Journal of Combinatorics
Volume 13, Number 2, 289–302, 2022

A Murnaghan Nakayama rule for schur functions in superspace

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The classical Murnaghan-Nakayama rule gives a combinatorial rule to find the coefficients $d^{(r)}_{\mu,\lambda}$ that arise in the expansion

$$p_r s_\mu = \sum_\lambda d^{(r)}_{\mu,\lambda} s_\lambda$$

Specifically, it states that

$$p_r s_\mu = \sum_{\substack{\mu\subseteq\lambda \\ \lambda/\mu\in RH(\lambda,r)}} sgn(\lambda/\mu) s_\lambda$$

where $RH(\lambda, r)$ is the set of all rim hooks of λ with r cells. The sign $sgn(\lambda/\mu)$ of a rim hook λ/μ is $(-1)^{\mathrm{row}(\lambda/\mu)-1}$ where $\mathrm{row}(\lambda/\mu)$ is the number of rows in λ/μ (see [5]).

In [6], (later cited in [5]), Remmel and Tiefenbruck combinatorially prove the classical Murnaghan-Nakayama rule using only the classical Pieri rules (rules that describe the product of an elementary symmetric function and complete symmetric function with a Schur function.) Generalized Demazure atoms are a generalization of symmetric functions introduced by Haglund, Mason and Remmel. Equipped with only the Pieri rules for generalized Demazure atoms, Remmel and Tiefenbruck prove a Murnaghan-Nakayama-type rule for Demazure atoms (using the same basic idea for the classical proof.) The author and Lapointe in [4] have defined Pieri rules for a different generalization of symmetric functions called *symmetric functions in superspace.* In this paper, we will also use the same basic idea in [6] to prove a Murnaghan-Nakayama-type rule for symmetric functions in superspace.

In the article [4], the Pieri rules for two particular generalizations of Schur functions in superspace were described and proved. These two Schur functions in superspace are called s_Λ and s^*_Λ. The first, s_Λ, is the limit of the Macdonald polynomial in superspace where $q, t \to 0$ and where Λ is a superpartition. The second, s^*_Λ, is the dual to s_Λ according to the scalar product $\langle\langle \_, \_ \rangle\rangle$ (see (6)).

The Murnaghan-Nakayama rule in superspace will give combinatorial interpretations of the coefficients

$$p_r s^*_\Lambda = \sum_\Omega d^{*(r)}_{\Lambda,\Omega} s^*_\Omega$$

where Λ and Ω are superpartitions.

In this paper, we will only consider the superstar Schur functions and leave the other rule for a later write-up.

1. Symmetric polynomials in superspace

A polynomial in superspace is a polynomial in the usual N commuting variables $z_1, \ldots, z_N$ and the N anticommuting variables $\theta_1, \ldots, \theta_N$ over a certain field, which will be taken to be $\mathbb{Q}$. A superpolynomial $P(z, \theta)$ is symmetric if the following is satisfied:

(1)

$$P(z_1, \ldots, z_N, \theta_1, \ldots, \theta_N) = P(z_{\sigma(1)}, \ldots, z_{\sigma(N)}, \theta_{\sigma(1)}, \ldots, \theta_{\sigma(N)}) \qquad \forall \sigma \in S_N$$

where S_N is the symmetric group on $\{1, \ldots, N\}$. The ring of polynomials in superspace has a natural grading with respect to the fermionic degree m (the total degree in the anticommuting variables.) We will denote Λ^m_N the ring of symmetric polynomials in superspace of fermionic degree m over the field $\mathbb{Q}$.

1.1. Superpartitions

A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a sequence of non-negative weakly decreasing integers called "parts" such that $0 = \lambda_k = \lambda_{k+1} = \ldots$ for some positive integer k. If λ_ℓ is the last non-zero part then often we will write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. We say that if $\sum_i \lambda_i = n$ that $|\lambda| = n$ or $\lambda \vdash n$ "λ partitions n". Each partition λ has an associated Young diagram with λ_i "cells" in the i^{th} row, from top to bottom. (We will use the English convention for drawing Young diagrams.) The cell (i, j) is in the i^{th} row and j^{th} column of the diagram. We say that the diagram μ is contained in λ, denoted $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i. We say that λ/μ is a horizontal (resp. vertical) $n - strip$ or skew row (resp. skew column) of size n if $\mu \subseteq \lambda$, $|\lambda| - |\mu| = n$, and the skew diagram λ/μ does not have two cells in the same column (resp. row.).

