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Diameters of graphs on reduced words of 12 and 21-inflations

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It is a classical result that any permutation in the symmetric group can be generated by a sequence of adjacent transpositions. The sequences of minimal length are called reduced words, and in this paper we study the graphs of these reduced words, with edges determined by relations in the underlying Coxeter group. Recently, the diameter has been calculated for the longest permutation $n \ldots 21$ by Reiner and Roichman as well as Assaf. In this paper we find inductive formulas for the diameter of the graphs of 12-inflations and many 21-inflations. These results extend to the associated graphs on commutation and braid classes. Also, these results give a recursive formula for the diameter of the longest permutation, which matches that of Reiner, Roichman and Assaf. Lastly, we make progress on conjectured bounds of the diameter by Reiner and Roichman, which are based on the underlying hyperplane arrangement, and find families of permutations that achieve the upper bound and lower bound of the conjecture. In particular permutations that avoid 312 or 231 have graphs that achieve the upper bound.

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1. Introduction

The symmetric group $\mathfrak{S}_n$ of $[n] := \{1, 2, \ldots, n\}$ can be generated by adjacent transpositions $s_i = (i, i+1)$. The shortest sequences of adjacent transpositions that achieve $\pi \in \mathfrak{S}_n$ are called reduced words of π, the collection of which is denoted by $R(\pi)$. Tits [29] showed that one can transform any reduced word of π into any other by a sequence of

1. commutation moves, where you exchange adjacent s_i and s_j if $|i-j| > 1$, and

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2. braid moves, where you exchange adjacent sequences $s_j s_{j+1} s_j$ and $s_{j+1} s_j s_{j+1}$.

The graphs G_π formed by vertices $R(\pi)$ and edges associated to a single commutation or a single braid move, called a commutation or braid edge respectively, have been well studied [4, 11, 10, 13, 23, 24]. Tits showed that the graph is connected [29]. Stanley [20] enumerated $R(\pi)$ using symmetric functions. These symmetric functions have led to connections to Schubert calculus, Demazure characters and flag skew Schur functions [1, 3, 5, 16]. The associated graph C_π, which is G_π contracted along commutation edges, has additionally received a lot of attention. Elnitsky proved that the vertices of C_π are in bijection with rhombic tilings of certain polygons and that the graph is bipartite [12]. The graph C_π also has connections to geometric representation theory [8].

Interestingly, the theory of permutation pattern enumeration and avoidance has found itself highly useful in describing properties of $R(\pi)$ and its associated graphs G_π and C_π [9, 15, 25, 26, 28]. Vexillary permutations, those avoiding 2143, have $R(\pi)$ enumerated by the number of standard Young tableaux of a single shape [20], and permutations avoiding 321 only have commutation edges [5].

The longest permutation $\delta_n = n(n-1)\cdots 1$ has been particularly well studied. This is because $R(\pi)$ is closely connected to the weak Bruhat order and type A hyperplane arrangements. Reiner found that in G_{δ_n}, the expected number of braid edges connected to a single vertex is one [17]. The expected number of braid edges connected to vertices in a singular commutation class, those vertices connected by commutation edges, is also one [19]. Tenner studied the expected number of commutation edges, which is more complex [27].

The diameter of G_{δ_n} was first been determined by Reiner and Roichman [18], using hyperplane arrangements, and later by Assaf [2], using posets and Rothe diagrams. The diameter of C_{δ_n} has also been calculated [14, 30]. Our paper continues this direction and finds the diameters of G_π, C_π and B_π in several cases, where B_π is G_π contracted along all braid edges. Our project is motivated by the conjectured upper and lower bounds for the diameter of G_π by Reiner and Roichman [18]. While we haven't verified the conjectured bounds in all cases, we are able to prove these bounds in all cases we have a formula for the diameter of G_π.

Our paper is organized as follows. We first introduce preliminary topics in Section 2 where we also define the two families of permutations we are studying, 12-inflations and 21-inflations. In Sections 3 and 4 we describe a way to encode the vertices $R(\pi)$, so we can more easily construct paths in G_π. With these paths we find exact recursive formulas for the diameters of G_π, C_π and B_π in the case of 12-inflations in Theorem 3.6, and we find recursive upper and lower bounds for the diameters of graphs G_π, C_π and B_π in the case of certain 21-inflations in Theorem 4.11. Also, we prove several graph isomorphisms using symmetries of the square in Section 5.1. We are then able to leverage our inductive formulas to find the exact diameters of G_π, C_π and B_π in the case of permutations π that avoid 231 or 312 in Section 5. Finally, in Section 6, we will connect our results back to Reiner and Roichman's conjectured upper and lower bounds. In particular we prove that all permutations π that avoid 231 or 312 satisfy this conjecture, and that these permutations achieve the upper bound of the conjecture in Theorem 6.6 and Theorem 6.7. Additionally, we describe another infinite family of permutations that achieves the lower bound of the conjecture in Theorem 6.10. We end our paper with Section 7 where we consider further directions.

2. Preliminaries

In this section we will give background information. The definitions are mainly obtained from Björner-Brenti [6], Bollobás [7], and Stanley [21, 22].

Let $\mathfrak{S}_n, n \geq 0$ denote the symmetric group on $[n] := \{1, 2, \dots, n\}$. A permutation $\pi \in \mathfrak{S}_n$ permutes $[n]$ by mapping $i \mapsto \pi_i$. We write π in one-line notation $\pi = \pi_1\pi_2 \dots \pi_n$. For $\pi \in \mathfrak{S}_n$ we say that n is the *size* of π and denote it by $|\pi|$.

The symmetric group $\mathfrak{S}_n$ is generated by the set of simple reflections $S = \{s_1, s_2, \dots, s_{n-1}\}$, where s_i interchanges the positions i and $i+1$. Thus, for $\pi \in \mathfrak{S}_n$, a decomposition $\pi = s_{i_1}s_{i_2}\dots s_{i_k}$ with letters in S is called a *reduced decomposition* for π if k is minimal. The word $i_1i_2\dots i_k$ is called a *reduced word* for π. We say that k is the *length* of π and denote it by $\ell(\pi)$. For a permutation $\pi \in \mathfrak{S}_n$ we will use $R(\pi)$ for the set of reduced words for π. Note that the notation for reduced words from permutations are similar, so we will rely on context to distinguish reduced words from permutations. Two reduced words for $\pi \in \mathfrak{S}_n$ can be related by

1. *short braid moves* or *commutation moves* by switching adjacent jk if $|j-k| > 1$ or
2. *braid moves* by exchanging the occurrences of $j(j+1)j$ and $(j+1)j(j+1)$ on consecutive indices.

We say two reduced words for $\pi \in \mathfrak{S}_n$ are in the same *commutation class* if we can obtain one from another by applying a sequence of commutation moves. *Braid classes* of π are defined in terms of the braid moves.

An *inversion* in π is a pair (i, j) such that $i < j$ and $\pi_i > \pi_j$. Since the number of inversions in π is equal to its length $\ell(\pi)$, the longest permutation in $\mathfrak{S}_n$ is $\delta_n = n \dots 21 \in \mathfrak{S}_n$ and the shortest permutation is the identity $\iota_n = 12 \dots n \in \mathfrak{S}_n$.

Example 2.1. For $\pi = 3421 \in \mathfrak{S}_4$ we have $|\pi| = 4$, $\ell(\pi) = 5$ and $R(\pi) = \{12312, 12132, 21232, 23123, 21323\}$. We see that there are $\ell(\pi) = 5$ inversions in π, which are $(1,3), (1,4), (2,3), (2,4)$ and $(3,4)$.

Consider two sequences of integers $u = u_1u_2 \dots u_n$ and $v = v_1v_2 \dots v_n$. We say that u and v are *order isomorphic* if $u_i \leq u_j$ ($u_i \geq u_j$) if and only if $v_i \leq v_j$ ($v_i \geq v_j$). For example 463 and 231 are order isomorphic. We say a permutation π *contains* a pattern σ if π has a subsequence that is order isomorphic to σ. We say a permutation π *avoids* σ if π does not contain σ.

Example 2.2. The permutation $\pi = 142635$ contains a pattern 231 because π has the subsequence 463 and avoids 321 since there is no decreasing subsequence of length three.

Consider a permutation $\pi \in \mathfrak{S}_n$. A *block* of π is a consecutive sequence $\pi_a\pi_{a+1} \dots \pi_b$ whose union of values forms a consecutive interval of integers, $\{\pi_a, \pi_{a+1}, \dots, \pi_b\} = [c, d] = \{c, c+1, \dots, d\}$ for some integers $c \leq d$. Let $\sigma \in \mathfrak{S}_k$ and $\pi_{(1)}, \pi_{(2)}, \dots, \pi_{(k)}$ be permutations of possibly different non-negative lengths. The inflation of $\pi_{(1)}, \pi_{(2)}, \dots, \pi_{(k)}$ by σ, written $\sigma[\pi_{(1)}, \pi_{(2)}, \dots, \pi_{(k)}]$, is σ, but we replace σ_i with a block order isomorphic to $\pi_{(i)}$ for $1 \leq i \leq k$. See Figure 1 for an example.

Let $G = (V, E)$ be a simple, connected graph with vertex set V and edge set E. The *distance* $d_G(u, v)$ between $u, v \in V$ is the number of edges in the shortest path between u and v. The *diameter* $\operatorname{diam}(G)$ of the graph G is the maximum distance between any two vertices in G. A subgraph $G' = (V', E')$ is called an *induced subgraph* if G' contains all edges of G that join two vertices in V'.

For $\pi \in \mathfrak{S}_n$ we define G_π to be the graph on $R(\pi)$ where edges come from commutation and braid moves, which we will refer to as *commutation edges* and *braid edges* respectively. Denote C_π to be the graph G_π where we contract along commutation edges, so the only edges remaining are braid edges and the vertices are commutation classes. Let B_π be the graph where we contract G_π along braid edges, so the only remaining edges remaining are commutation edges and the vertices are braid classes.

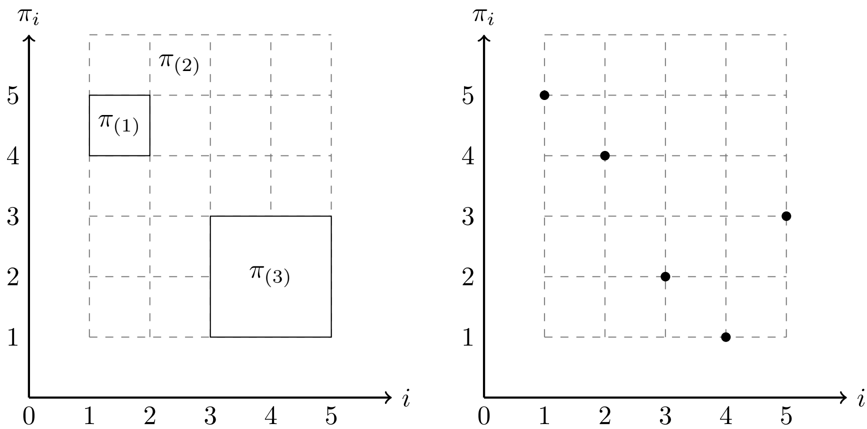

Figure 1: The inflation of $\pi_{(1)} = 21, \pi_{(2)} = \epsilon, \pi_{(3)} = 213$ by $\sigma = 231$ is $\pi = \sigma[\pi_{(1)}, \pi_{(2)}, \pi_{(3)}] = 54213$ where $\epsilon \in \mathfrak{S}_0$. The diagram is illustrated above.

Example 2.3. Let $\pi = 4231 \in \mathfrak{S}_4$. The graphs G_π, C_π, and B_π are in Figure 2. We see that $\operatorname{diam}(G_\pi) = 4, \operatorname{diam}(C_\pi) = 2$, and $\operatorname{diam}(B_\pi) = 2$.

Lemma 2.4. *Let $\pi \in \mathfrak{S}_n$.*

1. *If for any $r, r' \in R(\pi)$ we can find a path from r to r' with at most c commutation moves plus at most b braid moves, then diam$(G_\pi) \leq b+c$, diam$(C_\pi) \leq b$ and diam$(B_\pi) \leq c$.*
2. *If we can find a pair $r, r' \in R(\pi)$ where any path from r to r' has at least c commutation moves plus at least b braid moves, then diam$(G_\pi) \geq b + c$, diam$(C_\pi) \geq b$ and diam$(B_\pi) \geq c$.*

Proof. Suppose that for any $r, r' \in R(\pi)$ there exists a path from r to r' with at most c commutation moves plus at most b braid moves. Certainly the distance from r to r' is at most $b + c$. Because $d(r, r') \leq b + c$ for all $r, r' \in R(\pi)$, we know that $\operatorname{diam}(G_\pi) \leq b + c$. Now consider the graph C_π formed from G_π by contracting along all commutation edges. Let S and S' be two commutation classes in C_π and $v \in S$ and $v' \in S'$ be two vertices in G_π. There exists a path P from v to v' using at most c commutation moves and b braid moves. The associated path in C_π loses all commutation edges, because they are contracted, and keeps at most b braid edges so $d(S, S') \leq b$. Because $d(S, S') \leq b$ for all pairs of commutation classes we can conclude that $\operatorname{diam}(C_\pi) \leq b$. Similarly, we can conclude that $\operatorname{diam}(B_\pi) \leq c$.

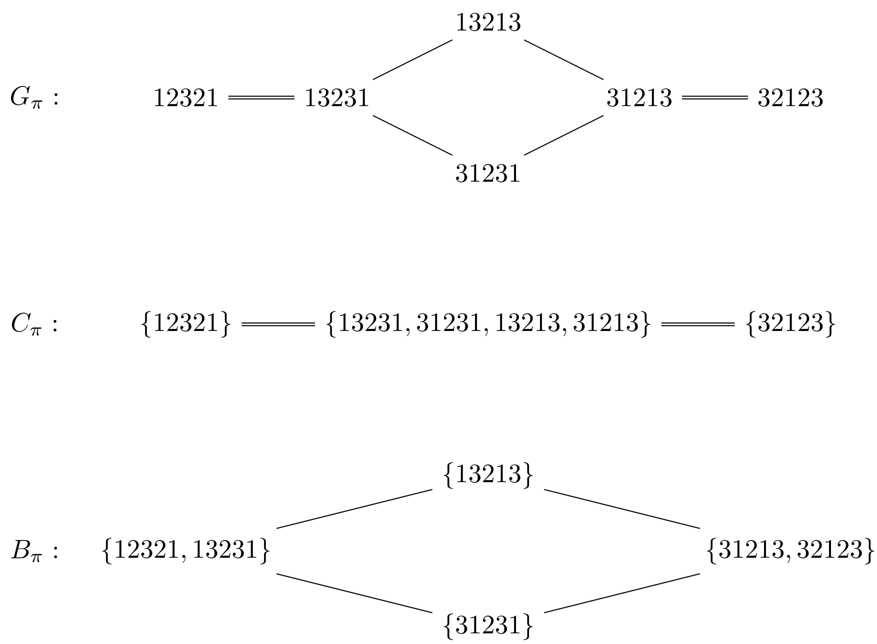

Figure 2: The graphs G_π, C_π, and B_π where $\pi = 4231 \in \mathfrak{S}_4$.

