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Maximizing the Edelman-Greene statistic

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The *Edelman-Greene statistic* of S. Billey and B. Pawlowski measures the "shortness" of the Schur expansion of a Stanley symmetric function. We show that the maximum value of this statistic on permutations of Coxeter length n is the number of involutions in the symmetric group S_n, and explicitly describe the permutations that attain this maximum. Our proof confirms a recent conjecture of C. Monical, B. Pankow, and A. Yong: we give an explicit combinatorial injection between certain collections of Edelman-Greene tableaux and standard Young tableaux.

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1. Introduction

Let S_n be the *symmetric group* on $[n] = \{1, 2, \ldots, n\}$. The group S_n can be embedded in S_{n+1} by the natural inclusion, and in this way define $S_\infty = \bigcup_{n=1}^{\infty} S_n$. Let $s_i \in S_\infty$ be the *simple transposition* swapping i and $i+1$. Each $w \in S_\infty$ is expressible as a product of simple transpositions; the minimum possible length of such an expression is the *Coxeter length* $\ell(w)$. An expression of w as a product of simple transpositions having length $\ell(w)$ is a *reduced decomposition* of w. Let $\mathsf{Red}(w)$ be the set of reduced decompositions of w. A permutation w is *totally commutative* if there exists $s_{i_1} \cdots s_{i_{\ell(w)}} \in \mathsf{Red}(w)$ with $|i_j - i_k| \geq 2$ for all $j \neq k$. Note that this is stricter than the definition of the similar sounding *fully commutative* [10]. For example, the permutation with one-line notation 2 3 1 5 4 is fully commutative but not totally commutative.

In their study of $\mathsf{Red}(w)$, P. Edelman and C. Greene [3] introduced a family of tableaux.

Definition 1. Fix a partition λ and $w \in S_\infty$. We say that S is an *Edelman-Greene tableau* (or *EG tableau*) of type (λ, w) if it is a filling of the cells of a Young diagram of shape λ such that the cells are strictly increasing on rows

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and columns, and that if the sequence $i_1, i_2, \ldots, i_{|\lambda|}$ results from reading the columns of the tableau from right-to-left, and within each column reading the entries from top-to-bottom, then $s_{i_1} s_{i_2} \cdots s_{i_{|\lambda|}} \in \mathsf{Red}(w)$. We denote the *reading decomposition* $\mathsf{Red}(S)$ to be the corresponding decomposition $s_{i_1} s_{i_2} \cdots s_{i_{|\lambda|}}$. Let $\mathsf{EG}(\lambda, w)$ be the set of these tableaux.

Now,

$$\mathsf{EG}(w) = \sum_{\lambda} a_{w,\lambda} \text{ , where } a_{w,\lambda} = |\mathsf{EG}(\lambda, w)| \tag{1}$$

is the *Edelman-Greene statistic* of S. Billey and B. Pawlowski [2].

Example 2. Let w be the permutation represented in one-line notation by 2 1 5 3 4. The set of reduced decompositions for w is $\{s_1 s_4 s_3, s_4 s_1 s_3, s_4 s_3 s_1\}$. If $\lambda = (3)$, then there is one EG tableau of type (λ, w), namely $\boxed{1}\boxed{3}\boxed{4}$, which corresponds to the reduced decomposition $s_4 s_3 s_1$. Similarly, if $\lambda = (2, 1)$, the only EG tableau of type (λ, w) is $\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$, which corresponds to the reduced decomposition $s_4 s_1 s_3$. As a result, $a_{w,(3)} = a_{w,(2,1)} = 1$, and $a_{w,\lambda} = 0$ for all other λ. As a result, $\mathsf{EG}(w) = 1 + 1 = 2$.

Define $\mathsf{inv}(n)$ to be the number of involutions in S_n, i.e. the number of permutations $w \in S_n$ such that w^2 is the identity permutation.

Our main result is the following:

Theorem 3. *For all $w \in S_\infty$,*

$$\mathsf{EG}(w) \leq \mathsf{inv}(\ell(w)) \tag{2}$$

and equality is attained if and only w is totally commutative.