In the classical theory, a skew shape λ/μ is a *rim hook* of λ if it contains no 2×2 subdiagram and any two consecutive cells are connected by an edge. A skew shape λ/μ is a *broken rim hook* of λ if it is a union of rim hooks.

Symmetric polynomials in superspace are naturally indexed by superpartitions. A superpartition Λ of degree $(n|m)$ is a pair $(\Lambda^\circ, \Lambda^*)$ of partitions Λ°, Λ^* such that:

- $\Lambda^* \subseteq \Lambda^\circ$
- $\Lambda^* \vdash n$
- Λ°/Λ^* is both a horizontal and a vertical m-strip.

Notice that if $\Lambda^\circ = \Lambda^* = \lambda$ then $\Lambda = (\lambda, \lambda)$ can be interpreted as an ordinary partition λ.

There is another way to write superpartitions that we will be using in this paper. We say that $\Lambda = (\Lambda^a, \Lambda^s)$ where Λ^a is an ordinary partition with m distinct parts (one of them possibly 0), and Λ^s is an ordinary partition (with possibly a string of zeros at the end) with $n - m$ parts. We will write $\Lambda^a = (\Lambda_1, \Lambda_2, \ldots, \Lambda_m)$ and $\Lambda^s = (\Lambda_{m+1}, \Lambda_{m+2}, \ldots, \Lambda_n)$. The correspondence between $(\Lambda^\circ, \Lambda^*)$ and (Λ^a, Λ^s) is given as follows: given $(\Lambda^\circ, \Lambda^*)$, the parts of Λ^a correspond to the parts of Λ° such that $\Lambda^\circ \neq \Lambda^*$, while the parts of Λ^s correspond to the parts of Λ° such that $\Lambda^\circ = \Lambda^*$.

We will draw the super-Young diagram of Λ to have squares in each cell of the Young diagram of Λ^* and circles for each cell of Λ°/Λ^*. When we describe the Pieri rules, the description of the superpartitions will be manipulations of these super-Young diagrams.

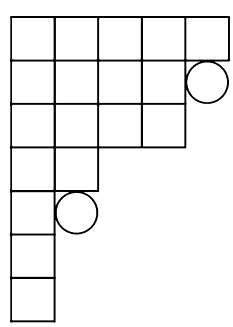


Figure 1: The super-Young diagram of Λ with $\Lambda^a = (4, 1), \Lambda^s = (5, 4, 2, 1, 1)$, $\Lambda^\circ = (5, 5, 4, 2, 2, 1, 1), \Lambda = (5, 4, 4, 2, 1, 1, 1)$.

1.2. Bases of symmetric polynomials in superspace

See [2].

1. The extension of the monomial symmetric functions m_Λ, defined by

$$m_\Lambda = {\sum_{\sigma \in S_N}}' \theta_{\sigma(1)} \dots \theta_{\sigma(m)} x_{\sigma(1)}^{\Lambda_1} \dots x_{\sigma(N)}^{\Lambda_N}, \tag{2}$$

where the sum is over the permutations of $\{1, \dots, N\}$ that produce distinct terms, and where the entries of $(\Lambda_1, \dots, \Lambda_N)$ are those of $\Lambda = (\Lambda^a; \Lambda^s) = (\Lambda_1, \dots, \Lambda_m; \Lambda_{m+1}, \dots, \Lambda_N)$;
2. the generalization of the power-sum symmetric functions $p_\Lambda = \tilde{p}_{\Lambda_1} \dots \tilde{p}_{\Lambda_m} p_{\Lambda_{m+1}} \dots p_{\Lambda_\ell}$,

$$\text{where } \tilde{p}_k = \sum_{i=1}^{N} \theta_i x_i^k \text{ and } p_r = \sum_{i=1}^{N} x_i^r, \text{ for } k \geq 0, r \geq 1; \tag{3}$$

(Comment to the reader: this paper will study the Murnaghan-Nakayama rule using p_r. The methods used in this paper do not seem to be compatible with $\tilde{p}_r$. I will be looking into it as future research. There is, however, an alternative definition of $\tilde{p}_r$ that is compatible with the methods used in this paper. This is another area of future research.)
3. the generalization of the elementary symmetric functions $e_\Lambda = \tilde{e}_{\Lambda_1} \dots \tilde{e}_{\Lambda_m} e_{\Lambda_{m+1}} \dots e_{\Lambda_\ell}$,

$$\text{where } \tilde{e}_k = m_{(0;1^k)} \text{ and } e_r = m_{(\emptyset;1^r)}, \text{ for } k \geq 0, r \geq 1; \tag{4}$$

