Journal of Combinatorics
Volume 14, Number 1, 139–153, 2023

Completing partial Latin squares with two filled rows and three filled columns*

Carl Johan Casselgren† and Herman Göransson

Consider a partial Latin square P where the first two rows and first three columns are completely filled, and every other cell of P is empty. It has been conjectured that all such partial Latin squares of order at least 8 are completable. Based on a technique by Kuhl and McGinn we describe a framework for completing partial Latin squares in this class. Moreover, we use our method for proving that all partial Latin squares from this family, where the intersection of the nonempty rows and columns form a Latin rectangle with three distinct symbols, are completable.

AMS 2000 subject classifications: Primary 05B15; secondary 05C15.
Keywords and phrases: Latin square, partial Latin square, completing partial Latin squares.

1. Introduction

Consider an $n \times n$ array P where each cell contains at most one symbol from $[n] = \{1, \dots, n\}$. P is called a *partial Latin square* if each symbol occurs at most once in every row and column. If no cell in P is empty, then it is a *Latin square*. An $r \times s$ array with entries from $\{1, \dots, n\}$, where $n = \max\{r, s\}$, is called a *Latin rectangle* if each symbol occurs at most once in every row and column, and no cell is empty. In all the preceding definitions n is referred to as the *order* of the (partial) Latin square or Latin rectangle, respectively.

The cell in position (i, j) in an array A is denoted by $(i, j)_A$, and the symbol in cell $(i, j)_A$ is denoted by $A(i, j)$. If $A(i, j) = k$, then k is an *entry* of cell $(i, j)_A$; we write $A(i, j) = \emptyset$ if $(i, j)_A$ is empty.

An $n \times n$ Latin square L is a *completion* of an $n \times n$ partial Latin square P if $L(i, j) = P(i, j)$ for each nonempty cell $(i, j)_P$ of P. P is *completable* if there is such a Latin square; otherwise, P is *non-completable*. The problem

*This paper is based on the Bachelor thesis [7] by Göransson written under the supervision of Casselgren.

†Casselgren was supported by a grant from the Swedish Research Council (2017-05077).

of completing partial Latin squares is a classic topic within combinatorics and several families of partial Latin squares have been proved to admit completions. Let us here just mention a few classic and recent results.

In general, it is an NP-complete problem to determine if a partial Latin square is completable [6]. Thus, it is natural to ask for completability of particular families of partial Latin squares. A classic result due to Ryser [12] states that if $n \geq r, s$, then every $n \times n$ partial Latin square whose nonempty cells form an $r \times s$ subrectangle Q is completable if and only if each of the symbols $1, \ldots, n$ occurs at least $r + s - n$ times in Q. Another classic result is Smetaniuk's proof [13] of Evans' conjecture [8], which states that every $n \times n$ partial Latin square with at most $n-1$ entries is completable. This was also independently proved by Andersen and Hilton [2].

Adams, Bryant and Buchanan [1] characterized which partial Latin squares with 2 completely filled rows and columns, and where all other cells are empty, are completable, and by results of Casselgren and Häggkvist [5], and Kuhl and Schroeder [10], all partial Latin squares of order at least 6 with all entries in one fixed column or row, or containing a prescribed symbol, are completable.

The result that all partial Latin squares with two filled rows and two filled columns of order at least 6 are completable was first proved in Buchanan's PhD thesis [4]. The shortened version in [1] is still over 25 pages long and also relies on a computer search for verifying completability for small orders. Quite recently, Kuhl and McGinn [9] gave a short proof of this result based on Smetaniuk's aforementioned proof of the famous Evans' conjecture. They also presented a conjecture on completing partial Latin squares with two filled rows and any number of filled columns. For the case of three filled columns their conjecture particularly implies the following.

Conjecture 1.1. *Every partial Latin square of order at least 8 with two completely filled rows and three completely filled columns, and where all other cells are empty, is completable.*

The non-completable partial Latin squares in Figure 1 show that the condition $n \geq 8$ in Conjecture 1.1 is necessary.

In this paper, we take the first step towards settling Conjecture 1.1 by proving it in the special case when the intersection of the nonempty rows and columns form a Latin rectangle of order 3; that is, it contains only three distinct symbols.

Our proof of this result employs methods from [9]. In fact, based on the techniques from that paper we shall present a general framework for completing partial Latin squares with two filled rows and three filled columns.