Suppose that there exists $r, r' \in R(\pi)$ so that any path from r to r' has at least c commutation moves plus at least b braid moves. Because $d(r, r') \geq b + c$ we can conclude that $\operatorname{diam}(G_\pi) \geq b + c$. Now consider C_π and the associated commutation equivalence classes $r \in S$ and $r' \in S'$. Suppose the contrary that $\operatorname{diam}(C_\pi) < b$. This means that there exists a path from S to S' that uses less than b braid steps. Lift this path to some path P in G_π from r to r'. Path P has at least b braid steps, so when we contract G_π to C_π at least one of these braid edges ends up contracted. Particularly, we then must have two $v, v' \in G_\pi$ connected by a braid edge e, where v and v' end up in the same commutation class. This means there exists a path Q from v to v' composed of only commutation moves. However, all cycles in G_π have an even number of braid moves and an even number of commutation moves [18]. The cycle Q together with e has an odd number of braid moves, which is a contradiction. Hence, $\operatorname{diam}(C_\pi) \geq b$ and similarly we can conclude that $\operatorname{diam}(B_\pi) \geq c$. $\square$

Let P be a poset. The *Hasse diagram* of a finite poset P is the graph whose vertices are the elements of P, whose edges are the cover relations, and such that if $x \lessdot y$ then y is drawn above x. We say P has a $\hat{1}$ if there exists an element $\hat{1} \in P$ such that $x \leq \hat{1}$ for all $x \in P$. Similarly, we say P has a $\hat{0}$ if there exists an element $\hat{0} \in P$ such that $\hat{0} \leq x$ for all $x \in P$. We call the elements $\hat{1}$ and $\hat{0}$, if they exist, the *maximum* and the *minimum* elements of P respectively. A subset $C = \{x_1, x_2, \ldots, x_n\}$ of P is called a *chain* if $x_1 < x_2 < \cdots < x_n$. A chain is called *maximal* if it is not contained in a longer chain of P. The *length* $\ell(C)$ of a finite chain is defined by $\ell(C) = |C| - 1$. If P is a poset such that all maximal chains have the same length n, then we say that P is *graded of rank* n. In this case there is a unique *rank function* $\rho : P \to \{0, 1, \ldots, n\}$ such that $\rho(x) = 0$ if x is a minimal element of P, and $\rho(y) = \rho(x) + 1$ if $x \lessdot y$ in P.

A *linear hyperplane* of $\mathbb{R}^d$ is a $(d-1)$-dimensional subspace $H = \{v \in \mathbb{R}^d : \alpha \cdot v = 0\}$ of $\mathbb{R}^d$ where $\alpha \in \mathbb{R}^d$ is a fixed nonzero vector and $\alpha \cdot v$ is the usual dot product. For an arrangement $\mathcal{A}$ of linear hyperplanes in $\mathbb{R}^d$, the *intersection poset* $L = L(\mathcal{A}) = \bigsqcup_{i=0}^{d} L_i$ is the graded poset of all nonempty intersections of hyperplanes, including $\mathbb{R}^d$ itself, ordered by reverse inclusion. For example, we observe that the minimum $\hat{0} = L_0 = \mathbb{R}^d$ element and the maximum $\hat{1} = L_d$ element. We also observe that $L_1 = \{H : H \in \mathcal{A}\}$ and $L_2 = \{H \cap K : H \neq K, H \in \mathcal{A}, K \in \mathcal{A}\}$ is the set of all codimension-two intersection subspaces, and so on. A *chamber* of an arrangement $\mathcal{A}$ is a connected component of the complement $X = \mathbb{R}^d - \bigcup_{H \in \mathcal{A}} H$ of the hyperplanes. For more details on these definitions related to hyperplane arrangements see [18].

Let $\mathcal{A}$ be the reflection arrangement in $\mathbb{R}^{n-1}$ of type A_{n-1}, which is associated with the symmetric group $\mathfrak{S}_n$. We identify the ambient space with the quotient of $\mathbb{R}^n = \{(x_1, \ldots, x_n) : x_i \in \mathbb{R}\}$ by the subspace spanned by $\{(1, 1, \ldots, 1)\}$. Then the hyperplanes in $\mathcal{A}$ are $H_{ij} := \{x_i = x_j\}$ for $1 \leq i < j \leq n$. Note that $\mathcal{A}$ is *central* and *essential*, meaning that $\bigcap_{H \in \mathcal{A}} H = \{\mathbf{0}\}$ where $\mathbf{0}$ denotes the origin of $\mathbb{R}^{n-1}$. The codimension-two intersection subspaces in L_2 are either $X_{ij,k\ell} := \{x_i = x_j, x_k = x_\ell\} = H_{ij} \cap H_{k\ell}$ or $X_{ijk} := \{x_i = x_j = x_k\} = H_{ij} \cap H_{jk}$. Let $\mathcal{C}$ be the set of chambers of $\mathcal{A}$. For two chambers c, c', define $L_1(c, c') := \{H \in L_1 : H \text{ separates } c \text{ from } c'\}$. We define a graph G_1 on $\mathcal{C}$ where two chambers $c, c' \in \mathcal{C}$ are joined if $|L_1(c, c')| = 1$. A *(reduced) gallery* from c to c' is a shortest path in the graph G_1. Fix a particular base chamber c_0 and let $\mathcal{R}$ denote the set of all galleries in G_1 from c_0 to $-c_0$. We define a graph G_2 on the set $\mathcal{R}$ where two galleries $r, r' \in \mathcal{R}$ are joined if they are separated by exactly one codimension-two intersection subspace X in L_2. For an intersection subspace X, a *localized arrangement* in $\mathbb{R}^d/X$ is $\mathcal{A}_X = \{H/X : H \in \mathcal{A}, X \subseteq H\}$, which can be identified with the interval $[\mathbb{R}^d, X]$ in the intersection poset L, i.e. $\{H : H \in \mathcal{A}, X \subseteq H\}$.

For $\pi \in \mathfrak{S}_n$, define $L_1(\pi) := L_1(c_0, \pi(c_0))$ and $L_2(\pi) := \{X \in L_2 : \mathcal{A}_X \subseteq L_1(\pi)\}$. We can interpret the set $L_1(\pi)$ as the usual (left) inversion set of π. We can also interpret $L_2(\pi)$ as the set $I_2(\pi) \cup I_3(\pi)$ where $I_2(\pi)$ is the set of all disjoint pairs of inversions $((i,j),(k,\ell))$ of π and $I_3(\pi)$ is the set of all triples of inversions $((i,j),(i,k),(j,k))$ of π.

3. 12-inflations

In this section we will describe the collection of reduced words for permutations that are formed from *12-inflations*, that is permutations equal to $\pi = 12[\alpha,\beta]$ for some permutations α and β. We will also find exact recursive formulas for the diameters of the graphs G_π, C_π and B_π for 12-inflations.

Let $u = u_1u_2\ldots u_k$ and $v = v_1v_2\ldots v_l$ be two sequences of integers. A *shuffle* of u and v is a sequence $w = w_1w_2\ldots w_{k+l}$ of integers with a subsequence $w_{i_1}w_{i_2}\ldots w_{i_k}$ equal to u and another subsequence $w_{j_1}w_{j_2}\ldots w_{j_l}$ equal to v where $\{i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_l\} = [k+l]$. Let $\mathrm{Shuff}(u,v)$ be the collection of all shuffles of u and v. In order to describe the set of reduced words of 12-inflations, $\pi = 12[\alpha,\beta]$, we will be shuffling reduced words of α and β. In these shuffles we will want to distinguish the letters that come from reduced words of α from those that come from reduced words of β. To do this we will write reduced words of $\alpha \in \mathfrak{S}_a$ in the alphabet $\underline{[a-1]} = \{\underline{1}, \underline{2}, \ldots, \underline{a-1}\}$ and reduced words of $\beta \in \mathfrak{S}_b$ in the alphabet $\overline{[b-1]} = \{\overline{1}, \overline{2}, \ldots, \overline{b-1}\}$. Denote these sets $\underline{R}(\alpha)$ and $\overline{R}(\beta)$ respectively.

Example 3.1. For $\alpha = 21$ and $\beta = 231$, we observe that $\underline{R}(\alpha) = \{\underline{1}\}$, $\overline{R}(\beta) = \{\overline{12}\}$ and $\mathrm{Shuff}(\underline{1}, \overline{12}) = \{\underline{1}\overline{12}, \overline{1}\underline{1}\overline{2}, \overline{12}\underline{1}\}$.

Let $\pi = 12[\alpha,\beta]$ and

$$U_{\alpha,\beta} = \bigcup_{u\in\underline{R}(\alpha), v\in\overline{R}(\beta)} \mathrm{Shuff}(u,v).$$

We will define a graph $G_{\alpha,\beta}$ with vertices $U_{\alpha,\beta}$ and edges formed from the following relations:

1. Commutation moves, which come from exchanging adjacent letters in the following cases.
   (a) $\underline{jk}$ if $|j-k| > 1$,
   (b) $\overline{jk}$ if $|j-k| > 1$
   (c) $\underline{j}\overline{k}$ for any j and k

2. Braid moves, which comes from exchanging the following occurrences on consecutive indices.

   (a) $\underline{j(j+1)j}$ and $\underline{(j+1)j(j+1)}$

   (b) $\overline{j(j+1)j}$ and $\overline{(j+1)j(j+1)}$

We will call moves like 1(a) and 2(a) *α-moves*, moves like 1(b) and 2(b) *β-moves* and moves in 1(c) *shift-moves*. We will show that $U_{\alpha,\beta}$ is in bijection with $R(\pi)$ by showing that the graph $G_{\alpha,\beta}$ is isomorphic to G_π.

First we will define the map between the vertices, $\eta : U_{\alpha,\beta} \to [a+b-1]^{\ell(\pi)}$, as follows, where $[k]^m$ is the set of all words length m with letters in $[k]$. Let $w \in U_{\alpha,\beta}$. Then $\eta(w) = r$ is defined by

$$r_i = \begin{cases} j & \text{if } w_i = \underline{j} \\ j+a & \text{if } w_i = \overline{j}. \end{cases}$$

Notice that all outputs of η are in $[a+b-1]^{\ell(\pi)}$. See Figure 3 for an example of G_π for a 12-inflation.

Example 3.2. For $\alpha = 21$ and $\beta = 321$, we see that $w = \overline{1}\underline{1}\overline{21} \in U_{\alpha,\beta}$ and $\eta(w) = 3143$.

We claim that η is a bijection between $U_{\alpha,\beta}$ and $R(\pi)$, and further that $G_{\alpha,\beta}$ is isomorphic to G_π. To prove this, we first state and prove the following lemma.

Lemma 3.3. *Let $\pi = 12[\alpha,\beta]$, $|\alpha| = a$ and $|\beta| = b$. The map η is injective. Also, if $w \in U_{\alpha,\beta}$ and $\eta(w) = r$ then we can describe exactly the commutation and braid moves of r with conditions on w.*

1. *We can perform a commutation move on $r_i r_{i+1}$ in r if and only if $w_i w_{i+1}$ equals*

   (a) $\underline{jk}$ for some $|j-k| > 1$,

   (b) $\overline{jk}$ for some $|j-k| > 1$ or

   (c) $\underline{j}\overline{k}$ or $\overline{k}\underline{j}$ for any j and k.

2. *We can perform a braid move on $r_i r_{i+1} r_{i+2}$ in r if and only if $w_i w_{i+1} w_{i+2}$ equals*

   (a) $\underline{j(j+1)j}$ or $\underline{j(j-1)j}$ for any j or

   (b) $\overline{j(j+1)j}$ or $\overline{j(j-1)j}$ for any j.

Proof. Let $w \in U_{\alpha,\beta}$ and $\eta(w) = r$. To show parts 1 and 2 it will suffice to show the following two notes. The first note is that $w_i - w_j = r_i - r_j$ if w_i and w_{i+1} are both in $\underline{[a-1]}$ or are both in $\overline{[b-1]}$. The second note is that $|r_i - r_j| > 1$ if one of w_i and w_{i+1} is in $\underline{[a-1]}$ and the other is in $\overline{[b-1]}$.

To show the first note, consider $w_i w_{i+1} = \underline{jk}$ for some j and k. Then $r_i r_{i+1} = jk$. If $w_i w_{i+1} = \overline{jk}$ for some j and k, then $r_i r_{i+1} = (j+a)(k+a)$. This justifies the first note. To prove the second note consider the case where $w_i w_{i+1} = \underline{j}\overline{k}$ for any j and k, then $r_i r_{i+1} = j(k+a)$. Since $j < a$ and $k+a > a$ we have that $|r_i - r_j| > 1$. Similarly if $w_i w_{i+1} = \overline{j}\underline{k}$ for any j and k, then $|r_i - r_j| > 1$. This proves part 1 and 2.

Next, we will show that η is injective. Suppose that $\eta(w) = \eta(w') = r$ and $w \neq w'$. This means that there exists an i with $w_i \neq w'_i$. If w_i and w'_i are both letters in $\underline{[a-1]}$, then because $w_i \neq w'_i$ the output of at index i must be different by our definition of η. This is the same if we had supposed that w_i and w'_i are both letters in $\overline{[b-1]}$. The last case is if w_i and w'_i are in separate sets, one in $\underline{[a-1]}$ and the other in $\overline{[b-1]}$. In this case the output at index i must also be different since letters in $\underline{[a-1]}$ map to numbers less than a, and letters in $\overline{[b-1]}$ map to numbers more than a. Hence, η must be injective. □

Theorem 3.4. *Let $\pi = 12[\alpha, \beta]$. The map η is a bijection between $U_{\alpha,\beta}$ and $R(\pi)$ and $G_{\alpha,\beta}$ is isomorphic to G_π.*

Proof. We have already shown that η is injective in Lemma 3.3. If we form a graph on the image of η by connecting vertices according to possible commutation and braid moves, Lemma 3.3 proves that we get a graph that is isomorphic to $G_{\alpha,\beta}$. Let I denote the graph that is formed from the image of η. Now, we will only have to show two things to prove that $G_{\alpha,\beta}$ is isomorphic to G_π, so also η is a bijection between $U_{\alpha,\beta}$ and $R(\pi)$. We will first show that there exists a specific $\tilde{w}$ that maps to $\tilde{r}$ that is in $R(\pi)$. Because G_π is a connected graph that can be generated by one single vertex $\tilde{r} \in R(\pi)$ by using commutation and braid moves, we can conclude that one connected component of I is isomorphic to G_π. The second thing we will show is that $G_{\alpha,\beta}$ is connected, which proves that $G_{\alpha,\beta}$ is isomorphic to G_π.

First, let us describe the specific $\tilde{w} \in U_{\alpha,\beta}$ where $\eta(\tilde{w}) = \tilde{r}$ is in $R(\pi)$. Pick some $\tilde{u} \in R(\alpha)$ and $\tilde{v} \in R(\beta)$. Certainly the concatenation $\tilde{w} = \underline{\tilde{u}_1\tilde{u}_2\dots\tilde{u}_{\ell(\alpha)}}\overline{\tilde{v}_1\tilde{v}_2\dots\tilde{v}_{\ell(\beta)}}$ is in $U_{\alpha,\beta}$. Further, $\eta(\tilde{w}) = \tilde{r} = \tilde{u}_1\tilde{u}_2\dots\tilde{u}_{\ell(\alpha)}(\tilde{v}_1 + a)(\tilde{v}_2 + a)\dots(\tilde{v}_{\ell(\beta)} + a)$ is in $R(\pi)$.

Next we will show that $G_{\alpha,\beta}$ is connected by describing a path from any $w \in U_{\alpha,\beta}$ to $\tilde{w}$. Since $w \in U_{\alpha,\beta}$ we know that w is formed from a shuffle of some $u \in \underline{R}(\alpha)$ and $v \in \overline{R}(\beta)$. Because we can perform commutation moves on $\underline{j}\overline{k}$ and $\overline{k}\underline{j}$ for any j and k we know that there is a path from w to the concatenation uv via commutation moves. Because we can perform commutation and braid moves on the letters $\underline{[a-1]}$ or the letters $\overline{[b-1]}$ and because G_α and G_β are connected there exist a sequences of commutation and braid moves to transform u to $\tilde{u}$ and v to $\tilde{v}$. Hence, there must be a path from uv to $\tilde{w}$ in $G_{\alpha,\beta}$, which completes the proof. □

We are now ready to prove an exact recursive formula for the diameters of G_π, C_π and B_π. It will be helpful to define a statistic on words in $U_{\alpha,\beta}$. Given $w \in U_{\alpha,\beta}$ let $\mathrm{shift}(w)$ count the number of pairs of indices $i < i'$ such that $w_i \in \underline{[a-1]}$ and $w_{i'} \in \overline{[b-1]}$.

Example 3.5. Given $\alpha = 21$, $\beta = 321$ and $w = \overline{1}\underline{1}\overline{21} \in U_{\alpha,\beta}$ we have that $\mathrm{shift}(w) = 2$.

Theorem 3.6. *Let* $\pi = 12[\alpha, \beta]$.