We offer three comparisons and contrasts of Theorem 3 with the literature.

First, B. Pawlowski has proved that $\mathbb{E}[\mathsf{EG}] \geq (0.072)(1.299)^m$, where the expectation is taken over $w \in S_m$ [7, Theorem 3.2.7]). More recently, C. Monical, B. Pankow, and A. Yong show that $\mathsf{EG}(w)$ is “typically” exponentially large on S_m [5, Theorem 1.1]. In comparison, Theorem 3 combined with a standard estimate for $\mathsf{inv}(n)$ [4] gives

$$\max\{\mathsf{EG}(w) : w \in S_\infty, \ell(w) = n\} \sim \left(\frac{n}{e}\right)^{\frac{n}{2}} \frac{e^{\sqrt{n}}}{(4e)^{\frac{1}{4}}}. \tag{3}$$

Second, in [6], maxima for the Littlewood-Richardson coefficients and their generalization, the Kronecker coefficients, were determined. We remark that the $a_{w,\lambda}$'s are also generalizations of the Littlewood-Richardson coefficients; this follows from [1, Corollary 2.4].

Finally, while the results of V. Reiner and M. Shimozono [8] (see specifically their Theorem 33) seem related to ours, we do not see any obvious way to get our result from theirs. In any case, our work does not depend on their paper and is combinatorial and self-contained.

This paper is structured as follows: in Section 2, the upper bound in Equation 2 is proved by describing a map with domain $\mathsf{EG}(\lambda, w)$, and proving that it is an injection. In Section 3, we exactly classify which permutations make Equation 2 attain equality.

2. Proof of the upper bound for $\mathsf{EG}(w)$

Recall that a *semistandard Young tableau* is a filling of the cells of a Young diagram of shape λ with positive integers such that the cells are weakly increasing along rows and strictly increasing on columns. The (countably infinite) set of semistandard Young tableaux of shape λ is given by $\mathsf{SSYT}(\lambda)$. In particular, $\mathsf{EG}(\lambda, w) \subset \mathsf{SSYT}(\lambda)$. Additionally, the *content* of $S \in \mathsf{SSYT}(\lambda)$ is the infinite tuple with i^{th} coordinate being the number of labels i in S.

Our proof of Theorem 3 is based on a specific relationship between EG tableaux and standard Young tableaux. A *standard Young tableau* is a semistandard Young tableau of shape λ with the numbers 1 through $|\lambda|$ each used exactly once. The set of standard Young tableaux of shape λ is given by $\mathsf{SYT}(\lambda)$, and denote $f^{\lambda} = |\mathsf{SYT}(\lambda)|$.

Figure 1 gives several examples of the well-known *standardization* map $\mathsf{std} : \mathsf{SSYT}(\lambda) \to \mathsf{SYT}(\lambda)$. Suppose $T \in \mathsf{SSYT}(\lambda)$ and k_i is the number of i's appearing in T. Now replace all 1's in T from left to right by $1, 2, \ldots, k_1$. Then replace all of the (original) 2's in T by $k_1 + 1, k_1 + 2, \ldots, k_1 + k_2$, *etc.* The result of this procedure is $\mathsf{std}(T)$.

If we restrict std to the subset of $\mathsf{SSYT}(\lambda)$ consisting of the (finitely many) tableaux with a given content μ, then it is easy to see that std is an injection. Now, content is not constant on $\mathsf{EG}(\lambda, w)$. Nevertheless, the conjecture of C. Monical, B. Pankow, and A. Yong [5, Conjecture 3.12] is the following, which we resolve here:

Theorem 4. *The map* $\mathsf{std} : \mathsf{EG}(\lambda, w) \to \mathsf{SYT}(\lambda)$ *is an injection.*

$$\mathsf{std}\left(\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline\end{array}\right)=\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline\end{array}\;,\qquad \mathsf{std}\left(\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 2&5\\\cline{1-2} 3\\\cline{1-1}\end{array}\right)=\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&7\\\cline{1-2} 4\\\cline{1-1}\end{array}$$

Figure 1: Two examples of the standardization map.