1	2	3	4	5
2	4	5	3	1
3	5	1		
4	3	2		
5	1	4		

1	2	3	4	5	6
2	6	1	5	4	3
3	5	4			
4	3	5			
5	4	6			
6	1	2			

1	2	3	4	5	6	7
2	1	7	6	4	5	3
3	7	2				
4	5	6				
5	6	4				
6	4	5				
7	3	1				

Figure 1: Non-completable partial Latin squares of order $5, 6$ and 7.

We then use this framework for giving a short proof of the fact that all such partial Latin squares where the intersection of the filled rows and columns form a Latin rectangle of order 3 are completable.

In Section 2 we review some material from [9] and introduce some additional tools, and in Section 3 we present our method for completing partial Latin squares with two filled rows and three filled columns and prove a special case of Conjecture 1.1.

2. Preliminaries

Two partial Latin squares P and P' are *isotopic* if P' can be obtained from P by permuting rows, permuting columns and/or permuting symbols in P. Note that if P and P' are isotopic, then P is completable if and only if P' is completable.

A partial Latin square P of order n can equivalently be described as a subset of $[n] \times [n] \times [n]$, where $(r, c, s) \in P$ if and only if $s = P(r, c)$. We shall swap freely between this representation and the array representation of partial Latin squares.

A *conjugate* of P is an array in which the coordinates of each triple (r, c, s) of P are uniformly permuted according to one of the following six ways:

$$(r, c, s), (c, r, s), (s, c, r), (c, s, r), (r, s, c), (s, r, c).$$

If P is a partial Latin square, then any conjugate of P is a partial Latin square as well. Moreover, any conjugate of P is completable if and only if P is completable.

An *intercalate* in an $n \times n$ partial Latin square L is a set

$$C = \{(r_1, c_1)_L, (r_1, c_2)_L, (r_2, c_1)_L, (r_2, c_2)_L\}$$

of cells in L such that

$$L(r_1, c_1) = L(r_2, c_2) = s_1 \text{ and } L(r_1, c_2) = L(r_2, c_1) = s_2.$$

A *swap on* C is the operation $L \mapsto L'$, where L' is an $n \times n$ partial Latin square with

$$L'(r_1, c_1) = L'(r_2, c_2) = s_2,\ L'(r_1, c_2) = L'(r_2, c_1) = s_1,$$

and $L'(i, j) = L(i, j)$ for all other (i, j).

The following well-known theorem was first proved by M. Hall [11].

Theorem 2.1. *Every partial Latin square of order n with $r \leq n$ completely filled columns and no other filled cells is completable.*

We shall need some further auxiliary results. The following lemma is a simple consequence of Hall's condition for matchings in bipartite graphs. Denote by $\delta(G)$ the minimum degree of a graph G.

Lemma 2.2. *If B is a balanced bipartite graph with parts V_1 and V_2, and $\delta(B) \geq \frac{|V_1|}{2}$, then B has a perfect matching.*

This lemma enables us to prove the following.

Lemma 2.3. *Let P be an $n \times n$ partial Latin square with r completely filled columns, one partially filled column with s filled cells and where all other columns are empty. If $n \geq 2r + s$, then P is completable.*

Proof. Without loss of generality, we assume that the cells in rows $1, \dots, n-s$ of the partially filled column c of P are empty, and that symbols $1, \dots, n-s$ do not appear in column c of P, where $r < c \leq n$.

We form a bipartite graph B with parts $V_1 = \{r_1, r_2, \dots, r_{n-s}\}$ and $V_2 = \{1, \dots, n-s\}$, and where $r_i j \in E(B)$ if and only if symbol j does not appear in row i of P. Now, $d_B(r_i) \geq n - s - r$, and $d_B(j) \geq n - s - r$, since there are at most r different symbols in each of the $n - s$ first rows of P, and each of the symbols $1, \dots, n - s$ appears in at most r different rows. Thus $\delta(B) \geq n - s - r \geq \frac{n-s}{2}$, by assumption; so by Lemma 2.2, B contains a perfect matching M. Now, for each empty cell $(i, c)_P$ in column c of P we assign the symbol j satisfying that $r_i j \in M$ to $(i, c)_P$. The obtained partial Latin square P' has $r + 1$ completely filled columns and all other cells of P' are empty. Thus, by Theorem 2.1, P' is completable, and so, P has a completion. $\square$

Finally, we shall need the following result proved by Häggkvist; see e.g. [3]. We denote by $\mathrm{PLS}(a, b; n)$ the set of all $n \times n$ partial Latin squares with a completely filled rows and b completely filled columns, and where all other cells are empty.

