sqrt{5}}{2}\right)^r$, $n = t/\sigma_r$, and μ_r and σ_r as in Proposition 10, we find that

$$\begin{aligned} \log(M_{Y'_r(t)}) &= \log[g_r(e^n)] - \log[g_r(1)] - \frac{t\mu_r}{\sigma_r} \\ &= k_0 + k_1t + k_2t^2 + O\left(\left(\frac{t}{\sigma_r}\right)^3\right), \end{aligned}$$

where $k_0 = 0$, $k_1 = 0$, and

$$\begin{aligned} k_2 &= \frac{((5 - \sqrt{5})^r + (5 - \sqrt{5})^r)^2}{2(3 + \sqrt{5})(20^r + (5 + \sqrt{5})^{2r})^2} \\ &\quad \left[-37 \cdot 20^r + (4\sqrt{5} - 21)(5 + \sqrt{5})^{2r} - (5 - \sqrt{5})^{2r}(21 + 4\sqrt{5}) \right. \\ &\quad \left. + r(36 \cdot 20^r + 13(5 - \sqrt{5})^{2r} + 13(5 + \sqrt{5})^{2r}) + 5 \cdot 20^r r^2 \right]^{-1} \\ &\quad \left[-37 \cdot 20^r(3 + \sqrt{5})(5 + \sqrt{5})^{2r} - (5 + \sqrt{5})^{4r}(43 + 9\sqrt{5}) \right. \\ &\quad \left. - 400^r(83 + 33\sqrt{5}) + r(3 + \sqrt{5})(13 \cdot 400^r + 36 \cdot 20^r(5 + \sqrt{5})^{2r} \right. \\ &\quad \left. + 13(5 + \sqrt{5})^{4r}) + r^2 5^{r+1}(3 + \sqrt{5})(2(5 + \sqrt{5}))^{2r} \right]. \end{aligned}$$

Lastly, note $\lim_{r \rightarrow \infty} \log(M_{Y'_r(t)}) = \frac{1}{2}t^2$. Thus Y'_r converges to the standard normal as $r \rightarrow \infty$. □

A.2.2. Type B Our first result is as follows.

Theorem 2 (Type B_r). *Let μ_r and σ_r^2 be defined as in Proposition 9. Then the random variable $Y'_{B_r} = (Y_{B_r} - \mu_r)/\sigma_r^2$ converges to the standard Gaussian distribution as $r \rightarrow \infty$.*

Proof. Recall

$$(44) \quad b_{\pm}(q) = \frac{(q^5 + q^4 + 5q^3 + 4q^2 + 4q) \pm (q^4 + q^3 + 3q^2 + 2q)\sqrt{q^2 + 4}}{2(q^2 + 4)}.$$

As with Type C, replacing e^n with $1+n+\frac{1}{2}n^2+O(n^3)$, Taylor expanding $y = \sqrt{x}$ about $x = 5$ and $z = 1/x$ about $x = 10$, and replacing $x = 5 + 2n + 2n^2$ and $x = 10 + 4n + 4n^2$, respectively, gives

$$(45) \quad b_+(e^n) = \frac{3}{2} + \frac{7}{2\sqrt{5}} + \left(3 + \frac{34}{5\sqrt{5}}\right)n + \left(\frac{7}{2} + \frac{194}{25\sqrt{5}}\right)n^2 + O(n^3)$$

Similarly,

$$(46) \quad b_-(e^n) = \frac{3}{2} - \frac{7}{2\sqrt{5}} + \left(3 - \frac{34}{5\sqrt{5}}\right)n + \left(\frac{7}{2} - \frac{194}{25\sqrt{5}}\right)n^2 + O(n^3).$$

Taylor expanding $y = 1/x$ about $x = \frac{3}{2} + \frac{7}{2\sqrt{5}}$, replacing $x = b_+(e^n)$ as in Equation (45), and multiplying the resulting expression to Equation (46) gives

$$(47) \quad \frac{b_-(e^n)}{b_+(e^n)} = \frac{1}{(15 + 7\sqrt{5})^3} \left[-20(15 + 7\sqrt{5}) \right. \\ \left. + 60(7 + 3\sqrt{5})n + (38 + 6\sqrt{5})n^2 \right] + O(n^3).$$