In order to prove Theorem 4, first recall that the simple transpositions satisfy:

$$s_i s_j = s_j s_i \quad \text{for } |i-j| \geq 2 \tag{4}$$

and

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \tag{5}$$

where (5) is the *braid relation*. Moreover, Tits' Lemma [12] states that any reduced decomposition can be transformed into any other reduced decomposition for the same permutation through a sequence of successive transformations (4) and (5). If $s_{i_1} s_{i_2} \cdots s_{i_k} \in \mathsf{Red}(w)$, define the *support* of w as $\mathsf{supp}(w) = \{i_1, i_2, \ldots, i_k\}$.

Lemma 5. $\mathsf{supp}(w)$ *is well-defined.*

Proof. This follows immediately from Tits' Lemma together with the fact that (4) and (5) preserve support. □

Knowing the structure of one reduced word for a permutation can give information about the structure of all other reduced words for that permutation.

Lemma 6. *For $w \in S_\infty$, if $|a-b| = 1$, and there exists a reduced decomposition of w such that all instances of s_a occur before all instances of s_b, then the same is true for all reduced decompositions of w.*

Proof. This holds by Tits' Lemma and examining (4) and (5). □

Let (x, y) be the matrix coordinates of a cell in λ. For $S \in \mathsf{SSYT}(\lambda)$, denote the label of cell $(x, y) \in \lambda$ by $S(x, y)$. A *descent* of $U \in \mathsf{SYT}(\lambda)$ is a label i such that $i-1$ is weakly east (and thus strictly north) of i. Let the *sweep map* of U, $\mathsf{sweep}(U)$ be the Young tableau of shape λ such that

$$(\mathsf{sweep}(U))(x, y) = |\{k : 1 \leq k \leq U(x, y), k \text{ is a descent in } U\}| + 1. \tag{6}$$

Figure 2 gives an example of the image of the sweep map.

$$S = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 7 \\ \hline 2 & 5 & 7 \\ \cline{1-3} 4 \\ \cline{1-1} 8 \\ \cline{1-1}\end{array}, \quad T = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 3 & 5 & 7 \\ \cline{1-3} 5 \\ \cline{1-1} 8 \\ \cline{1-1}\end{array}, \quad \mathsf{std}(S) = \mathsf{std}(T) = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 6 & 8 \\ \hline 2 & 5 & 7 \\ \cline{1-3} 4 \\ \cline{1-1} 9 \\ \cline{1-1}\end{array}$$

$$\mathsf{sweep}(\mathsf{std}(S)) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 3 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

Figure 2: An example of two semistandard Young tableaux and their images under std and sweep. Note that S and T are not EG tableaux.

Proposition 7. sweep *is a map from* $\mathsf{SYT}(\lambda)$ *to* $\mathsf{SSYT}(\lambda)$

Proof. Fix $U \in \mathsf{SYT}(\lambda)$. For a given cell $(x, y) \in \lambda$, $U(x, y) < U(x, y + 1)$, and so the number of descents less than or equal to $U(x, y)$ is at most the number of descents less than or equal to $U(x, y+1)$, and so by the definition of the sweep map, $(\mathsf{sweep}(U))(x, y) \leq (\mathsf{sweep}(U))(x, y + 1)$.

Additionally, $U(x, y) < U(x + 1, y)$. If none of $U(x, y) + 1, U(x, y) + 2, \ldots, U(x + 1, y)$ were descents, then each of those labels would be weakly northeast of the one before it, so $U(x + 1, y)$ would be weakly northeast of $U(x, y)$. This contradicts the fact that $(x + 1, y)$ is below (x, y). Therefore, one of $U(x, y) + 1, U(x, y) + 2, \ldots, U(x + 1, y)$ is a descent, and so by the definition of the sweep map, $(\mathsf{sweep}(U))(x, y) < (\mathsf{sweep}(U))(x + 1, y)$.