Theorem 2.4. *If $P \in \mathrm{PLS}(b, b; n)$ is a partial Latin square where the cells in the intersection of the filled rows and columns form a Latin square, then P is completable.*

2.1. Smetaniuk completion

A main ingredient in Smetaniuk's resolution of the Evans' conjecture is what we call the *Smetaniuk completion* of a partial Latin square. Below we briefly review this technique along with its generalization by Kuhl and McGinn [9].

If P is a partial Latin square of order n, then the set $D = \{(i, i)_P, i \in [n]\}$ is called the *forward diagonal* of P. A cell $(r, c)_P$ of P lies *below* D if $c < r$; the cell is *above* D if it is neither below D, nor in D.

For a partial Latin square P of order n, we define a new partial Latin square $T(P)$ of order $n + 1$ by setting

$$T(P) = \{(r + 1, c, s) : (r, c, s) \in P, c < r\} \cup \{(i, i, n + 1) : i \in [n + 1]\}.$$

Note that all cells above the forward diagonal of $T(P)$ are empty.

Theorem 2.5 (Smetaniuk completion [13])**.** *If P is a completable partial Latin square, then $T(P)$ is completable.*

In [9], the authors generalize the above ideas as follows. Let P be a partial Latin square of order n. If n is odd, then the *augmented forward diagonal D^2* of P is defined as the set

$$\begin{aligned} D^2 = \{&(i, i)_P, (i, i + 1)_P, (i + 1, i)_P, (i + 1, i + 1)_P : i \in \{4, 6, 8, \ldots, n - 1\}\} \\ &\cup \{(1, 1)_P, (2, 1)_P, (3, 2)_P, (3, 3)_P\}; \end{aligned}$$

if n is even, then the augmented forward diagonal is defined as the set

$$D^2 = \{(i, i)_P, (i, i + 1)_P, (i + 1, i)_P, (i + 1, i + 1)_P : i \in \{1, 3, 5, \ldots, n - 1\}\}.$$

The properties for a cell of lying below or above the augmented forward diagonal is defined analogously as above.

For a partial Latin square P of order n we define a partial Latin square $T^2(P)$ of order $n + 2$, with augmented forward diagonal D^2, by setting

(i) $T^2(P)(i,j) = P(i-2,j)$, if $(i,j)_{T^2(P)}$ lies below D^2 of $T^2(P)$,
(ii) $T^2(P)(i,j) \in \{n+1, n+2\}$ if $(i,j)_{T^2(P)} \in D^2$, and
(iii) the cells of $T^2(P)$ above D^2 are empty.

Note that the augmented forward diagonal of $T^2(P)$ is uniquely defined up to switching symbols on subarrays of D^2. As in [9], since this suffices for our purposes, we shall be content with this definition. Moreover, in [9] the authors worked with the (augmented) back diagonal rather than the (augmented) forward diagonal. By isotopy, this makes no difference for the purpose of completability. Thus, since the augmented forward diagonal is better suited for our purposes, we reformulate the results of [9] to this setting. Hence, by isotopy, we have the following.

Theorem 2.6. *[9] If P is a completable partial Latin square of order n, then $T^2(P)$ is completable (for any choice of the augmented forward diagonal satisfying (ii) that does not violate the Latin property).*

The proof of this theorem in [9] yields a Latin square which we shall refer to as the *Smetaniuk completion of* $T^2(P)$. Furthermore, when applying this theorem below, the augmented forward diagonal in the considered partial Latin squares will generally contain symbols 1 and 2; again, by isotopy, this of course makes no difference for the purpose of completability.

Observation 2.7. *[9] Let P be a Latin square of order n and let L be the Smetaniuk completion of $T^2(P)$ with augmented forward diagonal D^2. Then the following holds:*

(i) $L(i,j) = P(i-2,j)$ if cell (i,j) is below D^2 of L.
(ii) $L(i,j) \in \{n+1, n+2\}$ if $(i,j) \in D^2$.
(iii) For odd n, if $\{P(1,2), P(1,3)\} \cap \{P(2,4), P(2,5), P(3,4), P(3,5)\} = \emptyset$, then $L(3,4) = P(1,4)$ and $L(3,5) = P(1,5)$.

This observation implies the following.