4. the generalization of the homogeneous symmetric functions $h_\Lambda = \tilde{h}_{\Lambda_1} \dots \tilde{h}_{\Lambda_m} h_{\Lambda_{m+1}} \dots h_{\Lambda_\ell}$,

$$\text{where } \tilde{h}_k = \sum_{\Lambda \vdash (n|1)} (\Lambda_1 + 1) m_\Lambda \text{ and } h_r = \sum_{\Lambda \vdash (n|0)} m_\Lambda, \text{ for } k \geq 0, r \geq 1; \tag{5}$$

The relevant scalar product in this article is

$$\langle\langle p_\Lambda, p_\Omega \rangle\rangle = \delta_{\Lambda\Omega} z_{\Lambda^s} \tag{6}$$

where, as usual, $z_\lambda = 1^{n_\lambda(1)} n_\lambda(1)! 2^{n_\lambda(2)} n_\lambda(2)! \dots$ with $n_\lambda(i)$ is the number of parts of λ equal to i.

There are a few nice generalizations of Schur functions in superspace. In [2], s_Λ is defined as the special limit $q = t = 0$ of the Macdonald polynomials in superspace and s^*_Λ is the dual to s_Λ with respect to the scalar product (6), i.e., $\langle\langle s_\Lambda, s^*_\Omega \rangle\rangle = \delta_{\Lambda\Omega}$. I will refer to s_Λ as a super Schur function and s^*_Λ as a superstar Schur function. This paper will only prove the Murnaghan-Nakayama rule for s^*_Λ. The rule for s_Λ will be presented in a forthcoming paper using similar techniques as this paper.

(Comment to the reader: the Murnaghan-Nakayama rule for s_Λ can also be described using broken rim hooks in a similar way.)

2. Pieri rules

See [4] for Theorem 1 and see [1] for Theorem 2.

Theorem 1.[1] *Let Λ be a superpartition of fermionic degree m. Then for $r \geq 1$, we have*

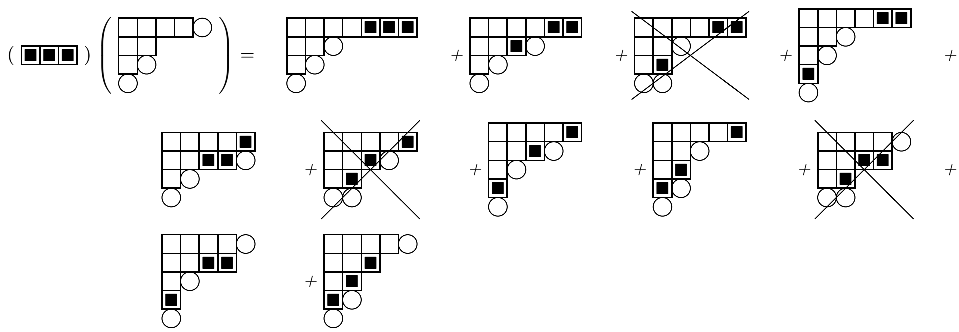
$$h_r s^*_\Lambda = \sum_\Omega s^*_\Omega \tag{7}$$

where the sum is over all superpartitions Ω of fermionic degree m obtained in the following way:

- *Select any partition λ such that λ/Λ^* is a horizontal r-strip. (This may or may not lead to a valid superpartition Ω.)*
- *Start with a super-Young diagram of Λ and add the cells of λ/Λ^* one by one to Λ from the left to the right.*
- *If a cell is added to an empty position of Λ then continue.*
- *If a cell is added to a position of Λ that has a circle, then move the circle down one row (towards the bottom of the diagram.) If at this point the circle is side-by-side with another circle, then this is not a valid superpartition and discard this construction. Otherwise, move on.*

[1]This is not exactly the way the theorem is stated in [4] but it is an equivalent construction.