(i) $diam(G_\pi) = diam(G_\alpha) + diam(G_\beta) + \ell(\alpha)\ell(\beta)$
(ii) $diam(C_\pi) = diam(C_\alpha) + diam(C_\beta)$
(iii) $diam(B_\pi) = diam(B_\alpha) + diam(B_\beta) + \ell(\alpha)\ell(\beta)$

Proof. By Theorem 3.4 we know that G_π is isomorphic to $G_{\alpha,\beta}$, so it suffices to prove this theorem on the graph $G_{\alpha,\beta}$. We will need to consider specific subgraphs of $G_{\alpha,\beta}$. The first is on the vertices $\{uv : u \in \underline{R}(\alpha), v \in \overline{R}(\beta)\}$, which we will call G_1. The second is on the vertices $\{vu : u \in \underline{R}(\alpha), v \in \overline{R}(\beta)\}$, which we will call G_2. Note that we can transform any two vertices in G_1 into each other by using at most $\mathrm{diam}(G_\alpha)$ α-moves plus at most $\mathrm{diam}(G_\beta)$ β-moves. Similarly, we can transform any two vertices in G_2 into each other by using at most $\mathrm{diam}(G_\alpha)$ α-moves plus at most $\mathrm{diam}(G_\beta)$ β-moves.

We claim that given any $w \in U_{\alpha,\beta}$ that any path from w to any vertex in G_2 takes at least $\mathrm{shift}(w)$ shift-moves, which are commutation moves. Note that G_2 contains exactly those vertices w with $\mathrm{shift}(w) = 0$. Also note that α-moves and β-moves do not change $\mathrm{shift}(w)$, but shift-moves change $\mathrm{shift}(w)$ exactly by one. This means any path from w to G_2 requires at least $\mathrm{shift}(w)$ shift-moves. Finally, note that there exists a path from w to a vertex in G_2 that takes exactly $\mathrm{shift}(w)$ shift-moves. Because G_1 contains exactly those vertices w with $\mathrm{shift}(w) = \ell(\alpha)\ell(\beta)$ we can similarly conclude that any path from w to any vertex in G_1 takes at least $\ell(\alpha)\ell(\beta) - \mathrm{shift}(w)$ shift-moves. Also, note that there does exist a path from w to a vertex in G_1 that uses exactly $\ell(\alpha)\ell(\beta) - \mathrm{shift}(w)$ shift-moves.

Next we will prove an upper bound for the diameter of G_π. Consider any pair of w and w'. We will show that there exists a path between w and w' of length at most $\text{diam}(G_\alpha)+\text{diam}(G_\beta)+\ell(\alpha)\ell(\beta)$. To do this we will describe two paths from w to w'. Let the first path, P_1, start at w that then takes $\text{shift}(w)$ shift-steps to get to a vertex h_2 in G_2. The path P_1 ends at w' with the $\text{shift}(w')$ shift-moves to get you from some vertex h_2' in G_2 to w'. We complete P_1 by connecting h_2 to h_2' with at most $\text{diam}(G_\alpha)$ α-moves plus at most $\text{diam}(G_\beta)$ β-moves. The length of P_1 is at most $\text{shift}(w)+\text{shift}(w')$ shift-moves plus $\text{diam}(G_\alpha)$ α-moves plus $\text{diam}(G_\beta)$ β-moves. We can similarly connect w to w' with another path, P_2, through G_1 that will have a length at most $2\ell(\alpha)\ell(\beta)-\text{shift}(w)-\text{shift}(w')$ shift-moves plus $\text{diam}(G_\alpha)$ α-moves plus $\text{diam}(G_\beta)$ β-moves. All together the cycle formed by combining the paths P_1 and P_2 has length at most

$$2\ell(\alpha)\ell(\beta) + 2\text{diam}(G_\alpha) + 2\text{diam}(G_\beta).$$

This implies either P_1 or P_2 has length at most $\ell(\alpha)\ell(\beta) + \text{diam}(G_\alpha) + \text{diam}(G_\beta)$ proving that $d(w,w') \leq \ell(\alpha)\ell(\beta)+\text{diam}(G_\alpha)+\text{diam}(G_\beta)$. Hence, $\text{diam}(G_\pi) \leq \ell(\alpha)\ell(\beta) + \text{diam}(G_\alpha) + \text{diam}(G_\beta)$.

Now we will prove a lower bound for the diameter of G_π. Pick two $u,u' \in R(\alpha)$ with $d(u,u') = \text{diam}(G_\alpha)$ and two $v,v' \in R(\beta)$ with $d(v,v') = \text{diam}(G_\beta)$. Consider $w = \underline{u}\overline{v}$ and $w' = \overline{v'}\underline{u'}$ and any path P between them. We have shown that this path will require at least $\ell(\alpha)\ell(\beta)$ shift-moves. We can also conclude that this path requires at least $\text{diam}(G_\alpha)$ α-moves, otherwise we could project our path onto G_α and have a path from u to u' that is shorter than $\text{diam}(G_\alpha)$. Similarly, we will have at least $\text{diam}(G_\beta)$ β-moves. Hence, the path P has length at least $\ell(\alpha)\ell(\beta) + \text{diam}(G_\alpha) + \text{diam}(G_\beta)$, proving that $\text{diam}(G_\pi) \geq \ell(\alpha)\ell(\beta) + \text{diam}(G_\alpha) + \text{diam}(G_\beta)$. Together with our upper bound we have proven our recursion on the diameter of G_π.

Using the ideas above we can show that for any $w,w' \in U_{\alpha,\beta}$ that we can find a path that uses at most $\text{diam}(C_\alpha)$ braid α-moves, at most $\text{diam}(C_\beta)$ braid β-moves and otherwise just uses commutation moves of different types. By Lemma 2.4 this proves that $\text{diam}(C_\pi) \leq \text{diam}(C_\alpha) + \text{diam}(C_\beta)$. Also, using similar ideas as above we can construct $w,w' \in U_{\alpha,\beta}$ where all paths from w to w' require at least $\text{diam}(C_\alpha)$ braid α-moves, at least $\text{diam}(C_\beta)$ braid β-moves and otherwise just uses commutation moves of different types. By Lemma 2.4 this proves that $\text{diam}(C_\pi) \geq \text{diam}(C_\alpha) + \text{diam}(C_\beta)$ and our recursion for $\text{diam}(C_\pi)$ is proven.

Again, using the ideas above we can show that for any $w,w' \in U_{\alpha,\beta}$ that we can find a path that uses at most $\ell(\alpha)\ell(\beta)$ shift-moves, at most $\text{diam}(B_\alpha)$

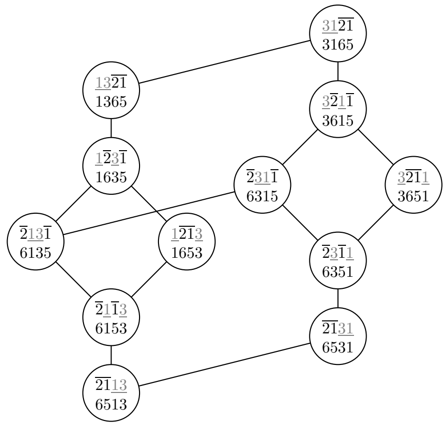

Figure 3: This is the graph G_π of the 12-inflation $\pi = 12[2143, 312] = 2143756$ with vertices given in both $U_{2143,312}$ and $R(\pi)$.

commutation α-moves, at most $\mathrm{diam}(B_\beta)$ commutation β-moves and otherwise just uses braid moves. By Lemma 2.4 this proves that $\mathrm{diam}(B_\pi) \leq \mathrm{diam}(B_\alpha) + \mathrm{diam}(B_\beta) + \ell(\alpha)\ell(\beta)$. Also, using similar ideas as above we can construct $w, w' \in U_{\alpha,\beta}$ where all paths from w to w' require at least $\ell(\alpha)\ell(\beta)$ shift-moves, at least $\mathrm{diam}(B_\alpha)$ commutation α-moves, at least $\mathrm{diam}(B_\beta)$ commutation β-moves and otherwise just uses braid moves. By Lemma 2.4 this proves that $\mathrm{diam}(B_\pi) \geq \mathrm{diam}(B_\alpha) + \mathrm{diam}(B_\beta) + \ell(\alpha)\ell(\beta)$ and our recursion for $\mathrm{diam}(B_\pi)$ is proven. $\square$

4. 21-inflations

In this section we describe the collection of reduced words for permutations that are formed from *21-inflations*, that is a permutation equal to $\pi = 21[\alpha, \beta]$ for some permutations α and β. The graphs of 21-inflations are more complex than 12-inflations, so we will only find an exact recursive formula for the diameters of G_π, C_π and B_π for 21-inflations $21[\alpha, 1]$ and $21[1, \beta]$. Because the longest permutation $\delta_n = n \ldots 21$ is a 21-inflation of the form $21[\alpha, 1]$ we have an exact recursive formula for the diameters of G_{δ_n}, C_{δ_n} and B_{δ_n}. For 21-inflations of the form $\pi = 21[\alpha, \iota_k]$ we provide recursive upper and lower bounds for the diameters of G_π, C_π and B_π.

4.1. Reduced words of 21-inflations

We will first describe a set that is in bijection with reduced words in $R(\pi)$ where $\pi = 21[\alpha, \beta]$, $|\alpha| = a$ and $|\beta| = b$. This set will contain certain shuffles of $R(\alpha)$, $R(\beta)$ and ballot sequences.

Let $x = x_1 x_2 \dots x_n$ be a sequence of positive integers. Define $N_j(x)$ to be the number of j's in x. We call x a *ballot sequence* if for all positive integers $j < k$ we have that $N_j(x_1 x_2 \dots x_m) \geq N_k(x_1 x_2 \dots x_m)$ for all $m \in [n]$. Since we will often be referring to prefixes $x_1 x_2 \dots x_m$ of $x = x_1 x_2 \dots x_n$, $m \leq n$, denote $x^{(m)} = x_1 x_2 \dots x_m$. Let $\mathrm{Ballot}_{a,b}$ be the collection of all ballot sequences that are rearrangements of the multiset $\{1^b, 2^b, \dots, a^b\}$ where i^b means there are b copies of i. A *reverse ballot sequence* is a sequence of positive integers at most L, for some L, where for all $j < k \leq L$ we have that $N_j(x^{(m)}) \leq N_k(x^{(m)})$ for all $m \in [n]$. Let $\mathrm{RBallot}_{a,b}$ be the collection of all reverse ballot sequences on the same multiset $\{1^b, 2^b, \dots, a^b\}$. There is a natural bijection that will be useful to us,

$$f : \mathrm{Ballot}_{a,b} \to \mathrm{RBallot}_{b,a} .$$

We will describe f with the composition of bijections

$$\mathrm{Ballot}_{a,b} \xrightarrow{f_1} \mathrm{Ballot}_{b,a} \xrightarrow{f_2} \mathrm{RBallot}_{b,a} .$$

With the first bijection f_1 we map $x \in \mathrm{Ballot}_{a,b}$ to $y \in \mathrm{Ballot}_{b,a}$ by sending letter x_i to $y_i = N_{x_i}(x^{(i)})$. The second bijection f_2 sends letter y_i to $z_i = b + 1 - y_i$ to get $z \in \mathrm{RBallot}_{b,a}$.

Later we will have words $w = w_1 w_2 \dots w_n$ with letters from $[a + b - 1] \cup \underline{[a-1]} \cup \overline{[b-1]}$. We can still define $N_j(w)$ to be the number of j's in w where we do not count $\underline{j}$ or $\overline{j}$. Also, if the subsequence of $w = w_1 w_2 \dots w_n$ formed from only letters in $[a + b]$ is a ballot sequence in $\mathrm{Ballot}_{a,b}$ then we can define $f(w)$ similarly and only apply the map to the subsequence that is the ballot sequence.

Example 4.1. The sequence 112323 is a ballot sequence in $\mathrm{Ballot}_{3,2}$ and 212211 is a reverse ballot sequence in $\mathrm{RBallot}_{2,3}$. We have $f(112323) = 212211$ and $f(11\overline{1}23\underline{2}2\underline{1}3) = 21\overline{1}22\underline{2}1\underline{1}1$.

We are now ready to start defining the set that is in bijection with $R(\pi)$ where $\pi = 21[\alpha, \beta]$. Given $x \in \mathrm{Ballot}_{a,b}$, $u \in \underline{R}(\alpha)$ and $v \in \overline{R}(\beta)$ define $\overline{\mathrm{Shuff}}(x, u, v)$ to the the collection of all $w \in \mathrm{Shuff}(x, u, v)$ such that

1. if $w_i = \underline{j}$ then $N_j(w^{(i-1)}) = N_{j+1}(w^{(i-1)})$ and

2. if $w_i = \overline{j}$ then $N_j(f(w)^{(i-1)}) = N_{j+1}(f(w)^{(i-1)})$

Example 4.2. Consider $x = 112323$ in $\mathrm{Ballot}_{3,2}$, $u = \underline{21}$ in $\underline{R}(312)$ and $v = \overline{1}$ in $\overline{R}(21)$. Then, $\overline{\mathrm{Shuff}}(x, u, v)$ contains elements like $\underline{21}\overline{1}112323$ and $1\underline{2}1\overline{1}2321\underline{1}3$.

We are going to find that these shuffles in $\overline{\mathrm{Shuff}}(x, u, v)$ encode reduced words of π so let

$$V_{\alpha,\beta} = \bigcup_{\substack{u \in \underline{R}(\alpha), v \in \overline{R}(\beta) \\ x \in \mathrm{Ballot}_{a,b}}} \overline{\mathrm{Shuff}}(x, u, v).$$

Remark 4.3. To give some intuition behind the definition of $V_{\alpha,\beta}$, underlined numbers will be associated to adjacent transpositions in the reflection sequence that exchange numbers greater than b. Meaning they are associated to inversions of $\alpha + b = (\alpha_1 + b)(\alpha_2 + b) \cdots (\alpha_a + b)$ where $|\alpha| = a$ and $|\beta| = b$. Overlined numbers will be associated to adjacent transpositions in the reflection sequence that exchange numbers at most b. Meaning they are associated to inversions of β. Nondecorated numbers are associated to the inversions between numbers greater than b and those at most b.

We will define a graph $H_{\alpha,\beta}$ with vertices $V_{\alpha,\beta}$ and edges formed from the following relations:

1. Commutation moves, which come from exchanging adjacent letters in the following cases.
   (a) $w_i w_{i+1} = \underline{jk}$ if $|j - k| > 1$
   (b) $w_i w_{i+1} = \overline{jk}$ if $|j - k| > 1$
   (c) $w_i w_{i+1} = jk$ if either $j > k$, or $j < k$ and $N_j(w^{(i-1)}) > N_k(w^{(i-1)})$
   (d) $w_i w_{i+1} = \underline{j}\overline{k}$ or $w_i w_{i+1} = \overline{k}\underline{j}$ for any j and k
   (e) $w_i w_{i+1} = \underline{j}k$ or $w_i w_{i+1} = k\underline{j}$ if $|j - k| > 1$ or $j > k$
   (f) $z_i z_{i+1} = \overline{j}k$ or $z_i z_{i+1} = k\overline{j}$ if $|j - k| > 1$ or $k < j$ where $f(w) = z$
2. Braid moves, which comes from exchanging the following occurrences on consecutive indices.
   (a) in w we exchange $\underline{j(j+1)j}$ and $\underline{(j+1)j(j+1)}$
   (b) in w we exchange $\overline{j(j+1)j}$ and $\overline{(j+1)j(j+1)}$
   (c) in w we exchange $\underline{j}j(j+1)$ and $j(j+1)\underline{j}$
   (d) in z we exchange $\overline{j}(j+1)j$ and $(j+1)j\overline{j}$ where $f(w) = z$

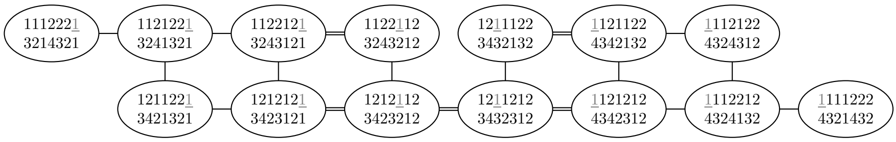

Figure 4: This is the graph G_π of the 21-inflation $\pi = 21[21, 123] = 54123$ with vertices given in both $V_{21,123}$ and $R(\pi)$.