Observation 2.8. *For odd n, let P be a Latin square of order n and let L be the Smetaniuk completion of $T^2(P)$ with augmented forward diagonal D^2. If*

$$\{P(1,2), P(1,3)\} \cap \{P(2,4), P(2,5), P(3,4), P(3,5)\} = \emptyset,$$

and the set $\{((2,4)_P, (2,5)_P, (3,4)_P, (3,5)_P\}$ is an intercalate, then the set $\{(1,4)_L, (1,5)_L, (2,4)_L, (2,5)_L\}$ is an intercalate on the same symbols.

2.2. Reducing partial Latin squares

Kuhl and McGinn [9] decribed a method for "reducing" elements of $\mathrm{PLS}(a, b; n)$. We sketch their method below; for a more elaborate exposition, see [9].

Let $a, b, j, k \in [n]$, let $P \in \mathrm{PLS}(a, b; n)$ and denote by C_j and R_k column j and row k, respectively, as subarrays of P. As for partial Latin squares, we shall often treat these subarrays as sets of ordered triples, i.e. $C_j = \{(i, j, s) : (i, j, s) \in P, i \in [n], s \in [n]\}$, and similarly for rows.

Henceforth, for $P \in \mathrm{PLS}(a, b; n)$, we shall assume that all nonempty cells of P are in the first a rows and first b columns of P. For any two columns C_j and C_k in P, we define the *column composition* $C_j \circ_l C_k$, where $l \leq a$, as a new column with the same elements as C_j except that the symbol in row l of $C_j \circ_l C_k$ is $P(l, k)$. A *row composition* is defined as a column composition in the row-column conjugate $P^{(rc)}$ of P.

Now, let $P \in \mathrm{PLS}(2, 3; n)$ and suppose α is a symbol not occurring in the 2×3 subarray in the upper left corner of P. Assume further that

$$P(j, 1) = P(k, 2) = P(l, 3) = P(1, q) = P(2, r) = \alpha.$$

If there is an $i \in [n] \setminus [2]$, such that $R_j \circ_1 R_i$, $R_k \circ_2 R_i$, $R_l \circ_3 R_i$ are Latin (i.e. contains no repeated symbols), then we say that α is a *row-replacable symbol* and that row R_i *replaces* α. If $i \in \{j, k, l\}$, then R_i *replaces itself*. Similarly, if there is a $p \in [n] \setminus [3]$ such that $C_q \circ_1 C_p$ and $C_r \circ_2 C_p$ are Latin, then α is a *column-replacable symbol*, and C_p *replaces* α. If $p \in \{q, r\}$, then C_p *replaces itself*.

If α is both row- and column-replacable, then we say that α is *replacable*. If α is replacable with R_i and C_p replacing α as above, then we define the *reduction* of A, denoted $R(P; R_i, C_p, \alpha)$, as the array obtained from P by

- removing rows R_j, R_k, R_l and columns C_q, C_r from P,
- adding the rows $R_j \circ_1 R_i$, $R_k \circ_2 R_i$, $R_l \circ_3 R_i$, and columns $C_q \circ_1 C_p$, $C_r \circ_2 C_p$, and finally
- removing C_p and R_i from P.

Note that, for the purpose of completability, we may by isotopy assume that $R(P; R_i, C_p, \alpha)$ is a partial Latin square; that is, the removed symbol is n and the last column and row are removed when forming $R(P; R_i, C_p, \alpha)$.

The following was proved in [9].

Lemma 2.9. *[9] Let $P \in \mathrm{PLS}(2, 3; n)$ where $n \geq 9$. If α is a symbol that does not occur in the intersection of the filled rows and columns of P, then there is a row replacing α.*

A partial Latin square $P \in \mathrm{PLS}(2,3;n)$ is *reducible* if there is a symbol α, a row R_i and a column C_j, such that row R_i replaces α and column C_j replaces α and itself; we say that the reduction $R(P;R_i,C_j,\alpha)$ is a *proper reduction* of P. For a sequence of partial Latin squares $A_1, A_2, \ldots, A_m$, where A_{i+1} is a proper reduction of A_i, $i = 1, \ldots, m-1$, we say that A_m is obtained by *successive reductions* of A_1 and that A_1 can be *successively reduced* to A_m.

The following is a main result of the method in [9], formulated here for partial Latin squares in $\mathrm{PLS}(2,3;n)$.

Theorem 2.10. *[9] If $P \in \mathrm{PLS}(2,3;n)$ is reducible and one of its proper reductions is completable, then P is completable.*

3. Completing partial Latin squares in $\mathrm{PLS}(2,3;n)$

In this section we describe our method for completing partial Latin squares in $\mathrm{PLS}(2,3;n)$. Throughout the rest of the paper, we assume that every partial Latin square from this family has all nonempty cells in the first two rows and first three columns.