We illustrate the rules by giving the expansion of $h_3 s^*_{(4,1,0),(2)}$.



$$h_3 s^*_{(4,1,0)(2)} = s^*_{(2,1,0)(7)} + s^*_{(3,1,0)(6)} + s^*_{(2,1,0)(6,1)} + s^*_{(4,1,0)(5)} + s^*_{(3,1,0)(5,1)} \\ + s^*_{(2,1,0)(5,2)} + s^*_{(4,1,0)(4,1)} + s^*_{(4,1,0)(3,2)}$$

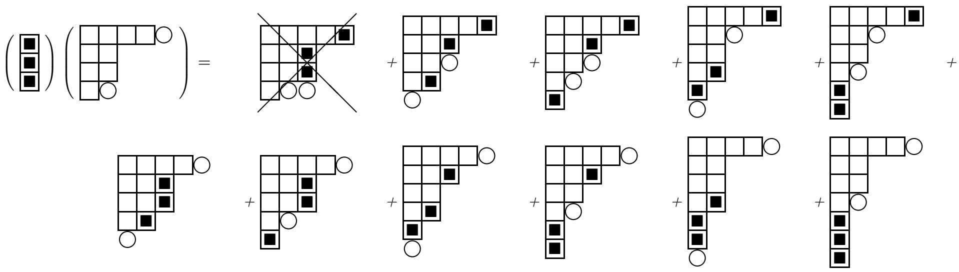
Theorem 2 ([1]). *Let* Λ *be a superpartition of fermionic degree* m. *Then for* $r \geq 1$, *we have*

$$e_r s^*_\Lambda = \sum_\Omega s^*_\Omega \tag{8}$$

where the sum is over all superpartitions Ω *of fermionic degree* m *obtained in the following way:*

- *Select any partition* λ *such that* λ/Λ^* *is a vertical* r*-strip. (This may or may not lead to a valid superpartition* Ω*.)*
- *Start with a super-Young diagram of* Λ *and add the cells of* λ/Λ^* *one by one to* Λ *from the top to bottom.*
- *If a cell is added to an empty position of* Λ *then continue.*
- *If a cell is added to a position of* Λ *that has a circle, then move the circle down one row (towards the bottom of the diagram.) If at this point the circle is side-by-side with another circle, then this is not a valid superpartition and discard this construction. Otherwise, move on.*

We illustrate the rules by giving the expansion of $e_3 s^*_{(4,1),(2,2)}$.



$$e_3 s^*_{(4,1)(2,2)} = s^*_{(2,1)(5,3,2)} + s^*_{(2,1)(5,3,1)} + s^*_{(2,0)(5,2,2,1)} + s^*_{(2,1)(5,2,1,1)} + s^*_{(4,0)(3,3,2)} + s^*_{(4,1)(3,3,1)} + s^*_{(4,0)(3,2,2,1)} + s^*_{(4,1)(3,2,1,1)} + s^*_{(4,0)(2,2,2,1,1)} + s^*_{(4,1)(2,2,1,1,1)}$$

3. Murnaghan-Nakayama rule for s^*_Λ

The goal of this section is to express the product $p_r s^*_\Lambda$ in terms of s^*_Ω. In particular, we will describe the coefficients $d^{*(r)}_{\Lambda,\Omega}$ in the expansion

$$p_r s^*_\Lambda = \sum_\Omega d^{*(r)}_{\Lambda,\Omega} s^*_\Omega. \tag{9}$$

Theorem 3.

$$p_r s^*_\Lambda = \sum_\Omega \overline{sgn}(\Omega^*/\Lambda^*) s^*_\Omega. \tag{10}$$

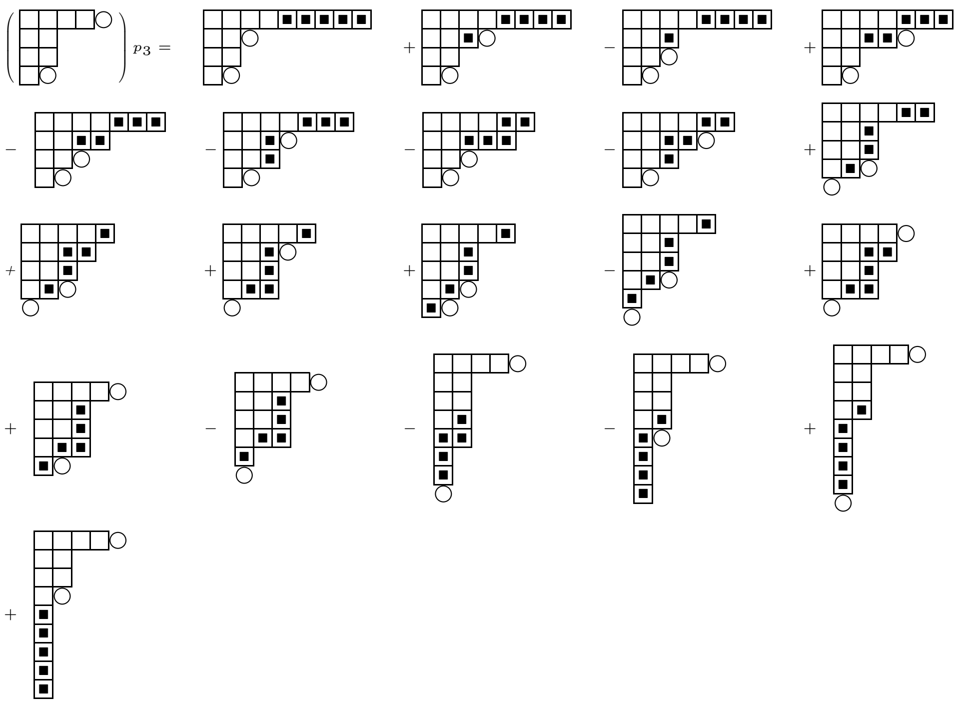
where the sum is over all superpartitions Ω *such that* Ω^*/Λ^* *is a broken rim hook with the following properties.*