Thus we have shown that $\mathsf{sweep}(U)$ is weakly increasing on rows and strictly increasing on columns, so it is a semistandard Young tableau of shape λ, and we are done. $\square$

In addition, the i^{th} *sweep* of U is

$$\mathsf{sweep}_i(U) := \{(x, y) \in \lambda : (\mathsf{sweep}(U))(x, y) = i\}. \tag{7}$$

Applying the sweep map to $\mathsf{std}(U)$ recovers some of the original information of U. For example, $\mathsf{sweep} \circ \mathsf{std}$ maps equal labels to equal labels.

Lemma 8. *If* $U \in \mathsf{SSYT}(\lambda)$ *and* $U(x, y) = U(c, d)$*, then* $(\mathsf{sweep}(\mathsf{std}(U)))(x, y) = (\mathsf{sweep}(\mathsf{std}(U)))(c, d)$.

Proof. Without loss of generality, assume that (x, y) is strictly northeast of (c, d). This means that $(\mathsf{std}(U))(x, y) > (\mathsf{std}(U))(c, d)$. None of $(\mathsf{std}(U))(c, d) + 1$, $(\mathsf{std}(U))(c, d) + 2$, $\ldots$, $(\mathsf{std}(U))(x, y)$ will be descents, and so we have that $(\mathsf{sweep}(\mathsf{std}(U)))(x, y) = (\mathsf{sweep}(\mathsf{std}(U)))(c, d)$. $\square$

Similarly, knowing the relative sizes of the labels of two cells after applying $\mathsf{sweep} \circ \mathsf{std}$ provides information on the relative ordering of the labels of the two cells in the preimage.

Lemma 9. *Say* $U \in \mathsf{SSYT}(\lambda)$ *and for* $i, j \in \mathbb{N}$, $(x, y) \in \mathsf{sweep}_i(\mathsf{std}(U))$ *and* $(c, d) \in \mathsf{sweep}_j(\mathsf{std}(U))$.

1. *If* $i = j$ *and* $y < d$, *then* $U(x, y) \leq U(c, d)$.
2. *If* $i < j$, *then* $U(x, y) < U(c, d)$

Proof. Proof of (1)*:* Since $(\mathsf{sweep}(\mathsf{std}(U)))(x, y) = (\mathsf{sweep}(\mathsf{std}(U)))(c, d)$, and (c, d) is to the right of (x, y), the definition of sweep says that $(\mathsf{std}(U))(x, y) < (\mathsf{std}(U))(c, d)$. Therefore, by the definition of standardization, $U(x, y) \leq U(c, d)$.
Proof of (2)*:* Since $(\mathsf{sweep}(\mathsf{std}(U)))(x, y) = i < j = (\mathsf{sweep}(\mathsf{std}(U)))(c, d)$, it follows from the definition of the sweep map that $(\mathsf{std}(U))(x, y) < (\mathsf{std}(U))(c, d)$. Hence, by the definition of standardization, $U(x, y) \leq U(c, d)$. However, by the contrapositive of Lemma 8, $U(x, y) \neq U(c, d)$, and we are done. □

Proof of Theorem 4: The theorem will be proved by contradiction. Assume for the sake of contradiction that there exists $S, T \in \mathsf{EG}(\lambda, w)$ such that $S \neq T$ and $\mathsf{std}(S) = \mathsf{std}(T)$. Since $S \neq T$,

$$D := \{(x, y) \in \lambda : S(x, y) \neq T(x, y)\} \tag{8}$$

is non-empty. Define $L = \max\{i : \mathsf{sweep}_i(\mathsf{std}(S)) \cap D \neq \emptyset\}$. Let

$$a := \max\{S(x, y) : (x, y) \in D\}, \text{ and } b := \max\{T(x, y) : (x, y) \in D\}. \tag{9}$$

There are two cases to consider: either $a = b$ or $a \neq b$.

First, we consider the case where $a = b$. By definition there exists $(x, y), (c, d) \in D$ be such that $S(x, y) = a$, and $T(c, d) = b$. By the definition of D, $T(x, y) \neq S(x, y) = T(c, d)$. Also, by the definition of b, $T(c, d) = b \geq T(x, y)$, and so $T(c, d) > T(x, y)$. By the definition of standardization, this means that $(\mathsf{std}(T))(c, d) > (\mathsf{std}(T))(x, y)$. However, similarly, $S(x, y) > S(c, d)$, which means that $(\mathsf{std}(S))(x, y) > (\mathsf{std}(S))(c, d)$. However, this contradicts the fact that $\mathsf{std}(S) = \mathsf{std}(T)$, and so Theorem 4 is proved in this case.