3.1. Reducibility

If P is an $n \times n$ partial Latin square where rows r_1 and r_2 are completely filled, then the (r_1,r_2)*-row-permutation* of P is the permutation $\sigma : [n] \to [n]$ defined by $\sigma(P(r_1,i)) = P(r_2,i)$ for every $i \in [n]$.

Consider the disjoint cycle representation of a row-permutation σ of $P \in \mathrm{PLS}(2,b;n)$. A *cycle type* of a cycle C of length m in this representation of σ is a sequence s of m 0s and 1s, where the ith element of s is 1 if the ith element of C appears in the upper left $1 \times b$ subarray of P; and 0 otherwise. Two cycle types are *equivalent* if one of them can be obtained from the other by permuting the elements in the sequence cyclically. If a cycle in the disjoint cycle representation of σ has a cycle type that is equivalent to s, then *s occurs in σ*.

For all non-equivalent cycle types s that occur in the permutation σ, let i_s be the number of times that s occurs in σ. The set of all ordered pairs (s, i_s), where s is a cycle type that occurs in σ, is called the *cycle type of σ*, or the *cycle type of P* if $P \in \mathrm{PLS}(2,3;n)$ and σ is the $(1,2)$-row-permutation of P.

Two cycle types A_1 and A_2 of row permutations are *equivalent* if they correspond to two different disjoint cycle representations of the same permutation. Note that A_1 and A_2 are equivalent if and only if there is a bijection

$\varphi : A_1 \to A_2$, such that $\varphi((s, i_s)) = (t, i_t)$ if and only if s and t are equivalent and $i_s = i_t$.

Two elements in a sequence s are called *adjacent* if one is immediately followed by the other.

Definition 3.1. Let $P \in \mathrm{PLS}(2, 3; n)$ be a partial Latin square, where $n \geq 8$. P is *completely reduced* if the cycle type of every cycle in the disjoint cycle representation of the $(1, 2)$-row-permutation of P is equivalent to one of the following sequences:

(i) 00,
(ii) 01,
(iii) 11,
(iv) 101,
(v) 111,
(vi) 1010,
(vii) 1110,
(viii) 10101,
(ix) 101010,

If P' is a completely reduced partial Latin square that is obtained from successive reduction of P, then P' is called a *complete reduction* of P.

Example 3.2. The partial Latin square P to the left in Figure 2 has cycle type $\{(111, 1), (000, 1)\}$, and the partial Latin square to the right is the complete reduction $R(P, R_6, C_5, 6)$ of P with cycle type $\{(111, 1), (00, 1)\}$.

1	2	3	4	5	6
2	3	1	5	6	4
3	5	4			
4	6	5			
5	4	6			
6	1	2			

1	2	3	4	5
2	3	1	5	4
3	5	4		
4	1	5		
5	4	2		

Figure 2: A partial Latin square in $\mathrm{PLS}(2, 3; n)$ with cycle type $\{(111, 1), (000, 1)\}$ (to the left), and a complete reduction of this partial Latin square (to the right).

We shall use the following simple lemma.

Lemma 3.3. *Let $P \in \mathrm{PLS}(2, 3; n)$, where $n \geq 8$. If P is not completely reduced, then the cycle type of at least one of the cycles of length at least 3 in the $(1, 2)$-row-permutation of P contains two adjacent zeros.*

The following theorem is now easy to prove.

Theorem 3.4. *A partial Latin square* $P \in \mathrm{PLS}(2,3;n)$*, where* $n \geq 9$*, has a proper reduction if and only if it is not completely reduced.*

Proof. If P is not completely reduced, then by the preceding lemma, there is a cycle C of length at least 3 in the $(1,2)$-row-permutation σ of P whose cycle type contains two adjacent zeros. This means that C contains a symbol s that is neither contained in the 2×3 subarray in the upper left corner of P, nor in an intercalate contained in the first two rows of P. Hence, the two columns containing s each replace s and themselves, respectively. Moreover, by Lemma 2.9, there is a row replacing s. Hence, P has a proper reduction.