- *Let* p *and* q *be the top-most and bottom-most row of* Ω^*/Λ^**, respectively. Then there is at least one cell from* Ω^*/Λ^* *in each row between* p *and* q*.*
- *Let* $\ell_1 < \cdots < \ell_m$ *be the rows with circles in* Λ *and* $d_1 < \cdots < d_m$ *be the rows with circles in* Ω*. The circle in row* d_i *in* Ω *can be interpreted as the result of moving the circle in row* ℓ_i *of* Λ*. Since circles can only move down,* $\ell_i \le d_i$ *for all* i*.*
- *If* $\ell_i \ne d_i$*, then the circle must have moved. The circle originally in row* ℓ_i *cannot pass the next circle down (*ℓ_{i+1}*), so* $d_i \le \ell_{i+1}$*.*
- *Let* i_p *be the minimum index such that* $p \le \ell_{i_p}$ *(if it exists) and* i_q *be the maximum index such that* $q + 1 \ge d_{i_q}$ *(if it exists.) See Figure 4.*

(In other words, i_p is the index of the top-most circle that was moved and i_q is the index of the bottom-most circle that has moved.)

- *We can partition the interval $[p, q]$ into the sub-intervals outside the paths of circles: $[p, \ell_{i_p}], [d_i, \ell_{i+1}]$ for $i_p \leq i \leq i_q$, and $\left[d_{i_q}, q\right]$, and sub-intervals on the paths of circles: (ℓ_i, d_i) for $i_p \leq i \leq i_q$. There are single rim hooks in each of the sub-intervals outside the paths of circles and broken rim hooks in each sub-interval on the paths of circles.*
- *The sign $\overline{sgn}(\Omega^*/\Lambda^*)$ is $(-1)^{q-p+(\#\ \text{of circles between } p \text{ and } q \text{ in } \Omega)}$.*

We illustrate the rules by giving the expansion of $p_5 s^*_{(4,1),(2,2)}$.



$$\begin{aligned}p_5 s^*_{(4,1)(2,2)} &= s^*_{(2,1)(9,2)} + s^*_{(3,0)(8,2)} - s^*_{(2,1)(8,3)} + s^*_{(4,1)(7,2)}\\ &- s^*_{(2,1)(7,4)} - s^*_{(3,1)(7,3)} - s^*_{(2,1)(6,5)} - s^*_{(4,1)(6,3)} + s^*_{(2,0)(6,3,3)}\\ &+ s^*_{(2,0)(5,4,3)} + s^*_{(3,0)(5,3,3)} + s^*_{(2,1)(5,3,3)} - s^*_{(2,0)(5,3,3,1)} + s^*_{(4,0)(4,3,3)}\\ &+ s^*_{(4,1)(3,3,3)} - s^*_{(4,0)(3,3,3,1)} - s^*_{(4,0)(2,2,2,2,1,1)} - s^*_{(4,1)(2,2,2,1,1,1)}\\ &+ s^*_{(4,0)(2,2,2,1,1,1,1)} + s^*_{(4,1)(2,2,1,1,1,1,1)}\end{aligned}$$

(Notes about the example above: Notice the two terms: $-s^*_{(2,1)(7,4)}$ and $-s^*_{(2,1)(6,5)}$ and how they both moved the top circle from row 1 down to row 3,

the former using a broken rim hook with two parts and the latter with a single rim hook (which can be interpreted as a broken rim hook with one part.) Also notice the similarities between $-s^*_{(4,1)(2,2,2,1,1,1)}$ *and* $s^*_{(4,0)(2,2,2,1,1,1,1)}$ *where you can have the same broken rim hook that moves the circle to different places.)*