Taylor expanding $y = \log(x)$ about $x = \frac{3}{2} + \frac{7}{2\sqrt{5}}$ and replacing $x = b_+(e^n)$ as in Equation (45) gives

$$(48) \quad \log(b_+(e^n)) = \frac{1}{5(15 + 7\sqrt{5})^2} \left[10(463 + 207\sqrt{5})n + (779 + 349\sqrt{5})n^2 \right] \\ - \log(2) - \log(5) + \log(15 + 7\sqrt{5}) + O(n^3).$$

Now, we can substitute Equation (48) into Equation (34), where $i_{\mathbf{g}} = 2$, to get

$$(49) \quad \log[M] = \frac{(76 + 34\sqrt{5} + (568 + 254\sqrt{5})r)n}{5(123 + 55\sqrt{5})} \\ + \frac{(-1157 - 517\sqrt{5} + (1599 + 715\sqrt{5})r)n^2}{50(123 + 55\sqrt{5})} \\ + (r - 2)\log(5 + \sqrt{5}) - (r - 1)\log(2) \\ - \log(5) + \log(15 + 7\sqrt{5}) + O(n^3).$$

Substituting Equations (47) and (35) (with $i_{\mathbf{g}} = 2$) into Equation (30),

we get an equation for $1 + S$:

$$(50) \quad 1 + S = 1 - \frac{4^{r+3}5^{r+2}(5 + \sqrt{5})^{-2(2+r)}}{(15 + 7\sqrt{5})^3} \left[3600 + 1610\sqrt{5} \right. \\ \left. - 10(161 + 72\sqrt{5})n(r + 1) + n^2 \left(-3182 - 14123\sqrt{5} \right. \right. \\ \left. \left. + (720 + 322\sqrt{5})r + (360 + 161\sqrt{5})r^2 \right) \right] + O(n^3).$$

Next, we can Taylor expand $y = \log(x)$ about

$$x = 1 - \frac{4^{r+3}5^{r+2}(5 + \sqrt{5})^{-2(2+r)}}{(15 + 7\sqrt{5})^3} (3600 + 1610\sqrt{5})$$

and replace $x = 1 + S$ as in Equation (50) to obtain

$$(51) \quad \log(1 + S) = \log \left[\frac{(15 + 7\sqrt{5})^3 - 4 \cdot 20^{r+2}(5 + \sqrt{5})^{-4-2r}(3600 + 1610\sqrt{5})}{(15 + 7\sqrt{5})^3} \right] \\ - \left[\frac{1}{(5 + \sqrt{5})^8(15 + 7\sqrt{5})^6} \right] \\ \left[\frac{1}{2 \cdot 20^r (360 + 161\sqrt{5}) - (5 + \sqrt{5})^{2r}(4935 + 2207\sqrt{5})^3} \right] \\ \left[2 \cdot 20^{r+7}n \left(\left(10 \cdot 400^r(162614600673847 \right. \right. \right. \\ \left. \left. \left. + 72723460248141\sqrt{5}) - 40 \cdot 20^r(5 + \sqrt{5})^{2r}(557288527109761 \right. \right. \right. \\ \left. \left. \left. + 249227005939632\sqrt{5})10(5 + \sqrt{5})^{4r}(7639424778862807 \right. \right. \right. \\ \left. \left. \left. + 3416454622906707\sqrt{5}) \right) (1 + r) \right) \\ + n^2 \left(22 \cdot 20^{2r}(162614600673847 + 72723460248141\sqrt{5} \right. \right. \\ \left. \left. - 2 \cdot 20^r(5 + \sqrt{5})^{2r}(23274560163131324 \right. \right. \\ \left. \left. + 10408699734234047\sqrt{5}) + (5 + \sqrt{5})^{4r}(150985072020448219 \right. \right. \\ \left. \left. + 67522576925084747\sqrt{5}) + 4 \cdot 20^r(5 + \sqrt{5})^{2r} \right. \right. \\ \left. \left. (1246135029698160 + 557288527109761\sqrt{5})r - 2(5 + \sqrt{5})^{4r} \right. \right. \\ \left. \left. (217082273114533535 + 7639424778862807\sqrt{5})r \right. \right. \\ \left. \left. + 2 \cdot 20^r(5 + \sqrt{5})^{2r}(1246135029698160 \right. \right.$$

$$+ 557288527109761\sqrt{5}r^2 - 2(5 + \sqrt{5})^{4r} \\ (217082273114533535 + 7639424778862807\sqrt{5}r^2) \Big] + O(n^3).$$