For the second case ($a \neq b$), assume without loss of generality that $b > a$. By Lemma 5, some cell in S is labeled b as well, so define

$$B = \{(x, y) \in \lambda : S(x, y) = b\} \text{ and } C = \min\{y : (x, y) \in B\}. \tag{10}$$

Claim 10. *All cells labeled b in S are also labeled b in T, and there exists at least one cell labeled b in T that is to the left of column C.*

Proof. Since $b > a$, $B \cap D = \emptyset$, and so if $(c, d) \in B$, $T(c, d) = b$ as well. In addition, by the definition of b there exists some cell $(x, y) \in D$ such that $T(x, y) = b$, so $(x, y) \notin B$. By Lemma 8, these cells must all be in the same sweep of $\mathsf{std}(T)$. We also know that, since $(x, y) \in D$, $S(x, y) \leq a < b$, so by Lemma 9, (x, y) must lie to the left of all cells in B, and so it must lie to the left of the column with index C, completing the proof. $\square$

Claim 11. *In T, all cells labeled b are in $\mathsf{sweep}_L(\mathsf{std}(T))$, and all cells labeled $a, a+1, \ldots, b$ in S are in $\mathsf{sweep}_L(\mathsf{std}(S))$.*

Proof. By the definition of L, there is some cell $(x, y) \in \mathsf{sweep}_L(\mathsf{std}(T)) \cap D$. By the definition of b, there exists some cell $(c, d) \in D$ such that $T(c, d) = b$ and $b \geq T(x, y)$, so by the contrapositive of Lemma 9, $(\mathsf{sweep}(\mathsf{std}(T)))(c, d) \geq (\mathsf{sweep}(\mathsf{std}(T)))(x, y) = L$. However, since $(c, d) \in D$, $(\mathsf{sweep}(\mathsf{std}(T)))(c, d) \leq L$, and so $(\mathsf{sweep}(\mathsf{std}(T)))(c, d) = L$. As a result, since $T(c, d) = b$, Lemma 8 implies that all cells labeled b in T must be in $\mathsf{sweep}_L(\mathsf{std}(T))$.

By the argument of the previous paragraph (replacing T with S and b with a), all cells labeled a in S must be in $\mathsf{sweep}_L(\mathsf{std}(S))$. By Claim 10, any cells in B are labeled b in T as well. Therefore, since all cells labeled b in T are in $\mathsf{sweep}_L(\mathsf{std}(T))$, all cells in B are also in $\mathsf{sweep}_L(\mathsf{std}(T)) = \mathsf{sweep}_L(\mathsf{std}(S))$. Additionally, the contrapositive of Lemma 9 implies that any cell labels between a and b in S must occur in $\mathsf{sweep}_L(\mathsf{std}(S))$ as well, completing the proof. $\square$

Claim 12. *In all columns with index at least C, no cell can be labeled $b-1$ in either S or T.*

Proof. By assumption $a \leq b - 1 < b$, so Claim 11 says that all cells labeled $b-1$ or b in S are in $\mathsf{sweep}_L(\mathsf{std}(S))$. By Lemma 9 all cells labeled $b-1$ in S must occur strictly to the left of all cells labeled b in S, which means none of them can be in a column with index at least C.

As a result, all s_{b-1}'s will occur after all s_b's in $\mathsf{Red}(S)$, and so by Lemma 6, the same is true for $\mathsf{Red}(T)$, since we assumed that $\mathsf{Red}(S), \mathsf{Red}(T) \in \mathsf{Red}(w)$. This means that all cells labeled $b-1$ in T must occur in some column weakly to the left of the leftmost occurrence of a cell labeled b in T. By Claim 10, this is strictly to the left of the column indexed C. Therefore, in all columns with index at least C, no cell can be labeled $b-1$ in either S or T, so the claim is true. $\square$

Define $G = (\bigcup_{i=L}^{\infty} \mathsf{sweep}_i(\mathsf{std}(S))) \cap \{(x, y) : y \geq C\}$.