Conversely, if there is a proper reduction of P, then, since there is a column replacing itself, there is a cycle of length at least 3 in σ that has a cycle type with adjacent zeros. Thus P is not completely reduced. □

It follows from this theorem that from any partial Latin square in $\mathrm{PLS}(2,3;n)$, $n \geq 9$, we can by successive reduction obtain a partial Latin square in $\mathrm{PLS}(2,3;8)$, or a partial Latin square in $\mathrm{PLS}(2,3;m)$ with a cycle type that is equivalent to one of the following cycle types:

(a) $\{(10,3),(00,k)\}$,
(b) $\{(10,1),(11,1),(00,k+1)\}$,
(c) $\{(10,1),(101,1),(00,k+1)\}$,
(d) $\{(10,1),(1010,1),(00,k)\}$,
(e) $\{(111,1),(00,k+2)\}$,
(f) $\{(1110,1),(00,k+1)\}$,
(g) $\{(10101,1),(00,k+1)\}$,
(h) $\{(101010,1),(00,k)\}$,

where $k \geq 1$.

Thus, for proving Conjecture 1.1 it suffices to show that all partial Latin squares in $\mathrm{PLS}(2,3;8)$ as well as all with a cycle type of type (a)-(h) can be completed. In the next section we shall verify the former statement and also prove that all partial Latin squares of the type (e) have completions.

Note that every partial Latin square in $\mathrm{PLS}(2,3;n)$ with $n \geq 9$ that is completely reduced, and has a cycle type containing $(111,1)$, must be of odd order.

3.2. Completing a particular family in $\mathbf{PLS(2,3;n)}$

In this section we prove that all partial Latin squares in $\mathrm{PLS}(2,3;n)$ with a specific cycle type are completable.

Theorem 3.5. *If $P \in \mathrm{PLS}(2,3;n)$ is a partial Latin square with cycle type $\{(111,1),(00,k+2)\}$, $k \geq 3$, then P is completable.*

Proof. In the proof we shall, by slight abuse of terminology, for simplicity allow partial Latin squares of order m to have a different symbol set than $\{1,\dots,m\}$.

Let P be a partial Latin square satisfying the conditions in the theorem; so $n = 3+2(k+2)$. In particular, P has odd order, since the $(1,2)$-row-permutation of P contains one cycle of length 3 and $k+2$ cycles of length 2. Moreover, by isotopy, we may assume that

- $P(1,i) = P(i,1) = i, i = 1,\dots,n$,
- $P(2,2) = 3, P(2,3) = 1$, and
- $P(2,2i) = 2i+1$ and $P(2,2i+1) = 2i$ for $i = 2,\dots,\frac{n-1}{2}$.

Consider the row-symbol conjugate $P^{(rs)}$ of P. Since the two first rows of P are completely filled, the augmented forward diagonal of $P^{(rs)}$ is completely filled with the symbols 1 and 2, and moreover

(i) $P^{(rs)}(1,1) = 1, P^{(rs)}(1,2) = s_1, P^{(rs)}(1,3) = 2$,
(ii) $P^{(rs)}(2,1) = 2, P^{(rs)}(2,2) = 1, P^{(rs)}(2,3) = s_2$, and
(iii) $P^{(rs)}(3,i) = 4-i$, for $i = 1,2,3$,

where s_1 and s_2 are some symbols from $\{3,\dots,n\}$.

Now, if $s_1 = s_2 = 3$, then the third row of P can be completed e.g. by applying Theorem 2.1 to a subarray of P, and the obtained partial Latin square is then completable by Theorem 2.4. Thus, P is completable. Consequently, it suffices to consider the following cases:

(a) s_1 and s_2 are distinct, and $3 \notin \{s_1,s_2\}$,
(b) s_1 and s_2 are distinct, and $3 \in \{s_1,s_2\}$,
(c) $s_1 = s_2 \neq 3$.

Suppose first that (a) holds, and assume without loss of generality that $s_1 = 4$ and $s_2 = 5$. We define the partial Latin square C of order $n-2$ by letting $(1,1,3),(1,2,4),(1,3,5) \in C$, and for $i = 2,\dots,n-2$ letting $(i,j,k) \in C$ if and only if $(i+2,j,k) \in P^{(rs)}$ and $(i+2,j)_{P^{(rs)}}$ is not contained in the augmented forward diagonal of $P^{(rs)}$. Then C is a partial Latin square on the symbols $[n] \setminus [2]$, which by Theorem 2.1 is completable. It thus follows from Theorem 2.6 that $T^2(C)$ has a Smetaniuk completion A, where symbols $1,2$ appear in the augmented forward diagonal. By possibly making some swaps on intercalates with symbols 1 and 2 in A, we obtain a completion of $P^{(rs)}$; so, by conjugacy, P is completable.