Given Equations (49) and (51), $n = t/\sigma_r$, μ_r and σ_r as in Proposition 9, and

$$g_r(1) = - \frac{5.12 \cdot 10^{11}(16692641 + 7465176\sqrt{5})}{(5 + \sqrt{5})^{12}(15 + 7\sqrt{5})^6} \\ \left(r \log \left(\frac{2}{5 + \sqrt{5}} \right) + 4 \log(5 + \sqrt{5}) - \log \left[8(15 + 7\sqrt{5}) \right] \right. \\ \left. - \log \left[320000(5 + \sqrt{5})^{-2(r+3)}(6460 + 2889\sqrt{5}) \right] \right. \\ \left. - \log \left[-20^r + (5 + \sqrt{5})^{2r} \right] + \log \left[(15 + 7\sqrt{5})^3 \right] \right),$$

we find that

$$\log(M_{Y'_r(t)}) = \log[g_r(e^n)] - \log[g_r(1)] - \frac{t\mu_r}{\sigma_r} \\ = k_0 + k_1t + k_2t^2 + O\left(\left(\frac{t}{\sigma_r}\right)^3\right)$$

where k_0 and k_1 simplify to 0 and

$$k_2 = - \left((779 + 349\sqrt{5}) \left((5 - \sqrt{5})^4(-5 + 3\sqrt{5}) - (5 + \sqrt{5})^r(5 + 3\sqrt{5}) \right)^2 \right) \\ \left[2(15 + 7\sqrt{5})^2 \left(-126 \cdot 20^r + (73 - 25\sqrt{5})(5 - \sqrt{5})^{2r} \right. \right. \\ \left. \left. + (73 + 25\sqrt{5})(5 + \sqrt{5})^{2r} + \left(72 \cdot 20^r + 13 \left((5 - \sqrt{5})^{2r}(-7 + 3\sqrt{5}) \right. \right. \right. \right. \\ \left. \left. \left. - (5 + \sqrt{5})^{2r}(7 + 3\sqrt{5}) \right) \right) r + 10 \cdot 20^r r^2 \right]^{-1} \\ - \left(13 \left((5 - \sqrt{5})^{2r}(-5 + 3\sqrt{5}) - (5 + \sqrt{5})^r(5 + 3\sqrt{5}) \right)^2 (r - 2) \right) \\ \cdot \left[20(-126 \cdot 20^r + (73 - 25\sqrt{5})(5 - \sqrt{5})^{2r} + (73 + 25\sqrt{5})(5 + \sqrt{5})^{2r} \right. \\ \left. + (72 \cdot 20^r + 13((5 - \sqrt{5})^{2r}(-7 + 3\sqrt{5}) \right. \\ \left. \left. - (5 + \sqrt{5})^{2r}(7 + 3\sqrt{5})))r + 10 \cdot 20^r r^2 \right]^{-1} \\ + \left[5 \cdot 20^{r+7} \left((5 - \sqrt{5})^r(-5 + 3\sqrt{5}) - (5 + \sqrt{5})^r(5 + 3\sqrt{5}) \right)^2 \right]$$

$$\begin{aligned}
 & \cdot \left(22 \cdot 20^{2r} (162614600673847 + 72723460248141\sqrt{5}) \right. \\
 & - 2 \cdot 20^r (5 + \sqrt{5})^{2r} (23274560163131324 + 10408699734234047\sqrt{5}) \\
 & + (5 + \sqrt{5})^{4r} (150985072020448219 + 67522576925084747\sqrt{5}) \\
 & + 4 \cdot 20^r (5 + \sqrt{5})^{2r} (1246135029698160 + 557288527109761\sqrt{5})r \\
 & - 2(5 + \sqrt{5})^{4r} (17082273114533535 + 7639424778862807\sqrt{5})r \\
 & \left. + (5 + \sqrt{5})^{2r} \left(2 \cdot 20^r (1246135029698160 + 557288527109761\sqrt{5}) \right) r^2 \right. \\
 & \left. - (5 + \sqrt{5})^{4r} (17082273114533535 + 7639424778862807\sqrt{5})r^2 \right) \\
 & \left[(5 + \sqrt{5})^8 (15 + 7\sqrt{5})^6 (2 \cdot 20^r (360 + 161\sqrt{5}) \right. \\
 & - (5 + \sqrt{5})^{2r} (4935 + 2207\sqrt{5}))^3 \\
 & \left(-126 \cdot 20^r + (73 - 25\sqrt{5})(5 - \sqrt{5})^{2r} + (73 + 25\sqrt{5})(5 + \sqrt{5})^{2r} \right. \\
 & + (72 \cdot 20^r + 13((5 - \sqrt{5})^{2r}(-7 + 3\sqrt{5})) \\
 & \left. \left. - (5 + \sqrt{5})^{2r}(7 + 3\sqrt{5}))r + 10 \cdot 20^r r^2 \right) \right]^{-1}.
 \end{aligned}$$