The proof will be in two steps. First we will describe the set of shapes Γ resulting from the product:

$$s_{(r-k,1^k)} s^*_\Lambda = \sum_\Gamma sgn(\Gamma^*/\Lambda^*) s^*_\Gamma, \tag{11}$$

then we will use the well-known identity:

$$p_r = \sum_{k=0}^{r-1} (-1)^k s_{(r-k,1^k)}. \tag{12}$$

to describe the set of shapes Ω resulting from the product:

$$p_r s^*_\Lambda = \sum_\Omega \overline{sgn}(\Omega^*/\Lambda^*) s^*_\Omega. \tag{13}$$

3.1. The product $s_{(r-k,1^k)} s^*_\Lambda$

In this section we will give a combinatorial description of the coefficients that appear in the superstar Schur function expansion of the product

$$s_{(r-k,1^k)} s^*_\Lambda. \tag{14}$$

It is shown in [6] that

$$s_{(r-k,1^k)} = \sum_{i=0}^{k} (-1)^{k-i} e_i h_{r-i}. \tag{15}$$

Theorems 1 and 2 describe the Pieri rules for $h_r s^*_\Lambda$ and $e_r s^*_\Lambda$. Recall that the Pieri rules for $h_{r-i} s^*_\Lambda$ result in adding a skew row of size $r-i$ adding the strip cell by cell from the left to the right and shifting the circles appropriately for each new cell and the Pieri rules for $e_i s^*_\Lambda$ result in adding a skew column of size i adding the strip cell by cell from the top to the bottom and shifting the circles appropriately for each new cell.

So, we can interpret $e_i h_{r-i} s^*_\Lambda$ as

$$e_i h_{r-i} s^*_\Lambda = \sum_\Gamma s^*_\Gamma \tag{16}$$

where the sum is over all superpartitions Γ such that Γ comes from Λ by first adding a skew row of size $r-i$ onto Λ according to the Pieri rules to get a shape Θ and then adding a skew column of size i onto Θ according to the Pieri rules to get Γ. If we put h's (for horizontal) in the cells of Θ^*/Λ^* and v's (for vertical) in the cells of Γ^*/Θ^*, then we would get a diagram D like the one in Figure 2. (We will call the cells in D with h's and v's "marked" cells.)

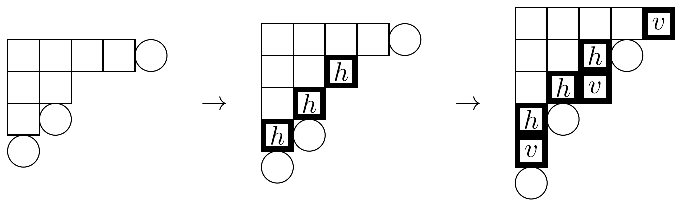


Figure 2: One possible term in the expansion of $e_3 h_3 s_{(4,1,0)(2)}$.

It is easy to see that Γ^*/Λ^* is a broken rim hook. (Refer to [5].)

Each circle has either moved down or stayed in its row. Furthermore, by the Pieri rules, each circle cannot move past the original row of the next circle down. Therefore, we can label the rows of the circles in Λ as $\ell_1, \ell_2, \ldots, \ell_m$ and the rows of the circles in D as $d_1, d_2, \ldots, d_m$. By the argument above, $\ell_1 \leq d_1 \leq \ell_2 \leq d_2 \leq \cdots \leq \ell_m \leq d_m$. We get equality $\ell_i = d_i$ when the i^{th} circle has not moved and $d_i = \ell_{i+1}$ when the $(i+1)^{st}$ circle has moved down by an h and the i^{th} circle has moved into its (now) unoccupied row. Notice that for a circle to move one row, i.e. $d_i - \ell_i = 1$ for some i, it could have been moved by an h or a v. Although, if a circle moves more than one row, then there must have been a v for each row it moved after the first. This brings us to a main property of all of these diagrams: if $d_i - \ell_i \geq 2$ for some i, then every row j must contain a v if j is between ℓ_i and d_i, $(\ell_i < j < d_i)$.