There is some intuition behind the choice of moves given above. We allow all possible commutation moves as long as the commutation move doesn't bring us outside the set $V_{\alpha,\beta}$. We allow all possible braid moves that occur purely on the letters in $\underline{[a-1]}$ or in $\overline{[b-1]}$ that keep us in the set $V_{\alpha,\beta}$, and we have two new kinds of braid moves, 2(c) and 2(d), that also keep us in the set $V_{\alpha,\beta}$. See Figure 4 for an example.

We can now define the map that will be a bijection between $R(\pi)$ for $\pi = 21[\alpha, \beta]$ and $V_{\alpha,\beta}$,

$$\psi : V_{\alpha,\beta} \to [a + b - 1]^{\ell(\pi)}.$$

For now, it will not be clear why the image of ψ is actually $R(\pi)$. This will become apparent later when we show $H_{\alpha,\beta}$ is isomorphic to G_π. For now it will suffice to know that the outputs are in $[a + b - 1]^{\ell(\pi)}$. Given $w \in \overline{\text{Shuff}}(x, u, v)$, and $f(w) = z$ since we will need z to compute $\psi(w)$ in some cases. We define $\psi(w) = r$ by

$$r_i = \begin{cases} j + b - N_j(w^{(i-1)}) - 1 = k + N_k(z^{(i-1)}) & \text{if } w_i = j \text{ or } z_i = k \\ j + b - N_j(w^{(i-1)}) & \text{if } w_i = \underline{j} \\ j + N_j(z^{(i-1)}) & \text{if } w_i = \overline{j}. \end{cases}$$

Example 4.4. Given $\alpha = 312$ and $\beta = 21$ we have $\psi(\underline{2}\underline{1}\overline{1}112323) = 431213423$ and $\psi(1\underline{2}1\overline{1}232\underline{1}3) = 241234213$.

We will show that the commutation moves and braid moves we have defined on $V_{\alpha,\beta}$ match with the possible commutation and braid moves of their associated outputs.

Lemma 4.5. *The map $\psi : V_{\alpha,\beta} \to [a + b - 1]^{\ell(\pi)}$ is injective. Also, if $\psi(w) = r$ with $f(w) = z$ then we can describe exactly the commutation and braid moves of r with conditions on w or $f(w) = z$.*

1. *We can perform a commutation move on $r_i r_{i+1}$ in r if and only if*

(a) $w_iw_{i+1} = \underline{jk}$ *if* $|j-k|>1$

(b) $w_iw_{i+1} = \overline{jk}$ *if* $|j-k|>1$

(c) $w_iw_{i+1} = jk$ *if either* $j>k$, *or* $j<k$ *and* $N_j(w^{(i-1)}) > N_k(w^{(i-1)})$

(d) $w_iw_{i+1} = \underline{j}\overline{k}$ *or* $w_iw_{i+1} = \overline{k}\underline{j}$ *for any* j *and* k

(e) $w_iw_{i+1} = \underline{j}k$ *or* $w_iw_{i+1} = k\underline{j}$ *if* $|j-k|>1$ *or* $j>k$

(f) $z_iz_{i+1} = \overline{j}k$ *or* $z_iz_{i+1} = k\overline{j}$ *if* $|j-k|>1$ *or* $k<j$.

2. *We can perform a braid move on* $r_ir_{i+1}r_{i+2}$ *in* r *if and only if*

(a) $w_iw_{i+1}w_{i+2} = \underline{j(j\pm 1)j}$

(b) $w_iw_{i+1}w_{i+2} = \overline{j(j\pm 1)j}$

(c) *either* $w_iw_{i+1}w_{i+2}$ *equals* $\underline{j}j(j+1)$ *or* $j(j+1)\underline{j}$

(d) *either* $z_iz_{i+1}z_{i+2}$ *equals* $\overline{j}(j+1)j$ *or* $(j+1)j\overline{j}$

Proof. Let $w \in V_{\alpha,\beta}$, $\psi(w) = r$ and $f(w) = z$. In the first part of this proof we will investigate the output r_ir_{i+1} for different choices of w_iw_{i+1}. The cases are recorded in Table 1 for reference. Looking at this table we can confirm part 1 of this lemma.

Case 1: We have $w_iw_{i+1} = \underline{jk}$ for some j and k. This means

$$r_ir_{i+1} = (j+b-N_j(w^{(i-1)}))(k+b-N_k(w^{(i)})).$$

Notice that $j \neq k$ because jk is an adjacent subsequence for some reduced word of α. We know that if $j<k$ then $N_j(w^{(i)}) \geq N_k(w^{(i)})$ and if $j>k$ then $N_j(w^{(i)}) \leq N_k(w^{(i)})$. Because $N_j(w^{(i-1)}) = N_j(w^{(i)})$ we can conclude that if $j<k$ then $r_i<r_{i+1}$ and if $j>k$ then $r_i>r_{i+1}$. Further because $N_j(w^{(i-1)}) = N_{j+1}(w^{(i-1)})$ and $N_k(w^{(i)}) = N_{k+1}(w^{(i)})$ we can conclude that $|j-k|=1$ if and only if $|r_i - r_{i+1}| = 1$.

Case 2: We have $w_iw_{i+1} = \overline{jk}$ for some j and k. This means

$$r_ir_{i+1} = (j+N_j(f(w)^{(i-1)}))(k+N_k(f(w)^{(i)})).$$

Notice that $j \neq k$ because jk is an adjacent subsequence for some reduced word of β. We know that if $j<k$ then $N_j(f(w)^{(i)}) \leq N_k(f(w)^{(i)})$ and if $j>k$ then $N_j(f(w)^{(i)}) \geq N_k(f(w)^{(i)})$. Because $N_j(f(w)^{(i-1)}) = N_j(f(w)^{(i)})$ we can conclude that if $j<k$ then $r_i<r_{i+1}$ and if $j>k$ then $r_i>r_{i+1}$. Further because $N_j(f(w)^{(i-1)}) = N_{j+1}(f(w)^{(i-1)})$ and $N_k(f(w)^{(i)}) = N_{k+1}(f(w)^{(i)})$ we can conclude that $|j-k|=1$ if and only if $|r_i - r_{i+1}| = 1$.

Case 3: We have $w_iw_{i+1} = jk$ for some j and k. This means

$$r_ir_{i+1} = (j + b - N_j(w^{(i-1)}) - 1)(k + b - N_k(w^{(i)}) - 1).$$

If $j = k$ then $N_j(w^{(i-1)}) = N_k(w^{(i)}) - 1$ so $r_ir_{i+1} = J(J-1)$ for some J. If $j < k$ then because $N_j(w^{(i-1)}) \geq N_k(w^{(i-1)}) = N_k(w^{(i)})$ we have that $r_i < r_{i+1}$. In this case where $j < k$ if $N_j(w^{(i-1)}) = N_k(w^{(i-1)})$, then $|r_i - r_{i+1}| = 1$. If instead $N_j(w^{(i-1)}) > N_k(w^{(i-1)})$, then $|r_i - r_{i+1}| > 1$. If $j > k$ then $N_j(w^{(i-1)}) + 1 = N_j(w^{(i)}) \leq N_k(w^{(i)})$ which implies that $r_i > r_{i+1}$ and $|r_i - r_{i+1}| > 1$.

Case 4: We have w_iw_{i+1} equals $\underline{j}\overline{k}$ or $\overline{k}\underline{j}$ for some j and k. In either case we have

$$\underline{j} \mapsto j + b - N_j(w^{(i-1)}) = J \text{ and } \overline{k} \mapsto k + N_k(f(w)^{(i-1)}) = K.$$

The values $N_1(w^{(i-1)}) \geq N_2(w^{(i-1)}) \geq \cdots$ form an integer partition $\lambda = \lambda_1\lambda_2\cdots$ with $\lambda_j = N_j(w^{(i-1)})$. Further the value $N_k(f(w)^{(i-1)})$ can be determined by a part of the complement λ' of λ, specifically $N_k(f(w)^{(i-1)}) = \lambda'_{b-k+1}$. Because of the structure of w we know that $N_j(w^{(i-1)}) = N_{j+1}(w^{(i-1)})$ and $N_k(f(w)^{(i-1)}) = N_{k+1}(f(w)^{(i-1)})$, so $\lambda_j = \lambda_{j+1}$ and $\lambda'_{b-k-1} = \lambda'_{b-k}$. This means that $\lambda_j \neq b - k$. Consider the case where $\lambda_j < b - k$ then $\lambda'_{b-k} > j$. Thus, $J > j + k$ and $K < j + k$. If instead $\lambda_j > b - k$ then $\lambda'_{b-k} < j$, so $J < j + k$ and $K > j + k$. This means that $|J - K| > 1$ in all cases.

Case 5: We have w_iw_{i+1} equals $\underline{j}k$ or $k\underline{j}$ for some j and k. Because of the conditions on w we can not have the case where $j\underline{j}$ or $\underline{j}(j+1)$. In either case we have

$$\underline{j} \mapsto j + b - N_j(w^{(i-1)}) = J \text{ and } k \mapsto k + b - N_k(w^{(i-1)}) - 1 = K.$$

If $j = k$ then $w_iw_{i+1} = \underline{j}j \mapsto J(J-1)$. If $j < k$ then $N_j(w^{(i-1)}) \geq N_k(w^{(i-1)})$. Because $N_k(w^{(i-1)}) = N_k(w^{(i+1)}) - 1$ we actually have $N_j(w^{(i-1)}) > N_k(w^{(i-1)})$, so $J < K$. In this case where $j < k$ if $k = j + 1$, then $|J - K| = 1$, which means that $w_iw_{i+1} = (j+1)\underline{j}$. If instead $k > j + 1$, then $|J - K| > 1$. Now suppose that $j > k$ so $N_j(w^{(i-1)}) \leq N_k(w^{(i-1)})$, which quickly implies that $J > K$ and $|J - K| > 1$.

Case 6: We have w_iw_{i+1} equals $\overline{j}k'$ or $k'\overline{j}$ for some j and k'. This is equivalent to the case where z_iz_{i+1} equals $\overline{j}k$ or $k\overline{j}$ for some j and k. Because of the conditions on z we can not have the case where $\overline{j}j$ or $(j+1)\overline{j}$. First consider the case where $j = k$ so $z_i z_{i+1} = j\overline{j} \mapsto J(J+1)$ for some J. In any other case $j \neq k$ and we have

$$\overline{j} \mapsto j + N_j(z^{(i-1)}) = J \text{ and } k \mapsto k + N_k(z^{(i-1)}) = K.$$

If $j < k$ then $N_j(z^{(i-1)}) \leq N_k(z^{(i-1)})$. This implies $J < K$. In this case where $j < k$ if $k = j+1$, then $|J - K| = 1$, which implies that $z_i z_{i+1} = \overline{j}(j+1) \mapsto J(J+1)$. If instead $k > j+1$, then $|J-K| > 1$. Now consider when $j > k$, then $N_j(z^{(i-1)}) \geq N_k(z^{(i-1)})$. Specifically because $N_k(z^{(i+1)}) = N_k(z^{(i-1)}) + 1$ we have $N_j(z^{(i-1)}) > N_k(z^{(i-1)})$. Hence, $J > K$ and $|J - K| > 1$.

Part 2: From the above arguments $w_i w_{i+1}$ or $z_i z_{i+1}$ maps to $r_i r_{i+1} = J(J+1)$ under the following circumstances: $w_i w_{i+1}$ equals $\underline{j(j+1)}$, $\overline{j(j+1)}$ or $j(j+1)$ where $N_j(w^{(i-1)}) = N_{j+1}(w^{(i-1)})$ or $z_i z_{i+1}$ equals $\overline{j}(j+1)$ or $j\overline{j}$. Also, from the above arguments $w_i w_{i+1}$ maps to $r_i r_{i+1} = J(J-1)$ under the following circumstances: $w_i w_{i+1}$ equals $\underline{j(j-1)}$, $\overline{j(j-1)}$, jj, $\underline{j}j$ or $j\underline{(j-1)}$. Considering all possible combinations to get $r_i r_{i+1} r_{i+2} = p(p \pm 1)p$, we will find only four do not give us a contradiction. We only do not get a contradiction for the cases described in 2(abcd) of the statement.

Injective: Finally, we will show that ψ is injective. Suppose that $\psi(w) = \psi(w') = r$ and $w \neq w'$. This means that there exists an i with $w^{(i-1)} = w'^{(i-1)}$ and $w_i \neq w'_i$. If w_i and w'_i are both letters in $\underline{[a-1]}$, then we have a contradiction. By our map $w_i \neq w'_i$ would imply two different outputs for the ith letter in r. We would similarly have a contradiction if $w_i \neq w'_i$ are both in $\overline{[b-1]}$ or $[a]$. If one of $w_i \neq w'_i$ is in $\underline{[a-1]}$ and the other is in $\overline{[b-1]}$ then we also have a contradiction. This is for the following reason. By the argument in Case 4 if there are adjacent $\underline{j}$ and $\overline{k}$ in w, then their outputs under ψ differ by at least two. This also implies that the output of $\underline{j}$ in $w^{(i-1)}\underline{j}$ and the output of $\overline{k}$ in $w^{(i-1)}\overline{k}$ differ by at least two for any j and k. Say $w_i = \underline{j}$ and $w'_i = k$, then $r_i = j + b - N_j(w^{(i-1)}) = k + b - N_k(w^{(i-1)}) - 1$. If $j = k$, then the outputs of w_i and w'_i must be different. If $j > k$ then $N_j(w^{(i-1)}) \leq N_k(w^{(i-1)})$ and we have a contradiction. If $j < k$ then $N_j(w^{(i-1)}) \geq N_k(w^{(i-1)})$ and we must have that $k = j + 1$. Because of our conditions on w if $w_i = j$ then $N_j(w^{(i-1)}) \geq N_{j+1}(w^{(i-1)})$, which means it is impossible for $w' = j + 1$. Finally consider the case where $w_i = \overline{j}$ and $w'_i = k'$ or equivalently $z_i = \overline{j}$ and $z'_i = k$, then $r_i = j + N_j(z^{(i-1)}) = k + N_k(z^{(i-1)})$. Suppose $j = k$. Because $z_i = \overline{j}$ we know that $N_j(z^{(i-1)}) = N_{j+1}(z^{(i-1)})$, which makes it impossible for $z'_i = j$. If $j > k$ then $N_j(z^{(i-1)}) \geq N_k(z^{(i-1)})$ implying that z_i and z'_i have two different outputs. We get a similar contradiction if $j < k$. Hence, in all cases we have proven we have a contradiction so ψ must be injective. $\square$

Table 1: This is the table of all cases of inputs and their associated outputs for ψ

Input $w_i w_{i+1}$	Output of ψ	
$\overline{jk}$	JK	$j-k=J-K$
$\underline{jk}$	JK	$j-k=J-K$
jj	$J(J-1)$	
$j(j+1)$	$J(J+1)$	$N_j(w^{(i-1)})=N_{j+1}(w^{(i-1)})$
$j(j+1)$	JK	$J<K-1$ and $N_j(w^{(i-1)})>N_{j+1}(w^{(i-1)})$
jk	JK	$j>k$ and $J-1>K$
$\underline{j}\overline{k}$ or $\overline{k}\underline{j}$	JK	$\lvert J-K\rvert>1$
$\underline{j}j$	$J(J-1)$	
$(j+1)\underline{j}$	$(J+1)J$	
$\underline{j}k$ or $k\underline{j}$	JK or KJ	$j<k-1$ and $J<K-1$
$\underline{j}k$ or $k\underline{j}$	JK or KJ	$j>k$ and $J-1>K$
$z_i z_{i+1}=j\overline{j}$	$J(J+1)$	
$z_i z_{i+1}=\overline{j}(j+1)$	$J(J+1)$	
$z_i z_{i+1}$ equals $\overline{j}k$ or $k\overline{j}$	JK or KJ	$j<k-1$ and $J<K-1$
$z_i z_{i+1}$ equals $\overline{j}k$ or $k\overline{j}$	JK or KJ	$j>k$ and $J-1>K$

We finally prove that the vertices $R(21[\alpha,\beta])$ are in bijection with $V_{\alpha,\beta}$ in the following theorem, which will help us easily construct paths in G_π.