Finally, note $\lim_{r \rightarrow \infty} \log(M_{Y'_r(t)}) = \frac{1}{2}t^2$. Thus, Y'_r converges to the standard normal distribution as $r \rightarrow \infty$. \square

A.2.3. Type D Next we prove Theorem 2 for the Lie algebra of type D_r .

Theorem 2 (Type D_r). *Let μ_r and σ_r^2 be defined as in Proposition 11. Then the random variable $Y_{D_r} = (Y_{D_r} - \mu_r) / \sigma_r^2$ converges to the standard Gaussian distribution as $r \rightarrow \infty$.*

Proof. Recall

$$\begin{aligned}
 (52) \quad d_{\pm}(q) &= \frac{(q^7 + 3q^6 + 10q^5 + 16q^4 + 25q^3 + 16q^2 + 4q)}{2(q^2 + 4)} \\
 &\quad \pm \frac{(q^6 + 3q^5 + 8q^4 + 12q^3 + 9q^2 + 2q) \sqrt{q^2 + 4}}{2(q^2 + 4)}.
 \end{aligned}$$

As with Types B and C, replacing e^n with $1 + n + \frac{1}{2}n^2 + O(n^3)$, Taylor expanding $y = \sqrt{x}$ about $x = 5$ and $z = 1/x$ about $x = 10$, and replacing $x = 5 + 2n + 2n^2$ and $x = 10 + 4n + 4n^2$, respectively, gives

$$(53) \quad d_+(e^n) = \frac{15}{2} + \frac{7\sqrt{5}}{2} + \left(22 + \frac{51}{\sqrt{5}} \right) n + \left(36 + \frac{829}{10\sqrt{5}} \right) n^2 + O(n^3)$$

Similarly,

$$(54) \quad d_-(e^n) = \frac{15}{2} - \frac{7\sqrt{5}}{2} + \left(22 - \frac{51}{\sqrt{5}}\right)n + \left(36 - \frac{829}{10\sqrt{5}}\right)n^2 + O(n^3)$$

Taylor expanding $y = 1/x$ about $x = \frac{15}{2} + \frac{7\sqrt{5}}{2}$, replacing $x = d_+(e^n)$ as in Equation (53), and multiplying the resulting expression to Equation (54) gives

$$(55) \quad \frac{d_-(e^n)}{d_+(e^n)} = \frac{1}{(15 + 7\sqrt{5})^3} \left[-20(15 + 7\sqrt{5}) + 20(7 + 3\sqrt{5})n \right. \\ \left. + (54 + 22\sqrt{5})n^2 \right] + O(n^3).$$

Taylor Expanding $y = \log(x)$ about $x = \frac{15}{2} + \frac{7\sqrt{5}}{2}$ and replacing $x = d_+(e^n)$ as in Equation (53) gives

$$(56) \quad \log(d_+(e^n)) = \frac{1}{5(15 + 7\sqrt{5})^2} \left[10(687 + 307\sqrt{5})n + 3(387 + 173\sqrt{5})n^2 \right] \\ - \log(2) + \log(15 + 7\sqrt{5}) + O(n^3).$$