Suppose now that (b) holds. By isotopy, we may assume $\{s_1, s_2\} = \{3, 4\}$. Suppose e.g. that $s_1 = 3$ and $s_2 = 4$ (the other case is similar). With this assumption, it is straightforward that by permuting the first three rows and columns, and symbols $1, 2, 3$ and 4, we can obtain, from $P^{(rs)}$ a partial Latin square that satisfies conditions (i)-(iii) and (c). We conclude that it suffices to consider the case when (c) holds.

So assume that (c) holds. Without loss of generality, we assume that $s_1 = s_2 = 4$. From $P^{(rs)}$, we shall define a sequence of partial Latin squares: each partial Latin square will contain an isotopism of the previous one, or will be obtained from it by a swap on an intercalate. We first define a partial Latin square B_1 by applying the permutation $(4\ 6)(5\ 7)$ to the rows and columns of $P^{(rs)}$ if $(i, 1, 4) \in P^{(rs)}$ for some $i \in \{4, 5\}$; otherwise, we set $B_1 = P^{(rs)}$.

Next, we put $S = \{B_1(4, i), B_1(5, i) : i \in [3]\}$ and pick a row $q \geq 6$ in B_1 such that $B_1(q, 3)\} \notin (S \cup \{3\})$ and $B_1(q, 1) \neq 4$; since B_1 has order at least 13, there is such a row q. We put $B_2 = B_1 \cup \{(q, 4, 4), (2, 4, B_1(q, 3))\}$, and note that the set $\{(q, 3)_{B_2}, (q, 4)_{B_2}, (2, 3)_{B_2}, (2, 4)_{B_2}\}$ is an intercalate in B_2. We swap on this intercalate to obtain B_3.

Next, we pick a symbol $\alpha \notin S \cup \{1, 2, 3, 4, B_3(2, 3)\}$; since $n \geq 13$ and

$$|S \cup \{1, 2, 3, 4, B_5(2, 3)\}| \leq 11,$$

there is indeed such a symbol α. We set $B_4 = B_3 \cup \{(2, 5, \alpha), (3, 4, \alpha), (3, 5, 4)\}$ and note that the cells

$$\{(2, 4)_{B_4}, (2, 5)_{B_4}, (3, 4)_{B_4}, (3, 5)_{B_4}\}$$

form an intercalate in B_4. We now permute the rows and columns according to $(1\ 3)$ and $(1\ 2)$ in B_4, respectively, and denote the obtained partial Latin square by B_5.

From B_5 we define a partial Latin square C of order $n - 2$ by letting

$$(1, 1, 4), (1, 2, 3), (1, 3, B_5(2, 3)), (2, 4, \alpha), (3, 4, 4) \in C,$$

and for $i = 2, \ldots, n - 2$ letting $(i, j, s) \in C$ if and only if $(i + 2, j, s) \in B_5$ and $(i + 2, j)_{B_5}$ is not contained in the augmented forward diagonal of B_5. Since $\{4, \alpha\} \cap S = \emptyset$, C is a partial Latin square on the symbols $[n] \setminus [2]$.

Now, since column 4 in C contains three nonempty cells, Lemma 2.3 implies that C has a completion C'. We define C_1 from C by filling column 4 in C as in column 4 in C', and, in addition, setting $C_1(2, 5) = 4$ and $C_1(3, 5) = \alpha$. Then, by construction, C_1 satisfies the following:

- row 2 in C_1 contains symbols $B_2(4,1)$, $B_2(4,2)$, $B_2(4,3)$, α, 4, which are all distinct by the choice of α and the construction of B_2;
- row 3 in C_1 contains symbols $B_2(5,1)$, $B_2(5,2)$, $B_2(5,3)$, α, 4 which are all distinct;
- for $i = 4, \ldots, n-2$, the first three cells in row i of C_1 agree with the first three cells in row $i+2$ in B_2.

Hence, C_1 is a partial Latin square over the symbols $[n] \setminus [2]$.

Again, by Lemma 2.3, there is a completion C_1' of C_1. Moreover, since C_1 is completable, it follows from Theorem 2.6 that $T^2(C_1)$ has a completion A with an augmented forward diagonal with the symbols 1 and 2; we choose this augmented forward diagonal of A so that it agrees with the augmented forward diagonal of B_5. Hence, the first three columns of A agree with B_5. Moreover, by Observations 2.7-2.8, the cells in positions $(1,4),(1,5),(2,4),(2,5)$ in A form an intercalate F on the symbols $\{4,\alpha\}$, as in B_5, and $A(q,4) = B_5(q,4) = B_2(q,3)$.