Theorem 4.6. *Let $\pi=21[\alpha,\beta]$, $|\alpha|=a$ and $|\beta|=b$. The map ψ is a bijection between $V_{\alpha,\beta}$ and $R(\pi)$ and proves that $H_{\alpha,\beta}$ is isomorphic to G_π.*

Proof. We have already shown that ψ is injective in Lemma 4.5. If we form a graph on the image of ψ by connecting vertices by commutation and braid moves, Lemma 4.5 proves that we get a graph that is isomorphic to $H_{\alpha,\beta}$. Call the graph on the image I. We will only have to show two things to prove $H_{\alpha,\beta}$ is isomorphic to G_π, so also ψ is a bijection between $V_{\alpha,\beta}$ and $R(\pi)$. The first thing is that we will find some specific $\tilde{w}$ that maps to $\tilde{r}$ that is in $R(\pi)$. Because G_π is a connected graph that can be generated by one single vertex $\tilde{r}$ by using commutation and braid moves, we can conclude that one connected component of I is isomorphic to G_π. The second thing we will show is that $H_{\alpha,\beta}$ is connected, which proves that $H_{\alpha,\beta}$ is isomorphic to G_π.

First we will show there exists some $\tilde{w} \in V_{\alpha,\beta}$ where $\psi(\tilde{w}) = \tilde{r}$ is in $R(\pi)$. Let $\tilde{u} \in R(\alpha)$, $\tilde{v} \in R(\beta)$ and $\tilde{x} = 1^b 2^b \dots a^b \in \text{Ballot}_{a,b}$. Let $\tilde{w}$ be the concatenation $\underline{\tilde{u}}\overline{\tilde{v}}\tilde{x}$, which is in $V_{\alpha,\beta}$. Let

$$\tilde{r} = (\tilde{u}_1 + b)(\tilde{u}_2 + b)\dots(\tilde{u}_{\ell(\alpha)} + b)\tilde{v}_1\tilde{v}_2\dots\tilde{v}_{\ell(\beta)}b(b-1)\dots 1(b+1)b\dots 2$$
$$\dots(a+b-1)(a+b-2)\dots a.$$

We can see that $\tilde{r}$ is in $R(\pi)$ by seeing the tranformation on the identity ι_{a+b}. After applying the first part, $(\tilde{u}_1+b)(\tilde{u}_2+b)\dots(\tilde{u}_{\ell(\alpha)}+b)$, the identity becomes $12\dots b(\alpha_1+b)(\alpha_2+b)\dots(\alpha_a+b)$. After applying $\tilde{v}_1\tilde{v}_2\dots\tilde{v}_{\ell(\beta)}$ we get $\beta_1\beta_2\dots\beta_b(\alpha_1+b)(\alpha_2+b)\dots(\alpha_a+b)$. The last portion of $\tilde{r}$ switches the α and β subsequences and we get $(\alpha_1+b)(\alpha_2+b)\dots(\alpha_a+b)\beta_1\beta_2\dots\beta_b = \pi$. By applying the map ψ we can show that $\psi(\tilde{w}) = \tilde{r}$.

Next we will show that $H_{\alpha,\beta}$ is connected by proving that from all $w \in V_{\alpha,\beta}$ we can describe a path to $\tilde{w}$. Let $w \in V_{\alpha,\beta}$, so w is formed from a shuffle of some $u \in \underline{R}(\alpha)$, $v \in \overline{R}(\beta)$ and $x \in \text{Ballot}_{a,b}$. We claim for now, and will prove later, that there is a path from w to uvy for some $y \in \text{Ballot}_{a,b}$. Because G_α is connected we can perform moves to change u to $\underline{\tilde{u}}$, and because G_β is connected we can perform moves to change v to $\overline{\tilde{v}}$. By the commutation rules given earlier for $V_{\alpha,\beta}$ we can transform y to $\tilde{x}$ by performing commutation moves that move the 1's to the front, then the 2's, then the 3's and so on. If we can prove our claim then we have proven that $H_{\alpha,\beta}$ is connected and thus is isomorphic to G_π.

Now we will prove our claim that there exists a path from w to uvy for some $y \in \text{Ballot}_{a,b}$. We will do so with a recursive algorithm starting with $w = u^{[0]}v^{[0]}y^{[0]}$ where $u^{[0]}$ and $v^{[0]}$ are empty and $y^{[0]} = w$. Suppose for some $i \geq 0$ that $u^{[i]}$ is a prefix of u, $v^{[i]}$ is a prefix of v and $y^{[i]}$ is a shuffle of the remaining letters of u and v in the same order and some ballot sequence so that $u^{[i]}v^{[i]}y^{[i]} \in V_{\alpha,\beta}$. In each step the number of letters in $y^{[i]}$ will decrease by one as we move one occurrence of $\underline{j}$ or $\overline{j}$ left and add it to $u^{[i]}$ or $v^{[i]}$. Once $y^{[i]}$ is a ballot sequence we have achieved the goal of our claim. We will now describe how you construct $u^{[i+1]}v^{[i+1]}y^{[i+1]}$.

Case 1: The first letter in $y^{[i]}$ that is not in $[a]$ is $\underline{j}$. Let $\mathbf{y}$ be the prefix of y containing all letters before $\underline{j}$. We can preform commutation moves on $\mathbf{y}$ to get $1^{m_1}2^{m_2}\dots a^{m_a}$ where $m_1 \geq m_2 \geq \dots$. We also have that $m_j = m_{j+1}$. Next we use commutation moves to transform the subsequence $j^{m_j}(j+1)^{m_j}$ into $(j(j+1))^{m_j}$. So we have just transformed $\mathbf{y}$ into $1^{m_1}2^{m_2}\dots(j-1)^{m_{j-1}}(j(j+1))^{m_j}(j+2)^{m_{j+2}}\dots a^{m_a}$. We can use commutation moves to move $\underline{j}$ left past $(j+2)^{m_{j+2}}\dots a^{m_a}$, braid moves to move $\underline{j}$ further left past $(j(j+1))^{m_j}$ and some more commutation moves to move $\underline{j}$ left past

$1^{m_1}2^{m_2}\dots(j-1)^{m_{j-1}}$. Finally, we use more commutation moves to move $\underline{j}$ left past $v^{[i]}$. This means that $u^{[i+1]} = u^{[i]}\underline{j}$, $v^{[i+1]} = v^{[i]}$ and $y^{[i+1]}$ is $y^{[i]}$ with that first occurrence of $\underline{j}$ removed and the prefix adjusted as described.

Case 2: The first letter in $y^{[i]}$ that is not in $[a]$ is $\overline{j}$. Let $\mathbf{y}$ be the prefix of y of all letters before $\overline{j}$, and let $\mathbf{z}$ be its image under f. We can preform commutation moves on $\mathbf{z}$ to get $b^{m_b}(b-1)^{m_{b-1}}\dots 1^{m_1}$ where $m_b \geq m_{b-1} \geq \dots$. We also have that $m_j = m_{j+1}$. We can use commutation moves to transform the subsequence $(j+1)^{m_j}j^{m_j}$ into $((j+1)j)^{m_j}$. So we have just transformed $\mathbf{z}$ into $b^{m_b}\dots(j+2)^{m_{j+2}}((j+1)j)^{m_j}(j-1)^{m_{j-1}}\dots 1^{m_1}$. We can use commutation moves to move $\overline{j}$ left past $(j-1)^{m_{j-1}}\dots 1^{m_1}$, braid moves to move $\overline{j}$ further left past $((j+1)j)^{m_j}$ and some more commutation moves to move $\overline{j}$ left past $b^{m_b}\dots(j+2)^{m_{j+2}}$. This means that $u^{[i+1]} = u^{[i]}$, $v^{[i+1]} = v^{[i]}\overline{j}$ and $y^{[i+1]}$ is $y^{[i]}$ with that first occurrence of $\overline{j}$ removed and the prefix adjusted as described, but expressed as a ballot sequence, rather than a reverse ballot sequence. $\square$

4.2. Diameters of the graphs for inflations $21[\alpha, \iota_b]$ and the longest permutations

In this section we prove recursive bounds for the diameters of the graphs for 21-inflations of the form $21[\alpha, \iota_b]$ for some permutation α. The recursive bounds will give us exact formulas for the diameters of graphs for $21[\alpha, 1]$. Because the longest permutation δ_n can be written as such an inflation we will have a recursion for the diameter of the graphs for δ_n as well.

Since the inflations we are considering are a subcollection of the inflations in Subsection 4.1 we can use Theorem 4.6 to describe a graph isomorphic to G_π. This graph is on the collection of vertices

$$V_{\alpha,\iota_b} = \bigcup_{x\in \mathrm{Ballot}_{a,b},\, u\in \underline{R}(\alpha)} \overline{\mathrm{Shuff}}(x, u, \emptyset).$$

The associated graph H_{α,ι_b} on V_{α,ι_b} will have edges from the following relations between its vertices.

1. Commutation moves or relations that come from exchanging adjacent
   (a) $w_iw_{i+1} = \underline{jk}$ if $|j-k| > 1$
   (b) $w_iw_{i+1} = jk$ if either $j > k$ or $j < k$ and $N_j(w^{(i-1)}) > N_k(w^{(i-1)})$
   (c) $w_iw_{i+1} = \underline{j}k$ or $w_iw_{i+1} = k\underline{j}$ if $|j-k| > 1$ or $j > k$
2. Braid moves or relations that come exchanging the following occurrences on consecutive indices

(a) in w we exchange $\underline{j(j+1)j}$ and $\underline{(j+1)j(j+1)}$
(b) in w we exchange $\underline{j}j(j+1)$ and $j(j+1)\underline{j}$

We will call moves of the type 1(a) and 2(a), which happen purely on the letters in $\underline{[a-1]}$, *α-moves.* We will call moves of type 1(b) *ballot-moves*, which happen purely on the letters in $[a]$. All the remaining moves we will call *shift-moves.*

Corollary 4.7. *The graph H_{α,ι_b} is isomorphic to G_π for $\pi = 21[\alpha, \iota_b]$.*

We are now ready to prove recursions on the diameters of graphs G_π, C_π and B_π where $\pi = 21[\alpha, \iota_b]$. It will be helpful to define a few statistics on V_{α,ι_b}. Given $w \in V_{\alpha,\iota_b}$ let

$$\begin{aligned}
\mathrm{Cshift}(w) &= |\{(i,i') : i < i', w_i = j, w_{i'} = \underline{k}, j \neq k, j \neq k+1\}|,\\
\mathrm{Bshift}(w) &= |\{(i,i') : i < i', w_i = j, w_{i'} = \underline{j}\}| \text{ and}\\
\mathrm{ballot}(w) &= |\{(i,i') : i < i', w_i = j, w_{i'} = k, j > k\}|.
\end{aligned}$$

Example 4.8. Let $\alpha = 321$, $b = 2$ and $w = \underline{1}1123\underline{2}2\underline{1}3$ in V_{α,ι_2}. Then $\mathrm{Cshift}(w) = 3$, $\mathrm{Bshift}(w) = 3$ and $\mathrm{ballot}(w) = 1$.

We first describe minimal paths between certain vertices in G_π in Lemma 4.9 and Lemma 4.10.

Lemma 4.9. *Let $w \in V_{\alpha,\iota_b}$ be a shuffle of $u \in \underline{R}(\alpha)$ and $x \in \mathrm{Ballot}_{a,b}$. Let $\tilde{x} = (12\dots a)^b \in \mathrm{Ballot}_{a,b}$.*

(a) There exists a path from w to $u\tilde{x}$ with $\mathrm{Cshift}(w) + \binom{a}{2}\binom{b}{2} - \mathrm{ballot}(w)$ commutation steps plus $\mathrm{Bshift}(w)$ braid steps.
(b) There exists a path from w to $\tilde{x}u$ with $\ell(\alpha)b(a-2) - \mathrm{Cshift}(w) + \binom{a}{2}\binom{b}{2} - \mathrm{ballot}(w)$ commutation steps plus $\ell(\alpha)b - \mathrm{Bshift}(w)$ braid steps.

Proof. Let $w \in V_{\alpha,\iota_b}$ be a shuffle of $u \in \underline{R}(\alpha)$ and $x \in \mathrm{Ballot}_{a,b}$. Let $\tilde{x} = (12\dots a)^b \in \mathrm{Ballot}_{a,b}$. We describe the path from w to $\tilde{w} = u\tilde{x}$ recursively. Let $w = u^{[0]}y^{[0]}$ where $u^{[0]}$ is empty and $y^{[0]} = w$. Suppose for some $i \geq 0$ that $u^{[i]}$ is a prefix of u and $y^{[i]}$ is a shuffle of the remaining letters of u in the same order and some ballot sequence so that $u^{[i]}y^{[i]} \in V_{\alpha,\iota_b}$. In each step the number of letters in $y^{[i]}$ will decrease by one as we move one occurrence of $\underline{j}$ left and add it to $u^{[i]}$. We will end with $y^{[k]}$, which is a pure ballot sequence and is equal to $\tilde{x}$. We will now describe how you construct $u^{[i+1]}y^{[i+1]}$.

Case 1: The word $y^{[i]}$ contains some letter $\underline{j}$, and say that $\underline{j}$ is the first from left to right and $\mathbf{y}$ is the prefix of $y^{[i]}$ before $\underline{j}$. If $\mathbf{y}$ is empty then let $u^{[i+1]} = u^{[i]}\underline{j}$ and $y^{[i+1]}$ be $y^{[i]}$ with the $\underline{j}$ removed. If $\mathbf{y}$ is not empty then $\mathbf{y}$ starts with 1 and has some maximal letter m_1. We can use ballot commutation moves to bring the first occurrences of $[m_1]$ to the front to form $(12\ldots m_1)$. Note that as we move $k \in [m_1]$ left into its place it is only shifting left past smaller letters and each shift increases $\mathrm{ballot}(\mathbf{y})$ by one. We have now transformed $\mathbf{y}$ into $(12\ldots m_1)\mathbf{y}'$. Repeat this process on $\mathbf{y}'$ until we transform $\mathbf{y}$ into $\mathbf{y}'' = (12\ldots m_1)(12\ldots m_2)\cdots(12\ldots m_k)$ where $m_1 \geq m_2 \geq \ldots \geq m_k$. By our conditions on w we know that $m_l \neq j$ for all l. Note that each of these commutation ballot moves have been increasing $\mathrm{ballot}(\mathbf{y})$ by one. Next we will take the $\underline{j}$ and shift it left through $\mathbf{y}''$. When we shift $\underline{j}$ left past a $l \neq j, j+1$ we are decreasing $\mathrm{Cshift}(w)$ by one, and when we shift $\underline{j}$ past the adjacent pair $j(j+1)$ we are decreasing $\mathrm{Bshift}(w)$ by one. We have just described how to shift $\underline{j}$ left past $\mathbf{y}''$, so we append $\underline{j}$ to the right side of $u^{[i]}$ to form $u^{[i+1]}$, and we remove the $\underline{j}$ from $y^{[i]}$ and adjust the prefix to $\mathbf{y}''$ to get $y^{[i+1]}$.

Case 2: The word $y^{[i]}$ does not contain any letter $\underline{j}$, so is a ballot sequence. If $y^{[i]}$ is not $\tilde{x}$, use the process above, that we used to transform $\mathbf{y}$ into $\mathbf{y}''$, to transform $y^{[i]}$ into $\tilde{x}$. With this our recursive process is complete.

Note that in the recursive process we have been increasing $\mathrm{ballot}(w)$ until $\mathrm{ballot}(w)$ equals $\mathrm{ballot}(\tilde{x}) = \binom{a}{2}\binom{b}{2}$. This means in this process we have used $\binom{a}{2}\binom{b}{2} - \mathrm{ballot}(w)$ commutation ballot-moves. Also in the recursive process we have been decreasing $\mathrm{Cshift}(w)$ until $\mathrm{Cshift}(w)$ equalled 0 by using $\mathrm{Cshift}(w)$ commutation shift-moves. Finally, in the recursive process we have been decreasing $\mathrm{Bshift}(w)$ until $\mathrm{Bshift}(w)$ equalled 0 by using $\mathrm{Bshift}(w)$ braid shift-moves. Hence, we have described a path from w to $u\tilde{x}$ with $\mathrm{Cshift}(w) + \binom{a}{2}\binom{b}{2} - \mathrm{ballot}(w)$ commutation steps plus $\mathrm{Bshift}(w)$ braid steps.