Now, we can substitute Equation (56) into Equation (34), where $i_g = 4$, to get

$$(57) \quad \log[M] = \frac{20n(-237 - 106\sqrt{5} + (284 + 127\sqrt{5})r)}{50(123 + 55\sqrt{5})} \\ + \frac{n^2(-3357 - 1501\sqrt{5} + 13(123 + 55\sqrt{5})r)}{50(123 + 55\sqrt{5})} \\ + 3\log 2 - 4\log(5 + \sqrt{5}) + \log(15 + 7\sqrt{5}) \\ + r(\log(5 + \sqrt{5}) - \log 2) + O(n^3).$$

Substituting Equations (55) and (35) (with $i_g = 4$) into Equation (30), we get an equation for $1 + S$:

$$(58) \quad 1 + S = 1 - \frac{4^{r+4}5^{r+3}(5 + \sqrt{5})^{-2(3+r)}}{(15 + 7\sqrt{5})^3} \left[64600 + 28890\sqrt{5} \right. \\ \left. - 10(2889 + 1292\sqrt{5})n(r - 3) + n^2(40806 + 18249\sqrt{5}) \right]$$

$$\left. -6(6460 + 2889\sqrt{5})r + (6460 + 2889\sqrt{5})r^2 \right] + O(n^3).$$

Next, we can Taylor expand $y = \log(x)$ about

$$x = 1 - \frac{4^{r+4}5^{r+3}(5 + \sqrt{5})^{-2(3+r)}}{(15 + 7\sqrt{5})^3} (64600 + 28890\sqrt{5})$$

and replace $x = 1 + S$ as in Equation (58) to obtain

(59)

$$\begin{aligned} \log(1 + S) &= \frac{1}{(5 + \sqrt{5})^{12}(15 + 7\sqrt{5})^6} \\ &\quad \left(5.12 \cdot 10^{11} \left(-1292 + \frac{2889}{\sqrt{5}} \right) \right. \\ &\quad \left. (215668928180 + 96450076809\sqrt{5}) \right. \\ &\quad \left. \log \left[320000(5 + \sqrt{5})^{-2(r+3)}(6460 + 2889\sqrt{5}) \right] \right. \\ &\quad \left. + \log \left[-20^r + (5 + \sqrt{5})^{2r} \right] - \log \left[(15 + 7\sqrt{5})^3 \right] \right. \\ &\quad - \frac{2^{2r+18}5^{r+9} \left(-1292 + \frac{2889}{\sqrt{5}} \right)}{(5 + \sqrt{5})^{12}(15 + 7\sqrt{5})^6 (20^r - (5 + \sqrt{5})^{2r})} \\ &\quad - \frac{(96450076809 + 43133785636\sqrt{5})(r - 3)n}{(5 + \sqrt{5})^{12}(15 + 7\sqrt{5})^6 (20^r - (5 + \sqrt{5})^{2r})} \\ &\quad - \frac{3 \cdot 2^{2r+17}5^{r+8} \left(-1292 + \frac{2889}{\sqrt{5}} \right) n^2}{(5 + \sqrt{5})^{12}(15 + 7\sqrt{5})^6 (20^r - (5 + \sqrt{5})^{2r})^2} \\ &\quad \left(2^{2r+1}5^r(96450076809 + 43133785636\sqrt{5}) \right. \\ &\quad \left. + (5 + \sqrt{5})^{2r}(454106630922 + 203082659155\sqrt{5}) \right. \\ &\quad \left. - 2(5 + \sqrt{5})^{2r}(215668928180 + 96450076809\sqrt{5})r \right) + O(n^3). \end{aligned}$$

Given Equations (57) and (59), $n = t/\sigma_r$, μ_r and σ_r as in Proposition 11, and

$$g_r(1) = - \frac{5.12 \cdot 10^{11}(16692641 + 7465176\sqrt{5})}{(5 + \sqrt{5})^{12}(15 + 7\sqrt{5})^6}$$

$$\begin{aligned} & \left(r \log \left[\frac{2}{5 + \sqrt{5}} \right] + 4 \log [5 + \sqrt{5}] - \log [8(15 + 7\sqrt{5})] \right. \\ & - \log \left[320000(5 + \sqrt{5})^{-2(r+3)}(6460 + 2889\sqrt{5}) \right] \\ & \left. - \log \left[-20^r + (5 + \sqrt{5})^{2r} \right] + \log \left[(15 + 7\sqrt{5})^3 \right] \right), \end{aligned}$$