It follows from (15) that

$$s_{(r-k,1^k)} s^*_\Lambda = \left(\sum_{i=0}^{k} (-1)^{k-i} e_i h_{r-i} \right) s^*_\Lambda. \tag{17}$$

Let's say that $\mathcal{D}_{r,k,\Lambda}$ is the set of all diagrams D obtained in the way described above from a superpartition Λ such that the number of v's in D

is less than or equal to k and the number of v's plus the number of h's is equal to r. The sum above can be interpreted as

$$s_{(r-k,1^k)} s^*_\Lambda = \sum_{D \in \mathcal{D}_{r,k,\Lambda}} s^*_{sh(D)}. \tag{18}$$

Here, $sh(D)$ is the shape of D and the sign of D, $sgn(D)$, is equal to $(-1)^{k-v(D)}$ where $v(D)$ is the number of v's. We will give a sign-reversing involution to give a combinatorial interpretation to the right side of the equation.

Given a diagram D, let c be the top right-most marked cell in D. Notice that c can be filled with either h or v. If c contains v then define $J(D)$ to be D with the entry in c changed to h. If c contains h and if D contains fewer than k v's then define $J(D)$ to be D with the entry in c changed to v. Otherwise, if c contains an h and D contains exactly k v's then $J(D) = D$.

It remains to be shown that $\mathcal{D}_{r,k,\Lambda}$ is closed under this function J. The Pieri rules for h are described as adding a new cell one by one from the left to the right so if the top-rightmost cell was h then it was the last h to be added. Conversely, the v cells are added from the top to the bottom so it the top-rightmost cell was a v then it was the first v to be added. Adding a single h cell to the top row has the same effect as adding a single v cell to the top row. This follows immediately from the fact that $h_1 = e_1$.

Now, it should be easy to see that J is a weight-preserving, sign-reversing involution, i.e. $J(J(D)) = D$ and if $J(D) \neq D$ then $sgn(J(D)) = -sgn(D)$. The fixed points of J are those diagrams D such that D has exactly k v's (and so has a sign of $+1$) and has a h in the top-rightmost cell. So now we can interpret the sum as the sum over all fixed points,

$$s_{(r-k,1^k)} s^*_\Lambda = \sum_{J(D)=D} sgn(D) s^*_{sh(D)}. \tag{19}$$

The fixed points of J: $\mathcal{D}_{r,k,\Lambda} \to \mathcal{D}_{r,k,\Lambda}$ are just those shapes D with exactly $r-k$ h's, k v's, such that the top right-most marked cell is an h, the marked cells form a broken rim hook (or possibly a single rim hook), and there is a v in each row strictly between ℓ_i and d_i if $d_i - \ell_i \geq 2$.

3.2. The product $p_r s^*_\Lambda$

Now we can use the identity (12) to get the following interpretation of $p_r s^*_\Lambda$:

$$p_r s^*_\Lambda = \sum_D \overline{sgn}(D) s^*_{sh(D)} \tag{20}$$

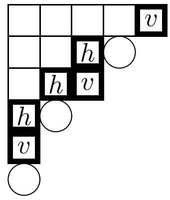


Figure 3: An element D of $\mathcal{D}_{6,4,(4,1,0)(1)}$. $sgn(D) = (-1)^{(4-3)}$, $sh(D) = (3,1,0)(5,3,1)$.

where the sum is over all diagrams D such that $D \in \mathcal{D}_{r,k,\Lambda}$ for some k, $0 \leq k < r$ and $J(D) = D$. The sign of D, $\overline{sgn}(D)$ is just $(-1)^{\#\text{ of }v\text{'s in }D}$.

Now we will define a second involution K. For a diagram D, just like before, let p be the first row that has a marked cell, let q be the last row with a marked cell, let ℓ_i be the row of the i^{th} circle of Λ and let d_i be the i^{th} circle of D. We will break the set of diagrams into two cases:

Case A: There is a broken rim hook between rows d_i and ℓ_{i+1} for some i, $1 \leq i < m$, before row ℓ_1 or after row d_m.

Let t be the first such row, i.e., t is the first row that is in between some d_i and ℓ_{i+1}, i.e., $d_i < t \leq \ell_{i+1}$ or possibley after row d_m such that the left-most cell of t is not directly touching sides with the left-most cell of row $t-1$. (Notice that t is outside the path of the circles.)

If the left-most cell in row t is an h then $K(D)$ will be the resulting diagram by changing it to a v and if the left-most cell in row t is a v then $K(D)$ will be the resulting diagram by changing it to an h.