We can describe the path from w to $w' = \tilde{x}u$ recursively using a very similar manner. Along this path the value of $\mathrm{ballot}(w)$ will increase until $\mathrm{ballot}(w)$ equals $\mathrm{ballot}(\tilde{x}) = \binom{a}{2}\binom{b}{2}$, so will use $\binom{a}{2}\binom{b}{2} - \mathrm{ballot}(w)$ commutation ballot-moves. The path moves letters $\underline{j}$ of u to the right, so along this path the value of $\mathrm{Cshift}(w)$ will increase to its maximum value $\mathrm{Cshift}(\tilde{x}u) = \ell(\alpha)b(a-2)$ by using $\ell(\alpha)b(a-2) - \mathrm{Cshift}(w)$ commutation shift-moves. Again, the path moves letters $\underline{j}$ of u to the right, so along this path the value of $\mathrm{Bshift}(w)$ will increase to its maximum value $\mathrm{Bshift}(\tilde{x}u) = \ell(\alpha)b$ by using $\ell(\alpha)b - \mathrm{Bshift}(w)$ braid shift-moves. Hence, there exists a path from w to $\tilde{x}u$ with $\ell(\alpha)b(a-2) - \mathrm{Cshift}(w) + \binom{a}{2}\binom{b}{2} - \mathrm{ballot}(w)$ commutation steps plus $\ell(\alpha)b - \mathrm{Bshift}(w)$ braid steps. $\square$

Lemma 4.10. *Let H_1 be the induced subgraph of H_{α,ι_b} on vertices $\{ux : u \in \underline{R}(\alpha), x \in \mathrm{Ballot}_{a,b}\}$, and H_2 be the induced subgraph on vertices $\{xu : u \in \underline{R}(\alpha), x \in \mathrm{Ballot}_{a,b}\}$.*

(a) Any path from $w \in H_1$ to $w' \in H_2$ requires at least $\ell(\alpha)b(a-2)$ commutation steps plus $\ell(\alpha)b$ braid steps.
(b) For any $w, w' \in V_{\alpha,\iota_b}$ where the ballot sequence in w' is $\tilde{x} = (12\ldots a)^b$, any path from w to w' has at least $\binom{a}{2}\binom{b}{2} - \mathrm{ballot}(w)$ ballot-steps.

Proof. Let H_1 be the subgraph of H_{α,ι_b} on vertices $\{ux : u \in \underline{R}(\alpha), x \in \mathrm{Ballot}_{a,b}\}$, and H_2 be the subgraph on vertices $\{xu : u \in \underline{R}(\alpha), x \in \mathrm{Ballot}_{a,b}\}$. Consider a path P from $w \in H_1$ to $w' \in H_2$. Note that $\mathrm{Cshift}(w') = \ell(\alpha)b(a-2)$, $\mathrm{Bshift}(w') = \ell(\alpha)b$ and $\mathrm{Cshift}(w) = \mathrm{Bshift}(w) = 0$. Also, note that any step in P will be either an α-move, ballot-move or a shift-move. A ballot or α-move won't change the value of Cshift or Bshift and a shift-move will only change Cshift or Bshift, but not both, by exactly one. This means in order to get from w to w' we require $\mathrm{Cshift}(w') = \ell(\alpha)b(a-2)$ commutation shift-moves and $\mathrm{Bshift}(w') = \ell(\alpha)b$ braid shift-moves.

Let $w, w' \in V_{\alpha,\iota_b}$ where the ballot sequence in w' is $\tilde{x} = (12\ldots a)^b$. Consider a path P from w to w'. Note that $\mathrm{ballot}(w') = \binom{a}{2}\binom{b}{2}$. Also, note that any step in P will be either an α-move, ballot-move or a shift-move. A shift-move or an α-move won't change the value of $\mathrm{ballot}(w)$ and a ballot-move will only change $\mathrm{ballot}(w)$ by exactly one. This means in order to get from w to w' we require $\binom{a}{2}\binom{b}{2} - \mathrm{ballot}(w)$ commutation ballot-moves. □

Theorem 4.11. *Let $\pi = 21[\alpha, \iota_b]$ and $|\alpha| = a$.*

(i) $\mathrm{diam}(G_\alpha) + \ell(\alpha)b(a-1) + \binom{a}{2}\binom{b}{2} \le \mathrm{diam}(G_\pi) \le \mathrm{diam}(G_\alpha) + \ell(\alpha)b(a-1) + 2\binom{a}{2}\binom{b}{2}$
(ii) $\mathrm{diam}(C_\pi) = \mathrm{diam}(C_\alpha) + \ell(\alpha)b$
(iii) $\mathrm{diam}(B_\alpha) + \ell(\alpha)b(a-2) + \binom{a}{2}\binom{b}{2} \le \mathrm{diam}(B_\pi) \le \mathrm{diam}(B_\alpha) + \ell(\alpha)b(a-2) + 2\binom{a}{2}\binom{b}{2}$.

Proof. By Corollary 4.7 we know that G_π is isomorphic to H_{α,ι_b}, so it suffices to prove this theorem on the graph H_{α,ι_b}. We will need to consider specific subgraphs of H_{α,ι_b}. The first is on the vertices $\{ux : u \in \underline{R}(\alpha), x \in \mathrm{Ballot}_{a,b}\}$, which we will call H_1. The second is on the vertices $\{xu : u \in \underline{R}(\alpha), x \in \mathrm{Ballot}_{a,b}\}$, which we will call H_2. In our proof we will be using two very particular ballot sequences $\tilde{x} = (12\ldots a)^b$ and $\tilde{y} = 1^b2^b\ldots a^b$.

Let $w, w' \in V_{\alpha,\iota_b}$ be shuffles of $u, u' \in \underline{R}(\alpha)$ and $x, x' \in \mathrm{Ballot}_{a,b}$ respectively. We can construct two different paths from w to w'. By Lemma 4.9

we can construct one path P_1 through H_1 by starting at w, proceeding to $u\tilde{x}$, then $u'\tilde{x}$ and finally w'. This path uses

$$\mathrm{Cshift}(w) + \mathrm{Cshift}(w') + 2\binom{a}{2}\binom{b}{2} - \mathrm{ballot}(w) - \mathrm{ballot}(w')$$

commutation steps,

$$\mathrm{Bshift}(w) + \mathrm{Bshift}(w')$$

braid steps, and $d(u, u')$ α-moves. By Lemma 4.9 we can construct a second path P_2 through H_2 by starting at w, proceeding to $\tilde{x}u$, then $\tilde{x}u'$ and finally w'. This path uses

$$2\ell(\alpha)b(a-2) - \mathrm{Cshift}(w) - \mathrm{Cshift}(w') + 2\binom{a}{2}\binom{b}{2} - \mathrm{ballot}(w) - \mathrm{ballot}(w')$$

commutation steps,

$$2\ell(\alpha)b - \mathrm{Bshift}(w) - \mathrm{Bshift}(w')$$

braid steps, and $d(u, u')$ α-moves. Either P_1 or P_2 is bounded above by the average, which has $\ell(\alpha)b(a-2) + \mathrm{ballot}(w) + \mathrm{ballot}(w')$ commutation steps and $\ell(\alpha)b$ braid steps. Note that $d(u, u')$ is bounded above by $\mathrm{diam}(G_\alpha)$. This means

$$d(w, w') \le \mathrm{diam}(G_\alpha) + \ell(\alpha)b(a-1) + 2\binom{a}{2}\binom{b}{2}.$$

This proves the upper bound for the diameter of G_π. Using the same argument we can show that we have found a path from w to w' that uses at most $\mathrm{diam}(B_\alpha) + \ell(\alpha)b(a-2) + 2\binom{a}{2}\binom{b}{2}$ commutation moves and at most $\mathrm{diam}(C_\alpha) + \ell(\alpha)b$ braid moves. This gives us an upper bound for both $\mathrm{diam}(C_\pi)$ and $\mathrm{diam}(B_\pi)$

Now let u, u' be two reduced words in $R(\alpha)$ with $d(u, u') = \mathrm{diam}(G_\alpha)$. Consider the vertices $w = \underline{u}\tilde{x}$ and $w' = \tilde{y}\underline{u}'$ of H_{α,ι_b} and P a path between them. Because w is in H_1 and w' is in H_2 by Lemma 4.10 we know that we have at least $\ell(\alpha)b(a-2)$ shift-steps that are commutation steps plus $\ell(\alpha)b$ shift-steps that are braid steps. The path P will also contain at least $\mathrm{ballot}(w') = \binom{a}{2}\binom{b}{2}$ ballot steps that are commutation steps. Finally, we must have at least $\mathrm{diam}(G_\alpha)$ α-steps else we could project our path onto G_α

and get a shorter path from u to u'. All together this means that any path from w to w' has length at least

$$\mathrm{diam}(G_\alpha) + \ell(\alpha)b(a-1) + \binom{a}{2}\binom{b}{2}.$$

We can similarly argue that any path w to w' has at least $\mathrm{diam}(C_\alpha) + \ell(\alpha)b$ braid moves and at least $\mathrm{diam}(B_\alpha) + \ell(\alpha)b(a-2) + \binom{a}{2}\binom{b}{2}$ commutation moves. This proves the lower bounds on C_π and B_π respectively. □

When $b = 1$ in Theorem 4.11, we get the following corollary.

Corollary 4.12. *Let $\pi = 21[\alpha, 1]$ and $|\alpha| = a$.*

(i) $\mathit{diam}(G_\pi) = \mathit{diam}(G_\alpha) + \ell(\alpha)(a-1)$
(ii) $\mathit{diam}(C_\pi) = \mathit{diam}(C_\alpha) + \ell(\alpha)$
(iii) $\mathit{diam}(B_\pi) = \mathit{diam}(B_\alpha) + \ell(\alpha)(a-2)$

Because $\delta_{n+1} = 21[\delta_n, 1]$, we get the following corollary.

Corollary 4.13. *We have the following recursions for the diameters $D(n) = \mathit{diam}(G_{\delta_n})$, $C(n) = \mathit{diam}(C_{\delta_n})$ and $B(n) = \mathit{diam}(B_{\delta_n})$ with $D(1) = C(1) = B(1) = 0$.*

(i) $D(n+1) = D(n) + (n-1)\binom{n}{2}$
(ii) $C(n+1) = C(n) + \binom{n}{2}$
(iii) $B(n+1) = B(n) + (n-2)\binom{n}{2}$

5. Diameters of the graphs for 312 or 231 avoiding permutations

In this section, we first describe how we can use symmetries on a square to justify cases where graphs G_π, C_π and B_π are isomorphic. Next, we describe 312 or 231 pattern avoiding permutations in terms of 12-inflations and 21-inflations. We then find exact recursive formulas for the diameters of the graphs G_π, C_π and B_π where the permutation π is 312-avoiding or 231-avoiding. In order to do this, we use Theorem 3.6 and Corollary 4.12, the recursive formulas for the diameters of G_π, C_π and B_π when $\pi = 12[\alpha, \beta]$ and $\pi = 21[\alpha, 1]$ for any permutations α and β.

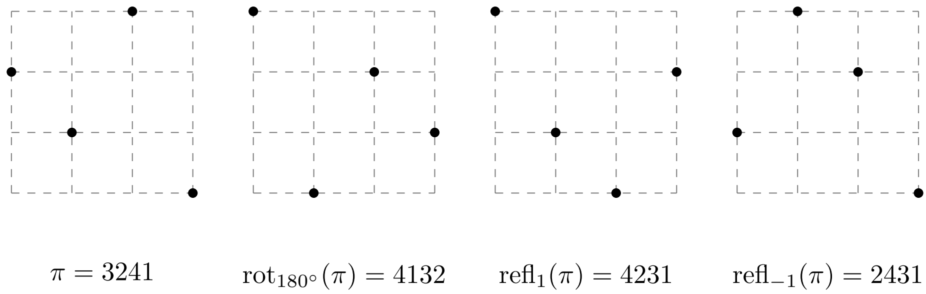

Figure 5: For $\pi = 3241$, we see that $\mathrm{rot}_{180^\circ}(\pi) = 4132$, $\mathrm{refl}_1(\pi) = 4231$, and $\mathrm{refl}_{-1}(\pi) = 2431$.

5.1. Symmetries

There are many pairs of permutations π and σ where G_π and G_σ are isomorphic. We will discuss three such cases related to symmetries on the square, the dihedral group. These graph isomorphisms will allow us to extend some of our previous results, and more quickly justify our families of permutations that achieve the upper bound of Conjecture 6.2.

There are eight operations in the dihedral group that preserve the square, and all of them can be described as a rotation rot_θ counter-clockwise by θ degrees or a reflection refl_m over a line with slope m going through the center of the square. If we perform the same operation on our box diagram of a permutation π, then the output is another box diagram of a permutation, which we will notate as $\mathrm{rot}_\theta(\pi)$ and $\mathrm{refl}_m(\pi)$. We will only be interested in three of these operations, rot_{180°, refl_1 and refl_{-1}.

Example 5.1. Let $\pi = 3241$ where $R(\pi) = \{1231, 1213, 2123\}$. Then $\mathrm{rot}_{180^\circ}(\pi) = 4132$ and $R(r_{180^\circ}(\pi)) = \{3213, 3231, 2321\}$. Also, $\mathrm{refl}_1(\pi) = 4213$ and $R(r_1(\pi)) = \{1321, 3121, 3212\}$. Finally, $\mathrm{refl}_{-1}(\pi) = 2431$ and $R(r_{-1}(\pi)) = \{3123, 1323, 1232\}$. We illustrate this in Figure 5.

Lemma 5.2. *Let $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathfrak{S}_n$.*

1. *Let $\sigma = \mathrm{rot}_{180^\circ}(\pi)$. Then, $\sigma = (n-\pi_n+1)(n-\pi_{n-1}+1)\ldots(n-\pi_1+1)$. Also, there is a bijection $R(\pi) \to R(\sigma)$ defined by $r_1r_2\cdots r_\ell \mapsto (n-r_1)(n-r_2)\cdots(n-r_\ell)$.*
2. *Let $\sigma = \mathrm{refl}_1(\pi)$. Then, $\sigma = \pi^{-1}$. Also, there is a bijection $R(\pi) \to R(\sigma)$ defined by $r_1r_2\cdots r_\ell \mapsto r_\ell r_{\ell-1}\cdots r_1$.*

3. *Let* $\sigma = \mathrm{refl}_{-1}(\pi)$. *Then,* $\sigma = (\mathrm{rot}_{180^\circ}(\pi))^{-1}$. *Also, there is a bijection* $R(\pi) \to R(\sigma)$ *defined by* $r_1 r_2 \cdots r_\ell \mapsto (n - r_\ell)(n - r_{\ell-1}) \cdots (n - r_1)$.

Proof. Let $\pi \in \mathfrak{S}_n$. First note that $\mathrm{rot}_{180^\circ} = \mathrm{refl}_\infty \circ \mathrm{refl}_0$. The reflection over the line with slope $m = 0$ takes π_i to $n - \pi_i + 1$, which is equivalent to $\mathrm{refl}_0(\pi) = \pi \circ \delta_n$. The reflection over the vertical line reverses the order of π, which is equivalent to $\mathrm{refl}_\infty(\pi) = \delta_n \circ \pi$. Thus $\mathrm{rot}_{180^\circ}(\pi) = (n - \pi_n + 1)(n - \pi_{n-1} + 1) \ldots (n - \pi_1 + 1)$. Further note that this means that $\mathrm{rot}_{180^\circ}(\pi) = \delta_n \circ \pi \circ \delta_n$. This implies that $\mathrm{rot}_{180^\circ}(s_i) = s_{n-i}$. If $\pi = s_{r_1} s_{r_2} \cdots s_{r_\ell}$, then

$$\mathrm{rot}_{180^\circ}(\pi) = \delta_n \pi \delta_n = (\delta_n s_{r_1} \delta_n)(\delta_n s_{r_2} \delta_n) \cdots (\delta_n s_{r_\ell} \delta_n) = s_{n-r_1} s_{n-r_2} \cdots s_{n-r_\ell},$$

which proves the stated bijection.