we find that

$$\begin{aligned} \log (M_{Y_r(t)}) &= \log [g_r(e^n)] - \log [g_r(1)] - \frac{t\mu_r}{\sigma_r} \\ &= k_0 + k_1 t + k_2 t^2 + O \left(\left(\frac{t}{\sigma_r} \right)^3 \right), \end{aligned}$$

where k_0 and k_1 simplify to 0 and

$$\begin{aligned} k_2 &= \frac{2.56 \cdot 10^{11}}{(5 + \sqrt{5})^{12}(15 + 7\sqrt{5})^6 (20^r - (5 + \sqrt{5})^{2r})^2} \\ & \left[23 \cdot 20^r + (3\sqrt{5} - 34)(5 + \sqrt{5})^{2r} - (5 - \sqrt{5})^{2r}(34 + 3\sqrt{5}) \right. \\ & \left. + \left(4 \cdot 20^r + 13 \left((5 - \sqrt{5})^{2r} + (5 + \sqrt{5})^{2r} \right) \right) r - 5 \cdot 20^r r^2 \right]^{-1} \\ & \left[389743431 \cdot 2^{6r+1} 5^{3r+1/2} + 1742985611 \cdot 8000^r \right. \\ & - 339763717 \cdot 2^{4r+1} \left((25 - 5\sqrt{5})^{2r} + (5(5 + \sqrt{5}))^{2r} \right) \\ & - 303893907 \cdot 5^{2r+1/2} 16^r \left((5 - \sqrt{5})^{2r} + (5 + \sqrt{5})^{2r} \right) \\ & + 383930743(5 + \sqrt{5})^{3r} \left(20^r (5 + \sqrt{5})^r - 2^{2r+1}(25 - 5\sqrt{5})^r \right) \\ & + 21462381 \cdot 5^{r+1/2} (5 + \sqrt{5})^{3r} 2^{(2r+3)} \left(2(5 + \sqrt{5})^r - 2(5 - \sqrt{5})^r \right) \\ & - 455572154(5 + \sqrt{5})^{4r} \left((5 - \sqrt{5})^{2r} + (5 + \sqrt{5})^{2r} \right) \\ & + 911144308(5 - \sqrt{5})^r (5 + \sqrt{5})^{5r} \\ & + 407476122\sqrt{5}(5 - \sqrt{5})^r (5 + \sqrt{5})^{5r} \\ & \left. - 203738061\sqrt{5}(5 + \sqrt{5})^{4r} \left((5 + \sqrt{5})^{2r} + (5 - \sqrt{5})^{2r} \right) \right] \\ & + r \left[-10264617 \cdot 4^{3r+2} 5^{3r+1/2} - 183619051 \cdot 2^{6r+1} 125^r \right. \\ & \left. + 12130911 \cdot 2^{4r+3} 5^{2r+1/2} (5 + \sqrt{5})^{2r} \right] \end{aligned}$$

$$\begin{aligned}
& + 217004333 \cdot \left(400^r \left((5 - \sqrt{5})^{2r} + (5 + \sqrt{5})^{2r} \right) + (5 + \sqrt{5})^{6r} \right) \\
& + 16692641 \cdot 2^{2r+2} 5^r (5 + \sqrt{5})^{3r} \left((5 + \sqrt{5})^r - 2(5 - \sqrt{5})^r \right) \\
& + 933147 \cdot 2^{2r+5} 5^{r+1/2} (5 + \sqrt{5})^{4r} + 97047288 \sqrt{5} (5 + \sqrt{5})^{6r} \\
& - 2(5 - \sqrt{5})^r (5 + \sqrt{5})^{3r} \left(9331472^{2r+5} 5^{r+1/2} \right. \\
& \left. + 13(5 + \sqrt{5})^{2r} (16692641 + 7465176\sqrt{5}) \right) \\
& + 13(5 - \sqrt{5})^{2r} \left(933147 \cdot 2^{4r+3} 5^{2r+1/2} \right. \\
& \left. + (5 + \sqrt{5})^{4r} (16692641 + 7465176\sqrt{5}) \right) \Big].
\end{aligned}$$

Finally, note $\lim_{r \rightarrow \infty} \log(M_{Y'_r(t)}) = \frac{1}{2}t^2$. Thus, Y'_r converges to the standard normal distribution as $r \rightarrow \infty$. \square

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