Claim 13. *For all $(x, y) \in G$, $S(x, y) = T(x, y) \geq b$.*

Proof. Since Claim 11 says that there is a cell in $\mathsf{sweep}_L(\mathsf{std}(S))$ labeled b in S, every cell in $\bigcup_{i=L+1}^{\infty} \mathsf{sweep}_i(\mathsf{std}(S))$ will have a label larger than b in S by Claim 9. The definition of C says that all cells in $\mathsf{sweep}_L(\mathsf{std}(S))$ in a column labeled at least C will have a label of b or more in S. As a result, all $(x, y) \in G$ have $S(x, y) \geq b$. Since $b > a$, none of these cells are in D, and so they have the same labels in T as well, completing the proof. ☐

Let $s_{i_1} s_{i_2} \cdots s_{i_{|\lambda|}} = \mathsf{Red}(S)$ and let $s_{j_1} s_{j_2} \cdots s_{j_{|\lambda|}} = \mathsf{Red}(T)$. Let I be the set of all indexes k such that s_{i_k} corresponds to a cell $(x, y) \in G$, and let $M = \max(I)$. By Claim 13, $i_k = j_k \geq b$ for all $k \in I$. By Claim 12, $i_a, j_a < b - 1$ for $a \leq M$, $a \notin I$, so s_{i_k} commutes with s_{i_a} for all such $a \leq M$, $a \notin I$ and $k \in I$. Therefore,

$$\prod_{k \in I} s_{i_k} \prod_{a \notin I} s_{i_a} = \mathsf{Red}(S) = \mathsf{Red}(T) = \prod_{k \in I} s_{j_k} \prod_{a \notin I} s_{j_a} \tag{11}$$

and so multiplying both sides by $(\prod_{k \in I} s_{i_k})^{-1}$ on the left results in $\prod_{a \notin I} s_{i_a} = \prod_{a \notin I} s_{j_a}$, and we denote the two sides $\mathsf{Red}(S')$ and $\mathsf{Red}(T')$ respectively. However, the definition of C says that $b \notin \mathsf{supp}(\mathsf{Red}(S'))$, but Claim 10 says that $b \in \mathsf{supp}(\mathsf{Red}(T'))$. This contradicts Lemma 5, and we are done. ☐

Example 14. To illustrate the above argument, in Figure 2, the cells corresponding to the fourth and fifth sweeps in S and T are the same, but not for the third sweep, so in this case, $L = 3$, $b = 5$, and $C = 2$.

This means that, by the fact that $S, T \in \mathsf{EG}(\lambda, w)$,

$$w = s_7 s_5 s_7 s_2 s_5 s_1 s_2 s_4 s_8 = s_7 s_5 s_7 s_3 s_5 s_1 s_3 s_5 s_8 \tag{12}$$

and by (4), this can be rewritten this as

$$w = s_7 s_5 s_7 s_5 s_2 s_1 s_2 s_4 s_8 = s_7 s_5 s_7 s_5 s_3 s_1 s_3 s_5 s_8 \tag{13}$$

and multiplying both permutations by $(s_7 s_5 s_7 s_5)^{-1}$ on the left results is

$$s_2 s_1 s_2 s_4 s_8 = s_3 s_1 s_3 s_5 s_8 \tag{14}$$

However, only one of the two permutations has s_5 in it, contradicting Lemma 5, and completing the proof.

Corollary 15.

$$a_{w,\lambda} \leq f^{\lambda} \tag{15}$$

Proof. This is immediate from Theorem 4. ☐

By Corollary 15,

$$\mathsf{EG}(w) = \sum_{|\lambda|=\ell(w)} a_{w,\lambda} \leq \sum_{|\lambda|=\ell(w)} f^{\lambda} = \mathsf{inv}(n), \tag{16}$$

where the last equality is a consequence of the Schensted correspondence (for example, in [9], Corollary 7.13.9).