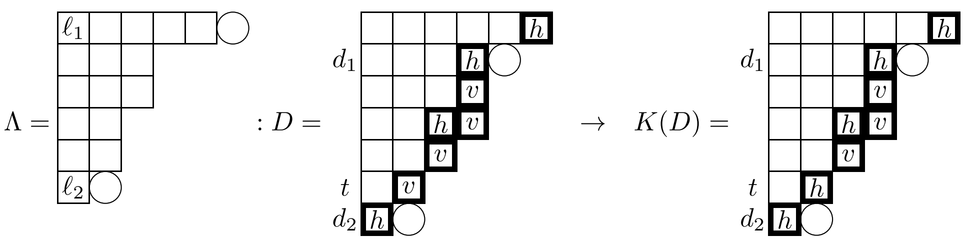


Figure 4: A superpartition Λ, a diagram D with $\overline{sgn}(D) = (-1)^4$ and $K(D)$ resulting from changing v to h in row t. Notice that $d_1 < t \leq \ell_2$.

Case B: There are no broken rim hooks between rows d_i and ℓ_{i+1}, before row ℓ_1 or after row d_m. (In other words, there are no broken rim hooks outside the paths of the cirles.)

In this case, $K(D) = D$.

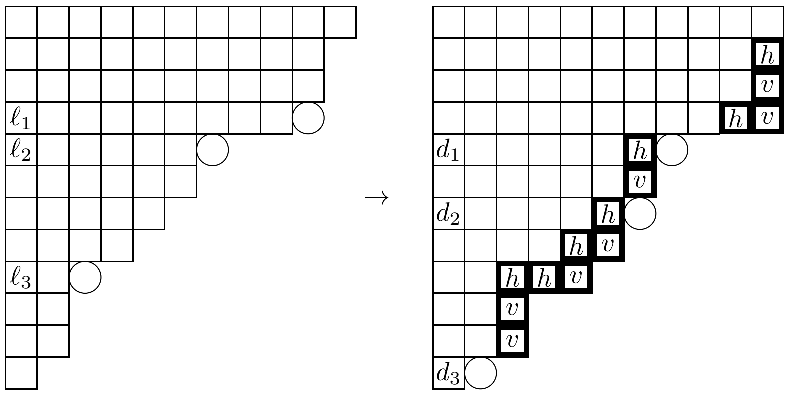


Figure 5: A fixed point of K where $\Lambda = (9,6,2)(11,10,10,6,5,4,2,2,1)$, $sh(D) = (7,6,1)(11,11,11,11,7,6,5,3,3)$, $\ell_1 = 4, \ell_2 = 5, \ell_3 = 9, d_1 = 5, d_2 = 7, d_3 = 12$, $p = 2, i_p = 1, q = 11, i_q = 2$ $\overline{sgn}(D) = (-1)^{11-2+2} = -1$.

It remains to be shown that K truly is an involution. Consider we have a diagram in Case A. Then the left-most cell of row t could be an g or a v since there is no cell to its right or above it. Furthermore, $t \notin (\ell_i, d_i]$ for any i. If $t \neq \ell_i$ for all i, then the left-most cell of t is not involved with moving any circles and by the same argument as with the involution J, since the h's are added in left to right and the v's are added in top to bottom, that cell could have been filled with either an h or a v. On the other hand, if $t = \ell_i$ for some i then the left-most cell of t moved the i^{th} circle down one row. This could have been accomplished with an h or a v. If this cell was an h then it moved the circle down one row and maybe later on, some v's moved it down more rows. If this cell was a v then it would be the first cell to move the circle down and maybe afterward, some more v's moved it down more rows since the v's are added from the top to the bottom.

Now, it should be easy to see that K is a weight-preserving, sign-reversing involution, i.e., $K(K(D)) = D$ and if $K(D) \neq D$ then $\overline{sgn}(K(D)) = -\overline{sgn}(D)$. In order to describe the fixed points of K, define i_p to be the minimum index such that $p \leq \ell_{i_p}$ and i_q to be the maximum index such that $q \geq d_{i_q}$. The fixed points are precisely those diagrams D that have at least one marked cell between rows p and q and consist of

- single rim hooks in the row intervals $[p, \ell_{i_p}], [d_i, \ell_{i+1}]$ for $i_p \leq i \leq i_q$, and $[d_{i_q}, q]$ (or possibly $[p, q]$ if either i_p or i_q doesn't exist) and
- broken rim hooks in the row intervals $[\ell_i, d_i]$ for $i_p \leq i_q$ (or nowhere if either i_p or i_q doesn't exist.)

- if $d_i > \ell_i$ the left-most cell of row ℓ_i is an h and all other rows in $(\ell_i, d_i]$ (if there are any) must contain a v.

The sign of D, $\overline{sgn}(D)$ is the number of v's.

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Received April 21, 2018