Now let $\sigma = \mathrm{refl}_1(\pi)$. The point (i, π_i) in the box diagram of π becomes (π_i, i) in the box diagram of σ. This implies that $\sigma = \pi^{-1}$. Further we have a bijection $R(\pi) \to R(\sigma)$ because if $r_1 r_2 \cdots r_\ell \in R(\pi)$ then $\pi = s_{r_1} s_{r_2} \cdots s_{r_\ell}$. Certainly $\pi^{-1} = s_{r_\ell} s_{r_{\ell-1}} \cdots s_{r_1}$, so $r_\ell r_{\ell-1} \cdots r_1 \in R(\sigma)$.

Finally let us consider $\sigma = \mathrm{refl}_{-1}(\pi)$. Note that refl_{-1} is equal to the composition $\mathrm{rot}_{180^\circ} \circ \mathrm{refl}_1$. Since we have already proven this for the maps rot_{180° and refl_1 we are done. ☐

Because of the bijections on reduced words defined in Lemma 5.2 we can see that for every commutation move and braid move made on $r \in R(\pi)$ there will be a corresponding move in its image in $R(\mathrm{rot}_\theta(\pi))$ or $R(\mathrm{refl}_m(\pi))$. This preserves the graph structure, the edge type and justifies why the corresponding graphs of π, $\mathrm{rot}_\theta(\pi)$ and $\mathrm{refl}_m(\pi)$ on reduced words, commutation classes and braid classes are isomorphic, for $\theta = 180^\circ$ and $m \in \{1, -1\}$.

Proposition 5.3. *For the following pairs of permutations,* π *and* σ*, the graphs* G_π*,* C_π *and* B_π *are isomorphic to* G_σ*,* C_σ *and* B_σ *respectively, so the diameters are also equal.*

1. π *and* $\sigma = \mathrm{rot}_{180^\circ}(\pi)$
2. π *and* $\sigma = \mathrm{refl}_1(\pi)$
3. π *and* $\sigma = \mathrm{refl}_{-1}(\pi)$

5.2. Diameters of the graphs for 312-avoiding permutations

Let π be a 312-avoiding permutation with $|\pi| = n$. Let $m \in [n]$ be such that $\pi_m = 1$. Since π is 312-avoiding, we should have $\pi_i < \pi_j$ for all $i < m < j$. Thus, we can write π as

$$\pi = 12[21[\pi', 1], \pi'']$$

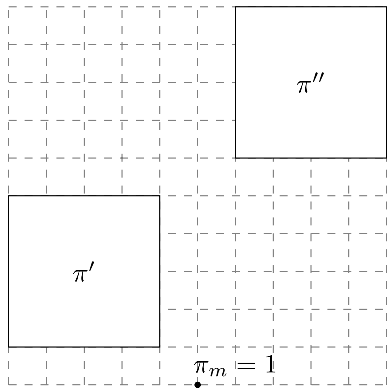

Figure 6: The structure of the 312-avoiding permutation $\pi = 12[21[\pi', 1], \pi'']$.

for two 312-avoiding permutations π' and π'' with $|\pi'| = m - 1$ and $|\pi''| = n - m$. Figure 6 would be helpful to see the structure of the permutation π. Hence, we have the following recursive formulas.

Theorem 5.4. *Let π be a 312-avoiding permutation in $\mathfrak{S}_n$. Then, $\pi = 12[21[\pi', 1], \pi'']$ for π' and π'' with $|\pi'| = m-1$ and $|\pi''| = n-m$. Moreover,*

(i) $diam(G_\pi) = diam(G_{\pi'}) + diam(G_{\pi''}) + (m-1)(\ell(\pi') + \ell(\pi'')) + \ell(\pi')(\ell(\pi'') - 1)$

(ii) $diam(C_\pi) = diam(C_{\pi'}) + diam(C_{\pi''}) + \ell(\pi')$

(iii) $diam(B_\pi) = diam(B_{\pi'}) + diam(B_{\pi''}) + (m-1)(\ell(\pi') + \ell(\pi'')) + \ell(\pi')(\ell(\pi'') - 2)$

Proof. By Theorem 3.6 and Corollary 4.12, we see that the diameter of G_π is

$$\begin{aligned}\mathrm{diam}(G_\pi) &= \mathrm{diam}(G_{21[\pi',1]}) + \mathrm{diam}(G_{\pi''}) + \ell(21[\pi',1])\ell(\pi'') \\ &= \mathrm{diam}(G_{\pi'}) + \mathrm{diam}(G_{\pi''}) + (|\pi'| - 1)\ell(\pi') + \ell(\pi'')(|\pi'| + \ell(\pi')) \\ &= \mathrm{diam}(G_{\pi'}) + \mathrm{diam}(G_{\pi''}) + (m-1)(\ell(\pi') + \ell(\pi'')) \\ &\quad + \ell(\pi')(\ell(\pi'') - 1).\end{aligned}$$

We also have the diameter of C_π as follows.

$$\begin{aligned}\mathrm{diam}(C_\pi) &= \mathrm{diam}(C_{21[\pi',1]}) + \mathrm{diam}(C_{\pi''}) \\ &= \mathrm{diam}(C_{\pi'}) + \mathrm{diam}(C_{\pi''}) + \ell(\pi').\end{aligned}$$

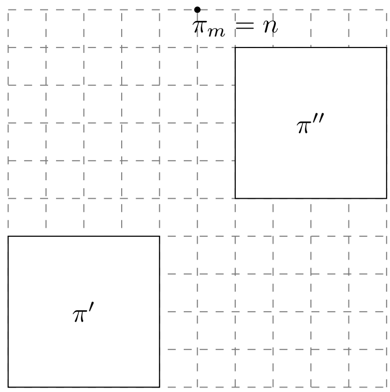

Figure 7: The structure of the 231-avoiding permutation $\pi = 12[\pi', 21[1, \pi'']]$.

Lastly, we see that the diameter of B_π is

$$\begin{aligned}
\mathrm{diam}(B_\pi) &= \mathrm{diam}(B_{21[\pi',1]}) + \mathrm{diam}(B_{\pi''}) + \ell(21[\pi',1])\ell(\pi'') \\
&= \mathrm{diam}(B_{\pi'}) + \mathrm{diam}(B_{\pi''}) + (|\pi'| - 2)\ell(\pi') + \ell(\pi'')(|\pi'| + \ell(\pi')) \\
&= \mathrm{diam}(B_{\pi'}) + \mathrm{diam}(B_{\pi''}) + (m-1)(\ell(\pi') + \ell(\pi'')) \\
&\quad + \ell(\pi')(\ell(\pi'') - 2).
\end{aligned}$$ □

5.3. Diameters of the graphs for 231-avoiding permutations

Let π be a 231-avoiding permutation with $|\pi| = n$. Let $m \in [n]$ be such that $\pi_m = n$. Since π is 231-avoiding, we should have $\pi_i < \pi_j$ for all $i < m < j$. Thus, we can write π as

$$\pi = 12[\pi', 21[1, \pi'']]$$

for two 231-avoiding permutations π' and π'' with $|\pi'| = m - 1$ and $|\pi''| = n - m$. Figure 7 would be helpful to see the structure of the permutation π. Hence, we have the following recursive formulas.

Theorem 5.5. *Let π be a 231-avoiding permutation in $\mathfrak{S}_n$. Then, $\pi = 12[\pi', 21[1, \pi'']]$ for π' and π'' with $|\pi'| = m - 1$ and $|\pi''| = n - m$. Moreover,*

(i) $diam(G_\pi) = diam(G_{\pi'}) + diam(G_{\pi''}) + (n-m)(\ell(\pi') + \ell(\pi'')) + \ell(\pi'')(\ell(\pi') - 1)$

(ii) $diam(C_\pi) = diam(C_{\pi'}) + diam(C_{\pi''}) + \ell(\pi'')$

(iii) $\mathit{diam}(B_\pi) = \mathit{diam}(B_{\pi'}) + \mathit{diam}(B_{\pi''}) + (n-m)(\ell(\pi') + \ell(\pi'')) + \ell(\pi'')(\ell(\pi') - 2)$

Proof. Suppose that $\pi \in \mathfrak{S}_n$ avoids 231. This means that $\pi = 12[\pi', 21[1, \pi'']]$. Then, $\sigma = \mathrm{rot}_{180^\circ}(\pi) = 12[21[\sigma'', 1], \sigma']$ where $\mathrm{rot}_{180^\circ}(\pi') = \sigma'$ and $\mathrm{rot}_{180^\circ}(\pi'') = \sigma''$. By Proposition 5.3 we know that G_π, C_π and B_π are isomorphic to G_σ, C_σ and B_σ, $G_{\pi'}$, $C_{\pi'}$ and $B_{\pi'}$ are isomorphic to $G_{\sigma'}$, $C_{\sigma'}$ and $B_{\sigma'}$ and $G_{\pi''}$, $C_{\pi''}$ and $B_{\pi''}$ are isomorphic to $G_{\sigma''}$, $C_{\sigma''}$ and $B_{\sigma''}$. Together with Theorem 5.4, we are done. $\square$

6. Connection to hyperplane arrangements

In this section we connect our results to Conjecture 6.2, Reiner and Roichman's conjecture. We prove that all permutations that avoid 231 or 312 satisfy the conjecture, and that these permutations achieve the upper bound of the conjecture. In addition, we found a set of permutations that achieve the lower bound of the conjecture.

As in Section 2, for an element $\pi \in \mathfrak{S}_n$, we define $I_2(\pi)$ to be the set of disjoint pairs of inversions $((i,j),(k,\ell))$ of π and define $I_3(\pi)$ to be the set of all triples of inversions $((i,j),(i,k),(j,k))$ of π. Then $L_2(\pi)$ can be interpreted as the union of $I_2(\pi)$ and $I_3(\pi)$. Note that $|I_3(\pi)|$ is the number of 321 patterns in π.

Example 6.1. Consider $\pi = 4312 \in \mathfrak{S}_4$. There are five inversions, those are $(1,2), (1,3), (1,4), (2,3)$, and $(2,4)$. Observe that there are two pairs of disjoint inversions, which are $((1,3),(2,4))$ and $((1,4),(2,3))$, and $|I_2(\pi)| = 2$. Note also that there are two 321 patterns, which are 431 and 432, and $|I_3(\pi)| = 2$, thus $|L_2(\pi)| = |I_2(\pi)| + |I_3(\pi)| = 4$.

Conjecture 6.2 (Reiner and Roichman)**.** *For $\pi \in \mathfrak{S}_n$,*

$$\frac{1}{2}|L_2(\pi)| \leq \mathit{diam}(G_\pi) \leq |L_2(\pi)|.$$

6.1. 12-inflations or 21-inflations

We first confirm the conjecture for 12-inflations of two permutations α and β and for 21-inflations of a permutation α and $\iota_b = 12\ldots b$.

Suppose $\pi = 12[\alpha, \beta] = \alpha_1\alpha_2\ldots\alpha_a(\beta_1 + a)(\beta_2 + a)\ldots(\beta_b + a) \in \mathfrak{S}_{a+b}$ is an 12-inflation of two permutations $\alpha \in \mathfrak{S}_a$ and $\beta \in \mathfrak{S}_b$. The disjoint pairs of inversions in π fall under one of the three cases: (i) disjoint pairs in α;

(ii) disjoint pairs in β; (iii) one inversion in α and the other inversion in β. Thus, we have

$$|I_2(\pi)| = |I_2(12[\alpha,\beta])| = |I_2(\alpha)| + |I_2(\beta)| + \ell(\alpha)\ell(\beta). \tag{1}$$

Since the 321 patterns in π are either 321 patterns in α or 321 patterns in β, we have

$$|I_3(\pi)| = |I_3(12[\alpha,\beta])| = |I_3(\alpha)| + |I_3(\beta)|. \tag{2}$$

By equation (1) and equation (2), we see that

$$\begin{aligned}|L_2(\pi)| &= |I_2(\pi)| + |I_3(\pi)| \\ &= |I_2(\alpha)| + |I_2(\beta)| + \ell(\alpha)\ell(\beta) + |I_3(\alpha)| + |I_3(\beta)| \\ &= |L_2(\alpha)| + |L_2(\beta)| + \ell(\alpha)\ell(\beta).\end{aligned}$$

The observation above together with the diameter formula $\mathrm{diam}(G_\pi) = \mathrm{diam}(G_\alpha) + \mathrm{diam}(G_\beta) + \ell(\alpha)\ell(\beta)$ in Theorem 3.6 prove the following proposition.

Proposition 6.3. *If Conjecture 6.2 is true for both α and β, then the conjecture is also true for the 12-inflation of α and β. Moreover, if both diam(G_α) and diam(G_β) hit the upper bound, then diam$(G_{12[\alpha,\beta]})$ also hits the upper bound.*

Proof. Suppose $\frac{1}{2}|L_2(\alpha)| \le \mathrm{diam}(G_\alpha) \le |L_2(\alpha)|$ and $\frac{1}{2}|L_2(\beta)| \le \mathrm{diam}(G_\beta) \le |L_2(\beta)|$. Then we see that

$$\begin{aligned}\mathrm{diam}(G_\pi) &= \mathrm{diam}(G_\alpha) + \mathrm{diam}(G_\beta) + \ell(\alpha)\ell(\beta) \\ &\le |L_2(\alpha)| + |L_2(\beta)| + \ell(\alpha)\ell(\beta) = |L_2(\pi)|\end{aligned}$$

and

$$\begin{aligned}\mathrm{diam}(G_\pi) &= \mathrm{diam}(G_\alpha) + \mathrm{diam}(G_\beta) + \ell(\alpha)\ell(\beta) \\ &\ge \frac{1}{2}|L_2(\alpha)| + \frac{1}{2}|L_2(\beta)| + \frac{1}{2}\ell(\alpha)\ell(\beta) = \frac{1}{2}|L_2(\pi)|.\end{aligned}$$

In the case of $\mathrm{diam}(G_\alpha) = |L_2(\alpha)|$ and $\mathrm{diam}(G_\beta) = |L_2(\beta)|$, we see that

$$\begin{aligned}\mathrm{diam}(G_\pi) &= \mathrm{diam}(G_\alpha) + \mathrm{diam}(G_\beta) + \ell(\alpha)\ell(\beta) \\ &= |L_2(\alpha)| + |L_2(\beta)| + \ell(\alpha)\ell(\beta) = |L_2(\pi)|. \qquad \square\end{aligned}$$

Suppose $\pi = 21[\alpha, \iota_b] = (\alpha_1 + b)(\alpha_2 + b) \dots (\alpha_a + b)12 \dots b \in \mathfrak{S}_{a+b}$ is the 21-inflation of a permutation $\alpha \in \mathfrak{S}_a$ and $\iota_b = 12 \dots b$. The disjoint pairs of inversions in π fall under one of the three cases: (i) disjoint pairs in α; (ii) one inversion (i, j) in α and the other inversion (r, s) for $r \in [a] - \{i, j\}, a + 1 \leq s \leq a + b$; (iii) two disjoint inversions $((i, r), (j, s))$ for $i, j \in [a]$ and $a + 1 \leq r, s \leq a + b$. Thus, we have

$$|I_2(\pi)| = |I_2(21[\alpha, \iota_b])| = |I_2(\alpha)| + \ell(\alpha)(a-2)b + 2\binom{a}{2}\binom{b}{2}. \tag{3}$$

Since the 321 patterns in π are either 321 patterns in α or a subword $(\alpha_i + b)(\alpha_j + b)k$ of π where (i, j) is an inversion in α and $a + 1 \leq k \leq a + b$, we have

$$|I_3(\pi)| = |I_3(21[\alpha, \iota_b])| = |I_3(\alpha)| + \ell(\alpha)b. \tag{4}$$

By equation (3) and equation (4), we see that

$$\begin{aligned}
|L_2(\pi)| &= |I_2(\pi)| + |I_3(\pi)| \\
&= |I_2(\alpha)| + \ell(\alpha)(a-2)b + 2\binom{a}{2}\binom{b}{2} + |I_3(\alpha)| + \ell(\alpha)b \\
&= |L_2(\alpha)| + \ell(\alpha)(a-1)b + 2\binom{a}{2}\binom{b}{2}.
\end{aligned}$$

The observation above together with the upper and lower bounds for the diameters of G_π, $\mathrm{diam}(G_\alpha) + \ell(\alpha)b(a-1) + \binom{a}{2}\binom{b}{2} \leq \mathrm{diam}(G_\pi) \leq \mathrm{diam}(G_\alpha) + \ell(\alpha)b(a-1) + 2\binom{a}{2}\binom{b}{2}$ in Theorem 4.11 prove the following proposition.