3. Classification of the maximizers of (2)

For a Young diagram of shape λ, define $w \in S_\infty$ to be λ-*maximal* if $a_{w,\lambda} = f^{\lambda}$. Note that this is equivalent to saying that $\mathsf{EG}(w, \lambda)$ and $\mathsf{SYT}(\lambda)$ are equally-sized sets. We now classify which w are λ-maximal for each fixed λ.

Theorem 16. *Consider a Young diagram of shape λ, and let w be a permutation.*

1. *If λ has only one row, w is λ-maximal if and only if there exists $i_1 < i_2 < \cdots < i_{|\lambda|}$ such that $w = s_{i_{|\lambda|}} s_{i_{|\lambda|-1}} \cdots s_{i_1}$.*
2. *If λ has only one column, w is λ-maximal if and only if $w = s_{i_1} s_{i_2} \cdots s_{i_{|\lambda|}}$ for some $i_1 < i_2 < \cdots < i_{|\lambda|}$.*
3. *If λ has more than one row and more than one column, w is λ-maximal if and only if $\ell(w) = |\lambda|$ and w is totally commutative.*

Proving Theorem 16 requires a few lemmas:

Lemma 17. *If w is totally commutative, then it is λ-maximal for all $|\lambda| = \ell(w)$.*

Proof. Let $i_1, \ldots, i_k$ be as in the definition of totally commutative. Then by (4),

$$s_{i_{\sigma(1)}} s_{i_{\sigma(2)}} \cdots s_{i_{\sigma(\ell(w))}} \in \mathsf{Red}(w) \text{ for all } \sigma \in S_{\ell(w)}. \tag{17}$$

For any $T \in \mathsf{SYT}(\lambda)$, replacing the label k with the k^{th} smallest element of $\mathsf{supp}(w)$ turns T into an element $T' \in \mathsf{EG}(w, \lambda)$. This mapping $T \mapsto T'$ is clearly an injection, so this and Corollary 15 combine to say that $a_{w,\lambda} = f^{\lambda}$. ☐

In [11], B. E. Tenner described *boolean permutations* as permutations with the property that $|\mathsf{supp}(w)| < \ell(w)$.

Lemma 18. *Boolean permutations are not λ-maximal for any λ.*

Proof. Assume for the sake of contradiction that there exists some Young diagram of shape λ such that w is λ-maximal. Fix an arbitrary $U \in \mathsf{EG}(w, \lambda)$. Since $|\mathsf{supp}(w)| < \ell(w) = |\lambda|$, there exists (x, y) and (c, d) such that $U(x, y) = U(c, d)$. Since U is strictly increasing on rows and columns, without loss of generality (c, d) is strictly northeast of (x, y), and in particular λ must have more than one row and more than one column. As a result, $(\mathsf{std}(U))(x, y) < (\mathsf{std}(U))(c, d)$. This is a contradiction, as then no element of $\mathsf{EG}(w, \lambda)$ maps to $S \in \mathsf{SYT}(\lambda)$, the unique element of $\mathsf{SYT}(\lambda)$ where cells are labeled 1 through $|\lambda|$ by going from left to right and top to bottom, but since w is λ-maximal, $\mathsf{std} : \mathsf{EG}(w, \lambda) \to \mathsf{SYT}(\lambda)$ is an injection between two equally sized finite sets by Theorem 4, so it should be a surjection. □

Proof of Theorem 16: The definition of λ-maximal immediately implies (1) and (2). Similarly, the reverse direction of (3) follows from Lemma 17. Therefore, the rest of this proof is devoted to proving the forward direction of (3).

Let λ have more than one row and more than one column, and let $w \in S_\infty$ be λ-maximal. By definition, $\ell(w) = |\lambda|$, and since $\mathsf{std} : \mathsf{EG}(w, \lambda) \to \mathsf{SYT}(\lambda)$ is an injection between two finite sets of the same size by Theorem 4, it is a bijection, and so std^{-1} is well-defined.