Now, if $A(2,4) = \alpha$, then we swap on the intercalate F to obtain the partial Latin square A'; otherwise, if $A(2,4) = 4$, we set $A' = A$. In A', the set

$$\{(2,3)_{A'}, (2,4)_{A'}, (q,3)_{A'}, (q,4)_{A'}\}$$

is an intercalate, and by swapping on this intercalate we obtain a completion of an isotopism of B_2. Now, since an isotopism of $P^{(rs)}$ is contained in B_2, $P^{(rs)}$ is completable, and so, P is completable. $\square$

Consider a partial Latin square P in $\mathrm{PLS}(2,3;n)$, where $n \geq 8$ and the 2×3 subarray in the upper left corner forms a Latin rectangle. By Theorem 3.4 one of the following must hold.

(i) There is a complete reduction P' of P with odd order at least 13.
(ii) There is a complete reduction P' of P with odd order 11.
(iii) There is a complete reduction P' of P with odd order 9.
(iv) P or a partial Latin square obtained from P by successive reduction is a partial Latin square of order 8.

Theorems 3.5 and 2.10 implies that every partial Latin square satisfying (i) is completable. The cases (ii)-(iv) have been settled by a computer search. As it turns out, every partial Latin square in $\mathrm{PLS}(2,3;n)$ with cycle type $\{(111,1),(00,k)\}$, where $n = 3+2k$ and $k \in \{3,4\}$, is completable. Additionally, this holds for all partial Latin squares in $\mathrm{PLS}(2,3;8)$ as well; for details, see [7]. Hence, since the cycle type of a partial Latin square in $\mathrm{PLS}(2,3;n)$, where the intersection of the filled rows and columns form a Latin rectangle, is (e), by Theorems 2.10 and 3.5 we have the following.

Corollary 3.6. *Every partial Latin square of order at least* 8 *with two filled rows and three filled columns, and where the intersection of the filled rows and columns form a Latin rectangle, is completable.*

References

[1] P. Adams, D. Bryant and M. Buchanan, Completing partial Latin squares with two filled rows and two filled columns, *Electronic Journal of Combinatorics* 15(1), R56 (2008), 26pp. MR2398848

[2] L.D. Andersen, A.J.W. Hilton, Thank Evans!, *Proc. London Math. Soc.* 47 (1983), pp. 507–522. MR0716801

[3] A.S. Asratian, T.M.J. Denley, R. Häggkvist, *Bipartite graphs and their applications*, Cambridge University Press, Cambridge, 1998. MR1639013

[4] M. Buchanan, *Embedding, existence and completion problems for Latin squares*, PhD thesis, University of Queensland, 2007.

[5] C.J. Casselgren, R Häggkvist, Completing partial Latin squares with one filled row, column and symbol, *Discrete Mathematics* 313 (2013), 1011–1017. MR3028194

[6] C.J. Colbourn, The complexity of completing partial Latin squares, *Discrete Applied Mathematics* 8 (1984), 25–30. MR0739595

[7] H. Göransson, *Completing partial Latin squares with two filled rows and three filled columns*, Bachelor thesis, Linköping University, 2020.

[8] T. Evans, Embedding incomplete Latin squares, *American Mathematical Monthly* 67 (1960), 958–961. MR0122728

[9] J. Kuhl, D. McGinn, On completing partial Latin squares with two filled rows and at least two filled columns, *Australasian Journal of Combinatorics* (2017), 186–201. MR3646034

[10] J. Kuhl, M.W. Schroeder, Completing Partial Latin Squares with One Nonempty Row, Column, and Symbol, *Electronic Journal of Combinatorics* 23 (2016), 13pp. MR3512645

[11] M. Hall, An existence theorem for Latin squares, *Bulletin of the American Mathematical Society* 51 (1945), 387–388. MR0013111

[12] H.J. Ryser, A combinatorial theorem with an application to Latin squares, *Proc. Amer. Math. Soc.* 2 (1951), 550–552. MR0042361

[13] B. Smetaniuk, A new construction for Latin squares I. Proof of the Evans conjecture, *Ars Combinatoria* 11 (1981), 155–172. MR0629869

Carl Johan Casselgren
Department of Mathematics
Linköping University
SE-581 83 Linköping
Sweden
E-mail address: carl.johan.casselgren@liu.se

Herman Göransson
Department of Mathematics
Linköping University
SE-581 83 Linköping
Sweden
E-mail address: herman.goransson@gmail.com

Received September 10, 2020