Proposition 6.4. *If Conjecture 6.2 is true for α, then the conjecture is also true for the 21-inflation of α and ι_b for any $b \geq 1$.*

Proof. Suppose $\frac{1}{2}|L_2(\alpha)| \leq \mathrm{diam}(G_\alpha) \leq |L_2(\alpha)|$. Then we see that

$$\begin{aligned}
\mathrm{diam}(G_\pi) &\leq \mathrm{diam}(G_\alpha) + \ell(\alpha)b(a-1) + 2\binom{a}{2}\binom{b}{2} \\
&\leq |L_2(\alpha)| + \ell(\alpha)b(a-1) + 2\binom{a}{2}\binom{b}{2} = |L_2(\pi)|
\end{aligned}$$

and

$$\mathrm{diam}(G_\pi) \geq \mathrm{diam}(G_\alpha) + \ell(\alpha)b(a-1) + \binom{a}{2}\binom{b}{2}$$

$$\geq \frac{1}{2}|L_2(\alpha)| + \ell(\alpha)b(a-1) + \binom{a}{2}\binom{b}{2} = \frac{1}{2}|L_2(\pi)|. \qquad \square$$

Let us consider a special case $\pi = 21[\alpha, 1]$ when $b = 1$. Observe that Conjecture 6.2 is true for π since it is a special case of the previous proposition. Moreover, we get the next proposition by the diameter formula $\mathrm{diam}(G_\pi) = \mathrm{diam}(G_\alpha) + \ell(\alpha)(a-1)$ in Corollary 4.12.

Proposition 6.5. *If* $\mathrm{diam}(G_\alpha)$ *hits the upper bound, then* $\mathrm{diam}(G_{21[\alpha,1]})$ *also hits the upper bound.*

Proof. Suppose $\mathrm{diam}(G_\alpha) = |L_2(\alpha)|$. Observe that

$$\mathrm{diam}(G_\pi) = \mathrm{diam}(G_\alpha) + \ell(\alpha)(a-1) = |L_2(\alpha)| + \ell(\alpha)(a-1) = |L_2(\alpha)|$$

and the proof follows. $\square$

6.2. Pattern avoiding permutations

We show that Conjecture 6.2 is true for any permutations π that avoid 231 or 312 patterns. To do this, it is sufficient to show that $\mathrm{diam}(G_\pi) = |L_2(\pi)|$.

Theorem 6.6. *If π is a 312-avoiding permutation, then* $\mathrm{diam}(G_\pi) = |L_2(\pi)|$.

Proof. As in Section 5, we can write $\pi = 12[21[\pi', 1], \pi'']$ for two 312-avoiding permutations π' and π'' with $\pi_m = 1, |\pi| = n, |\pi'| = m-1$, and $|\pi''| = n-m$.

The disjoint pairs of inversions in π fall under one of the five cases: (i) disjoint pairs in π'; (ii) disjoint pairs in π''; (iii) one inversion in π' and the other inversion in π''; (iv) one inversion (r, s) in π'' and the other inversion (i, m) for $i \in [|\pi'|] = [m-1]$; (v) one inversion (i, j) in π' and the other inversion (k, m) where $1 \leq k \leq m-1$ and $k \notin \{i, j\}$. Thus, we have

$$|I_2(\pi)| = |I_2(\pi')| + |I_2(\pi'')| + \ell(\pi')\ell(\pi'') + \ell(\pi'')|\pi'| + \ell(\pi')(|\pi'|-2). \tag{5}$$

Since the 321 patterns in π are either 321 patterns in π' or in π'' or a subword $(\pi'_i + 1)(\pi'_j + 1)1$ where (i, j) is an inversion in π', we have

$$|I_3(\pi)| = |I_3(\pi')| + |I_3(\pi'')| + \ell(\pi'). \tag{6}$$

By equation (5) and equation (6) we see that

$$|L_2(\pi)| = |L_2(\pi')| + |L_2(\pi'')| + (|\pi'|-1)\ell(\pi') + \ell(\pi'')(|\pi'| + \ell(\pi')).$$

We will prove the theorem by induction on the size of the permutations. (Base case) If $|\pi| \leq 2$, then $\operatorname{diam}(G_\pi) = |L_2(\pi)| = 0$. (Induction) Suppose $\operatorname{diam}(G_\pi) = |L_2(\pi)|$ for all permutations with the size less than n. Then,

$$\begin{aligned}\operatorname{diam}(G_\pi) &= \operatorname{diam}(G_{\pi'}) + \operatorname{diam}(G_{\pi''}) + (|\pi'|-1)\ell(\pi') + \ell(\pi'')(|\pi'| + \ell(\pi')) \\ &= |L_2(\pi')| + |L_2(\pi'')| + (|\pi'|-1)\ell(\pi') + \ell(\pi'')(|\pi'| + \ell(\pi')) \\ &= |L_2(\pi)|\end{aligned}$$

and the proof follows. $\square$

Theorem 6.7. *If π is a 231-avoiding permutation, then diam$(G_\pi) = |L_2(\pi)|$.*

Proof. Note that because $\mathrm{rot}_{180^\circ}(321) = 321$ a 321 pattern in π will map to a 321 pattern in $\mathrm{rot}_{180^\circ}(\pi)$, so $|I_3(\pi)| = |I_3(\mathrm{rot}_{180^\circ}(\pi))|$. Also, since $\mathrm{rot}_{180^\circ}(21) = 21$ any pair of disjoint inversions will map to another pair of disjoint inversions in $\mathrm{rot}_{180^\circ}(\pi)$, so $|I_2(\pi)| = |I_2(\mathrm{rot}_{180^\circ}(\pi))|$. This means that $|L_2(\pi)| = |L_2(\mathrm{rot}_{180^\circ}(\pi))|$.

Consider a permutation π that avoids 231. Because $\mathrm{rot}_{180^\circ}(231) = 312$ we know that $\sigma = \mathrm{rot}_{180^\circ}(\pi)$ avoids 312. By Proposition 5.3 the graphs G_π and G_σ are isomorphic, so since $|L_2(\pi)| = |L_2(\sigma)|$ Theorem 6.6 completes the proof. $\square$

6.3. Lower bound

We have seen that the following permutations hit the upper bound of Conjecture 6.2.

(i) $\pi = 12[\alpha, \beta]$ if $\operatorname{diam}(G_\alpha) = |L_2(\alpha)|$ and $\operatorname{diam}(G_\beta) = |L_2(\beta)|$.
(ii) $\pi = 21[\alpha, 1]$ if $\operatorname{diam}(G_\alpha) = |L_2(\alpha)|$.
(iii) A permutation π that is either 231-avoiding or 312-avoiding.

Remark 6.8. One can come up with a question: Are all the permutations that hit the upper bound of the conjecture constructed in these three ways? The answer is no because we have a counter example, $\pi = 2413 \in \mathfrak{S}_4$. The permutation π satisfies none of three conditions, but $\operatorname{diam}(G_\pi) = |L_2(\pi)|$.

It is natural for us to pay attention to the lower bound of the conjecture, and we look for permutations π such that $\operatorname{diam}(G_\pi) = \frac{1}{2}|L_2(\pi)|$. We make a list of all permutations $\pi \in \mathfrak{S}_n$ for $n = 4, 5, 6$ such that $\operatorname{diam}(G_\pi) = \frac{1}{2}|L_2(\pi)|$ in Table 2. We also express them in terms of 12-inflations and 21-inflations of $\iota_k = 12\ldots k$ for some $k \geq 1$. This observation suggests the following conjecture.

Table 2: All permutations π such that $\mathrm{diam}(G_\pi) = \frac{1}{2}|L_2(\pi)|$ in $\mathfrak{S}_n$ for $n = 4, 5, 6$ and expressions in terms of 12-inflations and 21-inflations of ι_k for some $k \geq 1$

$\mathfrak{S}_4$		$\mathfrak{S}_5$		$\mathfrak{S}_6$	
3412	$21[\iota_2, \iota_2]$	14523	$12[\iota_1, 21[\iota_2, \iota_2]]$	125634	$12[\iota_2, 21[\iota_2, \iota_2]]$
		34125	$12[21[\iota_2, \iota_2], \iota_1]$	145236	$12[12[\iota_1, 21[\iota_2, \iota_2]], \iota_1]$
		34512	$21[\iota_3, \iota_2]$	145623	$12[\iota_1, 21[\iota_3, \iota_2]]$
		45123	$21[\iota_2, \iota_3]$	156234	$12[\iota_1, 21[\iota_2, \iota_3]]$
				341256	$12[21[\iota_2, \iota_2], \iota_2]$
				345126	$12[21[\iota_3, \iota_2], \iota_1]$
				345612	$21[\iota_4, \iota_2]$
				451236	$12[21[\iota_2, \iota_3], \iota_1]$
				456123	$21[\iota_3, \iota_3]]$
				561234	$21[\iota_2, \iota_4]$

Conjecture 6.9. *Let $\pi \in \mathfrak{S}_n$ be a permutation. Then the permutation can be written as $\pi = 12[12[\iota_c, 21[\iota_a, \iota_b]], \iota_d]$ with $a \geq 2$ and $b \geq 2$ if and only if diam$(G_\pi) = \frac{1}{2}|L_2(\pi)|$.*

We state and prove the "only if" direction of the conjecture in the following theorem.

Theorem 6.10. *Let $\pi \in \mathfrak{S}_n$ be a permutation. If we can write*

$$\pi = 12[12[\iota_c, 21[\iota_a, \iota_b]], \iota_d]$$

with $a, b \geq 2$ and $c, d \geq 0$, then diam$(G_\pi) = \frac{1}{2}|L_2(\pi)|$.

Remark 6.11. For the permutation π in Theorem 6.10, we can have $\pi = 21[\iota_a, \iota_b]$ when $c = 0$ and $d = 0$.

To prove Theorem 6.10, we first state and prove the following lemmas.

Lemma 6.12. *diam$(G_{12[\alpha,\iota_b]})$ = diam$(G_{12[\iota_b,\alpha]})$ = diam(G_α) for any permutation α.*

Proof. By applying Theorem 3.6, we can see that

$$\mathrm{diam}(G_{12[\alpha,\iota_b]}) = \mathrm{diam}(G_\alpha) + \mathrm{diam}(G_{\iota_b}) + \ell(\alpha)\ell(\iota_b)$$

$$= \text{diam}(G_\alpha) + 0 + \ell(\alpha) \cdot 0$$
$$= \text{diam}(G_\alpha).$$

Similarly, we can show $\text{diam}(G_{12[\iota_b,\alpha]}) = \text{diam}(G_\alpha)$. $\square$

By the lemma above, we only need to work on $21[\iota_a, \iota_b] \in \mathfrak{S}_n$ from now.

Lemma 6.13. *Let* $\pi = 21[\iota_a, \iota_b] \in \mathfrak{S}_n$. *Then,* $|L_2(\pi)| = 2\binom{a}{2}\binom{b}{2}$.

Proof. Let $\pi = 21[\iota_a, \iota_b] = (1+b)(2+b)\dots(a+b)12\dots b$. Since there are not any 321 patterns in π, we see that $|I_3(\pi)| = 0$. Note that all inversions in π are of the form $(i, j+b)$ for $j \in [a], i \in [b]$. Thus, the number $|I_2(\pi)|$ of disjoint pairs of inversions in π is $ab(a-1)(b-1)/2 = 2\binom{a}{2}\binom{b}{2}$. Therefore, $|L_2(\pi)| = |I_2(\pi)| + |I_3(\pi)| = 2\binom{a}{2}\binom{b}{2}$. $\square$

Lemma 6.14. *Let* $\pi = 21[\iota_a, \iota_b] \in \mathfrak{S}_n$. *Then, diam*$(G_\pi) = \binom{a}{2}\binom{b}{2}$.

Proof. In Section 4.1 we discussed that the graph $G_\pi = G_{21[\iota_a,\iota_b]}$ is isomorphic to the graph H_{ι_a,ι_b}. The vertex set $V(H_{\iota_a,\iota_b})$ is the set $\text{Ballot}_{a,b} = \{1^b2^b\dots a^b, \dots, (12\dots a)^b\}$ of all ballot sequences and the edges are formed from exchanging adjacent letters $w_iw_{i+1} = jk$ of $w \in \text{Ballot}_{a,b}$ if either (i) $j > k$ or (ii) $j < k$ and $N_j(w^{(i-1)}) > N_k(w^{(i-1)})$. Observe that the graph H_{ι_a,ι_b} is a graded poset of rank $\binom{a}{2}\binom{b}{2}$ with a unique maximum $(123\dots a)^b$ and a unique minimum $1^b2^b3^b\dots a^b$ by the rank function $\rho(w) := \text{ballot}(w) = |\{(i,i') : i < i', w_i = j, w_{i'} = k, j > k\}|$. See Figure 8 for an example of the Hasse diagram of $G_{21[\iota_3,\iota_2]} \cong H_{\iota_3,\iota_2}$.

We claim that the diameter of G_π is the rank $\binom{a}{2}\binom{b}{2}$ of the poset G_π. Since the distance between the maximum and minimum is $\binom{a}{2}\binom{b}{2}$, we have $\text{diam}(G_\pi) \geq \binom{a}{2}\binom{b}{2}$. Suppose $u \neq v$ are any ballot sequences in $\text{Ballot}_{a,b}$. If u and v are in a same chain, then $d(u,v) \leq \binom{a}{2}\binom{b}{2}$. Assume u and v are not in a same chain. Then we can construct two different paths from u to v. Let the first path P starts at u to the maximum element and ends at v. Let the second path Q starts at u to the minimum element and ends at v. Either P or Q is bounded above by the rank of the poset $\binom{a}{2}\binom{b}{2}$, and we see $\text{diam}(G_\pi) \leq \binom{a}{2}\binom{b}{2}$. This shows $\text{diam}(G_\pi) = \binom{a}{2}\binom{b}{2}$. $\square$

Proof of Theorem 6.10. Suppose $\pi = 12[12[\iota_c, 21[\iota_a, \iota_b]], \iota_d]$ with $a, b \geq 2$ and $c, d \geq 0$. By Lemma 6.12, we have $\text{diam}(G_\pi) = \text{diam}(G_{21[\iota_a,\iota_b]})$. By Lemma 6.13 and Lemma 6.14, we have

$$\text{diam}(G_\pi) = \binom{a}{2}\binom{b}{2} = \frac{1}{2}|L_2(\pi)|$$

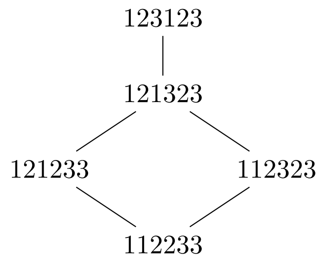

Figure 8: The Hasse diagram of $G_{21[\iota_3,\iota_2]} \cong H_{\iota_3,\iota_2}$.

and this completes the proof. $\square$

7. Future work

As we discussed in Remark 6.8, there are more families of permutations that achieve the upper bound of Conjecture 6.2. This should be one direction we can study in the future.

Open Problem 7.1. *Find the necessary and sufficient condition for permutations $\pi \in \mathfrak{S}_n$ to satisfy diam$(G_\pi) = |L_2(\pi)|$.*

After investigating the diameters of G_π for all permutations $\pi \in \mathfrak{S}_n$ for $n = 4, 5, 6$, we make the following conjecture.

Conjecture 7.2. *If $\pi \in \mathfrak{S}_n$ contains a pattern 3412, then diam$(G_\pi) < |L_2(\pi)|$.*

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