By Lemma 18, $\mathsf{supp}(w) = \{i_1, i_2, \ldots, i_{\ell(w)}\}$, where we can say $i_1 < i_2 < \cdots < i_{\ell(w)}$ without loss of generality. Therefore, std maps the label i_k to k and std^{-1} maps the label k to i_k for each k. Now assume for the sake of contradiction that w is not totally commutative. This means that $m := \min\{j : i_j + 1 = i_{j+1}\}$ exists.

Recall the definition of $\mathsf{Red}(U)$ for $U \in \mathsf{EG}(w, \lambda)$ from Definition 1.

Claim 19. *If $T \in \mathsf{SYT}(\lambda)$ is such that s_{i_m} occurs before $s_{i_{m+1}}$ in $\mathsf{Red}(\mathsf{std}^{-1}(T))$, then s_{i_m} occurs before $s_{i_{m+1}}$ in $\mathsf{Red}(\mathsf{std}^{-1}(T'))$ for all other $T' \in \mathsf{SYT}(\lambda)$.*

Proof. This follows immediately from Lemma 6 and the fact that each simple transposition occurs at most once in each element of $\mathsf{Red}(w)$. □

There are three cases to consider: $m = 1$, $m = |\lambda| - 1$ and λ is a rectangle, and the case where neither of the above is true.

Case 1: $(m = 1)$ Let $T, T' \in \mathsf{SYT}(\lambda)$ be such that $T(2, 1) = 2$ and $T'(1, 2) = 2$. As a result, because $T(1, 1) = T'(1, 1) = 1$, s_1 occurs before s_2 in $\mathsf{Red}(\mathsf{std}^{-1}(T))$, but s_2 occurs before s_1 in $\mathsf{Red}(\mathsf{std}^{-1}(T'))$. This contradicts Claim 19.

Case 2: $(m = |\lambda| - 1$ and λ is rectangular$)$ Say that λ is a $a \times b$ rectangle so that $m = ab - 1$. Let $T, T' \in \mathsf{SYT}(\lambda)$ be such that $T(a - 1, b) = m$ and

$T'(a, b-1) = m$. As a result, because $T(a, b) = T'(a, b) = m+1$, s_m occurs before s_{m+1} in $\mathsf{Red}(\mathsf{std}^{-1}(T))$, but s_{m+1} occurs before s_m in $\mathsf{Red}(\mathsf{std}^{-1}(T'))$. This once again contradicts Claim 19.
Case 3: (Neither Case 1 nor Case 2) There exists some $T \in \mathsf{std}(\lambda)$ such that the cell labeled m in T (denoted (a, b)) is strictly northeast of the cell $m+1$ in T (denoted (c, d)). From this, let $T' \in \mathsf{std}(\lambda)$ be identical to T except that $T'(a, b) = m+1$ and $T'(c, d) = m$. As before, s_m occurs before s_{m+1} in $\mathsf{Red}(\mathsf{std}^{-1}(T))$, but s_{m+1} occurs before s_m in $\mathsf{Red}(\mathsf{std}^{-1}(T'))$, contradicting Claim 19. This completes the proof. □

The above theorem allows us to characterize the permutations that maximize the Edelman-Greene statistic.

Corollary 20. $\mathsf{EG}(w) = \mathsf{inv}(\ell(w))$ *if and only if* w *is totally commutative.*

Proof. The reverse direction follows from Lemma 17. For the forward direction, consider three cases, based on the size of $\ell(w)$.
Case 1: ($\ell(w) = 1$) Any permutation with $\ell(w) = 1$ is totally commutative, making this case clear.
Case 2: ($\ell(w) = 2$) If $\ell(w) = 2$, then Lemma 18 says that $\mathsf{supp}(w) = \{i_1, i_2\}$, and $s_{i_1}s_{i_2} = s_{i_2}s_{i_1}$, so they commute and so w is totally commutative.
Case 3: ($\ell(w) \geq 3$) There exists some λ with $|\lambda| = \ell(w)$ and λ having at least two rows and at least two columns. Since $\mathsf{EG}(w) = \mathsf{inv}(\ell(w))$, w must be λ-maximal, and so by Theorem 16, w must be totally commutative. □

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